Contributions to Discrete Mathematics

Volume 20, Number 1, Pages 95–104
ISSN 1715-0868

SAVED BY THE ROOK: A CASE OF MATCHINGS AND HAMILTONIAN CYCLES

MARIÉN ABREU, JOHN BAPTIST GAUCI, AND JEAN PAUL ZERAFA

Abstract. The rook graph is a graph whose edges represent all the possible legal moves of the rook chess piece on a chessboard. The problem we consider is the following. Given any set M containing pairs of cells such that each cell of the $m_1 \times m_2$ chessboard is in exactly one pair, we determine the values of the positive integers m_1 and m_2 for which it is possible to construct a closed tour of all the cells of the chessboard which uses all the pairs of cells in M and some edges of the rook graph. This is an alternative formulation of a graph-theoretical problem presented in [1] involving the Cartesian product G of two complete graphs K_{m_1} and K_{m_2}, which is, in fact, isomorphic to the $m_1 \times m_2$ rook graph. The problem revolves around determining the values of the parameters m_1 and m_2 that would allow any perfect matching of the complete graph on the same vertex set of G to be extended to a Hamiltonian cycle by using only edges in G.

1. Introduction

The rook chess piece is allowed to move in a horizontal and vertical manner only—no diagonal moves are permissible. The rook graph represents all the possible moves of a rook on a chessboard, with its vertices and edges corresponding to the cells of the chessboard, and the legal moves of the rook from one cell to the other, respectively. All the legal moves of a rook on a $m_1 \times m_2$ chessboard give rise to the $m_1 \times m_2$ *rook graph*. In what follows we consider the following problem.

Problem 1.1. *Let G be a $m_1 \times m_2$ chessboard and let M be a set containing pairs of distinct cells of G such that each cell of G belongs to exactly one pair in M. Determine the values of m_1 and m_2 for which it is possible to construct a closed tour H visiting all the cells of the chessboard G exactly once, such that:*

Received by the editors October 8, 2021, and in revised form June 24, 2023.
2000 *Mathematics Subject Classification.* 05C45, 05C70, 05C76.
Key words and phrases. Perfect matching, Hamiltonian cycle, Cartesian product of complete graphs, line graph, complete bipartite graph.
Marién Abreu was partially supported by a grant of the group GNSAGA of INdAM.

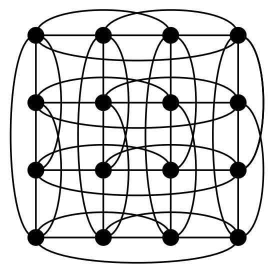

FIGURE 1. The 4×4 rook graph isomorphic to $K_4 \square K_4$

(i) *consecutive cells in H are either a pair of cells in M, or two cells in G which can be joined by a legal rook move; and*
(ii) *H contains all pairs of cells in M.*

In other words, given any possible choice of a set M as defined above, is a rook good enough to let one visit, exactly once, all the cells on a chessboard and finish at the starting cell, in such a way that each pair of cells in M is allowed to and must be used once? We remark that M can contain pairs of cells which are not joined by a legal rook move.

As many other mathematical chess problems, the above problem can be restated in graph theoretical terms. For a detailed exposition, we suggest the reader to [6]. We first give some definitions, and for definitions and notation not explicitly stated here, we refer the reader to [3]. All graphs considered in the sequel will be simple, that is, loops and multiple edges are not allowed. For any graph G with vertex set $V(G)$ and edge set $E(G)$, we let K_G denote the complete graph on the same vertex set $V(G)$ of G. Let G be of even order, that is, having an even number of vertices. A *Hamiltonian cycle* of a graph G is a cycle of G which visits every vertex of G. A *perfect matching* N of a graph G is a set of edges of G such that every vertex of G belongs to exactly one edge in N. This means that no two edges in N have a common vertex and that N is a set of independent edges covering $V(G)$. A Hamiltonian cycle of G can be considered as the disjoint union of two perfect matchings of G. A perfect matching of K_G is said to be a *pairing* of G. In what follows we shall consider Hamiltonian cycles of K_G composed of a pairing of G and a perfect matching of G. In order to distinguish between pairings of G, which may possibly contain edges not in G, and perfect matchings of G, we shall depict pairing edges as green, bold and dashed, and edges of a perfect matching of G as black and bold. To emphasise that pairings can contain edges in G, we shall depict such edges with a black thin line underneath the green, bold and dashed edge described above. This can be clearly seen in Figure 2.

Following [2], we say that a graph G has the *Pairing-Hamiltonian property* (the *PH-property* for short) if every pairing M of G can be extended to a Hamiltonian cycle H of K_G in which $E(H) - M \subseteq E(G)$. If a graph has the PH-property, then for simplicity we shall sometimes say that the graph

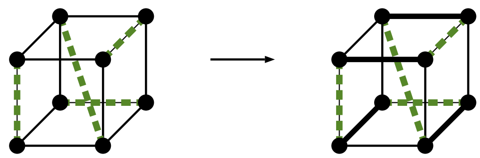

FIGURE 2. A pairing M in the cube $\mathcal{Q}_3$ which is not a perfect matching of $\mathcal{Q}_3$ and a Hamiltonian cycle of $K_{\mathcal{Q}_3}$ containing M

is PH. In order to provide the reader with some examples of graphs having the PH-property, we remark that the authors in [2], amongst other results, gave a complete characterisation of the cubic graphs, that is, graphs with all vertices having degree 3, having the PH-property. There are only three: the complete graph K_4, the complete bipartite graph $K_{3,3}$ and the 3-dimensional cube $\mathcal{Q}_3$ depicted in Figure 3. We note that in the first diagram of Figure 2, one of the green, bold and dashed edges is not an edge of $\mathcal{Q}_3$, and thus the diagram illustrates a possible pairing of $\mathcal{Q}_3$ which is not a perfect matching of $\mathcal{Q}_3$. As shown in Figure 2, this pairing can be extended to a Hamiltonian cycle of $K_{\mathcal{Q}_3}$ by using edges of $\mathcal{Q}_3$. The same argument can be repeated for all pairings of the three graphs shown in Figure 3; hence the reason why they have the PH-property. A similar property to the PH-property is the *PMH-property*, short for the *Perfect-Matching-Hamiltonian property*. See [1] for a more detailed introduction. A graph is said to have the PMH-property, if every perfect matching M of G can be extended to a Hamiltonian cycle H of K_G in which $E(H) - M \subseteq E(G)$. We note that in this case, H would also be a Hamiltonian cycle of G itself. In other words, the PMH-property is equivalent to the PH-property restricted to pairings of G which are also perfect matchings of G. Thus, the PMH-property is a somewhat weaker property than the PH-property.

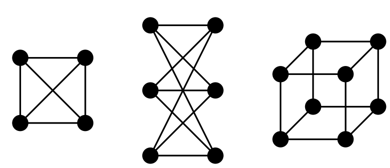

FIGURE 3. The only cubic graphs having the PH-property

The *Cartesian product* $G\square G'$ of two graphs G and G' is a graph whose vertex set is the Cartesian product $V(G) \times V(G')$ of $V(G)$ and $V(G')$. Two vertices (x, x') and (y, y') are adjacent precisely if $x = y$ and $x'y' \in E(G')$, or $xy \in E(G)$ and $x' = y'$. Thus,

$$V(G\square G') = \{(x, x') : x \in V(G) \text{ and } x' \in V(G')\},$$

and

$$E(G\square G') = \{(x, x')(y, y') : x = y, x'y' \in E(G'), \text{ or } xy \in E(G), x' = y'\}.$$

The $m_1 \times m_2$ *rook graph* is in fact isomorphic to the Cartesian product of the complete graphs K_{m_1} and K_{m_2}, denoted by $K_{m_1} \square K_{m_2}$.

Another result in [2] which we shall also be using later on is the following.

Theorem 1.2 ([2]). *The Cartesian product of a complete graph K_m (m even and $m \geq 6$) and a path P_q ($q \geq 1$) has the PH-property.*

However, this was not the first time that pairings extending to Hamiltonian cycles were studied. In 2007, Fink [4] proved what we believe is one of the most significant results in this area so far: for every $n \geq 2$, the n-dimensional hypercube is PH, thus answering a conjecture made by Kreweras [5]. The proof of the result, although technical, is very short and elegant.

With these notions in place, we can restate Problem 1.1 as follows.

Problem 1.3. *Let G be the $m_1 \times m_2$ rook graph, or equivalently $K_{m_1} \square K_{m_2}$. Determine for which values of m_1 and m_2 does G have the PH-property.*

Clearly, in order for $K_{m_1} \square K_{m_2}$ to admit a pairing, at least one of m_1 and m_2 must be even, and without loss of generality, in the sequel we shall tacitly assume that m_1 is even.

We recall that the *line graph* $L(G)$ of a graph G is the graph whose vertices correspond to the edges of G, and two vertices of $L(G)$ are adjacent if the corresponding edges in G are incident to a common vertex. The $m_1 \times m_2$ *rook graph*, or equivalently $K_{m_1} \square K_{m_2}$, can also be seen as the line graph of the complete bipartite graph K_{m_1,m_2}. The authors in [1] give some sufficient conditions for a graph G in order to guarantee that its line graph $L(G)$ has the PMH-property. Amongst other results, they show that the line graph of complete graphs K_n, for $n \equiv 0, 1 \pmod 4$, has the PMH-property, and that, by a similar reasoning, $L(K_{m,m})$ has the PMH-property for every even $m \geq 50$. In Section 2, we determine for which values m_1 and m_2 does $L(K_{m_1,m_2})$ admit the PH-property. This gives a complete solution to Problem 1.3.

2. Main result

In this section we give a complete solution to Problem 1.3, summarised in the following theorem.

Theorem 2.1. *Let m_1 be an even integer and let $m_2 \geq 1$. The $m_1 \times m_2$ rook graph does not have the PH-property if and only if $m_1 = 2$ and m_2 is odd.*

Proof. When $m_2 = 1$, $K_{m_1} \square K_1$ is K_{m_1} and the result clearly follows. Consequently, we shall assume that $m_2 > 1$. By Theorem 1.2, $K_{m_1} \square K_{m_2}$ is PH when $m_1 \geq 6$, since $K_{m_1} \square K_{m_2}$ contains $K_{m_1} \square P_{m_2}$, and, in general, if a graph contains a spanning subgraph which is PH, then the initial graph is itself PH.

So consider the cases when $m_1 = 2$ or 4. For $m_1 = 2$, $K_{m_1} \square K_{m_2}$ is PH if and only if $m_2 \equiv 0 \pmod 2$. In fact, if m_2 is odd, then the pairing consisting of the m_2-edge-cut between the two copies of K_{m_2} cannot be extended to a Hamiltonian cycle, as can be seen in Figure 4. If m_2 is even, then the result

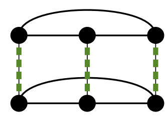

FIGURE 4. A pairing in $K_2 \square K_3$ which cannot be extended to a Hamiltonian cycle

follows once again by Theorem 1.2 when $m_2 \geq 6$. If $m_2 = 2$, then the result easily follows, and when $m_2 = 4$, $K_2 \square K_4$ is PH because the 3-dimensional cube $\mathcal{Q}_3$ is a spanning subgraph of $K_2 \square K_4$ and has the PH-property by Fink's result in [4] (also referred to previously).

What remains to be considered is the case when $m_1 = 4$ and $m_2 \geq 3$. The graph $K_4 \square K_4$ contains $C_4 \square C_4$, the 4-dimensional hypercube Q_4, which is PH ([4]), and for $m_2 \geq 6$ and m_2 even, the result follows once again by Theorem 1.2. Therefore, what remains to be shown is the case when $m_2 \geq 3$ and m_2 is odd, which is settled in the following technical lemma. □

Lemma 2.2. *For every odd $m \geq 3$, the $4 \times m$ rook graph has the PH-property.*

Proof. Let the $4 \times m$ rook graph $K_4 \square K_m$ be denoted by G. We let the vertex set of G be $\{a_i, b_i, c_i, d_i : i \in [m]\}$, such that for each i, the vertices a_i, b_i, c_i, d_i induce a complete graph on four vertices, denoted by K_4^i, and the vertices represented by the same letter induce a K_m. Let M be a pairing of G. We consider two cases:

Case 1. M does not induce a perfect matching in each K_4^i; and
Case 2. M induces a perfect matching in each K_4^i.

We start by considering *Case 1*, and without loss of generality assume that $|M \cap E(K_4^1)| < 2$. If we delete all the edges having exactly one end-vertex in K_4^1 from G, then we obtain two components G_1 and G_2 isomorphic to K_4^1 and $K_4 \square K_{m-1}$, respectively. Since G_1 is of even order and $M \cap E(G_1)$ is not a perfect matching of this graph, G_1 has an even number (two or four) of vertices which are unmatched by $M \cap E(G_1)$.

We pair these unmatched vertices such that $M \cap E(G_1)$ is extended to a perfect matching M_1 of G_1. By a similar reasoning, $M \cap E(G_2)$ does not induce a pairing of G_2 and the number of vertices in G_2 which are unmatched by $M \cap E(G_2)$ is again two or four. Without loss of generality, let a_1, b_1 be two vertices in G_1 unmatched by $M \cap E(G_1)$ such that $a_1b_1 \in M_1$, and let x, y be the two vertices in G_2 such that a_1x and b_1y are both edges in the pairing M of G. We extend $M \cap E(G_2)$ to a pairing M_2 of G_2 by adding the edge xy to $M \cap E(G_2)$, and we repeat this procedure until all vertices in G_2 are matched. Since $m - 1$ is even, G_2 has the PH-property and so M_2 can be extended to a Hamiltonian cycle H_2 of K_{G_2}. We extend H_2 to a Hamiltonian cycle of G containing M as follows. If $c_1d_1 \in M \cap E(G_1)$, then we replace the edge xy in H_2 by the edges $xa_1, a_1d_1, d_1c_1, c_1b_1, b_1y$, as in Figure 5. Otherwise, $c_1d_1 \in M_1 - (M \cap E(G_1))$, and so there exist two vertices u, v in G_2 such that c_1u and d_1v belong to the initial pairing M, and uv belongs to M_2. In this case, we replace the edges xy and uv in H_2 by the edges xa_1, a_1b_1, b_1y, and uc_1, c_1d_1, d_1v, respectively. In either case, H_2 is extended to a Hamiltonian cycle of G containing the pairing M, as required.

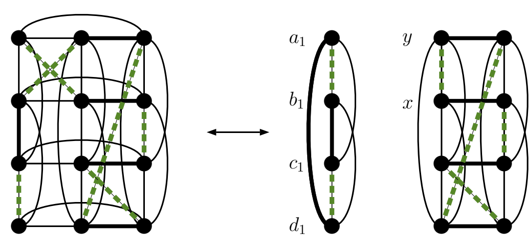

FIGURE 5. An illustration of the inductive step in Case 1 when $m_2 = 3$

Next, we move on to *Case 2*, that is, when M induces a perfect matching in each K_4^i. This case is true by Proposition 1 in [2], however, here we adopt a constructive and more detailed approach highlighting the very useful technique used in [4]. There are three different ways how M can intersect the edges of K_4^i, namely $M \cap E(K_4^i)$ can either be equal to $\{a_ib_i, c_id_i\}$, $\{a_ic_i, b_id_i\}$, or $\{a_id_i, b_ic_i\}$. The number of 4-cliques intersected by M in $\{a_ib_i, c_id_i\}$ is denoted by ν_{cd}^{ab}, and we define ν_{bd}^{ac} and ν_{bc}^{ad} in a similar way. Without loss of generality, we shall assume that $\nu_{cd}^{ab} \geq \nu_{bd}^{ac} \geq \nu_{bc}^{ad}$. We shall also assume that the first ν_{cd}^{ab} 4-cliques in $\{K_4^i : i \in [m]\}$ are the ones intersected by M in $\{a_ib_i, c_id_i\}$, and, if $\nu_{bc}^{ad} \neq 0$, then the last ν_{bc}^{ad} 4-cliques are the ones intersected by M in $\{a_id_i, b_ic_i\}$. This can be seen in Figure 6, in which "unnecessary" curved edges of G are not drawn so as to render the figure more clear.

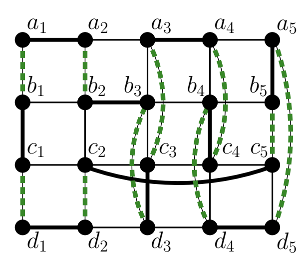

FIGURE 6. G when $\nu_{cd}^{ab} = 2$, $\nu_{bd}^{ac} = 2$ and $\nu_{bc}^{ad} = 1$

When $\nu_{cd}^{ab} = 1$, we have that $\nu_{bd}^{ac} = \nu_{bc}^{ad} = 1$, and in this case it is easy to see that M can be extended to a Hamiltonian cycle of K_G, for example

$(a_1, b_1, c_1, d_1, d_3, a_3, c_3, b_3, b_2, d_2, c_2, a_2)$. We remark that this is the only time when all the 4-cliques are intersected differently by M. Therefore, assume $\nu_{cd}^{ab} \geq 2$. First, let $\nu_{cd}^{ab} = 2$. If $\nu_{bc}^{ad} = 0$, then $\nu_{bd}^{ac} = 1$ and it is easy to see that M can be extended to a Hamiltonian cycle of K_G, for example $(a_1, b_1, b_2, a_2, a_3, c_3, b_3, d_3, d_2, c_2, c_1, d_1)$. The only other possibility is to have $\nu_{bd}^{ac} = 2$ and $\nu_{bc}^{ad} = 1$, and once again M can be extended to a Hamiltonian cycle of K_G, as Figure 6 shows.

Thus, we can assume that $\nu_{cd}^{ab} \geq 3$. Let $r = \nu_{cd}^{ab} + \nu_{bd}^{ac}$ and let r' be the largest even integer less than or equal to r. Moreover, let G_1 be the subgraph of G induced by the vertices $\{b_i, c_i : i \in [m]\}$ that is isomorphic to $K_2 \square K_m$ and let $M_1 = \{b_1b_2, \ldots, b_{r'-1}b_{r'}, c_1c_2, \ldots, c_{r'-1}c_{r'}, b_{r'+1}c_{r'+1}, \ldots, b_mc_m\}$. Clearly, M_1 is a pairing of G_1 which contains $M \cap E(G_1)$, and can be extended to a Hamiltonian cycle H_1 of K_{G_1} as follows: $(b_1, b_2, \ldots, b_{r'}, b_{r'+1}, c_{r'+1}, c_{r'+2}, b_{r'+2}, \ldots, b_mc_m, c_{r'}, c_{r'-1}, \ldots, c_1)$. This is depicted in Figure 7. We note that if $r' = m-1$, then we do not consider the index $r'+2$ in the last sequence of vertices forming H_1. Deleting the edges belonging to $M_1 - M$ from H_1 gives a collection of r disjoint paths $\mathcal{P} = \{P^i : i \in [r]\}$. We note that the union of all the end-vertices of the paths in $\mathcal{P}$ give $\{b_i, c_i : i \in [r]\}$. If we look at the example given in Figure 7, then the only path in $\mathcal{P}$ on more than two vertices is the path $b_8b_9c_9c_{10}b_{10}b_{11}c_{11}c_8$.

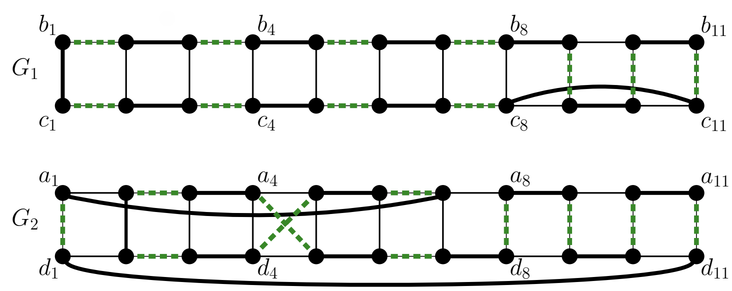

FIGURE 7. G_1 and G_2 when $\nu_{cd}^{ab} = 4$, $r = r' = 8$, and $m = 11$ in Case 2

Next, let G_2 be the subgraph of G induced by the vertices $\{a_i, d_i : i \in [m]\}$, which is isomorphic to $K_2 \square K_m$ as G_1. For every $i \in [r]$, we let u_i and v_i be the two end-vertices of the path P^i, and we let x_i and y_i be the two vertices in G_2 such that u_ix_i and v_iy_i both belong to M. We remark that $\{a_i, d_i : i \in [r]\} = \{x_i, y_i : i \in [r]\}$. Let $M_2 = \{x_1y_1, \ldots, x_ry_r\} \cup (M \cap E(G_2))$. If $r = m$, then $M \cap E(G_2)$ is empty, otherwise it consists of $\{a_{r+1}d_{r+1}, \ldots, a_md_m\}$. If ν_{cd}^{ab} is even as in Figure 7, then M_2 contains:

$$\{a_1d_1, a_2a_3, \ldots, a_{\nu_{cd}^{ab}-2}a_{\nu_{cd}^{ab}-1}, a_{\nu_{cd}^{ab}}d_{\nu_{cd}^{ab}+1}, d_2d_3, \ldots, d_{\nu_{cd}^{ab}-2}d_{\nu_{cd}^{ab}-1}, d_{\nu_{cd}^{ab}}a_{\nu_{cd}^{ab}+1}\}.$$

Otherwise, M_2 contains $\{a_1d_1, a_2a_3, \ldots, a_{\nu_{cd}^{ab}-1}a_{\nu_{cd}^{ab}}, d_2d_3, \ldots, d_{\nu_{cd}^{ab}-1}d_{\nu_{cd}^{ab}}\}$. Moreover, if r is even, then $a_rd_r \in M_2$. In either case, M_2 can be extended to a Hamiltonian cycle H_2 of K_{G_2}, as can be seen in Figure 7, which shows the case when ν_{cd}^{ab} and r are both even. We remark that the green, bold and dashed edges in the figure are the ones in M_1 and M_2. If for each $i \in [r]$, then we replace the edges x_iy_i in H_2 by x_iu_i, the path P^i, and v_iy_i as in Figure 8, a Hamiltonian cycle of K_G containing M is obtained, proving our theorem. □

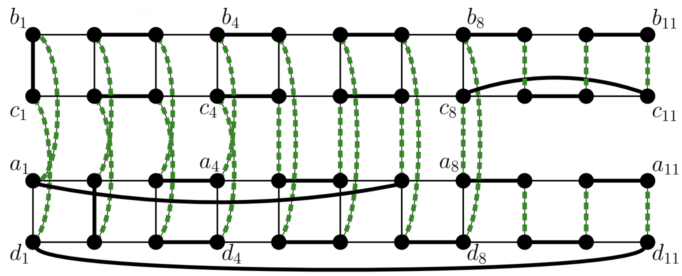

FIGURE 8. Extending H_1 and H_2 from Figure 7 to a Hamiltonian cycle of K_G containing M

3. VERTICAL AND DIAGONAL MOVES ONLY—NO HORIZONTAL MOVES PERMISSIBLE

In the next theorem we present a rather simple proof to show that the complete bipartite graph having equal part sizes is PH. Note that, if $m \neq n$, then $K_{m,n}$ has no perfect matchings.

Theorem 3.1. *For every $n \geq 2$, the complete bipartite graph $K_{n,n}$ has the PH-property.*

Proof. Let $\{u_1, \dots, u_n\}$ and $\{w_1, \dots, w_n\}$ be the partite sets of $K_{n,n}$. We proceed by induction on n. When $n = 2$, result holds since $K_{2,2} \simeq K_2 \square K_2$. So assume $n > 2$ and let M be a pairing of $K_{n,n}$. If $M = \{u_iw_i : i \in [n]\}$, then M easily extends to a Hamiltonian cycle of the underlying complete graph on $2n$ vertices. Thus, assume there exists $j \in [n]$ such that $u_jw_j \notin M$. Without loss of generality, let j be equal to n. Then, M contains the edges xu_n and yw_n, for some x and y belonging to the set $Z = \{u_i, w_i : i \in [n-1]\}$. We note that Z induces the complete bipartite graph $K_{n-1,n-1}$ with partite sets $\{u_1, \dots, u_{n-1}\}$ and $\{w_1, \dots, w_{n-1}\}$, which we denote by G'. The set of edges $M' = M \cup xy - xu_n - yw_n$ is a pairing of G', and so, by induction on n, M' can be extended to a Hamiltonian cycle H' of $K_{G'}$. This Hamiltonian cycle can be extended to a Hamiltonian cycle H of the underlying complete graph of $K_{n,n}$ by replacing the edge xy in H', by the edges xu_n, u_nw_n, w_ny. The resulting Hamiltonian cycle H clearly contains M, proving our theorem. □

Although the statement and proof of Theorem 3.1 are quite easy, they may lead to another intriguing problem. From Theorem 2.1 we know that the rook is not good enough to solve our problem on a $2 \times m_2$ chessboard when m_2 is odd. However, the above result shows that if the rook was somehow allowed to do only vertical and diagonal moves instead of vertical and horizontal moves only, then it would always be possible to perform a closed tour on a $2 \times m_2$ chessboard in such a way that each pair of cells in M is allowed to and must be used once, no matter the choice of M. In general, we denote the graph arising from all the possible vertical and diagonal moves on a $m_1 \times m_2$ chessboard as $\mathcal{V}_{m_1}\mathcal{D}_{m_2}$, with m_1 corresponding to the vertical axis of the chessboard. We recall that no horizontal moves are permissible, and in this sense, $\mathcal{V}_{m_1}\mathcal{D}_{m_2}$ can be seen as the graph complement of the disjoint union of m_1 paths each on m_2 vertices (see for example Figure 9).

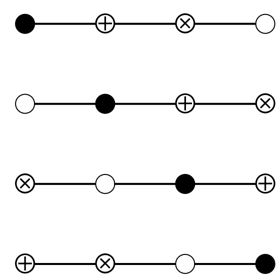

FIGURE 9. $\mathcal{V}_4\mathcal{D}_4$ is the graph complement of the above graph. In particular, each set of four vertices with the same vertex shape induces a 4-clique in $\mathcal{V}_4\mathcal{D}_4$.

As before, for $\mathcal{V}_{m_1}\mathcal{D}_{m_2}$ to be PH, at least one of m_1 or m_2 must be even. Moreover, we remark that when $m_2 \leq m_1$, the graph $\mathcal{V}_{m_1}\mathcal{D}_{m_2}$ contains $K_{m_1}\square K_{m_2}$ as a spanning subgraph. This can be seen in Figure 9, where each set of four vertices having the same vertex shape induces a 4-clique in $\mathcal{V}_4\mathcal{D}_4$, which together with all the vertical edges of $\mathcal{V}_4\mathcal{D}_4$ give $K_4\square K_4$. Finally, we also observe that $\mathcal{V}_{m_1}\mathcal{D}_{m_2}$ is isomorphic to the co-normal product of K_{m_1} and $\overline{K}_{m_2}$, where the latter is the empty graph on m_2 vertices. Recall that the *co-normal product* $G*G'$ of two graphs G and G' is a graph whose vertex set is the Cartesian product $V(G)\times V(G')$ of $V(G)$ and $V(G')$, and two vertices (x,x') and (y,y') are adjacent precisely if $xy \in E(G)$ or $x'y' \in E(G')$. Thus,

$$V(G*G') = \{(x,x') : x \in V(G) \text{ and } x' \in V(G')\},$$

and

$$E(G*G') = \{(x,x')(y,y') : xy \in E(G) \text{ or } x'y' \in E(G')\}.$$

We wonder for which values of m_1 and m_2 the graph $\mathcal{V}_{m_1}\mathcal{D}_{m_2}$ is PH.

REFERENCES

1. Marién Abreu, John Baptist Gauci, Domenico Labbate, Giuseppe Mazzuoccolo, and Jean Paul Zerafa, *Extending perfect matchings to Hamiltonian cycles in line graphs*, Electron. J. Comb. **28** (2021), no. 1, research paper p1.7, 13 (English).

2. Adel Alahmadi, Robert E. L. Aldred, Ahmad Alkenani, Rola Hijazi, Patrick Solé, and Carsten Thomassen, *Extending a perfect matching to a Hamiltonian cycle*, Discrete Math. Theor. Comput. Sci. **17** (2015), no. 1, 241–254 (English).
3. Reinhard Diestel, *Graph theory*, 2nd ed., Grad. Texts Math., vol. 173, Berlin: Springer, 2000 (English).
4. Jiří Fink, *Perfect matchings extend to Hamilton cycles in hypercubes*, J. Comb. Theory, Ser. B **97** (2007), no. 6, 1074–1076 (English).
5. Germain Kreweras, *Matchings and Hamiltonian cycles on hypercubes*, Bull. Inst. Comb. Appl. **16** (1996), 87–91 (English).
6. Allen J. Schwenk, *Which rectangular chessboards have a knight's tour?*, Math. Mag. **64** (1991), no. 5, 325–332 (English).

Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, Italy
E-mail address: `marien.abreu@unibas.it`

Department of Mathematics, University of Malta, Malta
E-mail address: `john-baptist.gauci@um.edu.mt`

St. Edward's College, Birgu (Città Vittoriosa), Cottonera, Malta; Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università degli Studi di Modena e Reggio Emilia, Italy; Faculty of Economics, Management & Accountancy, and Faculty of Education, University of Malta, Malta
E-mail address: `zerafa.jp@gmail.com`