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FURTHER ROGERS-RAMANUJAN TYPE IDENTITIES FOR MODIFIED LATTICE PATHS

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Abstract. Recently, the authors introduced the modified lattice paths which generalize Agarwal-Bressoud weighted lattice paths. Using these new objects they interpreted combinatorially two basic series identities which led to two new combinatorial Rogers-Ramanujan type identities. In this paper we obtain three more Rogers-Ramanujan type identities for modified lattice paths. This also leads to three new 3-way combinatorial identities.

1. Introduction

The theory of partitions is closely related to the theory of lattice paths. MacMahon [[20], Chapter VI] described a relationship between the both theories. Polya [21] has shown a connection between the lattice path from the origin to the point (p, q) and a partition, into parts limited in magnitude to p and in number to q. Agarwal [1] defined $n(x, y)$-reflected lattice paths as paths from $(0, 0)$ to (n, n) having the property that for each (x, y) in the path $(n - x, n - y)$ is also on the path and proved that the number of $2n(x, y)$-reflected lattice paths equals the number of partitions of $2n^2$ into at most $2n$ parts each $\leq 2n$ and the parts which are strictly less than $2n$ can be paired such that the sum of each pair is $2n$. Agarwal and Andrews [5] proved that the number of $n(y, x)$-reflected lattice paths equals the number of self-conjugate partitions with largest part $\leq n$. Agarwal and Bressoud [8] introduced a new class of weighted lattice paths and using them, they proved that the multiple basic series identity found by Agarwal, Andrews and Bressoud in [7] was indeed the analytic counterpart of the generalized colored

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partition theorem of Agarwal and Andrews [6]. These lattice paths were further used to interpret many more basic series identities combinatorially (see for instance [9]). Recently, the authors in [11] defined modified lattice paths which generalize Agarwal-Bressoud weighted lattice paths. Using these new objects they interpreted two basic series identities combinatorially which led to two new combinatorial Rogers-Ramanujan type identities. In this paper we obtain three more Rogers-Ramanujan type identities for modified lattice paths. First we recall the following definitions:

Definition 1.1. *(Agarwal and Andrews* [6]*) A partition with "$(n+t)$ copies of n", (also called an $(n+t)$-color partition), $t \geq 0$, is a partition in which a part of size n, $n \geq 0$, can occur in $(n+t)$ different colors denoted by subscripts $n_1, n_2, ..., n_{n+t}$.*

Example 1.2. *Partitions of 2 with "$(n+1)$ copies of n" are*

$$\begin{gathered}2_1\ ,\ 2_1+0_1\ ,\ 1_1+1_1\ ,\ 1_1+1_1+0_1,\\ 2_2\ ,\ 2_2+0_1\ ,\ 1_2+1_1\ ,\ 1_2+1_1+0_1,\\ 2_3\ ,\ 2_3+0_1\ ,\ 1_2+1_2\ ,\ 1_2+1_2+0_1.\end{gathered}$$

Note that zeros are permitted if and only if t is greater than or equal to one.

Definition 1.3. *(Agarwal and Andrews* [6]*) The weighted difference of two elements m_i and n_j, $m \geq n$, is defined by $m-n-i-j$ and is denoted by $((m_i - n_j))$.*

Definition 1.4. *(Agarwal and Sood* [10]*) Let m_i be a part in an $(n+t)$-color partition of a non-negative integer ν. We split the color 'i' into two parts — the green part and the red part and denote them by 'g' and 'r', respectively, such that $1 \leq g \leq i$, $0 \leq r \leq i-1$, and $g+r = i$. An $(n+t)$-color partition in which each part is split in this manner is called a split $(n+t)$-color partition.*

Example 1.5. *In 7_{3+2}, the green part is 3 and the red part is 2.*

Remark: If the red part is 0, we will not write it. Thus, for example, we will write 7_5 for 7_{5+0}.

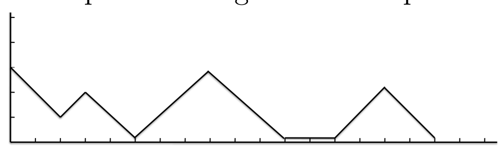
Definition 1.7. *(Agarwal and Bressoud* [8]*) Lattice paths are defined as paths of finite length lying in the first quadrant. They will begin on the y-axis or on the x-axis and terminate on the x-axis. Only three moves are allowed at each step:*
northeast: from (i, j) to $(i+1, j+1)$
southeast: from (i, j) to $(i+1, j-1)$, only allowed if $j > 0$,
horizontal: from $(i, 0)$ to $(i+1, 0)$, only allowed along x-axis.
The following terminology will be used in describing lattice paths:

a) Peak: Either a vertex on the y-axis which is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.
b) Valley: A vertex preceded by a southeast step and followed by a northeast step. Note that a southeast step followed by a horizontal step followed by a northeast step does not constitute a valley.

c) *Mountain: A section of the path which starts on either the x- or y-axis, which ends on the x-axis, and which does not touch the x-axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.*
d) *Plain: A section of the path consisting of only horizontal steps which starts either on y-axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.*
e) *Height of a vertex v: It is the y-coordinate of v.*
f) *Weight of a vertex v: It is the x-coordinate of v.*
g) *Weight of a path P: It is the sum of weights of all peaks in P.*

Example 1.8. *The following path has four peaks, two valleys, three mountains and one plain.*

Graph 1: A weighted lattice path



In the example given above, there are two peaks of height three and two of height two, one valley of height one and one of height zero. The weight of this path is $0 + 3 + 8 + 15 = 26$.

Definition 1.9. *(Sachdeva and Agarwal [11]) We divide the height 'h' of each peak into two parts — the lower part will be called a pillar and the upper part a beam and denote their heights by 'p' and 'b' respectively such that $1 \le p \le h$, $0 \le b \le h - 1$ and $h = p + b$. A lattice path wherein the heights of the peaks are divided into pillars and beams is called a modified lattice path. In a modified lattice path a pillar will be denoted by a 'dark line' and a beam by a 'light line'.*

A series involving factors like rising q-factorial $(a; q)_n$ defined by

$$(a; q)_n = \prod_{i=0}^{\infty} \frac{(1 - aq^i)}{(1 - aq^{n+i})}$$

where $|q| < 1$ and 'a' any constant, is called basic series (or q-series, or Eulerian series). The following two "sum-product" basic series identities are known as Rogers-Ramanujan identities:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \prod_{n=1}^{\infty} (1 - q^{5n-1})^{-1}(1 - q^{5n-4})^{-1}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \prod_{n=1}^{\infty} (1 - q^{5n-2})^{-1}(1 - q^{5n-3})^{-1}. \tag{1.2}$$

They were first discovered by Rogers [22] and rediscovered by Ramanujan in 1913. MacMahon [20] gave the following combinatorial interpretation of (1.1) and (1.2), respectively:

Theorem 1.10. *The number of partitions of n into parts with minimal difference 2 equals the number of partitions of n into parts which are congruent to $\pm 1 (\bmod 5)$.*

Theorem 1.11. *The number of partitions of n with minimal part 2 and minimal difference 2 equals the number of partitions of n into parts which are congruent to $\pm 2 (\bmod 5)$.*

Gordon [16] generalized Theorems (1.10) and (1.11) and Andrews [12] gave the analytic counterpart of Gordon's generalization. Partition theoretic interpretations of many more q-series identities like (1.1) and (1.2) have been given by several mathematicians. See, for instance, Göllnitz [14, 15], Gordon [17], Connor [13], Hirschhorn [19], Agarwal and Andrews [5], Subbarao [25], Subbarao and Agarwal [26].

In all these results ordinary partitions were used. Using $(n+t)$-color partitions several more basic series identities were interpreted combinatorially in [2, 3, 4, 8, 9, 18]. Very recently, using split $(n+t)$-color partitions, Sood and Agarwal [24] interpreted combinatorially the following three q-series identities from Slater's compendium [[23], I(29), I(51), I(50)]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1+q^{2n-1})(1+q^{6n-2})(1+q^{6n-4})(1-q^{6n})}{(1-q^{2n})} \tag{1.3}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q;q^2)_n}{(q;q)_{2n+1}} = \prod_{n=1}^{\infty} \frac{(1-q^{12n-4})(1-q^{12n-8})(1-q^{12n})}{(1-q^n)} \tag{1.4}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(q;q)_{2n+1}} = \prod_{n=1}^{\infty} \frac{(1-q^{12n-2})(1-q^{12n-10})(1-q^{12n})}{(1-q^n)} \tag{1.5}$$

in the following forms:

Theorem 1.12. *Let $A_1(\nu)$ denote the number of split n-color partitions of ν such that the parts and their subscripts have the same parity, the red part of the subscripts cannot exceed 1 and the weighted difference of any pair of parts is non-negative and even. Let $B_1(\nu)$ denote the number of partitions of ν such that the odd parts are distinct, even parts congruent to $\pm 2, \pm 4 (\bmod 12)$ and two copies of the parts which are congruent to $\pm 2 (\bmod 12)$ are used. Then*

$$A_1(\nu) = B_1(\nu), \forall \nu \geq 0.$$

Theorem 1.13. *Let $A_2(\nu)$ denote the number of split $(n+1)$-color partitions of ν such that the parts and their subscripts have the opposite parity, the red part of the subscripts cannot exceed 1, the smallest part is of the form i_{i+1}, the red part of the smallest part is 0 and the weighted difference of any pair of parts is non-negative and even. Let $B_2(\nu)$ denote the number of partitions of ν such that the parts not congruent to $\pm 0, \pm 4 (\bmod 12)$. Then*

$$A_2(\nu) = B_2(\nu), \forall \nu \geq 0.$$

Theorem 1.14. *Let $A_3(\nu)$ denote the number of split $(n+2)$-color partitions of ν such that the parts and their subscripts have the same parity, the red part of the subscripts cannot exceed* 1*, the smallest part is of the form i_{i+2}, the red part of the smallest part is* 0 *and the weighted difference of any pair of parts is non-negative and even. Let $B_3(\nu)$ denote the number of partitions of ν such that the parts not congruent to $\pm 0, \pm 2$(mod 12). Then*

$$A_3(\nu) = B_3(\nu), \forall \nu \geq 0.$$

In this paper we interpret identities (1.3)–(1.5) combinatorially by using modified lattice paths. This leads to three new 3-way combinatorial identities. In our next section we state our main results and prove them in Section 3. We conclude in the last section by posing an open problem.

2. The main results

We shall prove that identities (1.3)–(1.5) have their modified lattice paths theoretic interpretations in the following theorems, respectively:

Theorem 2.1. *Let $C_1(\nu)$ denote the number of modified lattice paths of weight ν which start at (0,0)*

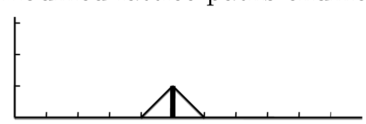
(i) have no valley above height 0,
(ii) no plain with odd length,
(iii) no beam with height > 1.

Then

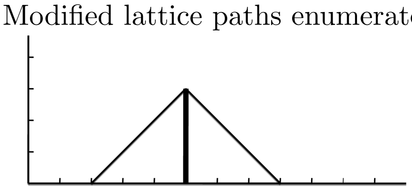
$$C_1(\nu) = A_1(\nu), \forall \nu \geq 0.$$

Example 2.2. $C_1(5) = 7$*, the relevant modified lattice paths are*

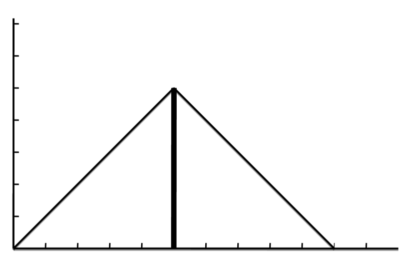
Graph 2: Modified lattice paths enumerated by $C_1(5)$



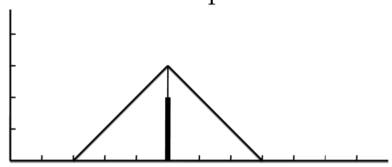
Graph 3: Modified lattice paths enumerated by $C_1(5)$



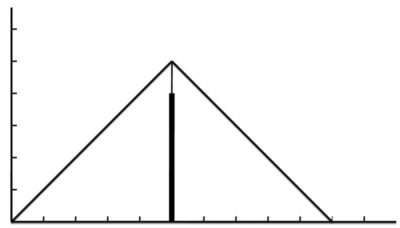
Graph 4: Modified lattice paths enumerated by $C_1(5)$



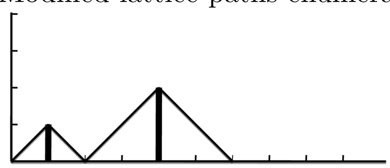
Graph 5: Modified lattice paths enumerated by $C_1(5)$



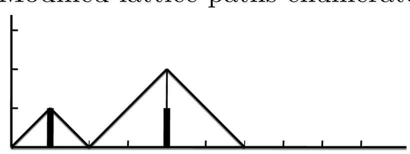
Graph 6: Modified lattice paths enumerated by $C_1(5)$



Graph 7: Modified lattice paths enumerated by $C_1(5)$



Graph 8: Modified lattice paths enumerated by $C_1(5)$



Theorem 2.3. *Let $C_2(\nu)$ denote the number of modified lattice paths of weight ν which start at (0,1)*

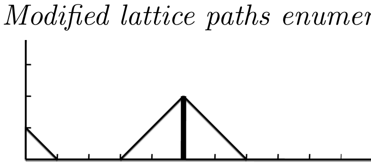
(i) have no valley above height 0,
(ii) no plain with odd length,
(iii) no beam with height > 1,
(iv) the first peak is supported by a pillar only.

Then

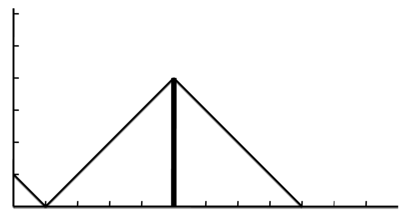
$$C_2(\nu) = A_2(\nu), \forall \nu \geq 0.$$

Example 2.4. $C_2(5) = 6$*, the relevant modified lattice paths are*

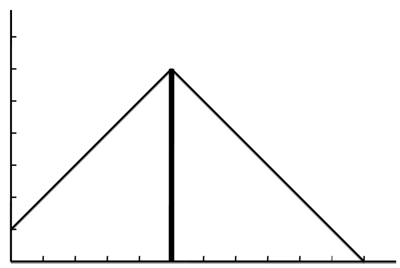
Graph 9: Modified lattice paths enumerated by $C_2(5)$



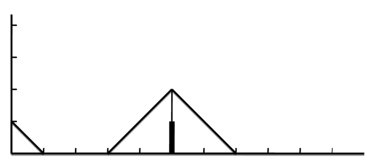
Graph 10: Modified lattice paths enumerated by $C_2(5)$



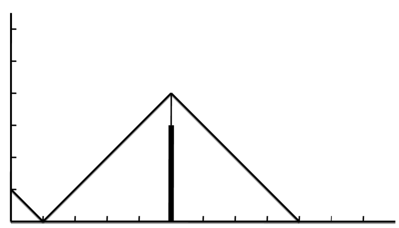
Graph 11: Modified lattice paths enumerated by $C_2(5)$



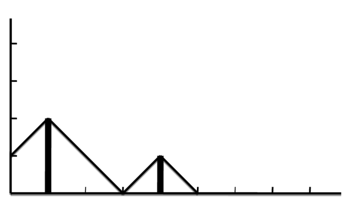
Graph 12: Modified lattice paths enumerated by $C_2(5)$



Graph 13: Modified lattice paths enumerated by $C_2(5)$



Graph 14: Modified lattice paths enumerated by $C_2(5)$



Theorem 2.5. *Let* $C_3(\nu)$ *denote the number of modified lattice paths of weight* ν *which start at (0,2)*

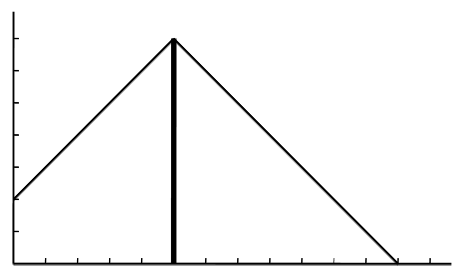
(i) have no valley above height 0,
(ii) no plain with odd length,
(iii) no beam with height > 1*,*
(iv) the first peak is supported by a pillar only.

Then

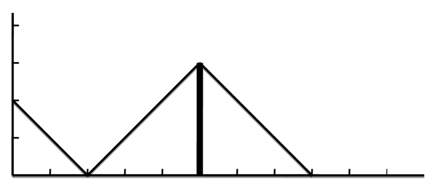
$$C_3(\nu) = A_3(\nu), \forall \nu \geq 0.$$

Example 2.6. $C_3(5) = 4$*, the relevant modified lattice paths are*

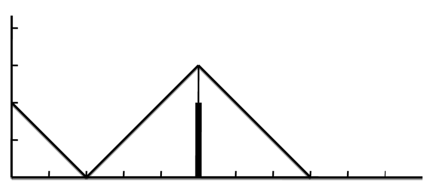
Graph 15: Modified lattice paths enumerated by $C_3(5)$



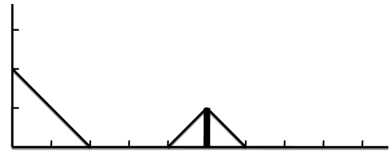
Graph 16: Modified lattice paths enumerated by $C_3(5)$



Graph 17: Modified lattice paths enumerated by $C_3(5)$



Graph 18: Modified lattice paths enumerated by $C_3(5)$



Theorems 1.12–1.14 and Theorems 2.1, 2.3, 2.5 lead to the following 3-way combinatorial identities :

Theorem 2.7. *For* $1 \leq k \leq 3$*,*

$$A_k(\nu) = B_k(\nu) = C_k(\nu), \forall \nu \geq 0.$$

3. Proof of Theorem 2.7

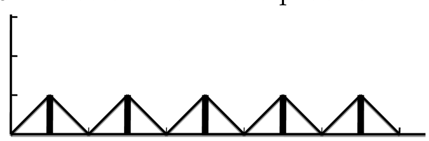
First we remark that for $1 \leq k \leq 3$, Sood and Agarwal have shown in [24] that the left-hand sides of the Equations (1.3)–(1.5) generate the sequences $A_k(\nu)$. Here we shall show that the left-hand sides of the Equations (1.3)–(1.5) also generate the sequences $C_k(\nu)$ and we shall also show bijectively that $A_k(\nu) = C_k(\nu)$. Furthermore, since each of these three cases is proved in a similar way we provide the details for $k = 1$ and sketch the changes required to treat the remainder.

3.1. **Proof of Theorem 2.1.** In

$$\frac{q^{m^2}(-q;q^2)_m}{(q;q)_{2m}} = \frac{q^{m^2}(-q;q^2)_m}{(q^2;q^2)_m(q;q^2)_m}$$

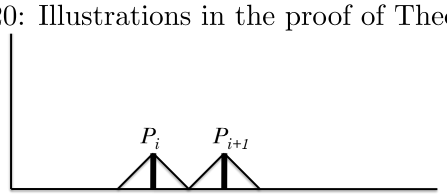
the factor q^{m^2} generates the lattice path having m peaks which start at $(0,0)$ and terminate at $(2m,0)$ such that each peak is supported by a pillar of height one. For instance, if we take $m = 5$, then the path will look like

Graph 19: Illustrations in the proof of Theorem 2.1



Next, we consider two successive peaks, say, P_i and P_{i+1} in Graph 19.

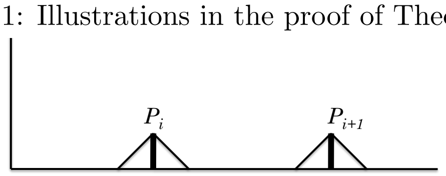
Graph 20: Illustrations in the proof of Theorem 2.1



$$P_i \equiv (2i-1,1)$$
$$P_{i+1} \equiv (2i+1,1)$$

The factor $1/(q^2;q^2)_m$ gives rise to m non-negative even parts, say, $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_m \geq 0$, which are encoded by inserting a_m horizontal steps before the first mountain and $a_i - a_{i+1}$ horizontal steps in front of the $(m-i+1)^{\text{st}}$ mountain, $1 \leq i \leq m-1$. It increases x-coordinate of the i^{th} peak by $a_m+(a_{m-1}-a_m)+(a_{m-2}-a_{m-1})+\cdots+(a_{m-i+1}-a_{m-i+2}) = a_{m-i+1}$ and the x-coordinate of the $(i+1)^{\text{th}}$ peak by a_{m-i}. It transforms Graph 20 to Graph 21.

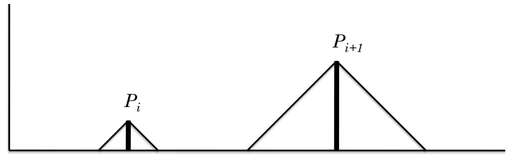
Graph 21: Illustrations in the proof of Theorem 2.1



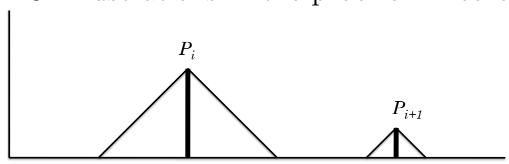
$$P_i \equiv (2i-1+a_{m-i+1},1)$$
$$P_{i+1} \equiv (2i+1+a_{m-i},1)$$

Now, the factor $1/(q;q^2)_m$ generates non negative multiples of $(2i-1)$, $1 \leq i \leq m$, say, $p_1 \times 1,\ p_2 \times 3, \cdots,\ p_m \times (2m-1)$. This is encoded by enlarging the height of the i^{th} pillar to $p_{m-i+1}+1$. Note that if we increase the height of a peak by one then its weight also increases by one and the weight of its successive peak increases by two. Graph 21 now becomes Graph 22 or Graph 23, depending on whether $p_{m-i} > p_{m-i+1}$ or $< p_{m-i+1}$. In the case when $p_{m-i} = p_{m-i+1}$, the new graph will look like Graph 21.

Graph 22: Illustrations in the proof of Theorem 2.1



Graph 23: Illustrations in the proof of Theorem 2.1

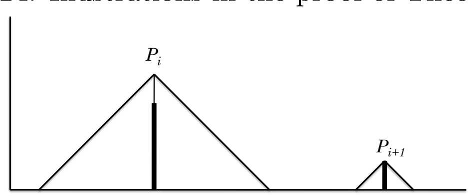


In Graph 22 (or Graph 23), the peaks P_i and P_{i+1} become

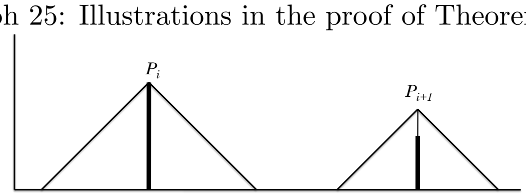
$$\begin{aligned} P_i \equiv &(2i-1+a_{m-i+1}+2(p_m+p_{m-1}+......+p_{m-i+2})+p_{m-i+1}, \\ &\qquad p_{m-i+1}+1), \\ P_{i+1} \equiv &(2i+1+a_{m-i}+2(p_m+p_{m-1}+......+p_{m-i+1})+p_{m-i}, p_{m-i}+1). \end{aligned}$$

The factor $(-q;q^2)_m$ introduces non negative multiples of distinct $(2i-1)$, $1 \le i \le m$, say, $b_1 \times 1,\ b_2 \times 3, \cdots,\ b_m \times (2m-1)$, where $b_1,\ b_2, \cdots,\ b_m$ are 0 or 1. This is encoded by putting a beam of height b_{m-i+1} on the i^{th} pillar. The Graph 22 (or Graph 23) will either not change or may change to three possible shapes. For instance, following are the three possibilities of Graph 23:

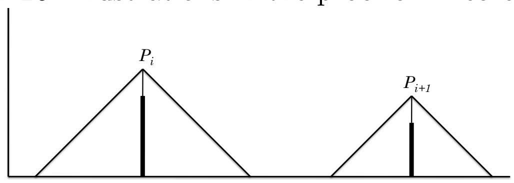
Graph 24: Illustrations in the proof of Theorem 2.1



Graph 25: Illustrations in the proof of Theorem 2.1



Graph 26: Illustrations in the proof of Theorem 2.1



In Graphs 24, 25, 26

$$\begin{aligned}P_i \equiv &(2i-1+a_{m-i+1}+2(p_m+p_{m-1}+......+p_{m-i+2})+p_{m-i+1}\\&+2(b_m+b_{m-1}+....+b_{m-i+2})+b_{m-i+1}, 1+p_{m-i+1}+b_{m-i+1})\\P_{i+1} \equiv &(2i+1+a_{m-i}+2(p_m+p_{m-1}+......+p_{m-i+1})+p_{m-i}\\&+2(b_m+b_{m-1}+....+b_{m-i+1})+b_{m-i}, 1+p_{m-i}+b_{m-i}).\end{aligned}$$

Each modified lattice path starting at $(0,0)$, with all valleys at height 0, no plain with odd length and no beam with height > 1 is uniquely generated in this manner. This proves that the left-hand side of equation (1.3) generates $C_1(\nu)$.

Next, we shall show that there is a bijection between the modified lattice paths enumerated by $C_1(\nu)$ and the split n-color partitions enumerated by $A_1(\nu)$. We do this by considering each path as the sequence of the weights of the peaks such that each weight is subscripted by the height of the respective peak. Further, the height of each peak is considered as the height of the supporting pillar which is associated with the green color plus the height of the supporting beam which is associated with the red color. In the final graph let us denote P_i and P_{i+1} by A_x and B_y $(B \geq A)$, respectively, then

$$\begin{aligned}A =&(2i-1)+a_{m-i+1}+2(p_m+p_{m-1}+......+p_{m-i+2})\\&+p_{m-i+1}+2(b_m+b_{m-1}+....+b_{m-i+2})+b_{m-i+1},\\x =&1+p_{m-i+1}+b_{m-i+1},\\B =&(2i+1)+a_{m-i}+2(p_m+p_{m-1}+....+p_{m-i+1})\\&+p_{m-i}+2(b_m+b_{m-1}+.....+b_{m-i+1})+b_{m-i},\\&\text{and}\\y =&1+p_{m-i}+b_{m-i}.\end{aligned}$$

The weighted difference of these two parts is

$$((B_y - A_x)) = B - A - x - y = a_{m-i} - a_{m-i+1},$$

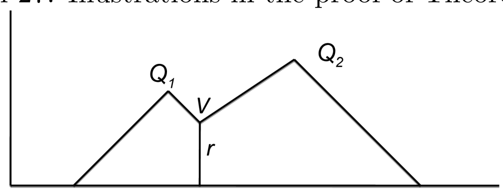
which is non-negative and even.

Next, we consider the split n-color part A_x. Note that the parity of both A and x depends on the parity of $p_{m-i+1}+b_{m-i+1}$. If $p_{m-i+1}+b_{m-i+1}$ is even (resp., odd) then both A and x are odd (resp., even). This shows that the parts and their subscripts have the same parity. Since the height of any beam can be 0 or 1, the red part in the corresponding split n-color partition cannot exceed 1. Since the lengths of the plains correspond to

$a_i's$, $1 \le i \le m$, which are non-negative and even, no plain can have odd length.

Conversely, we consider two parts of a partition enumerated by $A_1(\nu)$, say C_u and D_v (Note that the split subscripts are not needed here). Let $Q_1 \equiv (C, u)$ and $Q_2 \equiv (D, v)$ be the corresponding peaks in the associated lattice path.

Graph 27: Illustrations in the proof of Theorem 2.1

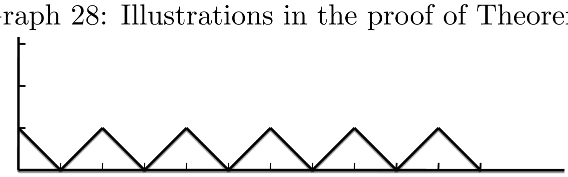


$$Q_1 \equiv (C, u)$$
$$Q_2 \equiv (D, v)$$

Suppose there is a plain between Q_1 and Q_2. Then the length of the plain would be $D-C-v-u$ which is the weighted difference of C_u and D_v and is, hence, non-negative and even. Thus, we get that there is no plain with odd length in the corresponding lattice path. Next, we prove by contradiction that there cannot be any valley above height 0. If there is a valley V of height r ($r > 0$) between the peaks Q_1 and Q_2, then there is a descent of $u-r$ from Q_1 to V and an ascent of $v-r$ from V to Q_2 (See Graph 27). This implies that $D = C + (u-r) + (v-r)$, or $D - C - u - v = -2r$. Using the fact that the weighted difference is non-negative, we get $r = 0$. Hence we get Theorem 2.1.

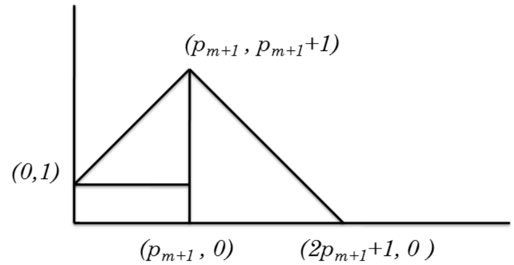
3.2. **Proof of Theorem 2.3.** This theorem is proved in a similar manner as Theorem 2.1. The only point of departure is that the extra factor of q^m puts a southeast step from (0,1) to (1,0) in front of the lattice path. Thus q^{m^2+m} generates a lattice path of one south-east step from (0,1) to (1,0) and m peaks starting from (1,0) and terminating at $(2m+1, 0)$. For $m = 5$, the path begins as

Graph 28: Illustrations in the proof of Theorem 2.3



The extra factor of $(1 - q^{2m+1})^{-1}$ introduces a non-negative multiple of $2m+1$, say, $p_{m+1} \times (2m+1)$. This is encoded by having the first peak grow to height $p_{m+1} + 1$ in the north-east direction. In the final graph the first peak looks like

Graph 29: Illustrations in the proof of Theorem 2.3



We say that the first peak is supported by a pillar of height $p_{m+1}+1$. So if we denote the $(i+1)th$ and $(i+2)th$ peaks by (A, x) and (B, y), respectively, then

$$\begin{aligned} A =&(2p_{m+1}+1)+(2i-1)+a_{m-i+1}+2(p_m+p_{m-1}+\ldots\ldots+p_{m-i+2})\\ &+p_{m-i+1}+2(b_m+b_{m-1}+\ldots.+b_{m-i+2})+b_{m-i+1}\ ,\\ x =&p_{m-i+1}+b_{m-i+1}+1\ ,\\ B =&(2p_{m+1}+1)+(2i+1)+a_{m-i}+2(p_m+p_{m-1}+\ldots.+p_{m-i+1})\\ &+p_{m-i}+2(b_m+b_{m-1}+\ldots..+b_{m-i+1})+b_{m-i}\ ,\\ \text{and}&\\ y =&p_{m-i}+b_{m-i}+1. \end{aligned}$$

The weighted difference is given by $B-A-x-y$ i.e. $a_{m-i}-a_{m-i+1}$ which is non-negative and even. Here, A and x and also B and y have opposite parity. This implies that even parts appear with odd subscripts and odd with even. The first part is $(p_{m+1})_{p_{m+1}+1}$, which is of the form i_{i+1} and shows that we are using $n+1$ copies of n.

3.3. **Proof of Theorem 2.5.** The proof is same as Theorem 2.1 except that the extra factor of q^{2m} puts two southeast steps $(0, 2)$ to $(1, 1)$ and $(1, 1)$ to $(2, 0)$ at the beginning of the path. The extra factor of $(1-q^{2m+1})^{-1}$ introduces a non-negative multiple of $(2m+1)$, say $p_{m+1}\times(2m+1)$, which causes the first peak grow in the north east direction to height $p_{m+1}+2$. We state that the first peak is supported by a pillar of height $p_{m+1}+2$. In this way, the weight of each subsequent peak will be increased by $2p_{m+1}+2$ which will not change the parity of the weight of any peak. So the parts and their subscripts have the same parity. The first part is $(p_{m+1})_{p_{m+1}+2}$ which is of the form i_{i+2} and shows that we are using $n+2$ copies of n. Proving Theorems 2.1, 2.3 and 2.5 completes the proof of Theorem 2.7.

To illustrate the bijections we have constructed to prove Theorems 2.1, 2.3, 2.5 we give the example for $\nu=5$ shown in Tables I–III.

Table I: Illustration of bijection to prove $C_1(5) = A_1(5)$

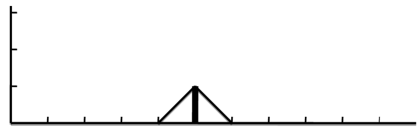
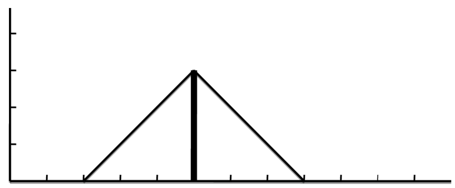
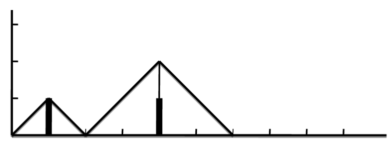
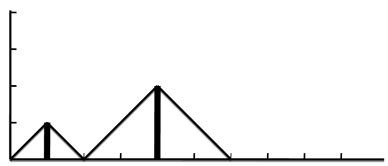
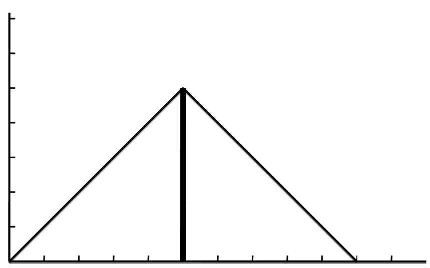
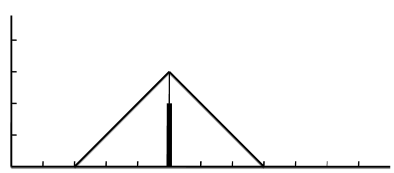
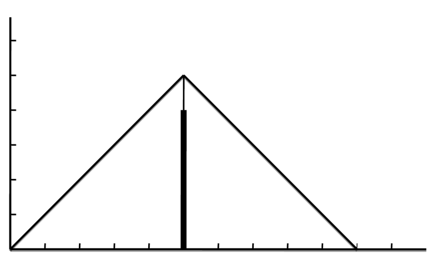
Split n-color partitions enumerated by $A_1(5)$	Corresponding modified lattice paths enumerated by $C_1(5)$
5_1	
5_3	
$4_{1+1} + 1_1$	
$4_2 + 1_1$	
5_5	
5_{2+1}	
5_{4+1}	

Table II: Illustration of bijection to prove $C_2(5) = A_2(5)$

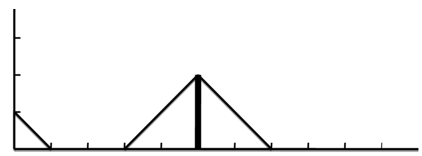
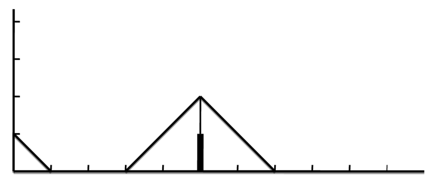
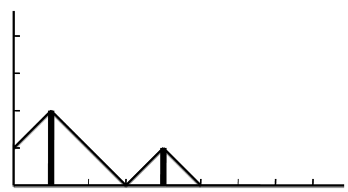
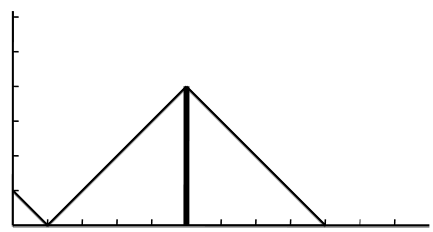
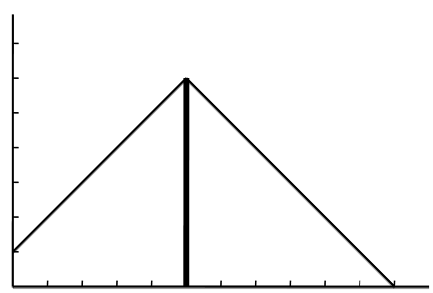
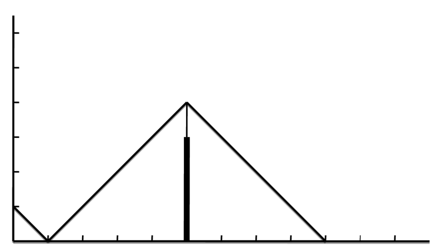
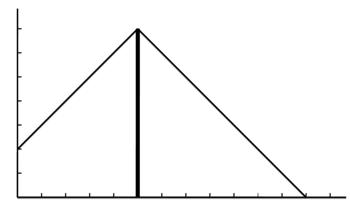
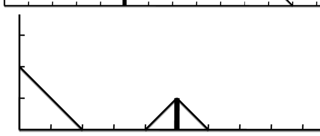
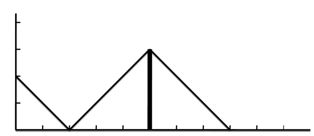
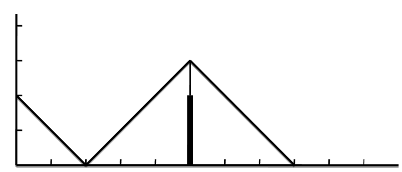
Split $(n+1)$-color partitions enumerated by $A_2(5)$	Corresponding modified lattice paths enumerated by $C_2(5)$
$5_2 + 0_1$	
$5_{1+1} + 0_1$	
$4_1 + 1_2$	
$5_4 + 0_1$	
5_6	
$5_{3+1} + 0_1$	

Table III: Illustration of bijection to prove $C_3(5) = A_3(5)$

Split $(n+2)$-color partitions enumerated by $A_3(5)$	Corresponding modified lattice paths enumerated by $C_3(5)$
5_7	
$5_1 + 0_2$	
$5_3 + 0_2$	
$5_{2+1} + 0_2$	

4. Conclusion

We hope that many more Rogers-Ramanujan type identities can be found for modified lattice paths. The most obvious question which arises from this work is: Do the three identities found in this paper lead to an infinite family of Rogers-Ramanujan type identities for modified lattice paths analogous to Agarwal-Bressoud's infinite family of Rogers-Ramanujan identities for weighted lattice paths in [8]?

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