Contributions to Discrete Mathematics

Volume 19, Number 2, Pages 36–55
ISSN 1715-0868

LATTICE PATHS IN CORRIDORS AND CYCLIC CORRIDORS

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Abstract. In this paper we use discrete Fourier transform and generating functions to count families of paths of a given length in a corridor. For example, we count Motzkin paths, colored Motzkin paths, Dyck paths, and Schröder paths.

1. Introduction and Notation

A *corridor* is a lattice that is within a strip bounded by the lines $x = 0, y = a$, and $y = b$, where $a < b$ are integers. In this paper we explore the behaviour of lattice paths within a corridor. For example, the corridor depicted in Figure 1 is formed by all integers points bounded by the lines $x = 0, y = 1$ and $y = 4$. Note that the figure also depicts a Motzkin path within the corridor. We say that the corridor is *cyclic* if the points that are on $y = a$ are also on $y = b$. Intuitively, the cyclic corridor can be thought as a circular cylinder.

A *Motzkin path* of length n is a lattice path in the first quadrant of the xy-plane from the point $(0, 0)$ to $(n, 0)$ using up-steps $U = (1, 1)$, horizontal steps $H = (1, 0)$, and down-steps $D = (1, -1)$. The Motzkin paths of length n are enumerated by the Motzkin numbers

$$m_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} c_k,$$

where

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

are the well-known Catalan numbers. A *colored* (a, b, c)*-Motzkin path* is a Motzkin path such that the up-steps, horizontal steps and down-steps are labeled or colored by a colors, b colors, and c colors, respectively (cf. [9]).

Received by the editors October 20, 2021, and in revised form May 4, 2022.
2000 *Mathematics Subject Classification.* 05A15, 05A19.
Key words and phrases. Corridor, lattice path, Dyck path, generating functions.

The *weight* of a colored (a, b, c)-Motzkin path is the product of colors (or weights) assigned to each step of the path (cf. [8]).

A *colored corridor* (a, b, c)*-Motzkin path* —*corridor Motzkin path*, for simplicity— of length n, is a colored (a, b, c)-Motzkin path within the corridor $[0, n] \times [1, h]$, from the initial point $(0, \ell)$ to the point (n, k). Let $\mathcal{M}_n^h(\ell, k)$ denote the set of all corridor Motzkin paths of length n from $(0, \ell)$ to the point (n, k) and let $\mathcal{M}^h(\ell, k) = \cup_{n\geq 0}\mathcal{M}_n^h(\ell, k)$. Let $m_n^h(\ell, k)$ denote the sum of the weights of all corridor Motzkin paths in $\mathcal{M}_n^h(\ell, k)$. For example, Figure 1 shows a corridor Motzkin path in $\mathcal{M}_{10}^4(2, 3)$ of weight $a^4b^3c^3$.

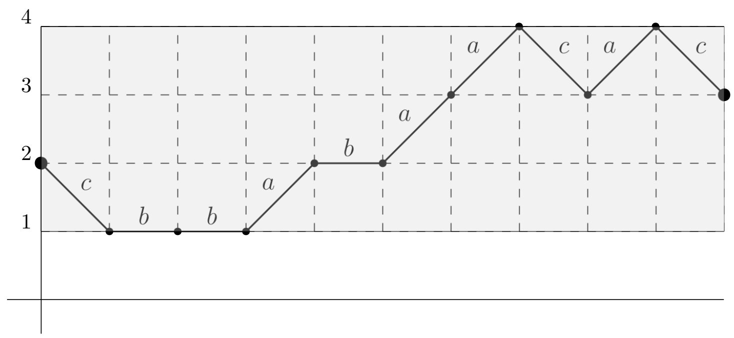

FIGURE 1. Corridor Motzkin path in $\mathcal{M}_{10}^4(2, 3)$ of weight $a^4b^3c^3$.

The corridor Motzkin paths have been studied extensively for the case $a = c = 1$ and $b = 0$. In this case, the paths are called *Dyck path in strips*, see for example [1, 2, 3, 4, 5, 6].

In this paper we use the Fourier method, introduced by Ault and Kicey [1, 2] and generating functions [7], to study lattice paths in strips. For example, we use a system of generating functions in two variables to count the total number Motzkin paths of length n that are within a corridor. We also use the discrete Fourier transform to count the number of Schröder paths that are within a corridor. In the end of the paper we give an approach to the paths in cyclic corridors using elementary number theory.

2. Colored Motzkin Path in Corridors

In this section we count the total number of corridor Motzkin paths of length n that are in a corridor. That is, we count all Motzkin paths, fully contained in a given corridor, starting at $(0, \ell)$ and ending at (n, k), where $1 \leq \ell, k \leq h$. To do this counting we solve a system of equations formed by generating functions. In the end of the section we give a connection between this counting and graph theory.

The length of a corridor Motzkin path P is denoted by $|P|$ and the number of horizontal steps of P is denoted by $\rho(P)$. We use $A_h^{(\ell,k)}(x, y)$ to denote the bivariate generating function defined by

$$A_h^{(\ell,k)}(x,y) := \sum_{P\in\mathcal{M}^h(\ell,k)} x^{|P|}y^{\rho(P)}.$$

Notice that the coefficient $[x^n y^s]A_h^{(\ell,k)}(x,y)$ is the sum of the weights of all corridor Motzkin paths in $\mathcal{M}_n^h(\ell,k)$ with exactly s horizontal steps.

We use $T_h(a,b,c)$ to denote the following tridiagonal matrix of size $h\times h$.

$$\begin{bmatrix} 1-bxy & -cx & & & & \\ -ax & 1-bxy & -cx & & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots \\ & & & & -ax & 1-bxy & -cx \\ & & & & & -ax & 1-bxy \end{bmatrix}_{h\times h}. \tag{2.1}$$

Moreover, we denote by $e_h(i)$ the column vector of size $h\times 1$ with a 1 in the i-th position and 0's everywhere.

Theorem 2.1. *Let $1\le \ell,k\le h$. Then*

$$A_h^{(\ell,k)}(x,y) = e_h(k)^T\cdot T_h(a,b,c)^{-1}\cdot e_h(\ell).$$

Proof. Let P be a path in $\mathcal{M}^h(\ell,k)$, with $h\ge 3$. For this proof there are four cases to consider (see Figure 2).

(1) $\ell > k = 1$. Here we have that P is non-empty and that the y-coordinate of the last point of P is 1. Consequently, the last step of P can be horizontal or a down-step. Therefore, we have the functional equation

$$A_h^{(\ell,1)}(x,y) = bxyA_h^{(\ell,1)}(x,y) + cxA_h^{(\ell,2)}(x,y)$$

(2) $2\le k\le h-1$ and $k\ne\ell$. This implies that P is non-empty and the last step of P can be an up-step, a horizontal or a down-step. So, we have the functional equation

$$A_h^{(\ell,k)}(x,y) = axA_h^{(\ell,k-1)} + bxyA_h^{(\ell,k)}(x,y) + cxA_h^{(\ell,k+1)}(x,y).$$

(3) $2\le k\le h-1$ and $k=\ell$. In this case, the y-coordinate of the first point and the last point are the same. Note that in this case the path P can be empty. Hence, we have the functional equation

$$A_h^{(\ell,\ell)}(x,y) = 1 + axA_h^{(\ell,\ell-1)}(x,y) + bxyA_h^{(\ell,\ell)}(x,y) + cxA_h^{(\ell,\ell+1)}(x,y).$$

(4) $1 < k = h$. Similarly to the first case, we obtain the functional equation

$$A_h^{(\ell,h)}(x,y) = axA_h^{(\ell,h-1)}(x,y) + bxyA_h^{(\ell,h)}(x,y).$$

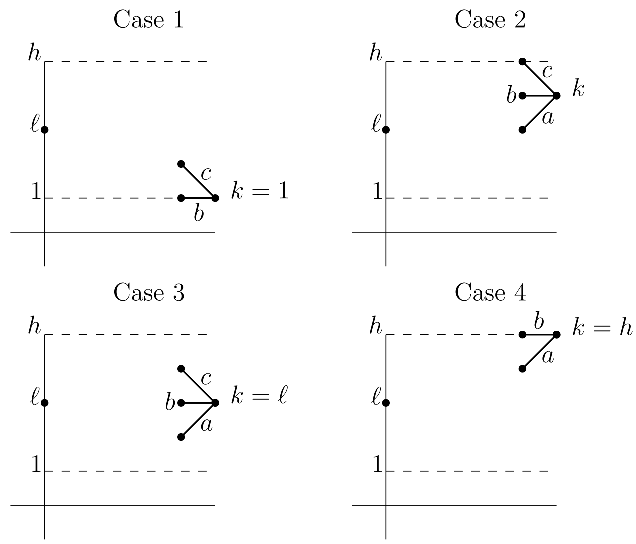

FIGURE 2. Decomposition of a corridor Motzkin path.

Combining the four functional equations above, we obtain the system of equations $T_h(a,b,c) \times \mathbb{A}_{h,\ell} = e_h(\ell)$, where $\mathbb{A}_{h,\ell} := [A_h^{(\ell,k)}(x,y)]_{h\times 1}$. Let $\mathbb{A}_h$ denote the matrix, whose (i,j)-th entry is given by the generating function $A_h^{(i,j)}(x,y)$, for each $1 \le i,j \le h$. It is easy to see that $T_h(a,b,c)\mathbb{A}_h = I_h$, where I_h is the identity matrix of size $h \times h$. Consequently, the generating function $A_h^{(\ell,k)}(x,y)$ is given by $A_h^{(\ell,k)}(x,y) = e_h(k)^T \cdot T_h(a,b,c)^{-1} \cdot e_h(\ell)$. □

For example, with $h = 4, \ell = 3$, and $k = 2$, we have

$$\begin{aligned} A_4^{(3,2)}(x,y) &= e_4(2)^T \cdot T_4(a,b,c)^{-1} \cdot e_4(3) \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1-bxy & -cx & 0 & 0 \\ -ax & 1-bxy & -cx & 0 \\ 0 & -ax & 1-bxy & -cx \\ 0 & 0 & -ax & 1-bxy \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \end{aligned}$$

This is equal to

$$\frac{cx - 2bcx^2y + b^2cx^3y^2}{1 - 3acx^2 + a^2c^2x^4 - 4bxy + 6abcx^3y + 6b^2x^2y^2 - 3ab^2cx^4y^2 - 4b^3x^3y^3 + b^4x^4y^4},$$

In particular, if we take the colors $(a,b,c) = (3,1,1)$, we have the generating function

$$\begin{aligned} &A_4^{(3,2)}(x,y) \\ =&\frac{x - 2x^2y + x^3y^2}{1 - 9x^2 + 9x^4 - 4xy + 18x^3y + 6x^2y^2 - 9x^4y^2 - 4x^3y^3 + x^4y^4} \\ =&x + 2yx^2 + \mathbf{(9 + 3y^2)}x^3 + (36y + 4y^3)x^4 + \cdots. \end{aligned}$$

Figure 3 shows the colored corridor $(3,1,1)$-Motzkin paths in $\mathcal{M}_3^4(3,2)$ corresponding to the bold coefficient in the above series.

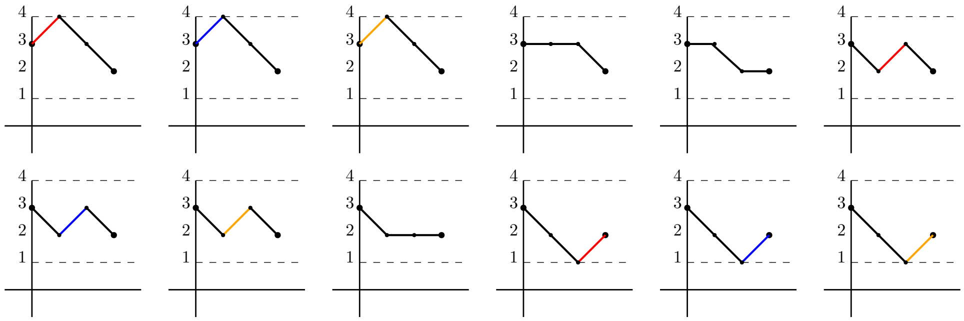

FIGURE 3. The corridor Motzkin paths in $\mathcal{M}_3^4(3,2)$ with the colors $(a,b,c) = (3,1,1)$.

We now illustrate the symmetries satisfied by the previous families of colored corridor Motzkin paths:

- *Symmetry.* Consider the case $a = c$, i.e., corridor paths where the same number of color choices is available for up-steps and down-steps. Hence, the tridiagonal matrix $T_h(a,b,a)$ is symmetric, as is its inverse matrix. We conclude that $A_h^{(\ell,k)}(x,y) = A_h^{(k,\ell)}(x,y)$, as the components of such matrix are the generating functions associated to these path families. From this relation follows that the coefficients of $A_h^{(\ell,k)}(x,1)$ and $A_h^{(k,\ell)}(x,1)$ coincide when $a = c$. We deduce that $m_n^h(\ell,k) = m_n^h(k,\ell)$, which is also readily seen by reversing the paths' direction as shown in Figure 4 (left).
- *Central symmetry.* Let $a = c$. By reflecting any path across the line $y = \frac{h-1}{2}$ as shown in Figure 4 (right), we have $m_n^h(\ell,k) = m_n^h(1+h-\ell, 1+h-k)$.

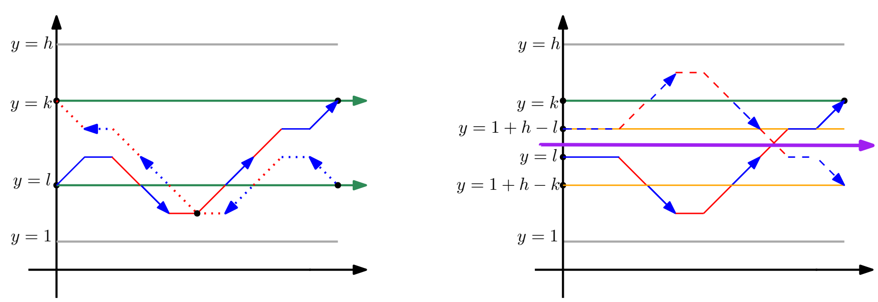

FIGURE 4. Directional and central symmetries of corridor Motzkin paths.

2.1. **Relationship with a graph walk.** Let $L_h(a,b,c)$ denote the graph with vertices $\{1,\dots,h\}$, where $h_1 \xrightarrow{w} h_2$ is a directed edge of $L_h(a,b,c)$ of weight $w=a$ if $h_2=h_1+1$, of weight $w=c$ if $h_2=h_1-1$, and weight $w=b$ if $h_1=h_2$. Figure 5 shows $L_4(a,b,c)$.

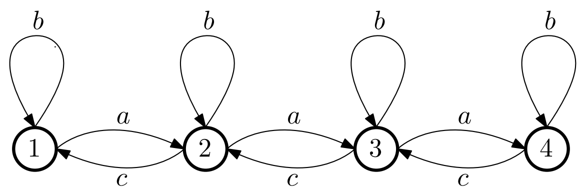

FIGURE 5. Graph $L_4(a,b,c)$.

Let $P \in \mathcal{M}_n^h(\ell,k)$, we define a weighted directed-walk (from the point of view of graph theory), using $L_h(a,b,c)$. The vertices of the weighted directed-walk are the y-coordinates of the ending points of the steps of P. So, $y_1 \xrightarrow{w} y_2$ is a directed edge if and only if y_1 and y_2 are y-coordinates of an initial point (from left to right) and the ending point of a step S in P, where w is the weight of S. For example, we see that the directed-walk associated to the path in Figure 1 is given by

$$2 \xrightarrow{c} 1 \xrightarrow{b} 1 \xrightarrow{b} 1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 3 \xrightarrow{a} 4 \xrightarrow{c} 3 \xrightarrow{a} 4 \xrightarrow{c} 3.$$

We observe that the number of colored corridor (a,b,c)-Motzkin paths of length n is equivalent to the sum of the weights of all paths on the line graph $L_h(a,b,c)$.

3. ENUMERATION OF CORRIDOR MOTZKIN PATHS VIA FOURIER METHODS

In this section we turn to the enumeration of corridor Motzkin paths via the discrete Fourier transform, a method discussed in [1]. We give a theorem that evaluates $m_n^h(\ell,k)$ for any $k\in\{1,\dots,h\}$.

The following definitions and notation can be found in [1]. However, here we write it again to make the section self-contained. Let $(u(x))_{x\in\mathbb{Z}}$ be an N-periodic sequence of complex numbers. The *discrete Fourier transform* (DFT) of u, denoted by $\mathcal{F}[u]$, is an N-periodic sequence of complex numbers $(U(\omega))_{\omega\in\mathbb{Z}}$ given by

$$\mathcal{F}[u](\omega) = U(\omega) = \sum_{x=0}^{N-1} u(x) e^{-\frac{2\pi i}{N}x\omega}.$$

In the same manner, the *inverse discrete Fourier transform* of a sequence $(U(\omega))_{\omega\in\mathbb{Z}}$ is defined by

$$\mathcal{F}^{-1}[U](x) = u(x) = \frac{1}{N}\sum_{\omega=0}^{N-1} U(\omega) e^{\frac{2\pi i}{N}\omega x}.$$

It is well-known that $\mathcal{F}$ is a linear operator on the space of periodic sequences, whose inverse is $\mathcal{F}^{-1}$. Thus, $\mathcal{F}[u]$ is a periodic sequence and for any integer k this holds

$$\begin{aligned}\mathcal{F}[u(x-k)](\omega) &= \sum_{x=0}^{N-1} u(x-k)e^{-\frac{2\pi i x\omega}{N}} \\ &= \sum_{x=-k}^{-k+N-1} u(x)e^{-\frac{2\pi i(x+k)\omega}{N}} \\ &= e^{-\frac{2k\pi i\omega}{N}} \sum_{x=-k}^{-k+N-1} u(x)e^{-\frac{2\pi i x\omega}{N}} \\ &= e^{-\frac{2k\pi i\omega}{N}} \mathcal{F}[u](\omega).\end{aligned}$$

In order to apply Fourier methods as in [1], a symmetry on the colors vector v is required. The condition $a = c$ allows us to obtain a recursive function $V_n(x)$ called *state* with initial value $V_0(x)$ and for $n \geq 1$

$$V_n(x) = aV_{n-1}(x-1) + bV_{n-1}(x) + aV_{n-1}(x+1).$$

Theorem 3.1. *Let n, k, and h be positive integers, with $1 \leq k \leq h$. Let us define*

$$g(\omega) := 2a\cos\left(\frac{\pi\omega}{d}\right) + b,$$

where $d = h+1$. Then the number of corridor Motzkin paths ending at (n, k) is given by

$$m_n^h(\ell, k) = \frac{1}{d}\sum_{\omega=0}^{2d-1} [g(\omega)]^n \sin\left(\frac{\pi\omega k}{d}\right)\sin\left(\frac{\pi\omega\ell}{d}\right).$$

Proof. Let $V_n(x)$ be the state function of the h-corridor, i.e., $V_n(x)$ is the number of corridor Motzkin paths of length n whose height (y-coordinate of their final point) is x. As in the previous remark, the following recurrence relation holds for $x \in \mathbb{N}_0$.

$$V_n(x) = aV_{n-1}(x-1) + bV_{n-1}(x) + aV_{n-1}(x+1).$$

When a path is reflected towards $y = 0$ the roles of U and D are swapped in the recurrence relation. Since the number of available colors for both U and D is a, the number of paths is preserved by this reflection, and therefore, the recurrence remains valid for $x < 0$.

Now, the initial state function satisfies

$$V_0(x) := \delta(x) = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{if } x \neq 0; \end{cases}$$

from which the state functions $V_n(x)$ can be obtained recursively. Notice that $V_n(x)$ is a $2d$-periodic complex sequence, whose Fourier transform can be obtained in this manner

$$\begin{aligned}\mathcal{F}[V_{n+1}(x)](\omega) =& a\mathcal{F}[V_n(x-1)](\omega) + b\mathcal{F}[V_n(x)](\omega) + a\mathcal{F}[V_n(x+1)](\omega)\\ =&(ae^{\frac{-\pi i\omega}{d}} + b + ae^{\frac{\pi i \omega}{d}})\mathcal{F}[V_n(x)](\omega)\\ =&\left(2a\cos\left(\frac{\pi\omega}{d}\right) + b\right)\mathcal{F}[V_n(x)](\omega)\\ =&g(\omega)\mathcal{F}[V_n(x)](\omega),\end{aligned}$$

where $g(\omega) = 2a\cos\left(\frac{\pi\omega}{d}\right) + b$.

The computations rely on Euler's formula, as well as the well-known property

$$\mathcal{F}[V_n(x-k)](\omega) = e^{\frac{-2k\pi i\omega}{2d}}\mathcal{F}[V_n(x)](\omega).$$

Furthermore, as specified in [1], the Fourier transform of the admissible delta function is $\mathcal{F}[V_0(x)](\omega) = -2i\sin\frac{\pi\omega\ell}{d}$. It follows that $\mathcal{F}[V_n(x)](\omega) = [g(\omega)]^n\left(-2i\sin\frac{\pi\omega\ell}{d}\right)$. So, we can recover $V_n(x)$ via the Fourier inversion transform. Thus,

$$\begin{aligned}V_n(x) &= \mathcal{F}^{-1}[\mathcal{F}[V_n(x)](\omega)](x)\\ &= \mathcal{F}^{-1}\left[[g(\omega)]^n\left(-2i\sin\frac{\pi\omega\ell}{d}\right)\right](x)\\ &= \frac{1}{2d}\sum_{\omega=0}^{2d-1}[g(\omega)]^n\left(-2i\sin\frac{\pi\omega\ell}{d}\right)e^{\frac{\pi i\omega x}{d}}.\end{aligned}$$

The real part of each summand is $[g(\omega)]^n\sin\left(\frac{\pi\omega x}{d}\right)\sin\left(\frac{\pi\omega\ell}{d}\right)$, thus the number of paths, i.e., the real part of this sum, corresponds to

$$m_n^h(\ell,x) = \frac{1}{d}\sum_{\omega=0}^{2d-1}[g(\omega)]^n\sin\left(\frac{\pi\omega x}{d}\right)\sin\left(\frac{\pi\omega\ell}{d}\right).$$

In particular, if we let $x = k$ for any $k\in\{1,\dots,h\}$, the result follows. □

The previous theorem provides an explicit formula for the number of corridor Motzkin paths. For example, let $h = 4, (a,b,c) = (2,3,2)$, and $\ell = 2$, and applying the previous theorem we obtain the values in Table 1. In Figure 6, are all different paths counted by $\mathcal{M}_3^4(2,3)$, with colors $(a,b,c) = (2,3,2)$. Thus, since $a = c = 2$, there are eight paths with steps UUD starting at $(0,2)$ and ending at $(3,3)$. Similarly we see that the total number of paths are 8, 18, 8, 18, 18, and 8, respectively. Thus, the total number of colored corridor (2,3,2)-Motzkin paths ending at $(3,2)$ is 78.

4. Schröder Corridor Paths

In this section we introduce the Schröder corridor path. The arguments used here are similar to the previous sections. A *Schröder path* of length n is a lattice path in the first quadrant of the xy-plane from $(0,0)$ to the point $(n,0)$ using up-steps $U = (1,1)$, horizontal steps $H_2 = (2,0)$, and

n	0	1	2	3	4	5	6	7	8
$k=4$	0	0	4	36	264	1800	11852	76524	488208
$k=3$	0	2	12	78	504	3226	20484	129318	813168
$k=2$	1	3	17	99	593	3603	22081	136083	841825
$k=1$	0	2	12	70	408	2410	14436	87470	534576

TABLE 1. Values of $\mathcal{M}_n^4(2,k)$ for $1 \le k \le 4$, and $n = 0, 1, \ldots, 8$.

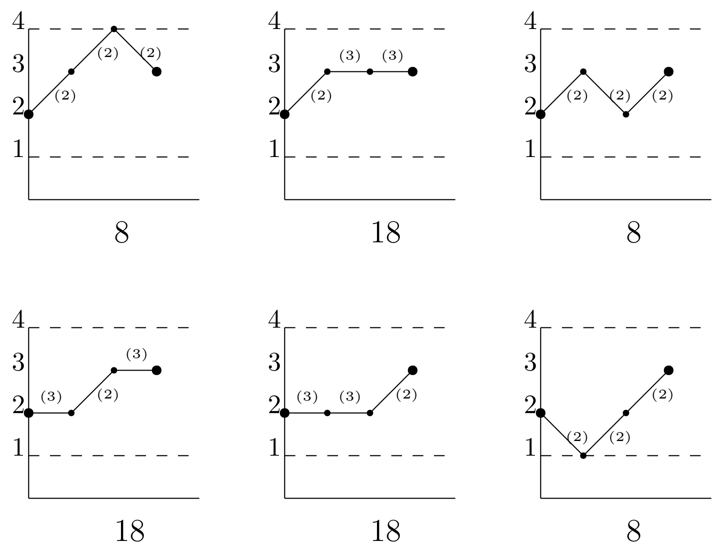

FIGURE 6. Different paths counted by $\mathcal{M}_3^4(2,3)$, with colors $(a,b,c) = (2,3,2)$.

down-steps $D = (1,-1)$. The Schröder paths of length n are enumerated by the sequence A006318 (see [10]).

Given a positive integer h, a *corridor Schröder path* of length n is a Schröder path within the corridor $[0,n] \times [1,h]$, from the initial point $(0,\ell)$ to the ending point (n,k), such that $1 \le \ell, k \le h$. We use $\mathcal{S}_n^h(\ell,k)$ to denote the set of all corridor Schröder paths of length n from the point $(0,\ell)$ to the point (n,k), and we let $\mathcal{S}^h(\ell,k) = \cup_{n\ge 0}\mathcal{S}_n^h(\ell,k)$ and let $s_n^h(\ell,k)$ to be the total number of all corridor Schröder paths in $\mathcal{S}_n^h(\ell,k)$.

We apply the discrete Fourier transform to give a trigonometric expression. For simplicity, we use $s_n(x)$ to denote the number of Schröder paths within the corridor $[0,n] \times [1,h]$, starting at the point $(0,\ell)$ and ending at the point (n,x). Notice that

$$s_n(x) = s_{n-1}(x-1) + s_{n-2}(x) + s_{n-1}(x+1).$$

Thus, letting $S_n = \mathcal{F}[s_n]$ and $\theta = \frac{\pi\omega}{d}$, we have the relation

$$\begin{aligned} S_n(\omega) &= e^{-i\theta} S_{n-1}(\omega) + S_{n-2}(\omega) + e^{i\theta} S_{n-1}(\omega) \\ &= 2\cos(\theta) S_{n-1}(\omega) + S_{n-2}(\omega), \end{aligned}$$

where $d = h+1$. In order to apply Fourier methods to this family of paths, we need to solve the previous recurrence relation and construct admissible versions s_0, s_1 of the initial state functions. Since

$$\alpha_1(\omega) = \cos\theta + \sqrt{1+\cos^2\theta} \quad \text{and} \quad \alpha_2(\omega) = \cos\theta - \sqrt{1+\cos^2\theta}$$

are the (real) roots of the characteristic polynomial $p(t) = t^2 - 2(\cos\theta)t - 1$ of $S_n(\omega)$, we have $S_n(\omega) = K_1\alpha_1^n(\omega) + K_2\alpha_2^n(\omega)$, for some real numbers K_1 and K_2 (to be found using the given initial values).

The initial admissible states $s_0(x)$ and $s_1(x)$ can be constructed as:

$$s_0(x) = \begin{cases} 1, & \text{if } x = \ell; \\ 0, & \text{otherwise}; \end{cases} \qquad s_1(x) = \begin{cases} 1, & \text{if } x = \ell \pm 1; \\ 0, & \text{otherwise}. \end{cases}$$

That is, $s_0(x) = \delta_\ell(x)$ and $s_1(x) = \delta_{\ell-1}(x) + \delta_{\ell+1}(x)$, where $\delta_j(k) = 1$ if $k = j$ and 0 otherwise. Let $\Delta_m := \delta_m(x) - \delta_{-m}(x)$ (in [1] it is the admissible extension of the delta function). So, we can take $s_0(x) = \Delta_\ell(x)$ and $s_1(x) = \Delta_{\ell-1}(x) + \Delta_{\ell+1}(x)$ as the admissible extensions of s_0 and s_1 (odd and $2d-$periodic). Therefore,

$$\begin{aligned} &S_0(\omega) = \mathcal{F}[\Delta_\ell(x)](\omega) = -2i\sin(\ell\theta), \quad \text{and} \\ &S_1(\omega) = \mathcal{F}[\Delta_{\ell-1}(x) + \Delta_{\ell+1}(x)](\omega) = -2i\left[\sin([\ell-1]\theta) + \sin([\ell+1]\theta)\right], \end{aligned}$$

where $S_1(\omega)$ can be further simplified as

$$\begin{aligned} S_1(\omega) &= -2i[\sin(\ell\theta - \theta) + \sin(\ell\theta + \theta)] \\ &= -2i[2\sin(l\theta)\cos(\theta)] \\ &= -4i\sin(l\theta)\cos(\theta). \end{aligned}$$

By the previous reasoning, the values of K_1 and K_2 are given by the solution of this system of equations

$$\begin{cases} K_1 + K_2 &= S_0(\omega) \\ K_1\alpha_1 + K_2\alpha_2 &= S_1(\omega). \end{cases}$$

Hence, we obtain that $K_1 = -C_1(\omega)i$ and $K_2 = C_2(\omega)i$, where

$$C_1(\omega) = \frac{\alpha_1(\omega)\sin(\ell\theta)}{\sqrt{1+\cos^2\theta}}, \qquad C_2(\omega) = \frac{\alpha_2(\omega)\sin(\ell\theta)}{\sqrt{1+\cos^2\theta}},$$

are real functions of ω. For simplicity, we use $C_1 := C_1(\omega)$ and $C_2 := C_2(\omega)$. Now we deduce a formula for $s_n(x)$ via the Fourier inversion transform

$$\begin{aligned} s_n(x) &= \mathcal{F}^{-1}[K_1\alpha_1^n + K_2\alpha_2^n] \\ &= \mathcal{F}^{-1}[K_1\alpha_1^n] + \mathcal{F}^{-1}[K_2\alpha_2^n] \\ &= -i\mathcal{F}^{-1}[C_1\alpha_1^n] + i\mathcal{F}^{-1}[C_2\alpha_2^n]. \end{aligned}$$

Since $C_1\alpha_1^n$ and $C_2\alpha_2^n$ are purely real numbers, we have

$$\Re(-i\mathcal{F}^{-1}[C_1\alpha_1^n]) = \Re\left((-i)\frac{1}{2d}\sum_{w=0}^{2d-1} C_1\alpha_1^n e^{i\theta x}\right) = \frac{1}{2d}\sum_{w=0}^{2d-1} C_1\alpha_1^n \sin(\theta x)\,.$$

Similarly,

$$\Re(i\mathcal{F}^{-1}[C_2\alpha_2^n]) = -\frac{1}{2d}\sum_{w=0}^{2d-1} C_2\alpha_2^n \sin(\theta x)\,,$$

where $\Re(c)$ denotes the real part of the complex number c. We conclude that for $d = h+1$, that

$$\begin{aligned} s_n(x) &= \frac{1}{2d}\sum_{w=0}^{2d-1} C_1\alpha_1^n \sin(\theta x) - \frac{1}{2d}\sum_{w=0}^{2d-1} C_2\alpha_2^n \sin(\theta x) \\ &= \frac{1}{2d}\sum_{w=0}^{2d-1} (C_1\alpha_1^n - C_2\alpha_2^n)\sin(\theta x) \\ &= \frac{1}{2d}\sum_{w=0}^{2d-1} \frac{\sin(\ell\theta)\sin(\theta x)}{\sqrt{1+\cos^2\theta}}[\alpha_1^{n+1} - \alpha_2^{n+1}]. \end{aligned}$$

Therefore, $s_n(x)$ is given by

$$\frac{1}{2d}\sum_{w=0}^{2d-1} \frac{\sin(\ell\theta)\sin(\theta x)}{\sqrt{1+\cos^2\theta}}[(\cos\theta + \sqrt{1+\cos^2\theta})^{n+1} - (\cos\theta - \sqrt{1+\cos^2\theta})^{n+1}].$$

For example, using the above formula for $h = 4, \ell = 2, x = 3 = k$, we obtain that the first few values of the sequence $s_n^4(2,3)$ are (the nonzero terms of this sequence are in A289803)

$$0,\ 1,\ 0,\ 5,\ 0,\ 23,\ 0,\ 103,\ 0,\ 456,\ 0,\ 2009,\ 0,\ 8833,\ 0,\ 38803,\dots.$$

Figure 7 shows all the corridor Schröder paths corresponding to 23 in the above sequence.

We have obtained a formula for $s_n(x)$ in terms of trigonometric functions. However, for more general path families, this method is more complicated. For example, if the horizontal step increases in length, more work is required to solve the Fourier transform recurrence (the characteristic polynomial is of degree greater than 2) and applying inversion, the explicit sum turns out to be cumbersome.

5. Dyck Paths in Cyclic Corridors

In this section we discuss Dyck paths within a corridor with cyclic properties. That is, informally, we think that the points are in a lattice embedded in a circular cylinder where we identify $y = 1$ and $y = h$ as the same line. In Figure 8, we can see that the points in line $y = 1$ and the points in line

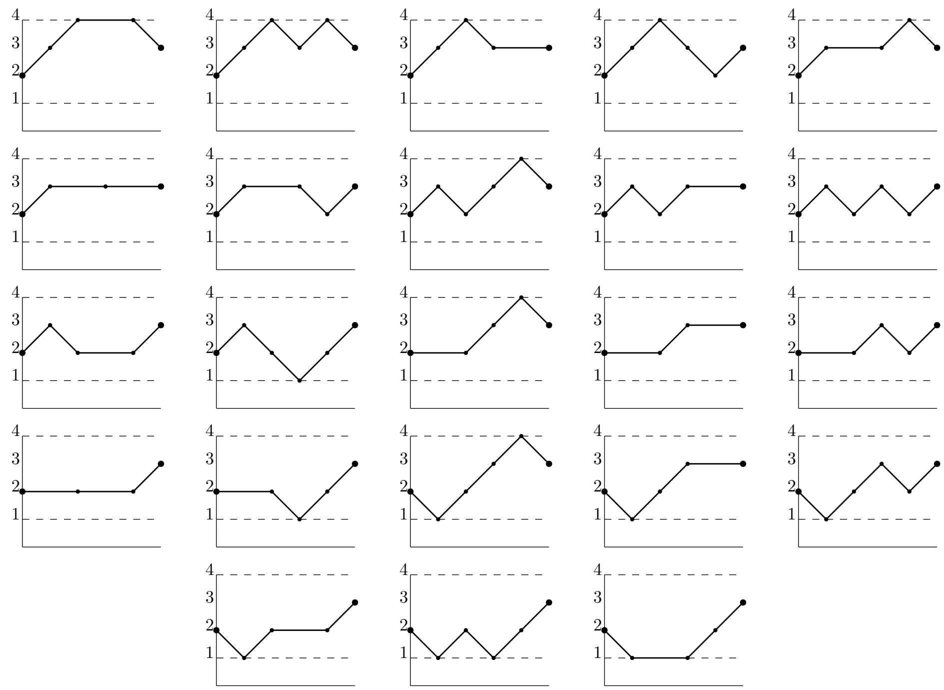

FIGURE 7. The corridor Schröder paths in $\mathcal{S}_5^4(2,3)$.

$y = 5$ are the same points. So, to avoid any ambiguity we only mark points in one of the two lines.

We would like to mention that the Section 2.6 in [1] is dedicated to cyclic corridors as well.

Let us now consider the problem of enumerating Dyck paths in a cyclic corridor. An *h-cyclic corridor Dyck path* is a Dyck path with step coordinates in $\{(a,b) : a \in \mathbb{N}_0 \text{ and } b \in \{1,2,\dots,h\}\}$. Let $(1, y_1, y_2, \dots, y_n)$ be the y-coordinates of an h-cyclic corridor Dyck path of length n, the j-th step is said to be a *teleport* if one of the following conditions hold:

- $j = 1$ and $y_1 = h - 1$,
- if $1 < j < n$ and $(y_{j-1}, y_j, y_{j+1}) = (h-1, h, 1)$ or $(y_{j-1}, y_j, y_{j+1}) = (2, 1, h-1)$.

We say that the points $(0,1)$ and (j, y_j) are *teleport* in the first and second cases, respectively. For example, Figure 8, shows a 5-cyclic corridor Dyck path of length 14. In this case, the path has four teleport steps $(0,1), (4,5), (11,5)$, and $(13,1)$ (identified in red). Note that $(6,1)$ is not a teleport point.

Let $t_{n,m}^{(h)}$ be the number of h-cyclic corridor Dyck paths of length n with exactly m teleport steps, from the initial point $(0,1)$. Let $\bar{t}_{n,m}^{(h)}$ be the auxiliary sequence that counts the number of h-cyclic corridor Dyck paths with initial step U (or equivalently, non-empty paths without an initial teleport

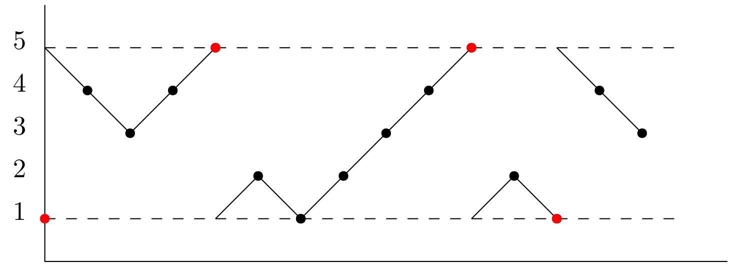

FIGURE 8. A 5-cyclic corridor Dyck path of length 14.

step). Notice that $t^{(h)}_{0,0} = 1$ and $\bar{t}^{(h)}_{0,0} = 0$. Figure 9, shows all 4-cyclic corridor Dyck path of length eight with exactly four teleport steps. That is, $t^{(4)}_{8,4} = 5$.

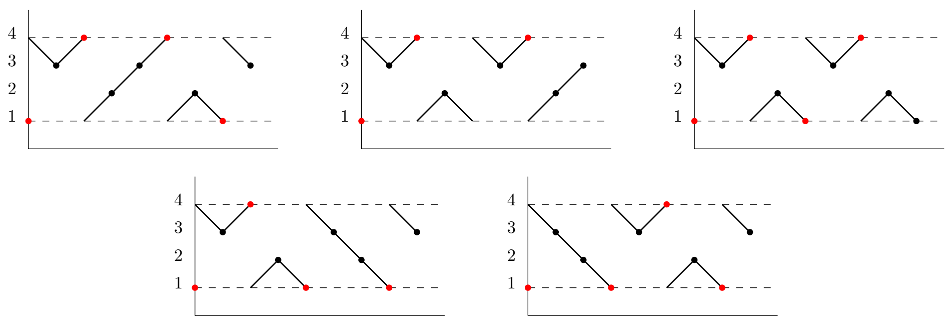

FIGURE 9. Lattice paths enumerated by $t^{(4)}_{8,4} = 5$.

We now give some lemmas that we need for the main theorem of this section. Let us first introduce some definitions. Let $\mathcal{C}^{(h)}$ be the set of h-cyclic corridor Dyck paths starting at the point $(0, 1)$, and let $\overline{\mathcal{C}}^{(h)}$ be the family of non-empty paths in $\mathcal{C}^{(h)}$ with initial step U. Now define four related m generating functions. For a given path P in $\mathcal{C}^{(h)}$, we use $|P|$ and $\texttt{Tel}(P)$ to denote the length and number of teleport steps of P.

$$T^{(h)}_m(x) := \sum_{n\geq 0} t^{(h)}_{n,m} x^n, \qquad \overline{T}^{(h)}_m(x) := \sum_{n\geq 0} \bar{t}^{(h)}_{n,m} x^n,$$

$$T^{(h)}(x, y) := \sum_{P\in\mathcal{C}^{(h)}} x^{|P|} y^{\texttt{Tel}(P)} = \sum_{n,m\geq 0} t^{(h)}_{n,m} x^n y^m, \quad \text{and}$$

$$\overline{T}^{(h)}(x, y) := \sum_{P\in\overline{\mathcal{C}}^{(h)}} x^{|P|} y^{\texttt{Tel}(P)} = \sum_{n,m\geq 0} \bar{t}^{(h)}_{n,m} x^n y^m.$$

Let $v^{(h)}_n(k)$ be the number of h-corridor Dyck paths starting at $(0, 1)$ and ending at (n, k). That is, $v^{(h)}_n(k) = m^h_n(1, k)$ with $a = 1 = c$ and $b = 0$, and

$c_n^{(h)} = \sum_{k=1}^{h} v_n^{(h)}(k)$. For the following lemma we consider the generating functions

$$Y^{(h)}(x) := \sum_{n\geq 0}(v_n^{(h)}(1) + v_n^{(h)}(h))x^n \quad \text{and} \quad G^{(h)}(x) = \sum_{n\geq 0} c_n^{(h)} x^n.$$

Lemma 5.1. *If $h \geq 4$, then the generating functions $Y^{(h)}(x)$ and $G^{(h)}(x)$ are given by these matrix equations*

$$Y^{(h)}(x) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \end{bmatrix}_h T_h(1,0,1)^{-1} e_h(1),$$

$$G^{(h)}(x) = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \end{bmatrix}_h T_h(1,0,1)^{-1} e_h(1),$$

where $T_h(1,0,1)$ is as defined in (2.1).

Proof. Let $B_k^{(h)}(x)$ be the generating function associated with the h-corridor Dyck paths with ending point (n,k). That is,

$$B_k^{(h)}(x) = \sum_{n\geq 0} v_n^{(h)}(k) x^n.$$

For any $1 < k < h$, a path enumerated by $B_k^{(h)}(x)$ comes from either a path ending at $(n, k-1)$ or a path ending at $(n, k+1)$. Hence,

$$B_k^{(h)}(x) = xB_{k-1}^{(h)}(x) + xB_{k+1}^{(h)}(x).$$

A path enumerated by $B_1^{(h)}(x)$ is either the empty path or comes from a path enumerated by $B_2^{(h)}(x)$. Analogously, the paths enumerated by $B_h^{(h)}(x)$ always come from a path enumerated by $B_{h-1}^{(h)}(x)$.

Therefore, we obtain the system of h linear equations:

$$\begin{aligned}
B_1^{(h)}(x) &= xB_2^{(h)}(x) + 1,\\
B_2^{(h)}(x) &= xB_1^{(h)}(x) + xB_3^{(h)}(x),\\
&\vdots\\
B_{h-1}^{(h)}(x) &= xB_{h-2}^{(h)}(x) + xB_h^{(h)}(x),\\
B_h^{(h)}(x) &= xB_{h-1}^{(h)}(x).
\end{aligned}$$

This system can be represented in matrix form as:

$$\begin{bmatrix} 1 & -x & & \\ -x & \ddots & \ddots & \\ & \ddots & \ddots & -x \\ & & -x & 1 \end{bmatrix}_{h\times h} \begin{bmatrix} B_1^{(h)}(x) \\ B_2^{(h)}(x) \\ \vdots \\ B_h^{(h)}(x) \end{bmatrix}_h = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since $Y^{(h)}(x) = B_1^{(h)}(x) + B_h^{(h)}(x)$ and $G^{(h)}(x) = B_1^{(h)}(x) + \cdots + B_n^{(h)}(x)$, we obtain the desired result. □

Theorem 5.2. *The bivariate generating functions* $T^{(h)}(x,y)$ *and* $\overline{T}^{(h)}(x,y)$ *are given by*

$$\overline{T}^{(h)}(x,y) = \frac{G^{(h)}(x)-1}{1-(Y^{(h)}(x)-1)y} \qquad \text{and} \qquad T^{(h)}(x,y) = 1+(1+y)\overline{T}^{(h)}(x,y).$$

Proof. Let P be a non-empty h-cyclic corridor Dyck path with exactly m teleport steps, whose first step is U. This path can be decomposed as $P_1P_2\cdots P_m$, where P_m is an h-cyclic corridor Dyck path and P_i, with $1 \le i \le m-1$, is a non-empty h-cyclic corridor Dyck path without teleport steps with ending point on either lines $y=1$ or $y=h$. See Figure 10 for a graphic representation.

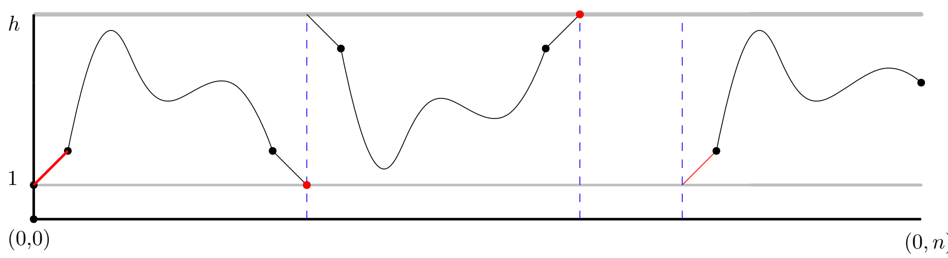

FIGURE 10. Decomposition of an h-cyclic corridor path with m teleport points.

From this decomposition we obtain the functional equation

$$\overline{T}_m^{(h)}(x) = (Y^{(h)}(x)-1)^m(G^{(h)}(x)-1).$$

Therefore,

$$\begin{aligned}\overline{T}^{(h)}(x,y) &= \sum_{m\ge 0} \overline{T}_m^{(h)}(x)y^m \\ &= \sum_{m\ge 0} (Y^{(h)}(x)-1)^m(G^{(h)}(x)-1)y^m \\ &= \frac{G^{(h)}(x)-1}{1-(Y^{(h)}(x)-1)y}.\end{aligned}$$

Let P be an h-cyclic corridor Dyck path with exactly m teleport steps. If the first step of P is U, then these kind of paths are counted by $\overline{T}^{(h)}(x,y)$. If the first step is U, then $(0,1)$ is a teleport step, counted by y. These paths are enumerated by $y\overline{T}^{(h)}(x,y)$. Therefore, we have the functional equation

$$T^{(h)}(x,y) = 1 + \overline{T}^{(h)}(x,y) + y\overline{T}^{(h)}(x,y).$$

□

From the above theorem we can obtain the generating functions:

$$T^{(2)}(x,y) = \frac{1}{1-x-xy},$$
$$T^{(3)}(x,y) = \frac{1+x+xy}{1-2x^2-2x^2y},$$
$$T^{(4)}(x,y) = \frac{1+xy}{1-x-x^2-x^2y},$$
$$T^{(5)}(x,y) = \frac{1+x-x^2+xy+x^2y}{1-3x^2-x^2y},$$
$$T^{(6)}(x,y) = \frac{1-x^2+xy}{1-x-2x^2+x^3-x^2y+x^3y}.$$

5.1. **Connection with the Fibonacci numbers.** The goal of this section is to analyze the case $h = 4$. This case has a special connection with the Fibonacci numbers F_n. Let

$$F(x) := \sum_{n\geq 0} F_n x^n = \frac{x}{(1-x-x^2)}$$

be the generating function of the Fibonacci sequence. Remember that $v_n^{(4)}(k)$ counts the number of 4-corridor Dyck paths starting at $(0,1)$ and ending at (n,k), and

$$c_n^{(4)} = \sum_{k=1}^{4} v_n^{(4)}(k).$$

In [1] it is proved that $c_n^{(4)} = F_{n+1}$, and therefore, $G^{(4)}(x) = 1/(1-x-x^2)$.

Proposition 5.3. *If $n \geq 0$, then $v_n^{(4)}(1) + v_n^{(4)}(4) = F_{n-1}$. Moreover, the generating function of this sequence is given by*

$$Y^{(4)}(x) := \sum_{n\geq 0}(v_n^{(4)}(1) + v_n^{(4)}(4))x^n = 1 + xF(x) = \frac{1-x}{1-x-x^2}.$$

Proof. These equations hold when $n \geq 1$

$$v_n^{(4)}(1) = v_{n-1}^{(4)}(2); \qquad v_n^{(4)}(2) = v_{n-1}^{(4)}(1) + v_{n-1}^{(4)}(3);$$
$$v_n^{(4)}(3) = v_{n-1}^{(4)}(4) + v_{n-1}^{(4)}(2); \qquad v_n^{(4)}(4) = v_{n-1}^{(4)}(3).$$

Therefore,

$$v_n^{(4)}(2) + v_n^{(4)}(3) = \sum_{\ell=1}^{4} v_{n-1}(\ell) = c_{n-1}^{(4)} = F_n.$$

Hence, $v_n^{(4)}(1) + v_n^{(4)}(4) = c_n^{(4)} - (v_n^{(4)}(2) + v_n^{(4)}(3)) = F_{n+1} - F_n = F_{n-1}$. The path of length $n = 0$ is the empty path. □

From Theorem 5.2 we obtain

$$\begin{aligned}
T^{(4)}(x,y) =& 1+(1+y)\left(\frac{G^{(4)}(x)-1}{1-(Y^{(4)}(x)-1)y}\right)\\
=& \frac{1+xy}{1-x-x^2-xy}\\
=& 1+x(y+1)+x^2(2y+2)+x^3\left(y^2+4y+3\right)\\
&+x^4\left(3y^2+8y+5\right)+x^5\left(y^3+8y^2+15y+8\right)\\
&+x^6\left(4y^3+19y^2+28y+13\right)\\
&+x^7\left(y^4+13y^3+42y^2+51y+21\right)\\
&+x^8\left(\mathbf{5y^4}+36y^3+89y^2+92y+34\right)+\cdots.
\end{aligned}$$

Notice that Figure 9 shows the 4-cyclic corridor Dyck paths of length 8 with exactly 4 teleport steps corresponding to the bold coefficient in the above series. That is, $t_{8,4}^{(4)}=5$. The array $[t_{n,k}^{(4)}]_{n,k\geq 0}$ corresponds to the array A119473. This array has a different combinatorial interpretation. The first few rows are

$$[t_{n,k}^{(4)}]_{n,k\geq 0}=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \dots\\
1 & 1 & 0 & 0 & 0 & 0 & \\
2 & 2 & 0 & 0 & 0 & 0 & \\
3 & 4 & 1 & 0 & 0 & 0 & \\
5 & 8 & 3 & 0 & 0 & 0 & \\
8 & 15 & 8 & 1 & 0 & 0 & \\
13 & 28 & 19 & 4 & 0 & 0 & \\
21 & 51 & 42 & 13 & 1 & 0 & \\
34 & 92 & 89 & 36 & 5 & 0 & \\
55 & 164 & 182 & 91 & 19 & 1 & \dots\\
\vdots & & & & & \vdots &
\end{pmatrix}$$

It is clear that $t_{n,0}^{(4)}=F_{n+1}$ $(n\geq 0)$. Moreover, the second column of the above array is given by $t_{n,1}^{(4)}=((n+4)F_n+2nF_{n-1})/5$. Indeed, expanding the generating function $T^{(4)}(x,y)$ we obtain

$$T^{(4)}(x,y)=\frac{1}{1-x-x^2}+\frac{x-x^3}{(1-x-x^2)^2}y+\frac{x^3(1-x^2)}{(1-x-x^2)^3}y^2+\cdots.$$

From the generating function of the Fibonacci sequence we obtain the expression for $t_{n,1}^{(4)}$.

6. An Approach with Elementary Number Theory

For simplicity, throughout this section we will work on the corridor $[0,n]\times[0,h]$. First, we observe that a teleport point with x-coordinate m $(0\leq m\leq$

n), can be interpreted as an identification by an equivalence relation between the points $(m, 0)$ and (m, h).

Following this idea, the lattice corridor $[0, n] \times [0, h]$ is naturally identified with the group $\mathbb{Z} \times \mathbb{Z}_h$, where $\mathbb{Z}_h$ is the set of integers modulo h. In order to go from the point $(0, 0)$ to (n, k) we need n total steps. Thus, if we use i up-steps and j down-steps, we have $i(1, 1) + j(1, -1) = (n, k)$. That is, $i + j = n$ and $i - j \equiv k \bmod h$. Therefore, we have the congruence

$$2j \equiv n - k \bmod h, \tag{6.1}$$

where $0 \le j \le n$. To give an explicit formula for our counting problem we analyze the congruence (6.1) depending on the behavior of h. That is, we have two distinguishable cases.

- *h is an odd integer.* In this case, $\gcd(2, h) = 1$. Therefore, the congruence (6.1) has a unique solution j^* mod h ($0 \le j^* < h$). Since $0 \le j \le n$, the other equivalent solutions are of the form $j = j^* + ht$, with
$$0 \le t \le \lfloor \frac{n - j^*}{h} \rfloor.$$
This includes all corridor paths from $(0, 0)$ to (n, k). Observe that j^* depends on n and k. Moreover, for each j of this form, there are $\binom{n}{j}$ ways of ordering j down-steps in an array of n steps. Therefore, the number of h-cyclic corridor Dyck paths starting at $(0, 0)$ and ending at (n, k) is given by
$$p_n^{(h)}(k) = \sum_{t=0}^{\lfloor \frac{n-j^*}{h} \rfloor} \binom{n}{j^* + ht}. \tag{6.2}$$
Moreover, we have
$$g_n^{(h)} := \sum_{k=0}^{h} p_n^{(h)}(k) = \sum_{k=0}^{h} \sum_{t=0}^{\lfloor \frac{n-j^*}{h} \rfloor} \binom{n}{j^* + ht} = \sum_{t=0}^{\lfloor \frac{n-j^*}{h} \rfloor} \sum_{k=0}^{h} \binom{n}{j^* + ht}.$$
Notice that for each fixed integer t, the solution j^* runs through every residue modulo h exactly once as k varies from 0 and h. Hence, this last sum can be rewritten as:
$$g_n^{(h)} = \sum_{t=0}^{\lfloor \frac{n-k}{h} \rfloor} \sum_{k=0}^{h-1} \binom{n}{k + ht} = \sum_{s=0}^{n} \binom{n}{s} = 2^n.$$
- *h is even integer.* If $n - k$ is odd, then the congruence (6.1) does not have a solution. Therefore, there are no paths from $(0, 0)$ to (n, k). So, we only consider the case when $n - k$ is an even integer. The congruence (6.1) can be rewritten as:
$$j \equiv \frac{n - k}{2} \bmod \frac{h}{2}.$$

Then the congruence (6.1) has two distinct solutions j_1 and j_2, with $0 \leq j_1 < j_2 < h$. Moreover, the other equivalent solutions are of the form $j = j_1 + ht_1$ with $0 \leq t_1 \leq \left\lfloor \frac{n-j_1}{h} \right\rfloor$, and $j = j_2 + ht_2$ with $0 \leq t_2 \leq \left\lfloor \frac{n-j_2}{h} \right\rfloor$. Note that $j_2 = j_1 + h/2$. Therefore,

$$p_n^{(h)}(k) = \sum_{t_1=0}^{\left\lfloor \frac{n-j_1}{h} \right\rfloor} \binom{n}{j_1 + ht_1} + \sum_{t_2=0}^{\left\lfloor \frac{n-j_2}{h} \right\rfloor} \binom{n}{j_2 + ht_2} = \sum_{t=0}^{2\left\lfloor \frac{n-j_1}{h} \right\rfloor} \binom{n}{j_1 + th/2},$$

and

$$g_n^{(h)} = \sum_{k=0}^{h} p_n^{(h)}(k) = \sum_{t_1=0}^{\left\lfloor \frac{n-j_1}{h} \right\rfloor} \sum_{k=0}^{h} \binom{n}{j_1 + ht_1} + \sum_{t_2=0}^{\left\lfloor \frac{n-j_2}{h} \right\rfloor} \sum_{k=0}^{h} \binom{n}{j_2 + ht_2}.$$

Notice that for $t = t_1 = t_2$ fixed, j_1 and j_2 pass through every residue modulo h exactly once as k varies from 0 to h, thus this expression can be rewritten as:

$$\sum_{t=0}^{\left\lfloor \frac{n-k}{h} \right\rfloor} \sum_{k=0}^{h-1} \binom{n}{k + ht} = \sum_{s=0}^{n} \binom{n}{s} = 2^n.$$

As an example, we analyze (6.1) when $h = 3$ with $n = 4$. Clearly, (6.1) becomes $j \equiv 2 + k \bmod 3$. So, if $k = 0$, we have that $j^* = 2$. This and (6.2) give $p_4^{(3)}(0) = 6$. Similarly, if $k = 1$ we have that $j^* = 0$ and that $p_4^{(3)}(1) = \binom{4}{0} + \binom{4}{3} = 5$. By symmetry, we have that $p_4^{(3)}(2) = p_4^{(3)}(1) = 5$. Figure 11 depicts all paths counted by $p_4^{(3)}(1) = 5$.

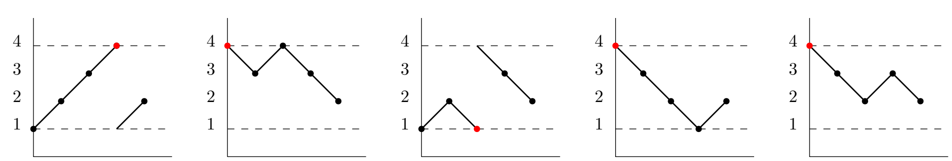

FIGURE 11. Lattice path enumerate by $p_4^{(3)}(1) = 5$.

7. Acknowledgement

The authors are grateful to the referee for the helpful suggestions and comments that help improve the paper.

We use *Mathematica*® to do experimentation and to verify the correctness of the results.

This paper was developed during the *Beyond Research: Experimental Mathematics in Number Theory and Combinatorics*, an undergraduate research experience. Hosted by the Universidad Nacional de Colombia, The Citadel, and Tulane University.

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