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METRIC PROPERTIES OF INCOMPARABILITY GRAPHS WITH AN EMPHASIS ON PATHS

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Abstract. We describe some metric properties of incomparability graphs. We consider the problem of the existence of infinite paths, either induced or isometric, in the incomparability graph of a poset. Among other things, we show that if the incomparability graph of a poset is connected and has infinite diameter, then it contains an infinite induced path. Furthermore, if the diameter of the set of vertices of degree at least 3 is infinite, then the graph contains as an induced subgraph either a comb or a kite.

1. Introduction and presentation of the results

In this paper, we highlight the special properties of incomparability graphs by considering the behavior of paths. We consider the problem of the existence of infinite paths, either induced or isometric, in the incomparability graph of a poset. We apply one of our results in the theory of hereditary classes of certain permutation classes that are well quasi-ordered by embeddability.

The graphs we consider are undirected, simple and have no loops. That is, a *graph* is a pair $G := (V, E)$, where E is a subset of $[V]^2$, the set of 2-element subsets of V. Elements of V are the *vertices* of G and elements of E its *edges*. Let the graph G be given, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. The *complement* of a graph $G = (V, E)$ is the graph G^c whose vertex set is V and edge set $E^c := [V]^2 \setminus E$.

Throughout, $P := (V, \leq)$ denotes an ordered set (poset), that is a set V equipped with a binary relation $\leq$ on V which is reflexive, antisymmetric

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and transitive. We say that two elements $x, y \in V$ are *comparable* if $x \le y$ or $y \le x$, otherwise we say they are *incomparable*. The *comparability graph*, respectively the *incomparability graph*, of a poset $P := (V, \le)$ is the undirected graph, denoted by $\mathrm{Comp}(P)$, respectively $\mathrm{Inc}(P)$, with vertex set V and edges the pairs $\{u, v\}$ of comparable distinct vertices (that is, either $u < v$ or $v < u$) respectively incomparable vertices.

A result of Gallai, 1967 [7], quite famous and nontrivial, characterizes comparability graphs among graphs in terms of obstructions: a graph G is the comparability graph of a poset if and only if it does not contain as an induced subgraph a graph belonging to a minimal list of finite graphs. Since the complement of a comparability graph is an incomparability graph, Gallai's result yields a similar characterization of incomparability graphs.

In this paper, we consider incomparability graphs as metric spaces by means of the distance of the shortest path. The metric properties of a graph, notably of an incomparability graph, and metric properties of its complement seem to be far apart. In general, metric properties of graphs are based on paths and cycles. It should be noted that incomparability graphs have no induced cycles of length at least five ([7]; for a short proof see after Lemma 3.1) while comparability graphs have no induced odd cycles but can have arbitrarily large induced even cycles.

In the sequel, we will illustrate the specificity of the metric properties of incomparability graphs by emphasising the properties of paths.

We start with few definitions. Let $G := (V, E)$ be a graph. If A is a subset of V, the graph $G_{\restriction A} := (A, E \cap [A]^2)$ is the *graph induced by G on A*. A *path* is a graph P such that there exists a one-to-one map f from the set $V(\mathrm{P})$ of its vertices into an interval I of the chain $\mathbb{N}$ of nonnegative integers in such a way that $\{u, v\}$ belongs to $E(\mathrm{P})$, the set of edges of P, if and only if $|f(u) - f(v)| = 1$ for every $u, v \in V(\mathrm{P})$. If I is finite, say $I = \{1, \dots, n\}$, then we denote that path by P_n; its *length* is $n - 1$ (so, if $n = 2$, P_2 is made of a single edge, whereas if $n = 1$, P_1 is a single vertex). We denote by P_∞ the one way infinite path i.e. $I = \mathbb{N}$. If x, y are two vertices of a graph $G := (V, E)$, we denote by $d_G(x, y)$ the length of a shortest path joining x and y if any, and $d_G(x, y) := \infty$ otherwise. This defines a distance on V, the *graphic distance*. A graph is *connected* if any two vertices belong to some path. The *diameter* of G, denoted by δ_G, is the supremum of the set $\{d_G(x, y) : x, y \in V\}$. If A is a subset of V, the graph G' induced by G on A is an *isometric subgraph* of G if $d_{G'}(x, y) = d_G(x, y)$ for all $x, y \in A$. The supremum of the length of induced finite paths of G, denoted by D_G, is sometimes called the (induced) *detour* of G [1].

The main results of the paper are presented in the next four subsections. Section 1.5 is devoted to an application of one of our main results (Theorem 1.4). The remaining sections contain intermediate results and proofs of our main results.

1.1. **Induced paths of arbitrarily large length in incomparability graphs and in arbitrary graphs.** We now consider the question of the existence of infinite induced paths in incomparability graphs with an infinite detour. In order to state our main result of this subsection, we need to introduce the notions of the direct sum and the complete sum of graphs. Let $G_n := (V_n, E_n)$ for $n \in \mathbb{N}$ be a family of graphs having pairwise disjoint vertex sets. The *direct sum* of $(G_n)_{n\in\mathbb{N}}$, denoted $\oplus_n G_n$, is the graph whose vertex set is $\bigcup_{n\in\mathbb{N}} V_n$ and edge set $\bigcup_{n\in\mathbb{N}} E_n$. The *complete sum* of $(G_n)_{n\in\mathbb{N}}$, denoted $\sum_n G_n$, is the graph whose vertex set is $\bigcup_{n\in\mathbb{N}} V_n$ and edge set $\bigcup_{i\neq j}\{\{v,v'\} : v \in V_i \wedge v' \in V_j\} \cup \bigcup_{n\in\mathbb{N}} E_n$.

A necessary condition for the existence of an infinite induced path in a graph is to have an infinite detour. On the other hand, the graphs consisting of the direct sum of finite paths of arbitrarily large length and the complete sum of finite paths of arbitrarily large length are (incomparability) graphs with an infinite detour and yet do not have an infinite induced path. We should mention that *in the case of incomparability graphs, having an infinite detour is equivalent to having a direct sum or a complete sum of finite paths of arbitrarily large length.* This is Theorem 2 from [20].

Theorem 1.1 ([20])**.** *Let G be the incomparability graph of a poset. Then G contains induced paths of arbitrarily large length if and only if G contains $\sum_{n\geq 1} \mathrm{P}_n$ or $\oplus_{n\geq 1}\mathrm{P}_n$ as an induced subgraph.*

For general graphs, the statement of Theorem 1.1 is false. Indeed, in [20] we exhibited uncountably many graphs of cardinality $\aleph_0$, containing finite induced paths of unbounded length and neither a direct sum nor a complete sum of finite paths of unbounded length. In particular, these graphs do not have an infinite induced path.

In the case of incomparability graphs of posets coverable by two chains, having an infinite detour is equivalent to the existence of an infinite induced path. Our first result is this.

Theorem 1.2. *Let P be a poset coverable by two chains (that is totally ordered sets). If* $\mathrm{Inc}(P)$*, the incomparability graph of P, is connected then the following properties are equivalent:*

(i) $\mathrm{Inc}(P)$ *contains the direct sum of induced paths of arbitrarily large length;*
(ii) the detour of $\mathrm{Inc}(P)$ *is infinite;*
(iii) the diameter of $\mathrm{Inc}(P)$ *is infinite;*
(iv) $\mathrm{Inc}(P)$ *contains an infinite induced path.*

A proof of Theorem 1.2 will be provided in section 5.

The implication $(i) \Rightarrow (iv)$ of Theorem 1.2 becomes false if the condition "coverable by two chains" is dropped (see Figure 1 for an example). Indeed,

Example 1.3. *There exists a poset with no infinite antichain whose incomparability graph is connected and embeds the direct sum of finite induced paths of arbitrarily large length and yet does not have an infinite induced path (See Figure 1).*

Example 1.3 and a proof that it verifies the required properties will be given in section 6.

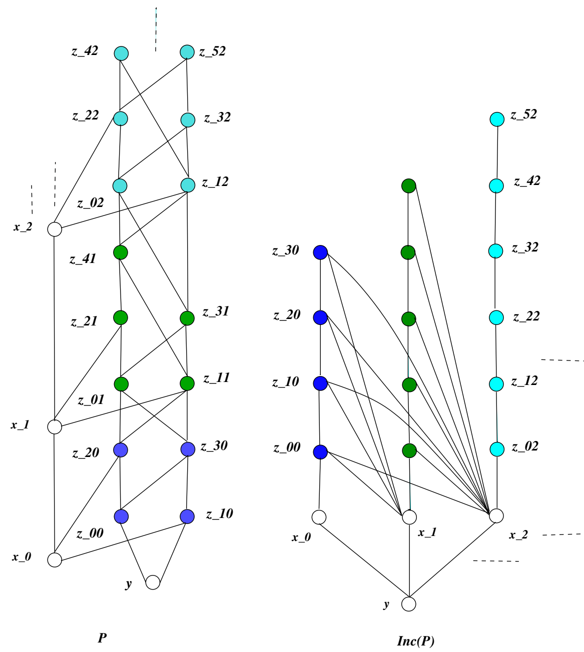

FIGURE 1. The Hasse diagram of a poset of width three and its incomparability graph that has a vertex y of infinite induced detour but no infinite induced path.

1.2. **Infinite induced paths, combs and kites.** We now consider the question of the existence of infinite induced paths in incomparability graphs with infinite diameter. In order to state our main result of this subsection, we need to introduce two types of graphs: comb and kite.

Let us recall that a graph $G := (V, E)$ is a *caterpillar* if the graph obtained by removing from V the vertices of degree one is a path (finite or not, reduced to one vertex or empty). A *comb* is a caterpillar such that every vertex is adjacent to at most one vertex of degree one. Incidentally, a path on three vertices is not a comb. It should be mentioned that caterpillars are incomparability graphs of interval orders coverable by two chains (see Lemma 14 of [25]).

We now give the definition of a *kite*. This is a graph obtained from an infinite path $\mathrm{P}_\infty := (x_i)_{i\in\mathbb{N}}$ by adding a new set of vertices Y (finite or infinite). We distinguish three types of kites (see Figure 2) depending on how the vertices of Y are adjacent to the vertices of P_∞.

A *kite of type* (1): every vertex of Y is adjacent to exactly two vertices of P_∞ and these two vertices are consecutive in P_∞. Furthermore, two distinct vertices of Y share at most one common neighbour in P_∞.

A *kite of type* (2): every vertex of Y is adjacent to exactly three vertices of P_∞ and these three vertices must be consecutive in P_∞. Furthermore, for all $x, x' \in Y$, if x is adjacent to x_i, x_{i+1}, x_{i+2} and x' is adjacent to $x_{i'}, x_{i'+1}, x_{i'+2}$ then $i+2 \leq i'$ or $i'+2 \leq i$.

A *kite of type* (3): every vertex of Y is adjacent to exactly two vertices of P_∞ and these two vertices must be at distance two in P_∞. Furthermore, for all $x, x' \in X$, if x is adjacent to x_i and x_{i+2} and x' is adjacent to $x_{i'}$ and $x_{i'+2}$ then $i+2 \leq i'$ or $i'+2 \leq i$.

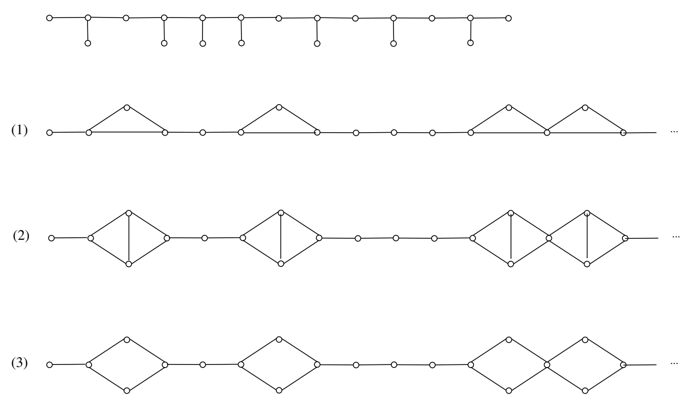

FIGURE 2. A comb and the three types of kites.

Theorem 1.4. *If G is a connected incomparability graph with infinite diameter. Then*

(1) *Every vertex of G has an induced path of infinite diameter starting at it.*
(2) *If the set of vertices of degree at least 3 in G has infinite diameter, then G contains an induced comb or an induced kite having an infinite diameter and infinitely many vertices of degree at least 3.*

Theorem 1.4 will be proved in section 9 (an important ingredient of its proof is Theorem 1.10 below).

1.3. **Infinite isometric paths in incomparability graphs.** A basic result about the existence of an infinite isometric path in a graph is König's lemma [9]. Recall that a graph is *locally finite* if every vertex has a finite degree.

Theorem 1.5 ([9]). *Every connected, locally finite, infinite graph contains an isometric infinite path.*

Moreover,

Theorem 1.6. *If a connected graph G has an infinite isometric path, then every vertex has an isometric path starting at it.*

Theorem 1.6 was proved by Watkins in the case of locally finite graphs (see [23], Lemma 3.2). The general case is contained in Theorem 3.5 and Lemma 3.7 of [16].

A necessary condition for a graph to have an infinite isometric path is to have infinite diameter. Note that a graph has an infinite diameter if and only if it has finite isometric paths of arbitrarily large length. The existence of such paths does not necessarily imply the existence of an infinite isometric path even if the graph is connected.

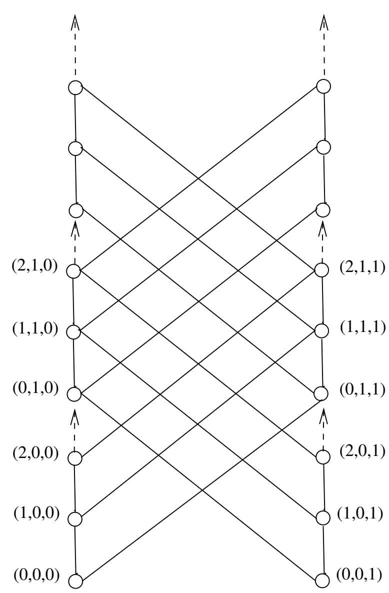

FIGURE 3. The Hasse diagram of a poset of width two whose incomparability graph is connected, has infinite diameter but no infinite isometric path.

Example 1.7. *There exists a poset coverable by two chains whose incomparability graph is connected, having infinite diameter and no isometric infinite path (see Figure 3).*

We provide Example 1.7 and a proof that it verifies the required properties in section 10.

We obtain a positive result in the case of incomparability graphs of interval orders with no infinite antichains. A poset P is an *interval order* if P is isomorphic to a subset $\mathcal{J}$ of the set $\mathrm{Int}(C)$ of nonempty intervals of a chain

C, ordered as follows: if $I, J \in \mathrm{Int}(C)$, then

$$I < J \text{ if } x < y \text{ for every } x \in I \text{ and every } y \in J. \tag{1.1}$$

Interval orders were considered in Fishburn [5, 4] and Wiener [24] in relation to the theory of measurement.

Theorem 1.8. *If P is an interval order with no infinite antichains so that* $\mathrm{Inc}(P)$ *is connected and has infinite diameter, then* $\mathrm{Inc}(P)$ *has an infinite isometric path.*

The proof of Theorem 1.8 will be provided in section 11.

The conclusion of Theorem 1.8 becomes false if the condition "no infinite antichains" is removed. Indeed,

Example 1.9. *There exists an interval order whose incomparability graph is connected, has an infinite diameter and no infinite isometric path.*

Example 1.9 and a proof that it verifies the required properties will be provided in section 11.

1.4. **Convexity and isometry of metric balls in incomparability graphs.** In this subsection, we compare the notions of order convexity and metric convexity with respect to the distance on the incomparability graph of a poset. Before stating our result we need a few definitions.

An *initial segment* of a poset $P := (V, \leq)$ is any subset I of V such that $x \in V$, $y \in I$ and $x \leq y$ imply $x \in I$. If X is a subset of V, the set

$$\downarrow X := \{y \in P : y \leq x \text{ for some } x \in X\}$$

is the least initial segment containing X, we say that it is *generated* by X. If X is a one element set, say $X = \{x\}$, we denote by $\downarrow x$, instead of $\downarrow X$, this initial segment and say that it is *principal*. *Final segments* are defined similarly.

Let $P := (V, \leq)$ be a poset. A subset X of V is *order convex* or *convex* if for all $x, y \in X$, $[x, y] := \{z : x \leq z \leq y\} \subseteq X$. For instance, initial and final segments of P are convex. Note that any intersection of convex sets is also convex. In particular, the intersection of all convex sets containing X, denoted $\mathrm{Conv}_P(X)$, is convex. This is the smallest convex set containing X. Note that

$$\mathrm{Conv}_P(X) = \{z \in P : x \leq z \leq y \text{ for some } x, y \in X\} = \downarrow X \cap \uparrow X.$$

Let $G := (V, E)$ be a graph. We equip it with the graphic distance d_G. A *ball* is any subset $B_G(x, r) := \{y \in V : d_G(x, y) \leq r\}$ where $x \in V, r \in \mathbb{N}$. A subset of V is *convex* w.r.t. the distance d_G if this is an intersection of balls. The *least convex subset* of G containing X is

$$\mathrm{Conv}_G(X) := \bigcap_{X \subseteq B_G(x,r)} B_G(x, r).$$

Let $X \subseteq V$ and $r \in \mathbb{N}$. Define

$$B_G(X, r) := \{v \in V : d_G(v, x) \leq r \text{ for some } x \in X\}.$$

With all needed definitions in hand, we are now ready to state the following theorem.

Theorem 1.10. *Let $P \coloneqq (V, \leq)$ be a poset, G be its incomparability graph, $X \subseteq V$ and $r \in \mathbb{N}$.*

(*a*) *If X is an initial segment, respectively a final segment, respectively an order convex subset of P then $B_G(X, r)$ is an initial segment, respectively a final segment, respectively an order convex subset of P. In particular, for all $x \in V$ and $r \in \mathbb{N}$, $B_G(x, r)$ is order convex;*

(*b*) *If X is order convex then the graph induced by G on $B_G(X, r)$ is an isometric subgraph of G. In particular, if X is included into a connected component of G then the graph induced by G on $B_G(X, r)$ is connected.*

It follows from Theorem 1.10 that *every ball in an incomparability graph G of a poset is order convex and that the graph induced on it is an isometric subgraph of G.*

The proof of Theorem 1.10 is provided in section 8.

1.5. **An application of Theorem 1.4 in the theory of well quasi order.** The purpose of this subsection is to provide an application of Theorem 1.4 in the theory of well quasi order. Let us first recall some notions from the Theory of Relations [6]. A graph G is *embeddable* in a graph G' if G is isomorphic to an induced subgraph of G'. The embeddability relation is a quasi order on the class of graphs. A class $\mathcal{C}$ of graphs, finite or not, is *hereditary* if it contains every graph which embeds in some member of $\mathcal{C}$. The *age* of a graph G is the collection of finite graphs, considered up to isomorphy, that embed in G (or alternatively, that are isomorphic to some induced subgraph of G). We recall that an age of finite graphs, and more generally a class of finite graphs, is *well quasi ordered* (w.q.o. for short) if it contains no infinite antichain, that is an infinite set of graphs G_n pairwise incomparable with respect to embeddability. There are several results about w.q.o. hereditary classes of graphs, for examples see [12, 11], [13] and [15].

We recall that a graph $G \coloneqq (V, E)$ is a *permutation graph* if there is a linear order $\leq$ on V and a permutation σ of V such that the edges of G are the pairs $\{x, y\} \in [V]^2$ which are reversed by σ. The study of permutations graphs became an important topic due to the Stanley-Wilf Conjecture, formulated independently by Richard P. Stanley and Herbert Wilf in the late 1980s, and solved positively by Marcus and Tardös [14] 2004. It was proved by Lozin and Mayhill 2011 [13] that a hereditary class of finite bipartite permutation graphs is w.q.o. by embeddability if and only there is a bound on the length of the double-ended forks (see Figure 4) it may contain (for an alternative proof see [19]). In [19], we extend the results of Lozin and Mayhill [13] and present an almost exhaustive list of properties of w.q.o. ages of bipartite permutation graphs. One of our results is a positive answer, in the case of an age of bipartite permutation graphs, to a long-standing unsolved question by the first author, of whether the following equivalence is true in general: an age is not w.q.o. if and only if it contains $2^{\aleph_0}$ subages (see subsection I-4 Introduction à la comparaison des âges, page 67, [17]). This result, Theorem 1.11 below, is a consequence of (2) of Theorem 1.4.

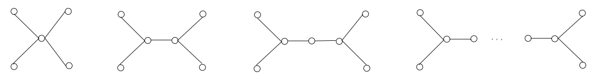

FIGURE 4. Double-ended forks: an antichain of finite graphs with respect to embeddability.

Theorem 1.11 ([19]). *Let $\mathcal{C}$ be an age that consists of finite bipartite permutation graphs. Then $\mathcal{C}$ is not w.q.o. if and only if it contains the age of a direct sum $\bigoplus_{i\in I} \mathrm{DF}_i$ of double ended forks of arbitrarily large length for some infinite subset I of $\mathbb{N}$. In particular, if $\mathcal{C}$ is not w.q.o., it contains $2^{\aleph_0}$ subages which are not w.q.o.*

A proof is given in [19]. For completeness, we provide the proof here.

Proof. The set of double-ended forks forms an infinite antichain, hence if $\mathcal{C}$ contains the direct sum $\bigoplus_{i\in I} \mathrm{DF}_i$ of double-ended forks of arbitrarily large length for some infinite subset I of $\mathbb{N}$, it is not w.q.o. Conversely, suppose $\mathcal{C}$ is not w.q.o. Then it embeds double-ended forks of unbounded length. This important result is due Lozin and Mayhill (see Theorem 7 in [13]). Let G be a graph with $\mathrm{Age}(G) = \mathcal{C}$. We consider two cases:
(1) Some connected component of G, say G_i, embeds double forks of unbounded length. In this case, the detour of G_i, that is the supremum of the lengths of induced paths in G_i, is unbounded. Since G_i is the incomparability graph of a poset of width at most two, its diameter is unbounded (See Corollary 4.6). In fact, since the vertices of degree 3 in the forks are end vertices of induced paths, the diameter of the set of vertices of degree 3 in G_i is unbounded. Thus from (2) of Theorem 1.4, G_i embeds an induced caterpillar or an induced kite with infinitely many vertices of degree at least 3. Since G is bipartite, it can only embed a kite of type (3). As it is easy to see, this caterpillar or that kite embeds a direct sum $\bigoplus_{i\in I} \mathrm{DF}_i$ of double-ended forks of arbitrarily large length, as required.
(2) If the first case does not hold, there are infinitely many connected components G_i, each embedding some double-ended fork DF_i, and the length of these double-ended forks is unbounded. This completes the proof of Theorem 1.11. □

The paper is organised as follows. In section 2 we present some prerequisites on graphs and posets. In section 3 we state a fundamental lemma on paths in incomparability graphs and some consequences. In section 4 we present few metric properties of posets of width 2. In section 5 we present the proof of Theorem 1.2. In section 6 we present Example 1.3. In section 7 we present various metric properties of incomparability graphs. In section

8 we present a proof of Theorem 1.10 and some consequences. In section 9 we give a proof of Theorem 1.4 (an important ingredient of the proof is Theorem 1.10). In section 10 we present Example 1.7. Finally, a proof of Theorem 1.8 and Example 1.9 are provided in section 11.

2. Graphs and Posets

2.1. **Posets.** Throughout, $P := (V, \leq)$ denotes an ordered set (poset). The *dual* of P denoted P^* is the order defined on V as follows: if $x, y \in V$, then $x \leq y$ in P^* if and only if $y \leq x$ in P. Let $P := (V, \leq)$ be a poset. We recall that two elements $x, y \in V$ are *comparable* if $x \leq y$ or $y \leq x$, otherwise, we say they are *incomparable*, denoted $x \parallel y$. A set of pairwise comparable elements is called a *chain*. On the other hand, a set of pairwise incomparable elements is called an *antichain*. The *width* of a poset is the maximum cardinality of its antichains (if the maximum does not exist, the width is set to be infinite). Dilworth's celebrated theorem on finite posets [2] states that the maximum cardinality of an antichain in a finite poset equals the minimum number of chains needed to cover the poset. This result remains true even if the poset is infinite but has finite width. If the poset P has width 2 and the incomparability graph of P is connected, the partition of P into two chains is unique (picking any vertex x, observe that the set of vertices at odd distance from x and the set of vertices at even distance from x form a partition into two chains). According to Szpilrajn [21], every order on a set has a linear extension. Let $P := (V, \leq)$ be a poset. A *realizer* of P is a family $\mathcal{L}$ of linear extensions of the order of P whose intersection is the order of P. Observe that the set of all linear extensions of P is a realizer of P. The *dimension* of P, denoted $\dim(P)$, is the least cardinal d for which there exists a realizer of cardinality d [3]. It follows from the Compactness Theorem of First Order Logic that an order is an intersection of at most n linear orders ($n \in \mathbb{N}$) if and only if every finite restriction of the order has this property. Hence the class of posets with dimension at most n is determined by a set of finite obstructions, each obstruction is a poset Q of dimension $n+1$ such that the deletion of any element of Q leaves a poset of dimension n; such a poset is said *critical*. For $n \geq 2$ there are infinitely many critical posets of dimension $n+1$. For $n = 2$ they have been described by Kelly [8]; for $n \geq 3$, the task is considered as hopeless.

2.1.1. *Comparability and incomparability graphs, permutation graph.* A graph $G := (V, E)$ is a *comparability graph* if the edge set is the set of comparabilities of some order on V. From the Compactness Theorem of First Order Logic, it follows that a graph is a comparability graph if and only if every finite induced subgraph is a comparability graph. Hence, the class of comparability graphs is determined by a set of finite obstructions. The complete list of minimal obstructions was determined by Gallai [7]. A graph $G := (V, E)$ is a *permutation graph* if there is a linear order $\leq$ on V and a permutation σ of V such that the edges of G are the pairs $\{x, y\} \in [V]^2$

which are reversed by σ. Denoting by $\leq_\sigma$ the set of oriented pairs (x, y) such that $\sigma(x) \leq \sigma(y)$, the graph is the comparability graph of the poset whose order is the intersection of $\leq$ and the opposite of $\leq_\sigma$. Hence, a permutation graph is the comparability graph of an order intersection of two linear orders, that is the comparability graph of an order of dimension at most two [3]. If the graph is finite, the converse holds. Hence, as it is well known, a finite graph G is a permutation graph if and only if G and G^c are comparability graphs [3]; in particular, a finite graph is a permutation graph if and only if its complement is a permutation graph. Via the Compactness Theorem of First Order Logic, an infinite graph is the comparability graph of a poset intersection of two linear orders if and only if each finite induced graph is a permutation graph (sometimes these graphs are called permutation graphs, while there is no possible permutation involved). For more about permutation graphs, see [10].

2.1.2. *Lexicographical sum.* Let I be a poset such that $|I| \geq 2$ and let

$$\{P_i := (V_i, \leq_i)\}_{i \in I}$$

be a family of pairwise disjoint nonempty posets that are all disjoint from I. The *lexicographical sum* $\sum_{i \in I} P_i$ is the poset defined on $\bigcup_{i \in I} V_i$ by $x \leq y$ if and only if

(a) There exists $i \in I$ such that $x, y \in V_i$ and $x \leq_i y$ in P_i; or
(b) There are distinct elements $i, j \in I$ such that $i < j$ in I, $x \in V_i$ and $y \in V_j$.

The posets P_i are called the *components* of the lexicographical sum and the poset I is the *index set.* If I is a totally ordered set, then $\sum_{i \in I} P_i$ is called a *linear sum.* On the other hand, if I is an antichain, then $\sum_{i \in I} P_i$ is called a *direct sum.* Henceforth we will use the symbol $\oplus$ to indicate a direct sum.

The decomposition of the incomparability graph of a poset into connected components is expressed in the following lemma which belongs to the folklore of the theory of ordered sets.

Lemma 2.1. *If $P := (V, \leq)$ is a poset, the order on P induces a total order on the set* Connect(P) *of connected components of* Inc(P)*, the incomparability graph of P, and P is the lexicographical sum of these components indexed by the chain* Connect(P)*. In particular, if $\preceq$ is a total order extending the order $\leq$ of P, each connected component A of* Inc(P) *is an interval of the chain $(V, \preceq)$.*

The next two sections introduce the necessary ingredients to the proof of Theorem 1.2.

3. A fundamental lemma

We state an improvement of I.2.2 Lemme, p.5 of [18].

Lemma 3.1. *Let x, y be two vertices of a poset P with $x < y$. If $x_0, \ldots, x_n$ is an induced path in the incomparability graph of P from x to y then $x_i < x_j$ for all $j - i \geq 2$.*

Proof. Induction on n. If $n \leq 2$ the property holds trivially. Suppose $n \geq 3$. Taking out x_0, induction applies to $x_1, \dots, x_n$. Similarly, taking out x_n, induction applies to $x_0, \dots, x_{n-1}$. Since the path from x_0 to x_n is induced, x_0 is comparable to every x_j with $j \geq 2$ and x_n is comparable to every x_j with $j < n-1$. In particular, since $n \geq 3$, x_0 is comparable to x_{n-1}. Necessarily, $x_0 < x_{n-1}$. Otherwise, $x_{n-1} < x_0$ and then by transitivity $x_{n-1} < x_n$ which is impossible since $\{x_{n-1}, x_n\}$ is an edge of the incomparability graph. Thus, we may apply induction to the path from $x_0, \dots, x_{n-1}$ and get $x_0 < x_j$ for every $j > 2$. Similarly, we get $x_1 < x_n$ and via the induction applied to the path from x_1 to x_n, $x_j < x_n$ for $j < n-1$. The stated result follows. □

An immediate corollary is this.

Corollary 3.2. *Let P be a poset such that* $\mathrm{Inc}(P)$ *is connected and let $a < b$. If (a,b) is a covering relation in P, then* $2 \leq d_{\mathrm{Inc}(P)}(a,b) \leq 3$.

Another consequence of Lemma 3.1 is that incomparability graphs have no induced cycles of length at least five [7]. Indeed, let P be a poset and let $x_0, \dots, x_l, x_0$ be an induced cycle of $\mathrm{Inc}(P)$. Suppose for a contradiction that $l \geq 4$. We will apply Lemma 3.1 successively to the induced paths $x_0, \dots, x_{l-1}$ and $x_1, \dots, x_l$ and will derive a contradiction. We may assume without loss of generality that $x_0 < x_{l-1}$. It follows from Lemma 3.1 applied to $x = x_0$ and $y = x_{l-1}$ that $x_0 < x_{l-2}$ (recall that $l \geq 4$) and $x_1 < x_{l-1}$. We now consider the induced path $x_1, \dots, x_l$. Then x_1 and x_l are comparable. It follows from $x_1 < x_{l-1}$ and Lemma 3.1 applied to $x = x_1$ and $y = x_l$ that $x_1 < x_l$. Hence, $x_{l-2} < x_l$. By transitivity we get $x_0 < x_l$ which is impossible.

Here is yet another consequence of Lemma 3.1.

Proposition 3.3. *Let $P := (V, \leq)$ be a poset. A sequence $a_0, \dots, a_n, \dots$ of vertices of V forms an induced path in* $\mathrm{Inc}(P)$ *originating at a_0 if and only if for all $i \in \mathbb{N}$, $a_i, a_{i+1}, a_{i+2}, a_{i+3}$ is an induced path of* $\mathrm{Inc}(P)$ *with extremities a_i, a_{i+3}.*

Proof. $\Rightarrow$ Obvious.
$\Leftarrow$ Suppose that for all $i \in \mathbb{N}$, $a_i, a_{i+1}, a_{i+2}, a_{i+3}$ is an induced path with extremities a_i, a_{i+3}. We prove by induction that for all $n \in \mathbb{N}$, $a_0, \dots, a_n$ is an induced path in G. Suppose $a_0, \dots, a_n$ is an induced path in G and assume without loss of generality that $a_0 < a_n$. Then $a_i < a_n$ for all $i \leq n-2$ (follows from Lemma 3.1). From $a_{n-2}, a_{n-1}, a_n, a_{n+1}$ is an induced path with extremities a_{n-2}, a_{n+1} and $a_{n-2} < a_n$ we deduce that $a_{n-2} < a_{n+1}$ and $a_{n-1} < a_{n+1}$. Therefore, $a_i < a_{n+1}$ for all $i \leq n-1$ proving that $a_0, \dots, a_n, a_{n+1}$ is an induced path in G. □

We should mention that the value of 3 from the previous proposition is the best possible. Indeed, if P is the direct sum of two copies of the chain of natural numbers, then $\mathrm{Inc}(P)$ is a complete bipartite graph and every path on 3 vertices is an induced path. Yet an infinite sequence of vertices that alternates between the copies of $\mathbb{N}$ does not constitute an infinite induced path of $\mathrm{Inc}(P)$.

4. Posets of width 2 and their distances

4.1. Posets of width 2 and bipartite permutation graphs. In this subsection, we recall some properties about posets of width at most 2 and permutation graphs. We start with a characterization of bipartite permutation graphs, next we give some properties of the graphic distance and the detour in comparability graphs of posets of width at most 2. We recall the existence of a universal poset of width at most 2 [18]. We describe the incomparability graph of a variant of this poset more appropriate for our purpose.

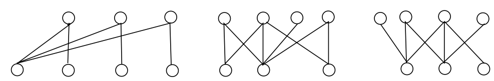

Figure 5. Critical posets of dimension 3 and height 2.

We note that a poset P of width at most 2 has dimension at most 2, hence its comparability graph is an incomparability graph. As previously mentioned, a finite graph G is a comparability and incomparability graph if and only if it is a permutation graph. Incomparability graphs of finite posets of width 2 coincide with bipartite permutation graphs. For arbitrary posets, the characterization is as follows.

Lemma 4.1. *Let G be a graph. The following are equivalent.*

(i) G is bipartite and is the comparability graph of a poset of dimension at most two;
(ii) G is bipartite and embeds no even cycles of length at least six and none of the comparability graphs of the posets depicted in Figure 5.
(iii) G is the incomparability graph of a poset of width at most 2.
(iv) G is a bipartite incomparability graph.

Proof. $(i) \Leftrightarrow (ii)$. If G is finite, this is Theorem 1 of [22]. Hence, the equivalence between (i) and (ii) holds for the restrictions of G to every finite set F of vertices. This gives immediately the implication $(i) \Rightarrow (ii)$. For the converse implication, we get that every finite induced subgraph of G is bipartite and the comparability graph of a poset of dimension at most two. The Compactness Theorem of First Order Logic implies that these properties extend to G.
$(iii) \Rightarrow (i)$. Suppose G is the incomparability graph of a poset of width at most 2. Then G has no 3-element cycles. Also, G has no induced odd cycles of length at least five (see [7], section 3.8, Table 5). This shows that G is bipartite. Since P is coverable by two chains it has order dimension two (the dimension of a poset is at most its width [2]) and therefore its incomparability graph is also a comparability graph [3]. Thus G is a comparability

graph of a poset of dimension at most two.
$(i) \Leftrightarrow (iv)$ follows from the fact that a graph G is the incomparability graph of a poset of dimension at most 2 if and only if this is the comparability graph of a poset of dimension at most 2 [3]. $(iii) \Leftrightarrow (iv)$. Implication $(iv) \Rightarrow (iii)$ is trivial. For the converse, suppose that G is a bipartite incomparability graph of a poset P, apply Dilworth's theorem [2] or pick any vertex x, and observe that the set of vertices at odd distance from x and the set of vertices at even distance from x form a partition of P into two chains, hence G is bipartite. □

We should mention the following result (this is essentially Lemma 14 from [25]) which states that a bipartite permutation graph without cycles must embed a caterpillar. A key observation is that if a vertex has at least three neighboring vertices in Inc(P), then at least one has degree one. Otherwise, Inc(P) would have a spider (see Figure 5) as an induced subgraph, which is impossible.

Lemma 4.2. *Let P be a poset of coverable by two chains. Then the following properties are equivalent.*

(i) The incomparability graph of P has no cycles of length three or four.
(ii) The incomparability graph of P has no cycle.
(iii) The connected components of the incomparability graph of P are caterpillars.

4.2. **Detour of bipartite permutation graphs.** We are going to evaluate the detour of connected components of the incomparability graph of a poset of width at most 2.

Let $P := (V, \leq)$ be a poset of width 2. Suppose that Inc(P) is connected. In this case, the partition of P into two chains is unique. An *alternating sequence* in P is any finite monotonic sequence $(x_0, \dots, x_i, \dots, x_n)$ of elements of V (i.e., increasing or decreasing) such that no two consecutive elements x_i and x_{i+1} belong to the same chain of the partition. The integer n is the *oscillation* of the sequence; x and y are its *extremities.*

We recall that the oscillation of an alternating sequence with extremities x, y is either 0 or at most $d_{\mathrm{Inc}(P)}$ (see I.2.4. Lemme p.6 of [18]). This allows to us define the following map. Let d_P be the map from $V \times V$ into $\mathbb{N}$ such that.

(1) $d_P(x, x) = 0$ for every $x \in V$;
(2) $d_P(x, y) = 1$ if x and y are incomparable;
(3) $d_P(x, y) = 2$ if x and y are comparable and there is no alternating sequence from x to y;
(4) $d_P(x, y) = n + 2$ if $n \neq 0$ and n is the maximum of the oscillation of alternating sequences with extremities x and y.

We recall a result of [18] II.2.5 Lemme, p. 6.

Lemma 4.3. *The map d_P is a distance on any poset P of width 2 such that the incomparability graph is connected. Moreover, for every $x, y \in P$ the* *following inequalities hold:*

$$0 \leq d_{\mathrm{Inc}(P)}(x,y) - d_P(x,y) \leq 2\lfloor d_{\mathrm{Inc}(P)}(x,y)/3 \rfloor. \tag{4.1}$$

We give a slight improvement of [18] I.2.3. Corollaire, p. 5.

Lemma 4.4. *Let P be poset of width 2 such that $\mathrm{Inc}(P)$ is connected. Let $n \in \mathbb{N}$, $r \in \{0,1\}$ and $x, y \in P$ such that $\mathrm{Inc}(P)$ contains an induced path of length $3n+r$ and extremities x and y. If $r \neq 1$ and $n \geq 1$ (resp. $r = 1$ and $n \geq 2$) then there is an alternating sequence with extremities x,y and oscillation n (resp. $n-1$).*

Proof. Since $n \geq 1$, x and y are comparable and we may suppose $x < y$. Let $x_0, \dots, x_{3n+r}$ be a path with $x_0 = x$, $x_{3n+r} = y$. According to Lemma 3.1 the sequence $x_0, \dots, x_{3i}, \dots, x_{3n}$ is alternating. If $r \neq 1$, we may replace x_{3n} by x_{3n+r} in the above sequence and get an alternating sequence with extremities x,y and oscillation n. If $r = 1$, we delete x_{3n} and replace $x_{3(n-1)}$ by x_{3n+r} in the above sequence. We get an alternating sequence of oscillation $n-1$. □

From Lemma 4.3, the oscillation between two vertices x and y of P is bounded above. With this lemma, the length of induced paths between x and y is bounded too, that is the detour $D_{\mathrm{Inc}(P)}(x,y)$ is an integer. In fact, we have:

Proposition 4.5. *Let P be poset of width 2 such that $\mathrm{Inc}(P)$ is connected and let $x, y \in P$. Then:*

(1) $d_{\mathrm{Inc}(P)}(x,y) = d_P(x,y) = D_{\mathrm{Inc}(P)}(x,y)$ *if either $x = y$, in which case this common value is 0, or x and y are incomparable, in which case this common value is 1.*
(2) $d_{\mathrm{Inc}(P)} \geq d_P(x,y) \geq \lfloor D_{\mathrm{Inc}(P)}(x,y)/3 \rfloor + \epsilon$ *where $\epsilon = 1$ if $D_{\mathrm{Inc}(P)}(x,y) \equiv 1 \mod 3$ and $\epsilon = 2$ otherwise.*

Proof. Assertion (1) is obvious. For (2), we may suppose $x < y$. The first inequality is embodied in Lemma 4.3. As observed above, $D_{\mathrm{Inc}(P)}(x,y)$ is bounded. We may write $D_{\mathrm{Inc}(P)}(x,y) = 3n + r$ with r be the remainder of $D_{\mathrm{Inc}(P)}(x,y) \mod 3$. Let $\alpha := \lfloor D_{\mathrm{Inc}(P)}(x,y)/3 \rfloor + \epsilon$. We have $\alpha = n+1$ if $r = 1$ and $\alpha = n+2$ otherwise. If $n = 0$ then since $x < y$, $r \neq 1$, hence $\alpha = 2$, since $d_P(x,y) = 2$, the inequality holds. We may suppose $n \geq 1$. If $r \neq 1$ then $\alpha = n+2$, while by definition of d_P and Lemma 4.4, $d_P(x,y) \geq n+2$. Hence, the second inequality holds. If $r = 1$ then $\alpha = n+1$. If $n = 1$ $d_P(x,y) \geq 2$ and the second inequality holds. Suppose $n \geq 2$. Then, by definition of $d_P(x,y)$ and by Lemma 4.4, $d_P(x,y) \geq n+1$. Thus second inequality holds. □

Corollary 4.6. *If a bipartite permutation graph has diameter at most k it contains no induced path of length $3k$.*

5. A proof of Theorem 1.2

Proof. The implication $(i) \Rightarrow (ii)$ is obvious. The implication $(ii) \Rightarrow (iii)$ follows from Proposition 4.5 given in Subsection 4.2. The implication $(iii) \Rightarrow (iv)$ follows from Theorem 1.4. The implication $(iv) \Rightarrow (i)$ is obvious. □

6. Example 1.3

Proof. Let $X := \{y, x_0, x_1, x_2, \dots\}$ and for every integer $i \geq 0$ let $Z_i := \{z_{0,i}, z_{1,i}, \dots, z_{i+3,i}\}$ be disjoint sets. We set $V := \bigcup_{i\geq 0} Z_i \cup X$ and $P := (V, \leq)$ where $\leq$ is the binary relation on V defined as follows: $X \setminus \{y\}$ is totally ordered by $\leq$ and $x_0 < x_1 < x_2 < \cdots < x_i < \dots$. For all $0 \leq i < j$, every element of Z_i is below every element of Z_j. For all $i \geq 0$, y is smaller than all elements in Z_i and is incomparable to x_i. For all $i \geq 0$, x_i is smaller than all element of $Z_i \setminus \{z_{0,i}\}$ and x_i is incomparable to all elements in $\bigcup_{j<i} Z_i \cup \{z_{0,i}\}$. For all integers $i \geq 0$ and for all $j \geq i+1$, x_i is smaller than all element in Z_j. Finally, the restriction of $\operatorname{Inc}(P)$ to Z_i is the induced path $z_{0,i}, z_{1,i}, \dots, z_{i+3,i}$ so that $z_{0,i} < z_{2,i} < z_{4,i} < \dots$ and $z_{1,i} < z_{3,i} < z_{5,i} < \dots$ (see Figure 1). It is not difficult to see that $\leq$ is an order relation and that the corresponding poset P can be covered by three chains.

Claim 1: The diameter of $\operatorname{Inc}(P)$ is 3.

Let a, b be two distinct vertices of $\operatorname{Inc}(P)$. If $a, b \in X$, then either $a = y$ or $b = y$ in which case $d_{\operatorname{Inc}(P)}(a,b) = 1$, or $y \notin \{a,b\}$ in which case $d_{\operatorname{Inc}(P)}(a,b) = 2$ (indeed, say $a = x_i$ and $b = x_j$ with $i < j$, then $a, z_{0,i}, b$ is an induced path in $\operatorname{Inc}(P)$). Suppose now $a \in X$ and $b \notin X$, say $b \in Z_i$ for some $i \geq 0$. If $a = y$, then $d_{\operatorname{Inc}(P)}(a,b) = 2$ (indeed, a, x_{i+1}, b is an induced path in $\operatorname{Inc}(P)$). Else if $a = x_j$ for some $j \geq 0$, then $d_{\operatorname{Inc}(P)}(a,b) = 1$ if $i < j$ and $d_{\operatorname{Inc}(P)}(a,b) = 3$ otherwise (indeed, $a, z_{0,j}, x_{i+1}, b$ is the shortest path joining a to b). Next we suppose that $\{a,b\} \cap X = \varnothing$. If $a, b \in Z_i$ for some $i \geq 0$, then $d_{\operatorname{Inc}(P)}(a,b) = 2$ (indeed, a, x_{i+1}, b is an induced path in $\operatorname{Inc}(P)$). Else if $a \in Z_i$ and $b \in Z_j$ for some $i \neq j$, then $d_{\operatorname{Inc}(P)}(a,b) = 2$ (indeed, a, x_{i+j}, b is an induced path in $\operatorname{Inc}(P)$).

Claim 2: An induced infinite path in $\operatorname{Inc}(P)$ contains necessarily finitely many elements of X.

Suppose an induced infinite path C contains infinitely many vertices from X. Since $\operatorname{Inc}(P)$ induces an independent set on $X \setminus \{y\}$ and C is connected we infer that C must meet infinitely many Z_i's. Hence, there exists some $x_i \in C$ which has degree at least 3 in C and this is not possible.

Claim 3: Deleting all vertices of X from $\operatorname{Inc}(P)$ leaves a disconnected graph.

Clearly, for all $i \geq 0$, Z_i is a connected component of $\operatorname{Inc}(P) \setminus X$.

Now suppose for a contradiction that $\operatorname{Inc}(P)$ embeds an infinite induced path C. It follows from Claim 2 that we can assume $V(C) \cap X = \varnothing$. Hence, C is an induced infinite path of $\operatorname{Inc}(P) \setminus X$. We derive a contradiction since all connected components of $\operatorname{Inc}(P) \setminus X$ are finite (indeed, the connected components of $\operatorname{Inc}(P) \setminus X$ are finite paths i.e. the subgraphs of $\operatorname{Inc}(P) \setminus X$ induced on the Z_i's).

Claim 4: The vertex y has an infinite induced detour.

Indeed, $\operatorname{Inc}(P)$ induces a path on $\{y, x_i,\} \cup Z_i$ of length $i+5$ for all $i \geq 0$. □

7. Order and metric convexities of incomparability graphs

In this section, we compare the notions of order convexity and metric convexity with respect to the distance on the incomparability graph of a poset.

We recall few definitions already provided in the introduction. Let $P := (V, \leq)$ be a poset. We recall $\mathrm{Conv}_P(X)$ is the smallest convex set containing X and that

$$\mathrm{Conv}_P(X) = \{z \in P : x \leq z \leq y \text{ for some } x, y \in X\} = \downarrow X \cap \uparrow X.$$

Let $G := (V, E)$ be a graph. We equip it with the graphic distance d_G. A *ball* is any subset $B_G(x, r) := \{y \in V : d_G(x, y) \leq r\}$ where $x \in V, r \in \mathbb{N}$. A subset of V is *convex* with respect to the distance d_G if this is an intersection of balls. The *least convex subset* of G containing X is

$$\mathrm{Conv}_G(X) := \bigcap_{X \subseteq B_G(x,r)} B_G(x, r).$$

Let $X \subseteq V$ and $r \in \mathbb{N}$. Define

$$B_G(X, r) := \{v \in V : d_G(v, x) \leq r \text{ for some } x \in X\}.$$

The proof of the following lemma is elementary and is left to the reader.

Lemma 7.1. *Let G be a graph, $X \subseteq V(G)$ and $r \in \mathbb{N}$. Then*

(1) $B_G(X, r) = B_G(B_G(X, 1), r - 1) = B_G(B_G(X, r - 1), 1)$ *for all* $r \geq 1$.
(2) $B_G(X \cup Y, r) = B_G(X, r) \cup B_G(X, r)$.

Lemma 7.2. *Let $P := (V, \leq)$ be a poset and G be its incomparability graph, $X \subseteq V$ and $r \in \mathbb{N}$. Then*

$$B_G(\downarrow X, r) = (\downarrow X) \cup B_G(X, r) = \downarrow B_G(X, r), \tag{7.1}$$

$$B_G(\uparrow X, r) = (\uparrow X) \cup B_G(X, r) = \uparrow B_G(X, r), \tag{7.2}$$

$$B_G(\uparrow X \cap \downarrow X, r) = B_G(\uparrow X, r) \cap B_G(\downarrow X, r), \tag{7.3}$$

$$B_G(\mathrm{Conv}_P(X), r) = \mathrm{Conv}_P(X) \cup B_G(X, r) = \mathrm{Conv}_P(B_G(X, r)). \tag{7.4}$$

Proof. We mention at first that all above equalities are clearly true for $r = 0$. We claim that it is enough to prove (7.1). Indeed, (7.2) is obtained from (7.1) applied to P^*. We now show how to obtain (7.3) using (7.1) and (7.2). The proof is by induction on r.
Basis step: $r = 1$.
Clearly,

$$B_G(\uparrow X \cap \downarrow X, 1) \subseteq B_G(\uparrow X, 1) \cap B_G(\downarrow X, 1).$$

Let $x \in B_G(\uparrow X, 1) \cap B_G(\downarrow X, 1)$. There are $y_1 \in \downarrow X$ and $y_2 \in \uparrow X$ such that x is equal to y_1 or incomparable to y_1 and similarly x is equal to y_2 or incomparable to y_2. Since $y_1 \in \downarrow X$ and $y_2 \in \uparrow X$ there are $x_1, x_2 \in X$ such that $y_1 \leq x_1$ and $x_2 \leq y_2$. If x is incomparable or equal to x_1 or to x_2, then $x \in B_G(X, 1) \subseteq B_G(\uparrow X \cap \downarrow X, 1)$ as required. If not, $x_2 \leq x \leq x_1$ (since x is equal to y_1 or incomparable to y_1 and x is equal to y_2 or incomparable to y_2), hence $x \in \downarrow X \cap \uparrow X \subseteq B_G(\downarrow X \cap \uparrow X, 1)$, as required.

Inductive step: Suppose $r > 1$. We have

$$\begin{aligned}
B_G(\uparrow X \cap \downarrow X, r) &= B_G(B_G(\uparrow X \cap \downarrow X, r-1), 1)\\
&= B_G(B_G(\uparrow X, r-1) \cap B_G(\downarrow X, r-1), 1)\\
&\qquad \text{(by the induction hypothesis)}\\
&= B_G(\uparrow B_G(X, r-1) \cap \downarrow B_G(X, r-1), 1)\\
&\qquad \text{(by equations (7.1) and (7.2))}\\
&= B_G(\uparrow B_G(X, r-1), 1) \cap B_G(\downarrow B_G(X, r-1), 1)\\
&\qquad \text{(follows from the basis step } r = 1)\\
&= \uparrow B_G(B_G(X, r-1), 1) \cap \downarrow B_G(B_G(X, r-1), 1)\\
&\qquad \text{(follows from (7.1) and (7.2))}\\
&= \uparrow B_G(X, r) \cap \downarrow (B_G(X, r))\\
&= B_G(\uparrow X, r) \cap (\downarrow B_G(X, r)) \text{ (follows from (7.1))}.
\end{aligned}$$

We now show how to obtain (7.4) using (7.1), (7.2) and (7.3).
From (7.1) and (7.2) we obtain

$$\begin{aligned}
B_G(\downarrow X, r) \cap B_G(\uparrow X, r) &= ((\downarrow X) \cup B_G(X, r)) \cap ((\uparrow X) \cup B_G(X, r))\\
&= \downarrow (B_G(X, r)) \cap \uparrow (B_G(X, r)).
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
B_G(\downarrow X, r) \cap B_G(\uparrow X, r) &= (\downarrow X \cap \uparrow X) \cup B_G(X, r))\\
&= \downarrow (B_G(X, r)) \cap \uparrow (B_G(X, r)).
\end{aligned}$$

Using (7.3) we have

$$B_G(\downarrow X \cap \uparrow X, r) = (\downarrow X \cap \uparrow X) \cup B_G(X, r)) = \downarrow (B_G(X, r)) \cap \uparrow (B_G(X, r)).$$

The required equalities follow by definition of the operator $Conv$.

We now prove (7.1).
Basis step: $r = 1$.
Since $X \subseteq \downarrow X$ we have $B_G(X, 1) \subseteq B_G(\downarrow X, 1)$. Hence, we have

$$B_G(\downarrow X, 1) \supseteq (\downarrow X) \cup B_G(X, 1).$$

From $X \subseteq B_G(X, 1)$ we deduce that $\downarrow X \subseteq \downarrow (B_G(X, 1))$. Hence,

$$(\downarrow X) \cup B_G(X, 1) \subseteq \downarrow B_G(X, 1).$$

Next, we prove that $B_G(\downarrow X, 1) \subseteq (\downarrow X) \cup B_G(X, 1)$. Let $x \in B_G(\downarrow X, 1)$. There exists then $y \in \downarrow X$ at distance at most 1 from x that is either $y = x$ or $y \parallel x$. If $y = x$ then $x \in \downarrow X$. Otherwise, since $y \in \downarrow X$ there is $y_1 \in X$ such that $y \leq y_1$. If y_1 is incomparable or equal to x then $x \in B_G(X, 1)$. Otherwise y_1

is comparable to x. Necessarily, $x \leq y_1$ since $x \parallel y$. Hence $x \in \downarrow X$.
Inductive step: Let $r > 1$. We suppose true the equalities

$$B_G(\downarrow X, r-1) = (\downarrow X) \cup B_G(X, r-1) = \downarrow B_G(X, r-1).$$

We apply the operator $T \longrightarrow B_G(T, 1)$ to each term of the previous equalities and obtain

$$B_G(B_G(\downarrow X, r-1), 1) = B_G((\downarrow X) \cup B_G(X, r-1), 1) = B_G(\downarrow B_G(X, r-1), 1).$$

We have

$$B_G(B_G(\downarrow X, r-1), 1) = B_G(\downarrow X, r) \text{ (see (1) of Lemma 7.1).}$$

Also,

$$\begin{aligned} B_G((\downarrow X) \cup B_G(X, r-1), 1) &= B_G(\downarrow X, 1) \cup B_G(B_G(X, r-1), 1) \\ &\quad \text{(see (2) of Lemma 7.1))} \\ &= B_G(\downarrow X, 1) \cup B_G(X, r) \\ &\quad \text{(see (2) of Lemma 7.1)} \\ &= (\downarrow X) \cup B_G(X, 1) \cup B_G(X, r) \\ &\quad \text{(follows from (7.1) with } r = 1\text{).} \\ &= (\downarrow X) \cup B_G(X, r). \end{aligned}$$

Finally we have

$$\begin{aligned} B_G(\downarrow (B_G(X, r-1)), 1) &= \downarrow B_G((B_G(X, r-1)), 1) \\ &= \downarrow (B_G(X, r)). \end{aligned}$$

□

8. A proof of Theorem 1.10 and some consequences

We now proceed to the proof of Theorem 1.10.

Proof. (a) Apply successively equations (7.1), (7.2) and (7.4) of Lemma 7.2. (b) Suppose $r = 1$. Let $G' := G_{\restriction B_G(X,1)}$ and $x, y \in B_G(X, 1)$. Let $n := d_G(x, y)$. Clearly, $n \leq d_{G'}(x, y)$. To prove that the equality holds, we may suppose that $2 \leq n < \infty$. We argue by induction on n. Let $u_0, \dots, u_n$ be a path in G connecting x and y. If $n \geq 4$, we have $x_0 < x_2 < x_n$ by Lemma 3.1. Since $B_G(X, 1)$ is convex, it contains x_2, hence, by induction, $d_G(x, x_2) = d_{G'}(x, x_2) = 2$ and $d_G(x_2, y) = d_{G'}(x_2, y) = n-2$, hence $d_G(x, y) = d_{G'}(x, y)$. Thus, to conclude, it suffices to solve the cases $n = 2$ and $n = 3$. Let $x', y' \in X$ with x' incomparable or equal to x and y' incomparable or equal to y. If u_{n-1} is incomparable or equal to y' then $x_{n-1} \in B_G(X, 1)$. From the induction, $d_{G'}(x, x_{n-1}) = d_G(x, x_{n-1})$ hence $d_{G'}(x, y) = d_G(x, y)$ as required. Hence, we may suppose u_{n-1} comparable to y', and similarly u_1 comparable to x'.

Also, if x' is incomparable or equal to x_2 then x, x', x_2 is a path in $B_G(X)$; if $n = 2$ we have $d_{G'}(x, y) = 2$ as required, if $n = 3$, then x, x', x_2, y is a path in

$B_G(X)$ and $d_{G'}(x,y) = 3$ as required. Thus we may suppose x' comparable to x_2 and, similarly, y' comparable to x_{n-2}.

Since x' is incomparable or equal to x_0 and, by Lemma 3.1, $x_0 < x_2$, we have $x' < x_2$. Similarly, we have $x_{n-2} < y'$. Since x_1 is comparable to x' and incomparable to x_2 we deduce $x' \leq x_1$ from $x' < x_2$. Similarly, we deduce $x_{n-1} \leq y'$. For $n = 2$ we have $x' \leq x_1, \leq y'$ and for $n = 3$, $x' \leq x_1, x_2 \leq y'$. By order convexity of X, $x_1 \in X$ (and also $x_2 \in X$ if $n = 3$, hence the path $x = x_0, x_1, x_2 = y$ if $n = 2$ and the path $x = x_0, x_1, x_2, x_3 = y$ if $n = 3$ is in $B_G(X)$ and thus $d_{G'}(x,y) = n$.

Suppose $r > 1$. Then from (a) above, $B_G(X,1)$ order convex. Via the induction hypothesis, $G_{\restriction B_G(B_G(X,1),r-1)}$ is an isometric subgraph of G. Since $B_G(X,r) = B_G(B_G(X,1), r-1)$, $G_{\restriction B_G(X,r)}$ is an isometric subgraph of G. □

As the proof of the Lemma 7.2 suggests, balls are not necessarily geodesically convex (for an example, look at the ball $B(x,1)$ in a four element cycle). A consequence of Theorem 1.10 is that the order convexity of balls is equivalent to the following inequality:

Corollary 8.1. *Let P be a poset and let G be its incomparability graph. Then*

$$d_G(u,v) \leq d_G(x,y) \text{ for all } x \leq u \leq v \leq y \text{ in } P. \tag{8.1}$$

Proof. The inequality above amounts to $d_G(u,v) \leq d_G(x,v) \leq d_G(x,y)$. We prove the first inequality; the second inequality follows by the same argument applied to the dual of P. We may suppose that $x < u < v$, otherwise nothing to prove. Let $n := d_G(v,x)$. By (a) of Theorem 1.10, $B_G(v,n)$ is order convex. Since $x, v \in B_G(v,n)$ and $x \leq u \leq v$, then $u \in B(v,n)$ amounting to $d_G(u,v) \leq n = d_G(x,v)$. Conversely, assuming that inequality (8.1) holds, observe that every ball $B_G(x,r)$ is order-convex. We may suppose $r \geq 1$, otherwise the conclusion is obvious. Let $u, v \in B_G(x,r)$ and $w \in P$ with $u < w < v$. If $x \parallel w$, then $d_G(x,w) = 1 \leq r$ hence $w \in B_G(x,r)$. If not, then either $x < w$ or $w < x$. In the first case, from $x < w < v$, inequality (8.1) yields $d_G(x,w) \leq d_G(x,v) \leq r$ hence $w \in B_G(x,r)$, whereas in the second case, from $u < w < x$, inequality (8.1) yields $d_G(w,x) \leq d_G(u,x) \leq r$ hence $w \in B_G(x,r)$. □

Corollary 8.2. $\delta_G(X) = \delta_G(\mathrm{Conv}_P(X)) = \delta_G(\mathrm{Conv}_G(X))$ *for every subset X of a poset P.*

Proof. Since by (a) of Theorem 1.10, each ball $B_G(x,r)$ is order convex, $\mathrm{Conv}_P(X) \subseteq \mathrm{Conv}_G(X)$. Hence

$$\delta_G(X) \leq \delta_G(\mathrm{Conv}_P(X)) \leq \delta_G(\mathrm{Conv}_G(X)).$$

The equality $\delta_G(X) = \delta_G(\mathrm{Conv}_G(X))$ is a general convexity property of metric spaces. Let $r := \delta_G(X)$. Let $x, y \in \mathrm{Conv}_G(X)$. We prove that $d_G(x,y) \leq r$. First $X \subseteq B_G(x,r)$. Indeed, let $z \in X$; since $\delta_G(X) = r$,

$X \subseteq B_G(x,r)$. Since $\mathrm{Conv}_G(X)$ is the intersections of balls containing X, we have $\mathrm{Conv}(X) \subseteq B_G(z,r)$, hence $z \in B_G(x,r)$. Next, from $X \subseteq B_G(x,r)$ we deduce $\mathrm{Conv}(X) \subseteq B_G(x,r)$ hence $y \in B_G(x,r)$ that is $d_G(x,y) \leq r$. □

Lemma 8.3. *Let $P := (V, \leq)$ be a poset and G be its incomparability graph. Let $x, y, z \in V$ be such that $x < z < y$. Then*

$$\max\{d_G(x,z), d_G(z,y)\} \leq d_G(x,y) \leq d_G(x,z) + d_G(z,y) \leq d_G(x,y) + 2.$$

Proof. The first inequality follows from Corollary 8.1. The second inequality is the triangular inequality. We now prove the third inequality. Let

$$p := d_G(x,z), \quad q := d_G(z,y), \quad r := d_G(x,y).$$

Claim 8.4. *Let $x_0 := x, \dots, x_r := y$ be a path from x to y. Then there exist $i \notin \{0, r\}$ such that z is incomparable to x_i.*

Subproof. By induction on r. Note that since $x < y$ we have $r \geq 2$. If $r = 2$, then necessarily z is incomparable to x_1. Suppose $r > 2$. Then $z \nleq x_1$. If z is incomparable to x_1, then we are done. Otherwise $x_1 < z$ and we may apply the induction hypothesis to x_1, y and the path $x_1, \dots, x_r = y$. This completes the proof of Claim 8.4. □

Let i be such in Claim 8.4. Then $x_0 := x, \dots, x_i, z$ is a path from x to z of length $i+1$ and $z, x_i, x_{i+1}, \dots, x_r$ is a path from z to y of length $r-i+1$. Then $p + q \leq i + 1 + r - i + 1 = r + 2$. The proof of the lemma is now complete. □

Lemma 8.5. *Let $x_0, \dots, x_n$ be an isometric path in a graph G with $n \geq 2$. There exists a vertex x_{n+1} such that $x_0, \dots, x_n, x_{n+1}$ is an isometric path in G if and only if $B_G(x_n, 1) \nsubseteq B_G(x_0, n)$.*

Proof. $\Rightarrow$ is obvious.
$\Leftarrow$ Suppose $B_G(x_n, 1) \nsubseteq B_G(x_0, n)$ and let $x_{n+1} \in B_G(x_n, 1) \setminus B_G(x_0, n)$.

Claim 8.6. $d_G(x_0, x_{n+1}) = n + 1$.

Indeed, since $x_{n+1} \in B_G(x_n, 1) \setminus B_G(x_0, n)$ we have $d_G(x_0, x_{n+1}) > n$. From the triangular inequality $d_G(x_0, x_{n+1}) \leq d_G(x_0, x_n) + d_G(x_n, x_{n+1}) = n + 1$.

Claim 8.7. $d_G(x_j, x_{n+1}) = n + 1 - j$ *for all* $0 \leq j \leq n$.

Indeed, From the triangular inequality

$$d_G(x_j, x_{n+1}) \leq d_G(x_j, x_n) + d_G(x_n, x_{n+1}) = n - j + 1.$$

Similarly,

$$d_G(x_0, x_{n+1}) \leq d_G(x_0, x_j) + d_G(x_j, x_{n+1})$$

and therefore

$$d_G(x_j, x_{n+1}) \geq d_G(x_0, x_{n+1}) - d_G(x_0, x_j) = n + 1 - j.$$

The equality follows. □

We could restate Lemma 8.5 as follows. There is an isometric path of length $n+1$ starting at some vertex x_0 if there is some $x_n \in B_G(x_0, n)$ such that $B_G(x_n, 1) \notin B_G(x_0, n)$.

Another consequence of the convexity of balls in an incomparability graph is the following:

Lemma 8.8. *Let G be the incomparability graph of a poset P. If a ball contains infinitely many vertices of a one way infinite induced path then it contains all vertices except may be finitely many vertices of that path.*

Proof. Let P_∞ be an infinite induced path of G and $(x_n)_{n\in\mathbb{N}}$ be an enumeration of its vertices, so that $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$. Without loss of generality we may suppose that $x_0 < x_2$ (otherwise, replace the order of P by its dual). By Lemma 3.1 we have $x_i < x_j$ for every $i+2 \leq j$. Let $B_G(x, r)$ be a ball of G containing infinitely many vertices of P_∞. Let $x_i \in \mathrm{P}_\infty \cap B(x, r)$. We claim that $x_j \in \mathrm{P}_\infty \cap B(x, r)$ for all $j \geq i+2$. Indeed, due to our hypothesis, we may pick $x_r \in \mathrm{P}_\infty \cap B(x, r)$ with $r \geq j+2$. We have $x_i < x_j < x_r$. Due to the convexity of $B(x, r)$ we have $x_j \in B(x, r)$. This proves our claim. □

Said differently:

Lemma 8.9. *If a one way infinite induced path P_∞ has an infinite diameter in the incomparability graph G of a poset then every ball of G with finite radius contains only finitely many vertices of P_∞.*

9. Induced infinite paths in incomparability graphs: A proof of Theorem 1.4

The proofs of (1) and (2) of Theorem 1.4 are similar. We construct a strictly increasing sequence $(y_n)_{n\in\mathbb{N}}$ of vertices such that $3 \leq d_G(y_n, y_{n+1}) < +\infty$ for all $n \in \mathbb{N}$ and we associate to each $n \in \mathbb{N}$ a finite path $\mathrm{P}_n := z_{(n,0)}, z_{(n,1)}, \dots, z_{(n,r_n)}$ of G of length $r_n := d_G(y_n, y_{n+1})$ joining y_n and y_{n+1}. We show first that the graph $G' := G_{\restriction \bigcup_{n\in\mathbb{N}} V(\mathrm{P}_n)}$ is connected and has an infinite diameter. Next, we prove that it is locally finite. Hence from Kőnig's Lemma (1.5), it contains an isometric path. This path yields an induced path of G. The detour via Kőnig's Lemma is because the union of the two consecutive paths P_n and P_{n+1} do not form necessarily a path. In the first proof, our paths have length 3. In the second proof, their end vertices have degree at least 3.

Lemma 9.1. *Let $P := (V, \leq)$ be a poset so that its incomparability graph G is connected and has infinite diameter. Let $x \in V$ be arbitrary. Then at least one of the sets*

$$d_G^+(x) := \{d_G(x, y) : x < y \in V\} \text{ or } d_G^-(x) := \{d_G(x, y) : y \in V \text{ and } y < x\}$$

is unbounded in $\mathbb{N}$. Furthermore, if $d_G^+(x) := \{d_G(x, y) : x < y \in V\}$ is unbounded in $\mathbb{N}$ and $z > x$, then $d_G^+(z) := \{d_G(z, y) : z < y \in V\}$ is unbounded in $\mathbb{N}$ (in particular, z cannot be maximal in P).

Proof. Suppose for a contradiction that the sets

$$d^+_G(x) := \{d_G(x,y) : x < y \in V\} \text{ and } d^-_G(x) := \{d_G(x,y) : y \in V \text{ and } y < x\}$$

are bounded. Let $r := \max d^+_G(x)$ and $r' := \max d^-_G(x)$. Then

$$V := B_G(x, \max\{2, r, r'\})$$

and therefore the diameter of G is bounded contradicting our assumption. Now let $z > x$ and suppose for a contradiction that

$$d^+_G(z) := \{d_G(z,y) : z < y \in V\}$$

is bounded and let $r := \max d^+_G(z)$. Let $x < y$. If $y \le z$ then $d(x,y) \le d(x,z)$ by Lemma 8.3; if $z \parallel y$ then $d(x,y) \le d(x,z) + 1$; if $z \le y$ then we have $d(x,y) \le d(x,z) + d(z,y) \le d(x,z) + r$, hence, the set $d^+_G(x)$ is bounded, contradicting our assumption. □

9.1. **Proof of (1) of Theorem 1.4.** We construct a sequence $(x_n)_{n\in\mathbb{N}}$ of vertices (see Figure 6). We pick $x_0 \in V$. According to Lemma 9.1, one of the sets

$$d^+_G(x_0) := \{d_G(x_0,y) : x_0 < y \in V\} \text{ and}$$
$$d^-_G(x_0) := \{d_G(x_0,y) : y \in V \text{ and } y < x_0\}$$

is unbounded. We may assume without loss of generality that the set $d^+_G(x_0)$ is unbounded. Choose an element $x_3 > x_0$ at distance three from x_0 in G and let x_0, x_1, x_2, x_3 be a path joining x_0 to x_1. Note that necessarily we have $x_0 < x_2$ and $x_1 < x_3$. Now suppose constructed a sequence $x_0, x_1, \dots, x_{3n}$ such that $x_0 < x_3 < \dots < x_{3n}$ and such that $x_{3i}, x_{3i+1}, x_{3i+2}, x_{3i+3}$ is a path of extremities x_{3i} and $x_{3(i+1)}$ for $i < n$. According to Lemma 9.1, the set $d^+_G(x_{3n})$ is unbounded. Hence, it contains a vertex $x_{3(n+1)}$ at distance three from x_{3n}. Let $x_{3n}, x_{3n+1}, x_{3n+2}, y_{3n+3}$ be a path of extremities x_{3n} and $x_{3(n+1)}$. By Lemma 3.1 we have necessarily:

$$x_{3n} < x_{3n+2} \text{ and } x_{3n+1} < x_{3n+3}. \tag{9.1}$$

Let P' be the poset induced on the set $V' := \{x_n : n \in \mathbb{N}\}$ and G' be the incomparability graph of P'. According to our construction, G' contains a spanning path (not necessarily induced), hence it is connected.

Claim 9.2. *$d_G(x_0, x_{3n}) \ge n + 2$ for every $n \ge 1$.*

Since $d_{G'}(x_0, x_{3n}) \ge d_G(x_0, x_{3n)})$, it follows that the diameter of G' is infinite.

Subproof. We prove the inequality of the claim by induction on $n \ge 1$. By definition, the inequality holds for $n = 1$. Suppose the inequality holds for n. It follows from Lemma 8.3 that

$$n + 5 \le d_G(x_0, x_{3n}) + d_G(x_{3n}, x_{3(n+1)}) \le d_G(x_0, x_{3(n+1)}) + 2$$

and therefore the inequality holds for $n + 1$. □

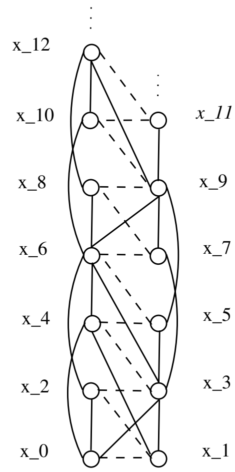

FIGURE 6.

Claim 9.3. *The incomparability graph of P' is locally finite, that is for all $x \in P'$, $\mathrm{inc}_{P'}(x) := \{y \in V' : x \parallel y\}$ is finite.*

In fact, $\mathrm{inc}_{P'}(x)$ has at most six elements.
We have

(a) $\mathrm{inc}_{P'}(x_{3n}) \subseteq \{x_{3n-1}, x_{3n+1}\}$ for $n \geq 1$.
(b) $\mathrm{inc}_{P'}(x_{3n+1}) \subseteq \{x_{3(n-2)+2}, x_{3(n-1)+1}, x_{3(n-1)+2}, x_{3n}, x_{3n+2}, x_{3(n+1)+1}\}$ for $n \geq 2$.
(c) $\mathrm{inc}_{P'}(x_{3n+2}) \subseteq \{x_{3n-1}, x_{3(n+1)}, x_{3(n+1)+1} x_{3(n+1)+2}, x_{3(n+2)+1}\}$ for $n \geq 1$.
(d) $\mathrm{inc}_{P'}(x_0) = \{x_1\}$, $\mathrm{inc}_{P'}(x_1) \subseteq \{x_0, x_2, x_4\}$, $\mathrm{inc}_{P'}(x_2) \subseteq \{x_1, x_3, x_5, x_7\}$ and $\mathrm{inc}_{P'}(x_4) \subseteq \{x_1, x_3, x_5, x_7\}$.

Subproof.

(a) Let $n \in \mathbb{N}$. By inequalities (9.1) stated above, we have

$$x_{3n-2} < x_{3n} < x_{3n+2}.$$

Let $n' \in \mathbb{N}$ be such that $n < n'$. By construction, $x_{3n} < x_{3n'}$. By inequalities (9.1) again we have $x_{3n'} < x_{3n'+2}$, hence $x_{3n} < x_{3n'+2}$. Since $x_{3n} < x_{3n'}$ and $x_{3n'+1}$ is incomparable to $x_{3n'}$ we infer that $x_{3n'+1} \nleq x_{3n}$. We have $d_G(x_{3n}, x_{3n'}) \geq 3$; indeed, if $n' = n+1$, $d_G(x_{3n}, x_{3n'}) = 3$ by construction, otherwise apply the first inequality of Lemma 8.3 with $x = x_{3n}$, $z = x_{3(n+1)}$ and $y = x_{3n'}$. Since $d_G(x_{3n}, x_{3n'}) \geq 3$ and x_3 is incomparable to x_{3n+1}, the vertices x_{3n} and $x_{3n'+1}$ cannot be incomparable; it follows that $x_{3n} < x_{3n'+1}$.

Since a poset and its dual have the same incomparability graph, we deduce that if $n' < n$, then $x_{3n'}, x_{3n'+1}, x_{3n'+2} < x_{3n}$. Hence, $\mathrm{inc}_{P'}(x_{3n}) \subseteq \{x_{3n-1}, x_{3n+1}\}$ for $n \geq 1$.

(b) Since $x_{3n-3} < x_{3n}$ and x_{3n} and x_{3n+1} are incomparable we infer that $x_{3n+1} \nleq x_{3n-3}$. It follows that $x_{3n-3} < x_{3n+1}$ because otherwise x_{3n-3},

x_{3n+1}, x_{3n} would be a path of length two contradicting our assumption that $d_G(x_{3n-3}, x_{3n}) = 3$. From Lemma 3.1, we deduce that if $k < 3n-4$, then $x_k < x_{3n-3}$ and hence $x_k < x_{3n+1}$. Hence, if $k < 3n-1$ and x_k is incomparable to x_{3n+1}, then $k \in \{3n-4, 3n-2\}$. Since $x_{3n+1} < x_{3n+3}$ it follows from Lemma 3.1 that if $k > 3n+4$, then $x_k > x_{3n+4}$ and hence $x_k \notin \mathrm{inc}_{P'}(x_{3n+1})$. Hence, x_{3n} and x_{3n+1} are possible elements incomparable to x_{3n+1}, hence the required inclusion.

(c) Since $x_{3n} < x_{3n+2}$ it follows from Lemma 3.1 that if x_k, for $k < 3n$, is incomparable to x_{3n+2} then $k \in \{3n-1, 3n+3\}$. Now observe that $x_{3n+2} < x_{3n+6}$ because otherwise $x_{3n+3}, x_{3n+2}, x_{3n+6}$ is a path of length two contradicting $d_G(x_{3n+3}, x_{3n+6}) = 3$. By duality we infer that if $k > 3n+4$, then x_k incomparable to x_{3n+2} implies $k \in \{3n+5, 3n+7\}$. The required inclusion readily follows.

(d) We have $x_0 < x_3$ and $x_0 < x_2$. Since $d_G(x_0, x_3) = 3$ and x_3 incomparable to x_4 we must have $x_0 < x_4$. From $\mathrm{inc}_{P'}(x_3) \subseteq \{x_2, x_4\}$ we deduce that x_1 is the only element incomparable to x_0. From $x_1 < x_3$ we deduce that $\mathrm{inc}_{P'}(x_1) \subseteq \{x_0\} \cup \mathrm{inc}_{P'}(x_3)$ and therefore $\mathrm{inc}_P(x_1) \subseteq \{x_0, x_2, x_4\}$. From $x_2 < x_6$ and $\mathrm{inc}_P(x_6) \subseteq \{x_5, x_7\}$ we derive $\mathrm{inc}_{P'}(x_2) \subseteq \{x_1, x_3, x_5, x_7\}$. Similarly, we have $\mathrm{inc}_{P'}(x_4) \subseteq \{x_1, x_3, x_5, x_7\}$. □

From Claim 9.2 and Claim 9.3, $\mathrm{Inc}(P')$ is connected, locally finite and has an infinite diameter. From Kőnig's Lemma, G' contains an infinite isometric path, hence G contains an infinite induced path. This completes the proof of (1). □

9.2. **Proof of (2) of Theorem 1.4.** We break the proof into two parts.

Claim 9.4. *If G is a connected incomparability graph of infinite diameter and if the set of vertices of degree at least 3 in G has infinite diameter, then G contains an infinite induced path such that the set of vertices of this path with degree at least 3 in G has an infinite diameter.*

Subproof. Let x be any vertex in G, $I := \mathrm{inc}_P(x) \cup \downarrow x$ and $F := \mathrm{inc}_P(x) \cup \uparrow x$. According to Theorem 1.10, I and F are order convex and $G_{\restriction I}$ and $G_{\restriction F}$ are isometric subgraphs of G. Since, trivially, $V(G) = I \cup F$, every vertex of degree at least 3 belongs to I or to F. Since the diameter in G of the set of vertices of degree at least 3 is infinite and $G_{\restriction I}$ and $G_{\restriction F}$ are isometric subgraphs we infer that the diameter in $G_{\restriction I}$ or in $G_{\restriction F}$ of the set of vertices of degree at least 3 is infinite. Choose y of degree at least 3. We may assume without loss of generality that the diameter in $G_{\restriction F}$ of the set of vertices of degree at least 3 is infinite. We start by showing that P contains an infinite chain of elements whose degree is at least 3 in G. Suppose constructed a sequence $y_0 := y < y_1 < \cdots < y_{n-1}$ of vertices of degree at least 3 such that $d_G(y_i, y_{i+1}) > 3$ for all $i \le n-2$. Let $y_n > x_0$ be a vertex of degree at least 3 such that $d_G(y_{n-1}, y_n) > \sum_{j=0}^{n-2} d_G(y_j, y_{j+1})$. This choice of y_n is possible since the diameter in $G_{\restriction F}$ of the set of vertices of degree at least 3 is infinite. Then y_{n-1} and y_n are comparable in P. It follows from Corollary 8.1 that

$y_{n-1} < y_n$. Hence, the sequence $(y_i)_{i\in\mathbb{N}}$ forms a chain in P. For all $n \in \mathbb{N}$, let $\mathrm{P}_n := z_{(n,0)}, z_{(n,1)}, \ldots, z_{(n,r_n)}$ be a path in G of length $r_n := d_G(y_n, y_{n+1})$ joining y_n and y_{n+1}. The graph $G' := G_{\restriction \cup_{i\in\mathbb{N}} V(P_i)}$ is connected and has infinite diameter.

Claim 9.4.a. *G' is locally finite.*

Subproof. It suffices to prove that for $n+2 \leq m$, every vertex of P_n is comparable to every vertex of P_m. Let $z_{n,i} \in \mathrm{P}_n$ and $z_{m,j} \in \mathrm{P}_n$.
Case 1: Suppose first $i = r_n - 1$.

(a) $z_{(n,r_n-1)} \leq y_m, z_{(m,1)}$. Indeed, $z_{(n,r_n-1)}$ and y_m are comparable, otherwise y_{n+1}, $z_{(n,r_n-1)}$, y_m form a path with extremities y_{n+1} and y_m hence $d_G(y_{n+1}, y_m) \leq 2$. This is impossible since
$$d_G(y_{n+1}, y_m) \geq d_G(y_{n+1}, y_{n+2}) \geq 4.$$
Furthermore, $z_{(n,r_n-1)} < y_m$, otherwise, since $y_{n+1} < y_m$, we obtain $y_{n+1} < z_{(n,r_n-1)}$ by transitivity, while these vertices are incomparable. Similarly, $z_{(n,r_n-1)}$ and $z_{(m,1)}$ are comparable otherwise y_{n+1}, $z_{(n,r_n-1)}$, $z_{(m,1)}$, y_m form a path with extremities y_n and y_m hence $d_G(y_{n+1}, y_m) \leq 3$, while this distance is at least 4. Necessarily, $z_{(n,r_n-1)} < z_{(m,1)}$, otherwise since $z_{(n,r_n-1)} < y_m$, we have $z_{m,1} < y_m$ which is impossible.

(b) By symmetry, $y_{n+1}, z_{(n,r_n-1)} \leq z_{m,1}$.

(c) $z_{(n,r_n-1)} \leq z_{(m,j)}$. We just proved it for $j = 0, 1$. If $j > 1$, this follows from $y_m < z_{m,i}$ by transitivity.

Case 2: Next, suppose $i = r_n$.
In this case $z_{(n,i)} = y_{n+1}$. If $j \geq 2$, we have
$$z_{(n,i)} = y_{n+1} < y_m = z_{(m,0)} < z_{(m,j)}.$$
If $j = 1$, this is just item (c) above.
Case 3: Finally, suppose that $i < r_n - 1$.
In this case, $z_{(n,i)} < y_{n+1} < z_{(m,j)}$. □

Since G' is connected, locally finite and has an infinite diameter, Kőnig's Lemma ensures that it contains an infinite isometric path P_∞. We claim that P_∞ contains an infinite number of vertices of degree at least 3 in G. Clearly, $V(\mathrm{P}_\infty)$ meets infinitely many P_i's. For each $i \in \mathbb{N}$ let $j_i \in V(\mathrm{P}_i)$ be the largest such that $z_{(i,j_i)} \in V(\mathrm{P}_\infty)$. Then the degree of $z_{(i,j_i)}$ is at least 3 in G. Indeed, if $z_{(i,j_i)} \in \{y_i, y_{i+1}\}$, then we are done. Otherwise $z_{(i,j_i)}$ is not an end vertex of P_i. Then $z_{(i,j_i)}$ must have a neighbour in P_∞ which is not in P_i and therefore must have degree three. So far we have proved that G' contains an infinite isometric path P_∞ containing infinitely many vertices of degree at least 3. Hence, G contains an infinite induced path P_∞ containing infinitely many vertices of degree at least 3. This proves our claim. □

Claim 9.5. *If G is a connected incomparability graph containing an infinite induced path such that the set of vertices of this path with degree at least 3 in G has an infinite diameter then G contains either a caterpillar or a kite.*

Subproof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of vertices of G with $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ forming an infinite induced path P_∞. Suppose that this path contains infinitely many vertices with degree at least 3 in G forming a set of infinite diameter in G.

Claim 9.5.a. *There is an infinite sequence $(y_n)_n$ of vertices in $V \setminus \mathrm{P}_\infty$ forming an independent set and a family of disjoint intervals $I_n := [l(n), r(n)]$ of $\mathbb{N}$ such that $\{l(n), r(n)\} \subseteq B_G(y_n, 1) \cap \mathrm{P}_\infty \subseteq I_n$ for all $n \in \mathbb{N}$.*

Subproof. Pick $x_{i_0} \in \mathrm{P}_\infty$ with degree at least 3 in G and set y_0 arbitrary in $B_G(x_{i_0}, 1) \setminus \mathrm{P}_\infty$. According to Lemma 8.9 the ball $B_G(y_0, 1)$ contains only finitely many vertices of P_∞. Let $l(0)$, resp., $r(0)$ be the least, resp., the largest integer k such that $x_k \in B_G(y_0, 1)$. Let $n > 0$. Suppose $(y_m)_m$, $I_m := [l(m), r(m)]$ be defined for $m < n$. By Lemma 8.9,

$$\mathrm{P}_\infty \cap \left(\bigcup_{m<n} B_G(y_m, 2) \right)$$

is finite, hence there is a vertex $x_{i_n} \in \mathrm{P}_\infty$ with degree at least 3 such that every vertex in the infinite subpath of P_∞ starting at x_{i_n} is at distance at least 3 of any y_m. Pick $y_n \in B(x_{i_n}, 1) \setminus \mathrm{P}_\infty$ and set $I_n = [l(n), r(n)]$ where $l(n)$, resp., $r(n)$ be the least, resp., the largest integer k such that $x_k \in B_G(y_n, 1)$. □

In order to complete the proof of Claim 9.5 we show that the graph G' induced on $\mathrm{P}_\infty \bigcup \{y_n : n \in \mathbb{N}\}$ contains a caterpillar or a kite. For that, we classify the vertices y_n. We say that y_n has *type* (0) if $l(n) = r(n)$ (that is y_n has just one neighbour on P_∞.) If the set Y_0 of vertices of type (0) is infinite then trivially $G_{\restriction \mathrm{P}_\infty \bigcup Y_0}$ is a caterpillar (see Figure 2). We say that y_n has *type* (1) if $r(n) = l(n) + 1$. Again, trivially, if the set Y_1 of vertices of type (1) is infinite then $G_{\restriction \mathrm{P}_\infty \bigcup Y_1}$ is a kite of *type* (1). We say that y_n has *type* (2) if $r(n) = l(n) + 2$. It has *type* (2.1) if $(y(n), x_{l(n)+1}) \in E(G)$ while it has *type* (2.2) if $(y(n), x_{l(n)+1}) \notin E(G)$. If for $i = 1, 2$ the set $Y_{2.i}$ of vertices of type $(2.i)$ is infinite then $G_{\restriction \mathrm{P}_\infty \bigcup Y_{2.i}}$ is a kite of type $(i+1)$ (see Figure 2). We say that y_n has type (3) if $r(n) \geq l(n) + 3$. It has type (3.1) if $(y(n), x_{l(n)+1}) \in E(G)$ while it has type (3.2) if $(y(n), x_{l(n)+1}) \notin E(G)$. If the set $Y_{3.i}$ of vertices of type $3.i$ is infinite delete from P_∞ the set $Y := \bigcup_{n \in Y_{3.i}} \{x_m : m \in \{l(n+2, \dots, r(n) - 1\}$. Then $G_{\restriction (\mathrm{P}_\infty \bigcup Y_{3.i}) \setminus Y}$ is a kite of type (2) if $i = 1$ or a caterpillar if $i = 2$ (see Figure 2). □

10. Example 1.7

We define the poset satisfying the conditions stated in Example 1.7. For a poset $P = (V, \leq)$ we set for every $x \in V$ we set

$$\mathrm{inc}_P(x) := \{y \in V : x \parallel y\}.$$

Let $P := (X, \leq)$ be the poset defined on $X := \mathbb{N} \times \mathbb{N} \times \{0, 1\}$ as follows. We let $(m, n, i) \leq (m', n', i')$ if

$$i = i' \text{ and } [n < n' \text{ or } (n = n' \text{ and } m \leq m')],$$

or

$$i \neq i' \text{ and } [n + 1 < n' \text{ or } (n + 1 = n' \text{ and } m \leq m')].$$

We set $A_n := \{(m, n, 1) : m \in \mathbb{N}\}$ for all $n \geq 0$ and $B_n := \{(m, n, 0) : m \in \mathbb{N}\}$ and note that $\bigcup_{n\in\mathbb{N}} A_n$ and $\bigcup_{n\in\mathbb{N}} B_n$ are two total orders of order type ω^2. In particular P is coverable by two chains and hence has width two.

Claim 10.1. $\leq$ *is an order relation.*

Proof. Reflexivity and antisymmetry are obvious. We now prove that $\leq$ is transitive. Let (m, n, i), (m', n', i'), (m'', n'', i'') be such that

$$(m, n, i) \leq (m', n', i') \leq (m'', n'', i'').$$

Note that since $\{i, i', i''\} \subseteq \{0, 1\}$ at least two elements of $\{i, i', i''\}$ are equal. if $i = i' = i''$ then clearly $(m, n, i) \leq (m'', n'', i'')$. Next we suppose that there are exactly two elements of $\{i, i', i''\}$ that are equal. There are three cases to consider.

Case 1: $i = i'$.

Since $(m, n, i) \leq (m', n', i')$ we have

$$n < n' \text{ or } (n = n' \text{ and } m \leq m'). \tag{10.1}$$

Since $i' \neq i''$ and $(m', n', i') \leq (m'', n'', i'')$ we have

$$n' + 1 < n'' \text{ or } (n' + 1 = n'' \text{ and } m' \leq m''). \tag{10.2}$$

If $n + 1 < n''$, then since $i \neq i''$ it follows that $(m, n, i) \leq (m'', n'', i'')$. Suppose $n'' \leq n + 1$. If $n' + 1 < n''$, then $n' < n$. It follows from (10.1) that $n = n'$ and hence $n + 1 < n''$ proving that $(m, n, i) \leq (m'', n'', i'')$. Else, $n'' \leq n' + 1$. It follows from (10.2) that $n' + 1 = n''$ and $m' \leq m''$. If $n < n'$, then $n + 1 < n''$ and once again we have $(m, n, i) \leq (m'', n'', i'')$. Otherwise it follows from (10.1) that $n = n'$ and $m \leq m'$. Hence, $n+1 = n''$ and $m \leq m''$ proving that $(m, n, i) \leq (m'', n'', i'')$.

Case 2: $i = i''$.

Since $(m, n, i) \leq (m', n', i')$ and $i \neq i'$ we have

$$n + 1 < n' \text{ or } (n + 1 = n' \text{ and } m \leq m'). \tag{10.3}$$

Since $(m', n', i') \leq (m'', n'', i'')$ and $i' \neq i''$ we have

$$n' + 1 < n'' \text{ or } (n' + 1 = n'' \text{ and } m' \leq m''). \tag{10.4}$$

We prove that $n < n''$. We suppose $n'' \leq n$ and we argue to a contradiction. We claim that none of $n + 1 < n'$ and $n' + 1 < n''$ can hold. Indeed, suppose $n + 1 < n'$. Then $n'' < n'$ and hence $n' + 1 < n''$ cannot be true. It follows from (10.4) that $n'+1 = n''$. But then $n'' = n'+1 > n' > n''$ which is impossible. Now suppose $n' + 1 < n''$. Then $n' + 1 < n < n + 1 < n'$ and this is impossible. It follows from (10.3) and (10.4) that $n+1 = n'$ and $m \leq m'$

and $n'+1 = n''$ and $m' \leq m''$. Hence, we have proved our claim that none of $n+1 < n'$ and $n'+1 < n''$ can hold. It follows from (10.3) and (10.4) that $n+1 = n'$ and $n'+1 = n''$, and in particular $n+2 = n''$. This contradicts $n'' \leq n$. Hence, $n < n''$ and therefore $(m,n,i) \leq (m'',n'',i'')$ since $i = i''$.

CASE 3: $i' = i''$.

Since $(m,n,i) \leq (m',n',i')$ and $i \neq i'$ we have

$$n+1 < n' \text{ or } (n+1 = n' \text{ and } m \leq m'). \tag{10.5}$$

Since $(m',n',i') \leq (m'',n'',i'')$ and $i' = i''$ we have

$$n' < n'' \text{ or } (n' = n'' \text{ and } m' \leq m''). \tag{10.6}$$

If $n+1 < n''$, then $(m,n,i) \leq (m'',n'',i'')$ since $i \neq i''$. We claim that none of $n+1 < n'$ and $n' < n''$ can hold. Suppose $n+1 < n'$. Then $n'' < n$ and it follows from (10.6) that $n' = n''$. But then $n'' \leq n+1 < n' = n''$ which is impossible. Suppose $n' < n''$. Then $n' < n+1$ and it follows from (10.5) that $n+1 = n'$. But then $n+1 = n' < n'' < n+1$ which is impossible. Hence, none of $n+1 < n'$ and $n' < n''$ can hold. It follows from (10.5) and (10.6) that $(n+1 = n'$ and $m \leq m')$ and $(n' = n''$ and $m' \leq m'')$. Therefore, $(n+1 = n''$ and $m \leq m'')$ proving that $(m,n,i) \leq (m'',n'',i'')$ as required.

□

Claim 10.2. *Let $j \in \mathbb{N}$. Then for all $x \in A_j$, $|B_{\mathrm{Inc}(P)}(x,1) \cap B_{j+1}|$ is finite.*

Proof. Let $x := (m,j,1) \in A_j$. Then

$$B_{\mathrm{Inc}(P)}(x,1) \cap B_{j+1} = \{(k,j+1,0) : 0 \leq k \leq m-1\}.$$

□

Claim 10.3. *Let $j \in \mathbb{N}$. Then for all $x \in B_j$, $|B_{\mathrm{Inc}(P)}(x,1) \cap A_{j+1}|$ is finite.*

Proof. Let $x := (m,j,0) \in B_j$. Then

$$B_{\mathrm{Inc}(P)}(x,1) \cap A_{j+1} = \{(k,j+1,1) : 0 \leq k \leq m-1\}.$$

□

Claim 10.4. *Let $j \in \mathbb{N}$. Then for all $x \in A_j$ and for all $y \in B_{\mathrm{Inc}(P)}(x,1) \cap B_{j+1}$,*

$$|B_{\mathrm{Inc}(P)}(y,1) \cap A_{j+2}| < |B_{\mathrm{Inc}(P)}(x,1) \cap B_{j+1}|.$$

Proof. Let $x := (m,j,1) \in A_j$. It follows from Claim 10.2 that $|N(x) \cap B_{j+1}| = m$. Let $y \in B_{\mathrm{Inc}(P)}(x,1) \cap B_{j+1}$, say $y = (m',j+1,0)$ and note that $m' < m$. Then it follows from Claim 10.3 that $|B_{\mathrm{Inc}(P)}(x,1) \cap B_{j+2}| = m'$. Since $m' < m$ we are done. □

Claim 10.5. *Let $j \in \mathbb{N}$. Then for all $x \in B_j$ and for all $y \in B_{\mathrm{Inc}(P)}(x,1) \cap A_{j+1}$,*

$$|B_{\mathrm{Inc}(P)}(y,1) \cap B_{2+1}| < |B_{\mathrm{Inc}(P)}(x,1) \cap A_{j+1}|.$$

Proof. Symmetry and Claim 10.4. □

Claim 10.6. *If there exists and infinite isometric path $(x_i)_{i\in\mathbb{N}}$ in* $\mathrm{Inc}(P)$ *starting at $x_0 = (0,0,1)$, then $x_{2n} \in A_{2n-1}$ and $x_{2n+1} \in B_{2n}$.*

Proof. From $B_0 = \mathrm{inc}_P(x_0) := \{y \in X : y$ incomparable to x in $P\}$ follows that $x_1 \in B_0$. Suppose for a contradiction that $x_2 \notin A_1$. Then $x_2 \in A_0$. In this case $x_3 \in B_1$ (this is because $x_0 < x_3$). But then $x_4 \in A_2$ because otherwise $x_4 \in A_1$ and hence the distance from x_0 to x_4 would be two which is not possible. By the same token $x_5 \in B_3$ and more generally $x_{2n+1} \in B_{2n-1}$ and $x_{2n-2} \in A_n$. This is impossible. Indeed, suppose $x_2 = (i,0,0)$ then $x_3 = (j,1,1)$ with $j < i$ and then $x_4 = (k,2,0)$ with $k < j$. Continuing this way we have a decreasing sequence of nonnegative integers. □

Claim 10.7. *Let $y \in B_{\mathrm{Inc}(P)}(x_0,1) \cap B_0$. Then the lengths of isometric paths starting at x_0 and going through y is bounded.*

Proof. Follows from Claims 10.4, 10.5 and 10.6. □

We conclude that there is no isometric path in $\mathrm{Inc}(P)$ starting at $(0,0,1)$. It follows from Theorem 1.6 that $\mathrm{Inc}(P)$ has no isometric infinite path.

11. Interval orders: A proof of Theorem 1.8 and Example 1.9

We recall that an order P is an *interval order* if P is isomorphic to a subset $\mathcal{J}$ of the set $\mathrm{Int}(C)$ of nonempty intervals of a chain C, ordered as follows: if $I, J \in \mathrm{Int}(C)$, then

$$I < J \text{ if } x < y \text{ for every } x \in I \text{ and every } y \in J. \tag{11.1}$$

The following proposition encompasses some known equivalent properties of interval orders. Its proof is easy and is left to the reader.

Proposition 11.1. *Let $P := (V, \leq)$ be a poset. The following propositions are equivalent.*

(i) P is an interval order.
(ii) P does not embed $2 \oplus 2$.
(iii) The set $\{(\downarrow x) \setminus \{x\} : x \in V\}$ is totally ordered by set inclusion.
(iv) The set $\{(\uparrow x) \setminus \{x\} : x \in V\}$ is totally ordered by set inclusion.

Lemma 11.2. *Let $P = (V, \leq)$ be an interval order and $x \in V$. Then the neighbours of x (in $\mathrm{Inc}(P)$) that lay on an induced path of length at least two in $\mathrm{Inc}(P)$ and starting at x and whose vertices are in $\mathrm{inc}_P(x) \cup \uparrow x$ form an antichain in P.*

Proof. Let $x := x_0, x_1, \ldots, x_n$ and $x := x'_0, x'_1, \ldots, x'_{n'}$ be two induced paths in $\mathrm{Inc}(P)$ with $n, n' \geq 2$ and whose vertices are in $\mathrm{inc}_P(x) \cup \uparrow x$. Note that necessarily $x < x_2$ and $x < x'_2$. Suppose for a contradiction that x_1 and x'_1 are comparable. Suppose $x_1 < x'_1$. Since $x < x_2$ and x_1 is incomparable to x and to x_2 and x is incomparable to x'_1 and P is an interval order we infer that x'_1 is comparable to x_2 and hence $x < x'_1$ or $x_1 < x_2$, which is impossible. The case $x'_1 < x_1$ can be dealt with similarly by considering the comparabilities $x'_1 < x_1$ and $x < x'_2$. □

11.1. **Proof of Theorem 1.8.** Let $x_0 \in P$ and set $I_0 := \mathrm{inc}_P(x_0) \cup \downarrow x_0$ and $F_0 := \mathrm{inc}_P(x_0) \cup \uparrow x_0$. Clearly, $V(G) = I_0 \cup F_0$. Furthermore, since the diameter of G is infinite and $G_{\restriction I_0}$ and $G_{\restriction F_0}$ are connected graphs we infer that the diameter in $G_{\restriction I_0}$ or in $G_{\restriction F_0}$ is infinite. We may assume without loss of generality that the diameter of $G_0 := G_{\restriction F_0}$ is infinite. Hence, the lengths of isometric paths in G_0 starting at x_0 are unbounded.

Claim 11.3. *There exists $x_1 \in \mathrm{inc}_P(x_0)$ such that the lengths of isometric paths in G_0 starting at x_0 and going through x_1 are unbounded.*

Proof. Since the antichains of P are finite, there are only finitely many neighbours of x_0 in G_0 laying on isometric paths starting at x_0 and of length at least two. Hence there must be a neighbour x_1 of x in G_0 such that the lengths of isometric paths in $G_{\restriction F_0}$ starting at x_0 and going through x_1 are unbounded.

Now suppose constructed an isometric path $x_0, \dots, x_n$ such that $x_i < x_j$ for all $j - i \geq 2$ and that the lengths of isometric paths starting at x_0 and going through $x_0, \dots, x_n$ are unbounded. From Lemma 11.2 we deduce that there are only finitely many neighbours of x_n that lay on such isometric paths. Applying Claim 11.3 to x_n we deduce that there exists $x_{n+1} > x_{n-1}$ such that $x_0, \dots, x_n, x_{n+1}$ is an isometric path of length $n+1$. □

We now proceed to the proof of Example 1.9.

Proof. We totally order the set $\mathbb{N} \times \mathbb{N}$ as follows: $(n, m) \leq (n', m')$ if $m < m'$ or $(m = m'$ and $n \leq n')$. Consider the set Q of intervals $X_{n,m} := [(n, m), (n, m+1)]$ ordered as in (1.1) above and set $G := \mathrm{Inc}(Q)$. Then $X_{n,m} \leq X_{n',m'}$ if and only if $m+1 < m'$ or $(m+1 = m'$ and $n \leq n')$. Equivalently, $\{X_{n,m}, X_{n',m'}\}$ is an edge of G if and only if $m = m'$ or $(m' = m+1$ and $n' < n)$ or $(m = m'+1$ and $n < n')$.

Claim 11.4. *G is connected and has infinite diameter.*

Subproof. Let $X_{n,m}$ and $X_{n',m'}$ be two elements of Q so that $n \leq n'$. We may suppose without loss of generality that $X_{n,m} \cap X_{n',m'} = \varnothing$. We may suppose without loss of generality that $m < m'$. Consider the sequence of intervals $X_{n,m}, X_{n',m}, X_{n+1,m+1}, X_{n,m+2}, X_{n',m+2}, \dots, X_{n',m'}$. This is easily seen to be a path in G proving that G is connected. □

Claim 11.5. *G has no isometric infinite path starting at $X_{0,0}$.*

Subproof. Let $X_{0,0} =: Y_0, \dots, Y_r, \dots$ be an isometric path. Then $Y_1 = X_{n_1,0}$ for some $n_1 \in \mathbb{N}$. Now Y_2 must intersect Y_1 but not Y_0. Hence, $Y_2 = X_{n_2,1}$ for some $n_2 < n_1$. Now Y_3 must intersect Y_2 but not Y_1. Suppose $Y_3 = X_{n',1}$. Then $n_1 < n'$. But then $X_{n'+1,0}$ intersects Y_3 and Y_0 and therefore the distance in G between Y_0 and Y_3 is two contradicting our assumption that $X_{0,0} =: Y_0, \dots, Y_n, \dots$ is isometric. Hence, we must have $Y_3 = X_{n_3,2}$ for some $n_3 < n_2$. An induction argument shows that $Y_r = X_{n_r,r-1}$ with $n_r < n_{r-1} < \cdots < n_1$. Since there are no infinite strictly decreasing sequences of positive integers the isometric path $X_{0,0} =: Y_0, \ldots, Y_r, \ldots$ must be finite. This completes the proof of Claim 11.5. □

It follows from Theorem 1.6 that G has no isometric path. □

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