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ON THE PERMANENT OF AN EVEN-DIMENSIONAL NONNEGATIVE POLYSTOCHASTIC TENSOR OF ORDER n

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Abstract. In this paper, we present an algorithm that allows us to compute the permanent of a tensor by using Laplace expansion. We prove that the permanent of a 4-dimensional polystochastic $(0,1)$-tensor of order n constructed using a special $n \times (n-1)$ row-Latin rectangle R with no transversals is positive. Also, we show that the permanent of an even-dimensional polystochastic $(0,1)$-tensor of order n constructed using the row-Latin rectangle R is positive. The result obtained here proves that each odd-dimensional Latin hypercube of order 4 has a transversal (Wanless' conjecture for odd-dimensional Latin hypercubes of order 4). We prove that the number of perfect matchings of the bipartite hypergraph associated to an even-dimensional polystochastic $(0,1)$-tensor of order 4 is positive. Furthermore, we extend some results concerning polystochastic $(0,1)$-tensors to nonnegative polystochastic tensors. Moreover, we prove that the permanent of a 4-dimensional nonnegative polystochastic tensor of order n constructed using the row-Latin rectangle R is positive. More generally, we show that the permanent of an even-dimensional nonnegative polystochastic tensor of order n constructed using the row-Latin rectangle R is positive. The result obtained here proves that the permanent of an even-dimensional nonnegative polystochastic tensor of order 4 is positive.

1. Introduction

A *d-dimensional tensor of order* $n_1 \times \cdots \times n_d$, $\mathcal{A} = (a_{i_1 \ldots i_d})_{n_1 \times \cdots \times n_d}$ is a multi-array of entries $a_{i_1 \ldots i_d} \in \mathbb{F}$, where $i_j = 1, \ldots, n_j$ for $j = 1, \ldots, d$ and $\mathbb{F}$ is a field. In this paper, we only consider real tensors, that is, those tensors for which $\mathbb{F} = \mathbb{R}$. When $n_1 = n_2 = \cdots = n_d = n$, we say that $\mathcal{A}$ is a square d-dimensional tensor of order n. A tensor $\mathcal{A}$ is called nonnegative if all its entries are at least zero. We refer the interested reader to [11] for more information on the theory of tensors.

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Some properties of permanents were extended from matrices to tensors by Dow and Gibson [4]. Many difficult problems in other fields can be stated equivalently as those which ask for the permanents of certain associated tensors. Recently, Taranenko studied the applications of the permanents of tensors to combinatorial designs, the number of Steiner systems, transversals in Latin hypercubes, 1-factors of hypergraphs, and MDS codes [13]. Some other applications of the permanents can be found in [1], [8] and [10]. The problem of finding a maximum size matching in a general hypergraph is NP-hard. Also, the problem of finding a perfect matching in a 3-partite hypergraph is one of Karp's 21 NP-complete problems [6]. Although the permanent of a tensor is important in other fields, its calculation lies in the list of NP-hard problems [13]. To the best of our knowledge, no algorithm exists for the computation of the permanent of a tensor. For this reason, we propose an algorithm that allows us to compute the permanent of a tensor. This algorithm helps us to avoid manual computation.

This paper is organized as follows. In Section 2, we review some basic definitions together with the known properties of the permanent of a tensor. Also, we present an algorithm that simplifies the calculation of the permanent, and provide some numerical examples for it. In Section 3, we first review the main definitions, and then examine the relation between the number of transversals of a d-dimensional Latin hypercube of order n and its corresponding $d+1$-dimensional polystochastic $(0,1)$-tensor of order n. Next, we prove that the permanent of a 4-dimensional polystochastic $(0,1)$-tensor of order n constructed using a special $n \times (n-1)$ row-Latin rectangle R with no transversals is positive. Also, we show that the permanent of an even-dimensional polystochastic $(0,1)$-tensor of order n constructed using the row-Latin rectangle R is positive. The result obtained here proves that each odd-dimensional Latin hypercube of order 4 has a transversal (Wanless' conjecture for odd-dimensional Latin hypercubes of order 4).

In Section 4, we apply our main result to the theory of hypergraphs; we prove that the number of perfect matchings of the bipartite hypergraph associated to an even-dimensional polystochastic $(0,1)$-tensor of order 4 is positive. In Section 5, we use a different technique to extend some results concerning polystochastic $(0,1)$-tensors to nonnegative polystochastic tensors. We prove that the permanent of a 4-dimensional nonnegative polystochastic tensor of order n constructed using the row-Latin rectangle R is positive. More generally, we show that the permanent of an even-dimensional nonnegative polystochastic tensor of order n constructed using the row-Latin rectangle R is positive. It is worth mentioning that if we use an $n \times (n-1)$ row-Latin rectangle with no transversals that is not isotopic to the special row-Latin rectangle R, then we can study these theorems similarly; we present a technique for doing so. The result obtained here proves that the permanent of an even-dimensional nonnegative polystochastic tensor of order 4 is positive. We conclude the paper with a brief conclusion in Section 6.

2. An algorithm for computing the permanent of a tensor

In this section, the permanent of a tensor and some of its properties are discussed. Specifically, we present an algorithm that allows us to compute the permanent of a tensor, and we use the algorithm to solve several numerical examples.

Definition 2.1 ([14])**.** *Let $\mathcal{A}$ be a square d-dimensional tensor of order n. Given $k \in \{0, 1, \dots, d\}$, a k-dimensional plane in $\mathcal{A}$ is a subtensor obtained by fixing $d-k$ indices and letting the other k indices vary from 1 to n. A 1-dimensional plane is said to be a line, and a $(d-1)$-dimensional plane is a hyperplane.*

Definition 2.2 ([4])**.** *Let $\mathcal{A}$ be a d-dimensional tensor of order $n_1 \times \cdots \times n_d$. The following sequence is called a diagonal of $\mathcal{A}$.*

$$(a_{1\sigma_2(1)\dots\sigma_d(1)}, a_{2\sigma_2(2)\dots\sigma_d(2)}, \dots, a_{n_1\sigma_2(n_1)\dots\sigma_d(n_1)}),$$

where σ_k is one-to-one function from $\{1, \dots, n_1\}$ to $\{1, \dots, n_k\}$ for $k = 2, \dots, d$.

Definition 2.3 ([4])**.** *Let $\mathcal{A}$ be a d-dimensional tensor of order $n_1 \times \cdots \times n_d$. The permanent of $\mathcal{A}$ is defined by*

$$per(\mathcal{A}) := \sum_{\sigma_k} \prod_{i=1}^{n_1} a_{i\sigma_2(i)\dots\sigma_d(i)}, \tag{2.1}$$

where the summation runs over all one-to-one functions σ_k from $\{1, \dots, n_1\}$ to $\{1, \dots, n_k\}$ and $k = 2, \dots, d$, with $per(\mathcal{A}) = 0$ if $n_1 > n_k$ for some k.

Definition 2.4. *For a tensor $\mathcal{A} = (a_{i_1\dots i_d})_{n_1\times\cdots\times n_d}$, the hyperplanes obtained by fixing $i_k, 1 \le k \le d$, are said to be hyperplanes of type k [4].*

Example 2.5. *If $\mathcal{A} = (a_{i_1 i_2 i_3})_{2\times 2\times 2}$, then*

$$per(\mathcal{A}) = a_{111}a_{222} + a_{121}a_{212} + a_{211}a_{122} + a_{221}a_{112}.$$

The following property is similar to the Laplace expansion of the permanent of a matrix [9].

Property 2.6 ([13])**.** *If $\mathcal{A}$ is a d-dimensional tensor of order n, then*

$$per(\mathcal{A}) = \sum_{i_2,\dots,i_d=1}^{n} a_{1,i_2,\dots,i_d} per(\mathcal{A}(1|i_2|\dots|i_d)),$$

where $\mathcal{A}(1|i_2|\dots|i_d)$ is a d-dimensional tensor of order $n-1$, which is obtained from $\mathcal{A}$ by removing hyperplanes i_k of type k for $k = 1, i_2, \dots, i_d$.

Property 2.6 is important and practical for the computation of the permanent of a tensor. As is known, the problem of computing the permanent of a tensor lies in the list of NP-complete problems [13]. In what follows, we use Property 2.6 to propose an algorithm for computing the permanent of a 4-dimensional tensor of order $n_1 \times n_2 \times n_3 \times n_4$ [7, 2].

Algorithm 1: The permanent of a 4-dimensional tensor.

Input: Natural numbers n_1, n_2, n_3, n_4 and a tensor $\mathcal{A} = (a_{i_1 i_2 i_3 i_4})_{n_1 \times n_2 \times n_3 \times n_4}$;
Function $p =$ `per_tens`$(\mathcal{A}, n_1, n_2, n_3, n_4)$;
if $n_1 > n_k$ *for some* k **then**
    $p = 0$;
else
    if $n_1 = 1$ **then**
        $p = \sum_{i_2=1}^{n_2} \sum_{i_3=1}^{n_3} \sum_{i_4=1}^{n_4} a_{1 i_2 i_3 i_4}$;
    else
        if $n_1, n_2, n_3, n_4 = n$ **then**
            $p =$ `per_tens_sq`$(\mathcal{A}, n)$;
        else
            $p =$ `per_tens_nonsq`$(\mathcal{A}, n_1, n_2, n_3, n_4)$;
        end
    end
end
⋮
Function $p =$ `per_tens_sq`$(\mathcal{A}, n)$;
if $n = 2$ **then**
    $p = a_{1111}a_{2222} + a_{1121}a_{2212} + a_{2111}a_{1222} + a_{1212}a_{2121} + a_{1211}a_{2122} + a_{1221}a_{2112} + a_{2211}a_{1122} + a_{2221}a_{1112}$;
else
    $p = 0$;
    for $i_2 = 1, \ldots, n$ **do**
        for $i_3 = 1, \ldots, n$ **do**
            for $i_4 = 1, \ldots, n$ **do**
                $p = p + a_{1 i_2 i_3 i_4}$`per_tens_sq`$(\mathcal{A}, 2:n, [1:i_2-1, i_2+1:n], [1:i_3-1, i_3+1:n], [1:i_4-1, i_4+1:n], n-1)$;
            end
        end
    end
end
⋮
Function $p =$ `per_tens_nonsq`$(\mathcal{A}, n_1, n_2, n_3, n_4)$;
if $n_1 = 2$ **then**
    $p = \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} \sum_{l=1}^{n_4} \left(a_{1jkl} \left(\sum_{\substack{m=1 \\ m \neq j}}^{n_2} \sum_{\substack{n=1 \\ k \neq j}}^{n_3} \sum_{\substack{s=1 \\ s \neq l}}^{n_4} a_{2mns} \right) \right)$;
else
    $p = 0$;
    for $i_2 = 1, \ldots, n_2$ **do**
        for $i_3 = 1, \ldots, n_3$ **do**
            for $i_4 = 1, \ldots, n_4$ **do**
                $p = p + a_{1 i_2 i_3 i_4}$`per_tens_nonsq`$(\mathcal{A}, 2:n_1, [1:i_2-1, i_2+1:n_2], [1:i_3-1, i_3+1:n_3], [1:i_4-1, i_4+1:n_4], n_1-1, n_2-1, n_3-1, n_4-1)$;
            end
        end
    end
end

This algorithm can be adapted to the case of a d-dimensional tensor by writing $d-1$ for loops. In the following examples, we present a few numerical experiments that illustrate the performance of the algorithm described above.

Example 2.7. *Let $\mathcal{A}$ be a 4-dimensional nonnegative tensor of order $4 \times 4 \times 4 \times 5$ with positive entries*

$$a_{1111} = a_{2222} = a_{3333} = a_{4444} = a_{4445} = a_{1234} = a_{2143} = a_{3321} = a_{4415} = a_{4412} = 1,$$

whose other entries are equal to 0. Using the algorithm above, we find that the permanent of $\mathcal{A}$ is equal to 4.

Example 2.8. *Let $\mathcal{A}$ be a tensor $\mathcal{A} = \mathrm{ones}(n_1, n_2, n_3, n_4)$, where $n_1 = 1$ and $n_2 = n_3 = n_4 = 2$ in the* MATLAB *command. Using the algorithm, we find that the permanent of $\mathcal{A}$ is equal to 8, which is the sum of the entries of $\mathcal{A}$.*

Example 2.9. *Let $\mathcal{A}$ be a tensor $\mathcal{A} = \mathrm{ones}(n_1, n_2, n_3, n_4)$ in the* MATLAB *command. We report numerical examples in the two cases nonsquare (case I) and square (case II) in the following table.*

TABLE 1. Some examples for the computation of the permanent

Case I		*Case II*	
$n_1 \times n_2 \times n_3 \times n_4$	per($\mathcal{A}$)	$n \times n \times n \times n$	per($\mathcal{A}$)
$5 \times 4 \times 5 \times 6$	0	$2 \times 2 \times 2 \times 2$	8
$2 \times 3 \times 4 \times 5$	1440	$3 \times 3 \times 3 \times 3$	216
$3 \times 3 \times 3 \times 4$	864	$4 \times 4 \times 4 \times 4$	13824
$4 \times 5 \times 5 \times 4$	345600	$5 \times 5 \times 5 \times 5$	1728000

3. THE PERMANENT OF A $(0,1)$-POLYSTOCHASTIC TENSOR

In this section, we first present some definitions that will be used in the proofs of our theorems. Next, we prove the aforementioned conjectures of Sun and Wanless in a special case. To do so, we first prove Sun's conjecture in a special case (a special case of Wanless' conjecture). Then, we prove Wanless' conjecture in a special case by using induction. Also, we prove conjectures of Sun and Wanless for odd-dimensional Latin hypercubes of order 4.

Definition 3.1 ([13]). *A nonnegative tensor is polystochastic if the sum of the entries in each of its lines is equal to 1. A 2-dimensional polystochastic tensor is known as a doubly stochastic matrix.*

Latin hypercubes generalize Latin squares to multidimensional arrays. The number of transversals in a Latin hypercube was described in [5].

Definition 3.2 ([13])**.** *A d-dimensional Latin hypercube Q of order n is a d-dimensional tensor of order n with the property that every line contains pairwise distinct elements of the set $\{1,\dots,n\}$. A 2-dimensional Latin hypercube is called a Latin square, and a 3-dimensional Latin hypercube is known as a Latin cube.*

Definition 3.3 ([14])**.** *A partial diagonal p of length k in a d-dimensional tensor of order n is a set $\{\alpha^1,\dots,\alpha^k\}$ of k indices such that for any i and j, α^i and α^j are distinct in all components. A partial diagonal p is positive if all entries of $\mathcal{A}$ with indices in p are greater than 0.*

Definition 3.4 ([13])**.** *A transversal in a Latin hypercube Q is a diagonal that all elements are distinct.*

There is a one-to-one correspondence between Latin hypercubes $Q = (q_{i_1\dots i_d})_{n\times\dots\times n}$ and polystochastic $(0,1)$-tensors $\mathcal{A} = (a_{i_1\dots i_{d+1}})_{n\times\dots\times n}$, in the sense that $q_{i_1\dots i_d} = i_{d+1}$ if and only if $a_{i_1\dots i_{d+1}} = 1$.

The permanent of a polystochastic $(0,1)$-tensor is equal to the number of transversals in its corresponding Latin hypercube [13]. Latin hypercubes generalize Latin squares to multidimensional arrays. The number of transversals in a Latin hypercube was described in [5].

Definition 3.5 ([14])**.** *A $k\times m$ row-Latin rectangle R is a table with k rows and m columns filled by m symbols in such a way that each row contains all the m symbols. A transversal in the rectangle R is the set of $\min\{k,m\}$ entries hitting each row, each column, and each symbol no more than once.*

Definition 3.6 ([3])**.** *We say that rectangles R and S are isotopic if S can be obtained by permuting the rows, columns, and symbols of R. The triple of permutations which achieves this is called an isotopism.*

The interested reader is referred to [3] for the theory of row-Latin rectangles. In the following example, we present a row-Latin rectangle with no transversals.

Example 3.7. *Let R be a row-Latin rectangle of the form below. We show that R has no transversals.*

$$\begin{array}{lllllllll}
1 & 2 & 3 & 4 & \dots & \dots & n-3 & n-2 & n-1\\
1 & 2 & 3 & 4 & \dots & \dots & n-3 & n-2 & n-1\\
\vdots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots\\
1 & 2 & 3 & 4 & \dots & \dots & n-3 & n-2 & n-1\\
n-1 & 1 & 2 & 3 & \dots & \dots & n-4 & n-3 & n-2\\
n-1 & 1 & 2 & 3 & \dots & \dots & n-4 & n-3 & n-2.
\end{array}$$

In fact, suppose that R has a transversal $\mathtt{T}$. We may assume, without loss of generality, that $\mathtt{T}$ has a 1 in the first column. Then, it must have a 2 in the second column, ..., and an $n-2$ in the $(n-2)$th column. But, then there is no possible choice for the $(n-1)$th column, since all the rows that contain $n-1$ have been used. Thus, R has no transversals.

The following conjecture has been proposed by Sun [12] in 2008. It was proved by Taranenko in the special case $n = 4$ [14].

Conjecture 3.8. *Every 3-dimensional Latin hypercube of order n has a transversal.*

To prove Theorem 3.11 below, we need the following lemma.

Lemma 3.9. *If A is a doubly stochastic $(0,1)$-matrix of order n that contains a positive partial diagonal of length $(n-2)$, then the partial diagonal can be extended to a positive partial diagonal of length $(n-1)$.*

Proof. This follows from the fact that, each doubly stochastic $(0,1)$-matrix is a permutation matrix. □

Lemma 3.9 is not valid for nonnegative matrices of order $n \geq 5$. For instance, matrix A below has a positive diagonal of length 3, which cannot be extended to a positive partial diagonal of length 4.

$$A = \begin{bmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{2}{4} \\ 0 & \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \\ 0 & 0 & \frac{2}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} & 0 & 0 \\ \frac{2}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \end{bmatrix}.$$

The following conjecture is equivalent to Sun's conjecture.

Conjecture 3.10. *The permanent of a 4-dimensional polystochastic $(0,1)$-tensor of order n is positive.*

In what follows, we use the special row-Latin rectangle R to present an algorithm for the study of Conjecture 3.10.

Theorem 3.11. *The permanent of a 4-dimensional polystochastic $(0,1)$-tensor of order n that is constructed using the special $n \times (n-1)$ row-Latin rectangle R, mentioned in Example 3.7, is positive.*

Proof. We show that a 4-dimensional polystochastic $(0,1)$-tensor of order n with permanent equal to 0 cannot be constructed by using the special row-Latin rectangle R. To see this, we try to construct such a tensor, and we observe that this is not indeed possible. We exert the algorithm below step by step for the construction. (See Table 2.) This algorithm has five steps. In step 1, we define a doubly stochastic matrix B, and we select a positive diagonal for it. In step 2, we define the doubly stochastic matrices B_i and we select a positive diagonal for B_i, where $i = 0, \ldots, n-1$, using a row-Latin rectangle R with no transversals. In step 4, we define the doubly stochastic matrices C_k, where $k = 1, \ldots, n-1$, and we extend a positive partial diagonal of length $n-2$ to a positive partial diagonal of length $n-1$ for C_k, where $k = 1, \ldots, n-1$. In step 5, we consider the vertical lines $(*, n-2, k, k)$, where $k = 1, \ldots, n-1$ and $* = 0, \ldots, n-1$, and we set equal to 1 the entries on these lines in order for $\mathcal{A}$ to remain polystochastic. In each step, we set equal to 1 some entries of $\mathcal{A}$, since $\mathcal{A}$ is a $(0,1)$-tensor by our hypothesis. At the end of step 5, we observe that it is not possible to construct a 4-dimensional polystochastic $(0,1)$-tensor of order n with permanent equal to 0.

Step 1: Consider the 2-dimensional plane B of the form

$$B = \mathcal{A}(*_1, *_2, 0, 0) \qquad *_1, *_2 = 0, \dots, n-1.$$

Since $\mathcal{A}$ is a polystochastic tensor, B is a doubly stochastic matrix. Therefore, being a doubly stochastic matrix, B has a positive diagonal [9]. Without loss of generality, we set the entries of $\mathcal{A}$ with indices $(i, i, 0, 0)$ equal to 1, where $i = 0, 1, \dots, n-1$, and we also consider these entries as a positive diagonal for B.

We denote these positive entries by 1_{s1} in Table 2. Notice that at the end of this step, we check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step 2: Consider the 2-dimensional plane B_i of the form

$$B_i = \mathcal{A}(i, i, *_1, *_2), \qquad i = 0, \dots, n-1, \qquad *_1, *_2 = 0, \dots, n-1.$$

As before, B_i is a doubly stochastic matrix, and hence has a positive diagonal. To select a positive diagonal for B_i, we consider an $n \times (n-1)$ row-Latin rectangle R with no transversals. (Since a transversal in R gives a positive diagonal for $\mathcal{A}$, and our goal is to construct a polystochastic $(0,1)$-tensor with permanent equal to 0, we consider a row-Latin rectangle without any transversals.) We assume that the entries of $\mathcal{A}$ with indices $\{(i, i, \beta_i^j, \gamma_i^j)\}_{j=1}^n$ are equal to 1, and also form a positive diagonal for B_i containing $a_{i,i,0,0}$, where β_i^j and γ_i^j are determined according to the rectangle R as follows.

> The entry in the $(i+1)$th row and the β_i^jth column of R is γ_i^j, where $i = 0, \dots, n-1$.

In what follows, we introduce a row-Latin rectangle R, and we use it to choose β_i^j and γ_i^j. To do so, we consider R in the form below.

$$\begin{array}{llllllllll}
1 & 2 & 3 & 4 & \dots & \dots & n-3 & n-2 & n-1 \\
1 & 2 & 3 & 4 & \dots & \dots & n-3 & n-2 & n-1 \\
\vdots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots \\
1 & 2 & 3 & 4 & \dots & \dots & n-3 & n-2 & n-1 \\
n-1 & 1 & 2 & 3 & \dots & \dots & n-4 & n-3 & n-2 \\
n-1 & 1 & 2 & 3 & \dots & \dots & n-4 & n-3 & n-2.
\end{array}$$

We observed in Example 3.7 that R has no transversals. We consider a positive diagonal for B_i, where $i = 0, \dots, n-1$, according to the row-Latin rectangle R and the discussion above, as follows.

For example, to choose a positive diagonal for B_0, we look at the first row of R. In this row, we see that the entry located in the first column is equal to 1 (that is, $R_{11} = 1$). Hence, we let $a_{0011} = 1$. Also, in this row of R, we see that $R_{1,n-1} = n-1$. Therefore, we let $a_{0,0,n-1,n-1} = 1$. Similarly, to select a positive diagonal for B_{n-1}, we look at the last row of R. In this row, we see that the entry located in the first column is equal to $n-1$ (that is, $R_{n1} = n-1$). Hence, we let $a_{n-1,n-1,1,n-1} = 1$. Also, in this row of R, we see that $R_{n,n-1} = n-2$. Therefore, we let $a_{n-1,n-1,n-1,n-2} = 1$.

Thus, we set equal to 1 the entries of $\mathcal{A}$ with indices

$$(i,i,0,0),(i,i,1,1),(i,i,2,2),\dots,(i,i,n-1,n-1),$$

for $i = 0,\dots,n-3$, and

$$(i,i,0,0),(i,i,1,n-1),(i,i,2,1),(i,i,3,2),\dots,(i,i,n-1,n-2),$$

for $i = n-2, n-1$. We choose these entries as a positive diagonal for B_i, where $i = 0,\dots,n-1$, and denote them by 1_{s2} in Table 2. So far, the entries of $\mathcal{A}$ with the following indices are equal to 1. (We set these entries equal to 1 in steps 1 and 2.)

$$\begin{array}{ccccc}
(0,0,0,0) & (0,0,1,1) & (0,0,2,2) & \dots & (0,0,n-1,n-1),\\
(1,1,0,0) & (1,1,1,1) & (1,1,2,2) & \dots & (1,1,n-1,n-1),\\
\vdots & \vdots & \vdots & \dots & \vdots\\
(n-3,n-3,0,0) & (n-3,n-3,1,1) & (n-3,n-3,2,2) & \dots & (n-3,n-3,n-1,n-1),\\
(n-2,n-2,0,0) & (n-2,n-2,1,n-1) & (n-2,n-2,2,1) & \dots & (n-2,n-2,n-1,n-2),\\
(n-1,n-1,0,0) & (n-1,n-1,1,n-1) & (n-1,n-1,2,1) & \dots & (n-1,n-1,n-1,n-2),
\end{array}$$

Also, we assume that for all other indices of the form

$$(i,i,\lambda,\mu), \qquad i = 0,1,\dots,n-1, \quad \lambda,\mu = 1,\dots,n-1,$$

the entries of $\mathcal{A}$ are equal to 0. We denote these entries by 0_{s2} in Table 2. We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step 3: We let $a_{*,n-2,k,k} = a_{*,n-1,k,k} = 0$, where $* = 0,\dots,n-3$ and $k = 0,\dots,n-1$, since these entries are on the same horizontal line as $a_{*,*,k,k} = 1$, where $* = 0,\dots,n-3$. We denote these 0 entries by 0_{s3} in Table 2.

Step 4: Consider the 2-dimensional plane C_k of the form

$$C_k = \mathcal{A}(*_1,*_2,k,k), \qquad k = 1,\dots,n-1, \qquad *_1,*_2 = 0,1,\dots,n-1.$$

As before, since $\mathcal{A}$ is a polystochastic tensor, C_k is a doubly stochastic matrix. We set equal to 1 the entries of $\mathcal{A}$ with the following indices in steps 1 and 2.

$$(0,0,k,k),(1,1,k,k),(2,2,k,k),\dots,(n-3,n-3,k,k), \qquad k = 1,\dots,n-1. \tag{3.1}$$

The entries of $\mathcal{A}$ with the indices mentioned in (3.1) form a positive partial diagonal of length $n-2$ for C_k, where $k = 1,\dots,n-1$. We can extend each of these positive partial diagonals of length $n-2$ to a positive partial diagonal of length $n-1$ by Lemma 3.9. Thus, we let $a_{n-2,n-1,k,k} = 1$ for $k = 1,\dots,n-1$, and extend each of these positive partial diagonals of length $n-2$ to a positive partial diagonal of length $n-1$ in the form

$$(0,0,k,k),(1,1,k,k),\dots,(n-3,n-3,k,k),(n-2,n-1,k,k),$$

for $k = 1,\dots,n-1$. We denote these entries by 1_{s4} in Table 2. At the end of this step, we check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step 5: Now, we consider the vertical lines of $\mathcal{A}$ with indices $(*, n-2, k, k)$, where $k = 1, \ldots, n-1, * = 0, \ldots, n-1$. We set equal to 0 the entries with indices $(*, n-2, k, k)$, where $* = 0 \ldots, n-2$ and $k = 1, \ldots, n-1$, in steps 2 and 3. In order for $\mathcal{A}$ to remain polystochastic, we have to let $a_{n-1,n-2,k,k} = 1$ for $k = 1, \ldots, n-1$. We denote these entries by 1_{s5} in Table 2. But, this gives a positive diagonal for $\mathcal{A}$ of the form

$$a_{0,0,0,0}, a_{1,1,1,1}, \ldots, a_{n-3,n-3,n-3,n-3}, a_{n-2,n-1,n-2,n-2}, a_{n-1,n-2,n-1,n-1}.$$

Therefore, we cannot construct a 4-dimensional polystochastic $(0,1)$ -tensor of order n with permanent equal to 0 by using special row-Latin rectangle R. The reason is that if $a_{n-1,n-2,n-1,n-1} = 0$, then $\mathcal{A}$ cannot be polystochastic. Thus, the permanent of each 4-dimensional polystochastic $(0,1)$-tensor of order n that is constructed using the special $n \times (n-1)$ row-Latin rectangle R, mentioned in Example 3.7, is positive. □

In the proof of Theorem 3.11, we can use any $n \times (n-1)$ row-Latin rectangle R' with no transversals and not necessarily isotopic to R. In this case, we can study Theorem 3.11 in the following form. Similar to the proof of Theorem 3.11, we first choose a positive diagonal for each doubly stochastic submatrix of $\mathcal{A}$ by using R' to set equal to 1 some entries of $\mathcal{A}$ that do not form a positive diagonal for it. Then, in order for $\mathcal{A}$ to be a polystochastic tensor, we consider each line of $\mathcal{A}$ whose all entries are equal to 0, and we set equal to 1 one entry of each of these lines in such a way that the positive entries considered so far do not form a positive diagonal for $\mathcal{A}$. Eventually, after these selections, we may obtain a positive diagonal for $\mathcal{A}$.

In Table 2, the 4-dimensional polystochastic $(0,1)$-tensor of order n after case 1. Here, "1" denotes an entry equal to 1, "0" shows an entry equal to 0, and dots are used to denote insignificant entries. The indices determine the steps in which the entries are specified.

Table 2. The table associated to Theorem 5.1

$\begin{smallmatrix} 1_{s1} & . & . & \cdots & . & . & . & . \\ . & 1_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} & .0_{s2} \\ . & 0_{s2} & 1_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} & 0_{s2} \\ . & 0_{s2} & 0_{s2} & \ddots & 0_{s2} & 0_{s2} & 0_{s2} & 0_{s2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 1_{s2} & 0_{s2} & 0_{s2} \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 1_{s2} & 0_{s2} \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} & 1_{s2} \end{smallmatrix}$	$\begin{smallmatrix} . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \end{smallmatrix}$	$\cdots$	$\begin{smallmatrix} . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \end{smallmatrix}$	$\begin{smallmatrix} 0_{s3} & . & . & \cdots & . & . & . & . \\ . & 0_{s3} & . & \cdots & . & . & . & . \\ . & . & 0_{s3} & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & 0_{s3} & . & . \\ . & . & . & \cdots & . & . & 0_{s3} & . \\ . & . & . & \cdots & . & . & . & 0_{s3} \end{smallmatrix}$	$\begin{smallmatrix} 0_{s3} & . & . & \cdots & . & . & . & . \\ . & 0_{s3} & . & \cdots & . & . & . & . \\ . & . & 0_{s3} & \cdots & . & . & . & . \\ . & . & . & \ddots & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & 0_{s3} & . & . \\ . & . & . & \cdots & . & . & 0_{s3} & . \\ . & . & . & \cdots & . & . & . & 0_{s3} \end{smallmatrix}$
$\begin{smallmatrix} . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \end{smallmatrix}$	$\begin{smallmatrix} 1_{s1} & . & . & \cdots & . & . & . & . \\ . & 1_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} & .0_{s2} \\ . & 0_{s2} & 1_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} & 0_{s2} \\ . & 0_{s2} & 0_{s2} & \ddots & 0_{s2} & 0_{s2} & 0_{s2} & 0_{s2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 1_{s2} & 0_{s2} & 0_{s2} \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 1_{s2} & 0_{s2} \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} & 1_{s2} \end{smallmatrix}$	$\cdots$	$\begin{smallmatrix} . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \end{smallmatrix}$	$\begin{smallmatrix} 0_{s3} & . & . & \cdots & . & . & . & . \\ . & 0_{s3} & . & \cdots & . & . & . & . \\ . & . & 0_{s3} & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & 0_{s3} & . & . \\ . & . & . & \cdots & . & . & 0_{s3} & . \\ . & . & . & \cdots & . & . & . & 0_{s3} \end{smallmatrix}$	$\begin{smallmatrix} 0_{s3} & . & . & \cdots & . & . & . & . \\ . & 0_{s3} & . & \cdots & . & . & . & . \\ . & . & 0_{s3} & \cdots & . & . & . & . \\ . & . & . & \ddots & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & 0_{s3} & . & . \\ . & . & . & \cdots & . & . & 0_{s3} & . \\ . & . & . & \cdots & . & . & . & 0_{s3} \end{smallmatrix}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$\begin{smallmatrix} . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \end{smallmatrix}$	$\begin{smallmatrix} . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \end{smallmatrix}$	$\cdots$	$\begin{smallmatrix} 1_{s1} & . & . & \cdots & . & . & . & . \\ . & 1_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} & .0_{s2} \\ . & 0_{s2} & 1_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} & 0_{s2} \\ . & 0_{s2} & 0_{s2} & \ddots & 0_{s2} & 0_{s2} & 0_{s2} & 0_{s2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 1_{s2} & 0_{s2} & 0_{s2} \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 1_{s2} & 0_{s2} \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} & 1_{s2} \end{smallmatrix}$	$\begin{smallmatrix} 0_{s3} & . & . & \cdots & . & . & . & . \\ . & 0_{s3} & . & \cdots & . & . & . & . \\ . & . & 0_{s3} & \cdots & . & . & . & . \\ . & . & . & \ddots & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & 0_{s3} & . & . \\ . & . & . & \cdots & . & . & 0_{s3} & . \\ . & . & . & \cdots & . & . & . & 0_{s3} \end{smallmatrix}$	$\begin{smallmatrix} 0_{s3} & . & . & \cdots & . & . & . & . \\ . & 0_{s3} & . & \cdots & . & . & . & . \\ . & . & 0_{s3} & \cdots & . & . & . & . \\ . & . & . & \ddots & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & 0_{s3} & . & . \\ . & . & . & \cdots & . & . & 0_{s3} & . \\ . & . & . & \cdots & . & . & . & 0_{s3} \end{smallmatrix}$
$\begin{smallmatrix} . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \ddots & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \end{smallmatrix}$	$\begin{smallmatrix} . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \ddots & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \end{smallmatrix}$	$\cdots$	$\begin{smallmatrix} . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \ddots & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \end{smallmatrix}$	$\begin{smallmatrix} 1_{s1} & . & . & \cdots & . & . & . & . \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} & 1_{s2} \\ . & 1_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} & 0_{s2} \\ . & 0_{s2} & 1_{s2} & \ddots & 0_{s2} & 0_{s2} & 0_{s2} & 0_{s2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ . & 0_{s2} & 0_{s2} & \cdots & 1_{s2} & 0_{s2} & 0_{s2} & 0_{s2} \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 1_{s2} & 0_{s2} & 0_{s2} \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 1_{s2} & 0_{s2} \end{smallmatrix}$	$\begin{smallmatrix} 0_{s3} & . & . & \cdots & . & . & . & . \\ . & 1_{s4} & . & \cdots & . & . & . & . \\ . & . & 1_{s4} & \cdots & . & . & . & . \\ . & . & . & \ddots & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & 1_{s4} & . & . \\ . & . & . & \cdots & . & . & 1_{s4} & . \\ . & . & . & \cdots & . & . & . & 1_{s4} \end{smallmatrix}$
$\begin{smallmatrix} . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \ddots & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \end{smallmatrix}$	$\begin{smallmatrix} . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \ddots & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \end{smallmatrix}$	$\cdots$	$\begin{smallmatrix} . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \ddots & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \\ . & . & . & \cdots & . & . & . & . \end{smallmatrix}$	$\begin{smallmatrix} . & . & . & \cdots & . & . & . & . \\ . & 1_{s5} & . & \cdots & . & . & . & . \\ . & . & 1_{s5} & \cdots & . & . & . & . \\ . & . & . & \ddots & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ . & . & . & \cdots & . & 1_{s5} & . & . \\ . & . & . & \cdots & . & . & 1_{s5} & . \\ . & . & . & \cdots & . & . & . & 1_{s5} \end{smallmatrix}$	$\begin{smallmatrix} 1_{s1} & . & . & \cdots & . & . & . & . \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} & 1_{s2} \\ . & 1_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} & 0_{s2} \\ . & 0_{s2} & 1_{s2} & \ddots & 0_{s2} & 0_{s2} & 0_{s2} & 0_{s2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ . & 0_{s2} & 0_{s2} & \cdots & 1_{s2} & 0_{s2} & 0_{s2} & 0_{s2} \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 1_{s2} & 0_{s2} & 0_{s2} \\ . & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 1_{s2} & 0_{s2} \end{smallmatrix}$

The conjecture below was proposed by Wanless [16] in 2011.

Conjecture 3.12. *Every odd-dimensional Latin hypercube of order n has a transversal.*

The following theorem is equivalent to Wanless' conjecture.

Conjecture 3.13. *The permanent of an even-dimensional polystochastic $(0,1)$-tensor of order n is positive.*

In what follows, we use the special row-Latin rectangle R to present an algorithm for the study of Conjecture 3.13.

Theorem 3.14. *The permanent of an even-dimensional polystochastic $(0,1)$ -tensor of order n that is constructed using the special $n \times (n-1)$ row-Latin rectangle R, mentioned in Example 3.7, is positive.*

Proof. We show that the permanent of each $2s$-dimensional polystochastic $(0,1)$-tensor of order n, where $s \in \mathbb{N}$, is positive. We prove this theorem by induction. Since 2-dimensional polystochastic $(0,1)$-tensors are doubly stochastic matrices, the statement of the theorem is true for 2-dimensional polystochastic $(0,1)$-tensors of order n (that is, $L = 1$) by [9].
The induction hypothesis: Assume that the statement of the theorem is true for $2s-2$-dimensional polystochastic $(0,1)$-tensors of order n (that is, $L = s-1$). Thus, the permanent of each $2s-2$-dimensional polystochastic $(0,1)$-tensor, where $s \in \mathbb{N}$, is positive.

We must prove that the statement of the theorem is also true for $2s$-dimensional polystochastic $(0,1)$-tensors of order n (that is, $L = s$). To see this, we show that a $2s$-dimensional polystochastic $(0,1)$-tensor of order n, where $s \in \mathbb{N}$, with permanent equal to 0 cannot be constructed. In fact, we try to construct such a tensor, and we observe that this construction is not possible.

We follow the algorithm below step by step to construct a $2s$-dimensional polystochastic $(0,1)$-tensor of order n with permanent equal to 0. This algorithm has five steps. In step 1, we define a doubly stochastic matrix B, and we select a positive diagonal for B. In step 2, we define the $2s-2$-dimensional polystochastic tensors B_i and we select a positive diagonal for B_i, where $i = 0, \dots, n-1$, using a row-Latin rectangle R with no transversals. In step 4, we define the doubly stochastic matrices C_k, where $k = 1, \dots, n-1$, and we extend a positive partial diagonal of length $n-2$ to a positive partial diagonal of length $n-1$ for C_k, where $k = 1, \dots, n-1$. In step 5, we consider the vertical lines $(*, n-2, k, \dots, k)$, where $k = 1, \dots, n-1$ and $* = 0, \dots, n-1$, and we set equal to 1 the entries on these lines in order for $\mathcal{A}$ to remain polystochastic. In each step, we set some entries of $\mathcal{A}$ equal to 1, since $\mathcal{A}$ is a $(0,1)$-tensor by our hypothesis. At the end of step 5, we observe that we cannot construct a $2s$-dimensional polystochastic $(0,1)$-tensor of order n with permanent equal to 0.

Step 1: Consider the 2-dimensional plane B of the form

$$B = \mathcal{A}(*_1, *_2, \underbrace{0, \ldots, 0}_{2s-2}) \qquad *_1, *_2 = 0, \ldots, n-1.$$

Since $\mathcal{A}$ is a polystochastic tensor, B is a doubly stochastic matrix. Hence, by [9], B has a positive diagonal. Without loss of generality, assume that the entries of $\mathcal{A}$ with indices $(i, i, 0, 0, \ldots, 0)$ are equal to 1, where $i = 0, \ldots, n-1$, and form a positive diagonal for B.

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step 2: Consider the $2s-2$-dimensional plane $\mathcal{B}_i$ of the form

$$\mathcal{B}_i = \mathcal{A}(i, i, *_1, \ldots, *_{2s-2}), \quad i = 0, \ldots, n-1, \quad *_1, \ldots, *_{2s-2} = 0, \ldots, n-1.$$

Then, $\mathcal{B}_i$ is a $2s-2$-dimensional tensor of order n. Since $\mathcal{A}$ is a polystochastic $(0,1)$-tensor, $\mathcal{B}_i$ is a $2s-2$-dimensional polystochastic $(0,1)$-tensor of order n. The induction hypothesis allows us to conclude that the permanent of $\mathcal{B}_i$ is positive, where $i = 0, \ldots, n-1$. To select a positive diagonal for $\mathcal{B}_i$, we consider an $n \times (n-1)$ row-Latin rectangle R with no transversals. (Since a transversal in R gives a positive diagonal for $\mathcal{A}$, and our goal is to construct a polystochastic $(0,1)$-tensor with permanent equal to 0, we consider a row-Latin rectangle without any transversals.) We assume that the entries of $\mathcal{A}$ with indices $\{(i, i, \beta_i^j, \gamma_i^j, \beta_i^j, \gamma_i^j, \ldots, \beta_i^j, \gamma_i^j)\}_{j=1}^n$ are equal to 1, and also form a positive diagonal for $\mathcal{B}_i$ containing $a_{i,i,0,0,\ldots,0}$, where β_i^j and γ_i^j are determined according to the rectangle R as follows.

The entry in the $(i+1)$th row and the β_i^jth column of R is γ_i^j, where $i = 0, \ldots, n-1$.

In what follows, we introduce a row-Latin rectangle R, and we use it to choose β_i^j and γ_i^j.

We choose the row-Latin rectangle R mentioned in Example 3.7. (We know that R has no transversals.) We consider a positive diagonal for $\mathcal{B}_i$, where $i = 0, \ldots, n-1$, according to the row-Latin rectangle R and the discussion above, as follows.

For example, to select a positive diagonal for $\mathcal{B}_0$, we look at the first row of R. In this row, we see that the entry located in the first column is equal to 1 (that is, $R_{11} = 1$). Hence, we let $a_{0011\ldots1} = 1$. Also in this row of R, we see that $R_{1,n-1} = n-1$. Therefore, we let $a_{0,0,n-1,n-1,\ldots,n-1} = 1$.

For instance, to choose a positive diagonal for $\mathcal{B}_{n-1}$, we look at the last row of R. In this row, we see that the entry located in the first column is equal to $n-1$ (that is, $R_{n1} = n-1$). So, we let $a_{n-1,n-1,1,n-1,1,n-1,\ldots,1,n-1} = 1$. Also in this row of R, we see that $R_{n,n-1} = n-2$. Therefore, we let

$$a_{n-1,n-1,n-1,n-2,n-1,n-2,\ldots,n-1,n-2} = 1.$$

Thus, we set equal to 1 the entries of $\mathcal{A}$ with indices

$$(i,i,0,0,\dots,0),(i,i,1,1,\dots,1),(i,i,2,2,\dots,2),$$
$$\dots,(i,i,n-1,n-1,\dots,n-1),$$

where $i = 0,\dots,n-3$, and

$$(i,i,0,\dots,0),(i,i,1,n-1,1,n-1,\dots,1,n-1),(i,i,2,1,2,1,\dots,2,1),$$
$$(i,i,3,2,3,2,\dots,3,2),\dots,(i,i,n-1,n-2,n-1,n-2,\dots,n-1,n-2),$$

where $i = n-2, n-1$. We choose these entries as a positive diagonal for B_i, where $i = 0,\dots,n-1$.

So far, the entries of $\mathcal{A}$ with the following indices are equal to 1. (We set these entries equal to 1 in steps 1 and 2.)

$$\begin{array}{llll}
(0,0,\dots,0) & (0,0,1,\dots,1) & (0,0,2,\dots,2) & \dots \\
(1,1,0,\dots,0) & (1,1,\dots,1) & (1,1,2,\dots,2) & \dots \\
\vdots & \vdots & \vdots & \dots \\
(n-3,n-3,0,\dots,0) & (n-3,n-3,1,\dots,1) & (n-3,n-3,2,\dots,2) & \dots \\
(n-2,n-2,0,\dots,0) & (n-2,n-2,1,n-1,1,n-1,\dots,1,n-1) & (n-2,n-2,2,1,2,1,\dots,2,1) & \dots \\
(n-1,n-1,0,\dots,0) & (n-1,n-1,1,n-1,1,n-1,\dots,1,n-1) & (n-1,n-1,2,1,2,1,\dots,2,1) & \dots
\end{array}$$

$$\begin{array}{ll}
\dots & (0,0,n-1,\dots,n-1), \\
\dots & (1,1,n-1,\dots,n-1), \\
\dots & \vdots \\
\dots & (n-3,n-3,n-1,\dots,n-1), \\
\dots & (n-2,n-2,n-1,n-2,n-1,n-2,\dots,n-1,n-2), \\
\dots & (n-1,n-1,n-1,n-2,n-1,n-2,\dots,n-1,n-2).
\end{array}$$

Also, we assume that for all other indices of the form

$$(i,i,\lambda_1,\lambda_2,\dots,\lambda_{2s-2}), \quad i = 0,1,\dots,n-1, \quad \lambda_1,\lambda_2\dots,\lambda_{2s-2} = 1,\dots,n-1,$$

the entries of $\mathcal{A}$ are equal to 0. Notice that at the end of this step, we check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step 3: We let $a_{*,n-2,k,\dots,k} = a_{*,n-1,k,\dots,k} = 0$, where $* = 0,\dots,n-3$ and $k = 0,\dots,n-1$, since these entries are on the same horizontal line as $a_{*,*,k,k,\dots,k} = 1$, where $* = 0,\dots,n-3$.

Step 4: Consider the 2-dimensional plane C_k of the form

$$C_k = \mathcal{A}(*_1,*_2,\underbrace{k,\dots,k}_{2s-2}), \qquad k = 1,\dots,n-1, \qquad *_1,*_2 = 0,1,\dots,n-1.$$

As before, C_k is a doubly stochastic matrix. We set equal to 1 the entries of $\mathcal{A}$ with the following indices in steps 1 and 2.

$$(0,0,k,k,\dots,k),(1,1,k,k,\dots,k),(2,2,k,k,\dots,k),\dots,$$
$$(n-3,n-3,k,k,\dots,k), \quad k = 1,\dots,n-1. \tag{3.2}$$

So, the entries of $\mathcal{A}$ with indices mentioned in (3.2) form a positive partial diagonal of length $n-2$ for C_k, where $k = 1,\dots,n-1$. We can extend each of these positive partial diagonals of length $n-2$ to a positive partial diagonal of length $n-1$ by Lemma 3.9. Hence, we let $a_{n-2,n-1,k,k,\dots,k} = 1$, where $k = 1, \dots, n-1$, and extend each of these positive partial diagonals of length $n-2$ to a positive partial diagonal of length $n-1$ of the form

$$(0, 0, k, k, \dots, k), (1, 1, k, k, \dots, k), (2, 2, k, k, \dots, k), \dots,$$
$$(n-3, n-3, k, k, \dots, k), (n-2, n-1, k, k, \dots, k), \qquad k = 1, \dots, n-1.$$

At the end of this step, we check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step 5: Now, we consider the vertical lines of $\mathcal{A}$ with indices $(*, n-2, k, k, \dots, k)$, where $k = 1, \dots, n-1, * = 0, \dots, n-1$. We set equal to 0 the entries with indices $(*, n-2, k, k, \dots, k)$, where $* = 0, \dots, n-2$ and $k = 1, \dots, n-1$, in steps 2 and 3. In order for $\mathcal{A}$ to remain polystochastic, we have to let $a_{n-1,n-2,k,k,\dots,k} = 1$ for $k = 1, \dots, n-1$. But, this gives a positive diagonal for $\mathcal{A}$ with the following indices.

$$(0, 0 \dots, 0), (1, 1 \dots, 1), \dots, (n-3, n-3, \dots, n-3),$$
$$(n-2, n-1, n-2, n-2 \dots, n-2), (n-1, n-2, n-1, n-1 \dots, n-1).$$

So, we cannot construct a $2s$-dimensional polystochastic $(0,1)$-tensor of order n with permanent equal to 0. Thus, the permanent of each even-dimensional polystochastic $(0,1)$-tensor of order n that is constructed using the special $n \times (n-1)$ row-Latin rectangle R, mentioned in Example 3.7, is positive. □

In the proof of Theorem 3.14, we can use any $n \times (n-1)$ row-Latin rectangle R' with no transversals and not necessarily isotopic to R. In this case, we can study Theorem 3.14 in the following form. Similar to the proof of Theorem 3.14, we first choose a positive diagonal for each 2-dimensional plane and $2s-2$-dimensional plane of $\mathcal{A}$ by using R' to set equal to 1 some entries of $\mathcal{A}$ that do not form a positive diagonal for it. Then, in order for $\mathcal{A}$ to be a polystochastic tensor, we consider each line of $\mathcal{A}$ whose all entries are equal to 0, and we set equal to 1 one entry of each of these lines in such a way that the positive entries considered so far do not form a positive diagonal for $\mathcal{A}$. Eventually, after these selections, we may obtain a positive diagonal for $\mathcal{A}$.

According to the relation between $d+1$-dimensional polystochastic $(0,1)$-tensors of order n and d-dimensional Latin hypercubes of the same order, we obtain the following corollary (special case of Wanless' conjecture).

Corollary 3.15. *The number of transversals in an odd-dimensional Latin hypercube of order* 4 *is positive.*

Proof. The number of transversals in an odd-dimensional Latin hypercube of order n is equal to the permanent of its corresponding even-dimensional polystochastic $(0,1)$-tensor of order n. We know that the row-Latin rectangle R is the only $n \times (n-1)$ row-Latin rectangle with no transversals for $n = 4$ [14]. Also, By Theorem 3.14, we know that the permanent of each even-dimensional polystochastic $(0,1)$-tensor of order n is positive. Thus, the number of transversals in an odd-dimensional Latin hypercube of order 4 is positive. □

Taranenko proved Corollary 3.15 by using Quasigroups [15].

4. Hypergraphs

As is known, for $d \geq 3$, it is an NP-complete problem to determine whether a given bipartite hypergraph contains a perfect matching. In this section, we discuss the application of the permanents of tensors to hypergraphs, and we give a positive answer to this question for the bipartite hypergraph associated to an even-dimensional polystochastic $(0,1)$-tensor of order 4.

Definition 4.1 ([13]). *A pair $H = (X, W)$ is called a hypergraph with vertex set X and hyperedge set W, where each hyperedge $w \in W$ is a subset of the vertices in X. A hypergraph H is called k-uniform if each of its hyperedges consists of k vertices.*

Definition 4.2 ([4]). *Let $\mathcal{A}$ be a d-dimensional tensor of order n. The generalization of the bipartite graph of a matrix to the tensor is the hypergraph $H(\mathcal{A}) = (V, W)$ with the vertex set*

$$V = \{v_j^k : k = 1, \ldots, d, j = 1, 2, \ldots, n_k\},$$

and the edge set

$$W = \{(v_{i_1}^1, v_{i_2}^2, \ldots, v_{i_d}^d) : a_{i_1 i_2 \ldots i_d} \neq 0)\}.$$

Proposition 4.3 ([4]). The bipartite hypergraph introduced above is a simple hypergraph.

Definition 4.4 ([4]). *For a simple hypergraph $H = (V, W)$, a subset M of the edges of H is said to be a matching if the edges in M are pairwise disjoint. A matching M is said to be perfect if M is a partition of the vertices of H. Observe that if $\mathcal{A}$ is a square d-dimensional tensor of order n, then the perfect matchings of $H(\mathcal{A})$ are the matchings of $H(\mathcal{A})$ of cardinality n. Let $\mathcal{A}$ be a d-dimensional tensor of order n. A nonzero term in the expansion (2.1) of the permanent of $\mathcal{A}$ corresponds to a matching $\{\{v_i^1, v_{\sigma_2(i)}^2, \ldots, v_{\sigma_d(i)}^d\} : i = 1, \ldots, n\}$ of the hypergraph $H(\mathcal{A})$ of cardinality n. Indeed, per$(\mathcal{A})$ is the sum of all products of n entries of $\mathcal{A}$ corresponding to the edges in the matching of $H(\mathcal{A})$ of cardinality n. If $\mathcal{A}$ is a d-dimensional $(0,1)$-tensor of order n, then per$(\mathcal{A})$ is the number of the matchings of $H(\mathcal{A})$ of cardinality n. Clearly, if $\mathcal{A}$ is a d-dimensional $(0,1)$-tensor of order n, then per$(\mathcal{A})$ is the number of the perfect matchings of $H(\mathcal{A})$ of cardinality n.*

We know that the row-Latin rectangle R is the only $n \times (n-1)$ row-Latin rectangle with no transversals for $n = 4$ [14]. Thus, using Theorem 3.14 we obtain the following corollary.

Corollary 4.5. *The number of perfect matchings of the bipartite hypergraph associated to an even-dimensional polystochastic* $(0,1)$*-tensor of order* 4 *is positive.*

5. The permanent of a nonnegative polystochastic tensor

In this section, we generalize the theorems established in Section 3 to nonnegative tensors. We prove that the permanent of a 4-dimensional nonnegative polystochastic tensor of order n that is constructed using the special row-Latin rectangle R is positive. To help the readers better understand the proof of the theorem, we present an example of a 4-dimensional nonnegative polystochastic tensor of order 6. Moreover, we utilize induction on d to show that the permanent of an even-dimensional nonnegative polystochastic tensor of order n that is constructed using the special row Latin rectangle R is positive.

In the following theorem, we use the special row-Latin rectangle R to prove that the permanent of each 4-dimensional nonnegative polystochastic tensor of order n is positive. Taranenko proved this theorem for the special case $n = 4$ [13].

Theorem 5.1. The permanent of a 4-dimensional nonnegative polystochastic tensor $\mathcal{A}$ of order n that is constructed using the special $n \times (n-1)$ row-Latin rectangle R, mentioned in Example 3.7, is positive.

Proof. To prove the theorem, we try to construct a 4-dimensional nonnegative polystochastic tensor of order n with permanent equal to 0, and observe that such a construction is not possible. To construct a tensor with the aforementioned properties, we follow the algorithm below step by step. The proof contains three steps. Step III contains four steps, namely, I-III, II-III, III-III and IV-III. Step I-III itself contains two steps: I-I-III and II-I-III. Step II-III has five steps, which are I-II-III, II-II-III, III-II-III, IV-II-III and V-II-III. Step III-III has three steps: I-III-III, II-III-III and III-III-III. In each step, we check the permanent of the tensor under construction.

Step I: Consider a matrix B of the form

$$\mathcal{A}(*_1, *_2, 0, 0), \qquad *_1, *_2 = 0, 1, \dots, n-1.$$

Since $\mathcal{A}$ is a polystochastic tensor, B is a doubly stochastic matrix. Thus, B has a positive diagonal [9]. Without loss of generality, we let $a_{i,i,0,0} > 0$, where $i = 0, 1, \dots, n-1$. It is clear that these positive entries form a positive diagonal for B. Notice that at the end of this step, we check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step II: Let B_i be composed of those entries of $\mathcal{A}$ which are of the form

$$\mathcal{A}(i, i, *_1, *_2), \qquad i = 0, 1, \dots, n-1, \qquad *_1, *_2 = 0, 1, \dots, n-1.$$

Each B_i is a doubly stochastic matrix, and so has a positive diagonal. To select a positive diagonal of B_i, we consider an $n \times (n-1)$ row-Latin rectangle R whose number of transversals is equal to 0. (Recall that our goal is to construct a nonnegative polystochastic tensor with permanent equal to 0, and it is clear that a transversal in R gives a positive diagonal for $\mathcal{A}$. Thus, we consider a row-Latin rectangle R without any transversal.) We assume that the entries of $\mathcal{A}$ with indices $\{(i,i,\beta_i^j,\gamma_i^j)\}_{j=1}^n$ are positive numbers, and also that they form a positive diagonal for B_i containing $a_{i,i,0,0}$, where β_i^j and γ_i^j are determined according to the rectangle R in the following form.

The entry in the $(i+1)$th row and the β_i^jth column of R is γ_i^j, where $i=0,\dots,n-1$.

Now, we choose a row-Latin rectangle R as follows. We know the number of transversals in R is equal to 0 by Example 3.7.

$$\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & \dots & \dots & n-3 & n-2 & n-1 \\
1 & 2 & 3 & 4 & \dots & \dots & n-3 & n-2 & n-1 \\
\vdots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots \\
1 & 2 & 3 & 4 & \dots & \dots & n-3 & n-2 & n-1 \\
n-1 & 1 & 2 & 3 & \dots & \dots & n-4 & n-3 & n-2 \\
n-1 & 1 & 2 & 3 & \dots & \dots & n-4 & n-3 & n-2.
\end{array}$$

For example, to select a positive diagonal for B_0, we look at the first row of R. In this row, the entry located in the first column is 1 (that is, $R_{11}=1$). Thus, we let $a_{0,0,1,1}>0$. Since $R_{1,n-1}=n-1$, we let $a_{0,0,n-1,n-1}>0$. Also, to select a positive diagonal for B_{n-1}, we look at the last row of R (that is, the nth row of R). Since $R_{n1}=n-1$, we let $a_{n-1,n-1,1,n-1}>0$. Also, in this row of R, $R_{n,n-1}=n-2$. Hence, we let $a_{n-1,n-1,n-1,n-2}>0$.

Therefore, a positive diagonal for B_i, where $i=0,1,\dots,n-1$, is formed by the entries of $\mathcal{A}$ with indices

$$(i,i,0,0),(i,i,1,1),(i,i,2,2),\dots,(i,i,n-1,n-1), \quad i=0,1,\dots,n-3$$

and

$$(i,i,0,0),(i,i,1,n-1),(i,i,2,1),(i,i,3,2),\dots,(i,i,n-1,n-2),$$

for $i=n-2,n-1$. So far, the entries of $\mathcal{A}$ with the following indices are positive. (These entries are considered to be positive in steps I and II.)

$$\begin{array}{lllll}
(0,0,0,0) & (0,0,1,1) & (0,0,2,2) & \dots & (0,0,n-1,n-1), \\
(1,1,0,0) & (1,1,1,1) & (1,1,2,2) & \dots & (1,1,n-1,n-1), \\
\vdots & \vdots & \vdots & \dots & \dots \\
(n-3,n-3,0,0) & (n-3,n-3,1,1) & (n-3,n-3,2,2) & \dots & (n-3,n-3,n-1,n-1), \\
(n-2,n-2,0,0) & (n-2,n-2,1,n-1) & (n-2,n-2,2,1) & \dots & (n-2,n-2,n-1,n-2), \\
(n-1,n-1,0,0) & (n-1,n-1,1,n-1) & (n-1,n-1,2,1) & \dots & (n-1,n-1,n-1,n-2).
\end{array}$$

Assume that for all other indices of the form below, the entries of $\mathcal{A}$ are equal to 0.

$$(i,i,\lambda,\mu), \qquad i=0,1,\dots,n-1, \quad \lambda,\mu=1,2,\dots,n-1.$$

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A})=0$.

Step III: By step II, we know that $a_{n-1,n-1,i,i}=a_{n-2,n-2,i,i}=0$ for $i=1,2,\dots,n-1$. In order for $\mathcal{A}$ to remain polystochastic, we consider the vertical lines $(*, n-1, i, i)$ and $(*, n-2, i, i)$, where $i = 1, 2, \ldots, n-1$. (For each i, let $* = 0, 1, \ldots, n-1$.) Thus, in order for $\mathcal{A}$ to remain polystochastic, we should put a positive number 1 or at least two positive entries less than 1) on each vertical line $(*, n-1, i, i)$ and each vertical line $(*, n-2, i, i)$, where $i = 1, 2, \ldots, n-1$. Now, we consider four cases.

Step I-III: In this case of step III, we want to set equal to 1 one of the entries on each vertical line $(*, n-1, i, i)$ and each vertical line $(*, n-2, i, i)$, where $i = 1, 2, \ldots, n-1$.

Step I-I-III: We cannot set equal to 1 the entries with indices

$$(*, n-1, i, i), \quad * = 0, 1, \ldots, n-3, \quad i = 1, 2, \ldots, n-1,$$

because the entries $a_{*,n-1,i,i}$ lie on the same horizontal line as the positive entries $a_{*,*,i,i}$, where $* = 0, 1, \ldots, n-3$ and $i = 1, 2, \ldots, n-1$. For example, if we set $a_{0,n-1,1,1} = 1$, then the summation of the entries on the horizontal line that contains the entries $a_{0,n-1,1,1}$ and $0 < a_{0,0,1,1} < 1$ will be greater than 1, which contradicts the fact that $\mathcal{A}$ is polystochastic.

Similarly, we cannot set equal to 1 the entries with indices

$$(*, n-2, i, i), \quad * = 0, 1, \ldots, n-3, \quad i = 1, 2, \ldots, n-1,$$

because the entries $a_{*,n-2,i,i}$ lie on the same horizontal line as the positive entries $a_{*,*,i,i}$, where $* = 0, 1, \ldots, n-3$ and $i = 1, 2, \ldots, n-1$.

Step II-I-III: According to step I-III, step I-I-III, and the fact that $a_{n-1,n-1,i,i} = 0$ for $i = 1, 2, \ldots, n-1$, in order for $\mathcal{A}$ to remain polystochastic on the vertical lines $(*, n-1, i, i)$, we have to let $a_{n-2,n-1,i,i} = 1$ for $i = 1, 2, \ldots, n-1$.

Similarly, according to step I-III, step I-I-III, and the fact that $a_{n-2,n-2,i,i} = 0$ for $i = 1, 2, \ldots, n-1$, in order for $\mathcal{A}$ to remain polystochastic on the vertical lines $(*, n-2, i, i)$, we have to let $a_{n-1,n-2,i,i} = 1$ for $i = 1, 2, \ldots, n-1$.

But, this gives a positive diagonal for the 4-dimensional nonnegative polystochastic tensor $\mathcal{A}$ of order n of the form

$$a_{0,0,0,0}, a_{1,1,1,1}, \ldots, a_{n-3,n-3,n-3,n-3}, a_{n-2,n-1,n-2,n-2}, a_{n-1,n-2,n-1,n-1}.$$

Thus, we cannot construct an $n \times n \times n \times n$ nonnegative polystochastic tensor with permanent equal to 0 in this case.

Step II-III: In this case of step III, we set equal to 1 one of the entries on each vertical line $(*, n-1, i, i)$, and we put exactly two positive entries (less than 1) on each vertical line $(*, n-2, i, i)$, where $i = 1, 2, \ldots, n-1$.

According to step II-I-III, we let $a_{n-2,n-1,i,i} = 1$ for $i = 1, 2, \ldots, n-1$. Since the entries with indices $(*, n-1, i, i)$ and $(n-2, n-1, i, i)$ are on the same vertical line, $a_{*,n-1,i,i} = 0$ for $* = 0, 1, \ldots, n-3$. Also, since the entries with indices $(n-2, *, i, i)$ and $(n-2, n-1, i, i)$ are on the same horizontal line, $a_{n-2,*,i,i} = 0$ for $* = 0, 1, \ldots, n-3$ and $i = 1, 2, \ldots, n-1$.

Now, we consider the vertical lines $(*, n-2, i, i)$ for $i = 1, \ldots, n-1$ and $* = 0, 1, \ldots, n-1$. We choose arbitrary row blocks k and m, and set positive numbers (less than 1) for the entries of $\mathcal{A}$ with indices

$$(k, n-2, i, i), \qquad (m, n-2, i, i), \qquad k, m \in \{0, 1, \dots, n-3\}, \quad k \neq m.$$

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Hence, the entries

$$a_{k,k,i,i}, a_{m,m,i,i}, \qquad i = 1, 2, \dots, n-1 \tag{5.1}$$

are positive numbers (less than 1), because the entries mentioned in (5.1) are on the same horizontal line as the entries $0 < a_{k,n-2,i,i} < 1$ and $0 < a_{m,n-2,i,i} < 1$.

Since our goal is to construct a tensor with permanent equal to 0, we set equal to 0 the entries of $\mathcal{A}$ in the last row block that forms a positive diagonal for $\mathcal{A}$. Thus, we let

$$a_{n-1,n-2,i,i} = a_{n-1,k,i,i} = a_{n-1,m,i,i} = 0, \qquad i = 1, 2, \dots, n-1, \tag{5.2}$$

because, if we set positive numbers for the entries mentioned in (5.2), then $per(\mathcal{A}) > 0$.

Clearly, by setting positive numbers for the entries with indices

$$(n-1, l, i, i), \quad l = 0, 1, \dots, n-3, \quad l \neq k, m, \quad i = 1, 2, \dots, n-1, \tag{5.3}$$

the permanent of $\mathcal{A}$ will be equal to 0.

Step I-II-III: Now, we consider the horizontal lines

$$(n-1, *, i, i), \qquad * = 0, 1, \dots, n-1, \quad i = 1, 2, \dots, n-1.$$

In order for $\mathcal{A}$ to remain polystochastic, each horizontal line $(n-1, *, i, i)$ must has exactly one entry equal to 1 or at least two positive entries (less than 1). We choose these positions to put the positive entries among the positions mentioned in (5.3), because we want the permanent of $\mathcal{A}$ to remain equal to 0.

Now, if we set equal to 1 one of the entries on each horizontal line, for example, if we set

$$a_{n-1,s,i,i} = 1, \qquad s \in \{0, 1, \dots, n-3\}, \qquad s \neq k, m,$$

then we conclude that $a_{s,s,i,i} = 0$, becuase $a_{s,s,i,i}$ and $a_{n-1,s,i,i}$ are on the same vertical line. But, this contradicts $a_{s,s,i,i} > 0$ (by step II).

Therefore, we have to put at least two positive entries on each horizontal line $(n-1, *, i, i)$ in the positions mentioned in (5.3). Without loss of generality, we choose column blocks s_1 and s_2 of l and put positive numbers (less than 1) for the entries with indices

$$(n-1, s_1, i, i), (n-1, s_2, i, i), \quad s_1, s_2 \in \{0, 1, \dots, n-3\}, \quad s_1, s_2 \neq k, m,$$

where $i = 1, 2, \dots, n-1$. In what follows, in relation (5.3), the column blocks different from $s_1, s_2, k, m, n-2, n-1$ in the last row block are indexed by $s_3, s_4, \dots, s_{\max v}$ in the form

$$\mathcal{A}(n-1, s_v, i, i), \qquad s_v \neq n-2, n-1, k, m, s_1, s_2,$$

where $s_3, s_4, \ldots, s_{\max v}$ are shown by s_v. We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Since $0 < a_{n-1,s_1,i,i} < 1$ and $a_{s_1,s_1,i,i}$ lie on the same vertical line, and the same is also true for $0 < a_{n-1,s_2,i,i} < 1$ and $a_{s_2,s_2,i,i}$, the entries $a_{s_1,s_1,i,i}$ and $a_{s_2,s_2,i,i}$ cannot be equal to 1 and are positive numbers (less than 1). Therefore, each horizontal line $(s_1, *, i, i)$, where $* = 0, 1, \ldots, n-1$ and $i = 1, 2, \ldots, n-1$, must have at least one more positive entry (less than 1). The same is also true for the horizontal lines $(s_2, *, i, i)$.

Step II-II-III: We consider the horizontal lines $(s_1, *, i, i)$, where $* = 0, 1, \ldots, n-1$ and $i = 1, 2, \ldots, n-1$. It is clear that the permanent of $\mathcal{A}$ remains equal to 0 if we set positive numbers (less than 1) for the entries with indices

$$(5.4)\quad (s_1, s_v, i, i), \qquad s_v = s_2, s_3, \ldots, s_{\max v}, \qquad s_v \neq k, m, n-2, n-1, s_1.$$

Next, we put exactly one positive number (less than 1) on each horizontal line $(s_1, *, i, i)$. (This position is selected from the indices mentioned in (5.4).) For example, we put a positive number (less than 1) in the position (s_1, j_1, i, i), where $i = 1, 2, \ldots, n-1$. This implies that the entries $a_{j_1,j_1,i,i}$ are positive numbers (less than 1) for $i = 1, 2, \ldots, n-1$. Now, we set equal to 0 the entries with indices $(s_1, *, i, i)$, where $* \neq s_1, s_2, \ldots, s_{\max v}$, because, if we set positive numbers for these entries, then the permanent of $\mathcal{A}$ does not remain equal to 0.

At the end of this step, we check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step III-II-III: Now, we consider the horizontal lines $(s_2, *, i, i)$, where $* = 0, 1, \ldots, n-1$ and $i = 1, 2, \ldots, n-1$. It is clear that the permanent of $\mathcal{A}$ remains equal to 0 if we set positive numbers (less than 1) for the entries of $\mathcal{A}$ with indices

$$(5.5)\quad (s_2, s_v, i, i), \quad s_v = s_1, s_3, s_4, \ldots, s_{\max v}, \quad s_v \neq k, m, n-2, n-1, s_2.$$

Next, we put exactly one positive number (less than 1) on each horizontal line $(s_2, *, i, i)$. (This position is selected from the indices mentioned in (5.5).) For example, we put a positive number (less than 1) in the position (s_2, j_2, i, i), where $i = 1, 2, \ldots, n-1$. This implies that the entries $a_{j_2,j_2,i,i}$ are positive numbers (less than 1) for $i = 1, 2, \ldots, n-1$. In what follows, we set equal to 0 the entries with indices $(s_2, *, i, i)$, where $* \neq s_1, s_2, \ldots, s_{\max v}$, because by setting positive numbers for these entries, the permanent of $\mathcal{A}$ will not remain equal to 0.

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

We repeat the previous process for each $s_v = s_3, s_4, \ldots, s_{\max v}$. Notice that $s_v \neq n-1, n-2, k, m$. In what follows, consider the horizontal lines $(s_{\max v}, *, i, i)$, where $* = 0, 1, \ldots, n-1$ and $i = 1, 2, \ldots, n-1$, and repeat the process above for $s_{\max v}$. Similarly, we set positive numbers (less than 1) for the entries $a_{s_{\max v}, j_{\max v}, i, i}$, where $j_{\max v} \in \{s_1, s_2, \ldots, s_{\max v-1}\}$. This allows us to conclude that the entries $a_{j_{\max v}, j_{\max v}, i, i}$ are positive numbers (less than 1) for $i = 1, 2, \ldots, n-1$. Now, we set equal to 0 the entries with indices $(s_{\max v}, *, i, i)$, where $* \neq s_1, s_2, \ldots, s_{\max v}$, because, if we consider positive numbers for these entries, then the permanent of $\mathcal{A}$ will not remain equal to 0.

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step IV-II-III: By step II-III, we know that $0 < a_{k,k,i,i} < 1$, where k is mentioned in (5.1). Since we want $\mathcal{A}$ to be a polystochastic tensor, we must put at least one positive entry on each vertical line $(*, k, i, i)$, where $i = 1, 2, \ldots, n-1$. It is clear that if we set positive numbers for the entries with indices $(*, k, i, i)$, where $* = 0, 1, \ldots, n-3, * \neq m$, then the permanent of $\mathcal{A}$ is not 0. (We recall that $a_{n-1,k,i,i} = a_{n-2,k,i,i} = 0$ by step II-III.) Thus, we let $a_{*,k,i,i} = 0$, where $* = 0, 1, \ldots, n-3, * \neq m$. Also, if we set positive numbers for $a_{m,k,i,i}$, where m is mentioned in step II-III and $i = 1, 2, \ldots, n-1$, then the permanent of $\mathcal{A}$ remains 0. Hence, we put positive numbers (less than 1) in positions (m, k, i, i), where $i = 1, 2, \ldots, n-1$.

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Now, we know that the entries $a_{m,m,i,i}$, with m mentioned in (5.1), are positive numbers (less than 1) (by step II-III). Since we want $\mathcal{A}$ to be a polystochastic tensor, we must put at least one positive entry on each vertical line $(*, m, i, i)$, where $i = 1, 2, \ldots, n-1$ and $* = 0, 1, \ldots, n-1$. It is clear that if we set positive numbers for the entries with indices $(*, m, i, i)$, where $* = 0, 1, \ldots, n-3, * \neq k$, then the permanent of $\mathcal{A}$ is not 0. (We recall that $a_{n-1,m,i,i} = a_{n-2,m,i,i} = 0$ by step II-III.) Thus, we let $a_{*,m,i,i} = 0$ for $* = 0, 1, \ldots, n-3, * \neq k$. If we set positive numbers for the entries $a_{k,m,i,i}$, where k is mentioned in step II-III and $i = 1, 2, \ldots, n-1$, then the permanent of $\mathcal{A}$ remains 0. Hence, we put positive numbers (less than 1) in positions (m, k, i, i), where $i = 1, 2, \ldots, n-1$.

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step V-II-III: Now, we consider the block $(k, k, *_1, *_2)$, where $*_1, *_2 = 0, 1, \ldots, n-1$ and k is mentioned in step II-III. Since we want $\mathcal{A}$ to be plystochastic, the summation of the entries on each vertical and horizontal line of this block must be equal to 1. We know that the entries $a_{k,k,i,i}$ are positive numbers (less than 1) for $i = 1, 2, \ldots, n-1$ (by (5.1)). Thus, in order for $\mathcal{A}$ to remain polystochastic, we must put at least one positive number (less than 1) on each vertical line and each horizontal line of this block. For example, consider the vertical line $(k, k, *, 1)$, where $* = 0, 1, \ldots, n-1$. Since $0 < a_{k,k,1,1} < 1$ and $a_{k,k,l,1} = 0$, where $l \neq 0, 1$ (by step II), in order for $\mathcal{A}$ to remain polystochastic, we have to set a positive number (less than 1) for $a_{k,k,0,1}$. In what follows, consider the vertical lines $(k, k, *, z)$, where $z = 2, 3, \ldots, n-1$. Similarly, for these vertical lines, we set positive numbers (less than 1) for the entries $a_{k,k,0,z}$, where $z = 2, 3, \ldots, n-1$.

Now, consider the horizontal line $(k, k, 1, *)$, where $* = 0, 1, \ldots, n-1$. Since $0 < a_{k,k,1,1} < 1$ and $a_{k,k,1,l} = 0$, where $l \neq 0, 1$ (by step II), in order for $\mathcal{A}$ to remain polystochastic, we have to set a positive number (less than 1) for $a_{k,k,1,0}$. In what follows, consider the horizontal lines $(k, k, z, *)$, where

$z = 2, 3, \dots, n-1$ and $* = 0, 1, \dots, n-1$. Similarly, for these horizontal lines, we set positive numbers (less than 1) for the entries $a_{k,k,z,0}$, where $z = 2, 3, \dots, n-1$. Thus, the following entries are positive and less than 1.

$$a_{k,k,0,1}, a_{k,k,0,2}, \dots, a_{k,k,0,n-1}$$
$$a_{k,k,1,0}, a_{k,k,2,0}, \dots, a_{k,k,n-1,0}.$$

Consider the block $(m, m, *_1, *_2)$, where $*_1, *_2 = 0, 1, \dots, n-1$ and m is mentioned in step II-III, because we know that the positive entries on this block are less than 1. For the vertical and horizontal lines in this block, similar to what we did for the block $(k, k, *_1, *_2)$, we set positive numbers (less than 1) for the entries

$$a_{m,m,0,1}, a_{m,m,0,2}, \dots, a_{m,m,0,n-1}$$
$$a_{m,m,1,0}, a_{m,m,2,0}, \dots, a_{m,m,n-1,0}.$$

Now, we check the permanent of $\mathcal{A}$. If $per(\mathcal{A}) > 0$, then the proof of step II-III is complete. If $per(A) = 0$, we consider the block $(s_1, s_1, *_1, *_2)$, where $*_1, *_2 = 0, 1, \dots, n-1$ and s_1 is mentioned in step I-II-III, because we know that the positive entries on this block are less than 1. We set positive values (less than 1) for the following entries, similar to what we did for $a_{k,k,0,*}$ and $a_{k,k,*,0}$ with $* = 1, 2, \dots, n-1$.

$$a_{s_1,s_1,0,1}, a_{s_1,s_1,0,2}, \dots, a_{s_1,s_1,0,n-1}$$
$$a_{s_1,s_1,1,0}, a_{s_1,s_1,2,0}, \dots, a_{s_1,s_1,n-1,0}.$$

Now, we check the permanent of $\mathcal{A}$. If $per(\mathcal{A}) > 0$, then the proof of step II-III is complete. Otherwise, we consider the block $(s_2, s_2, *_1, *_2)$ and set positive numbers (less than 1) for the entries

$$a_{s_2,s_2,0,1}, a_{s_2,s_2,0,2}, \dots, a_{s_2,s_2,0,n-1}$$
$$a_{s_2,s_2,1,0}, a_{s_2,s_2,2,0}, \dots, a_{s_2,s_2,n-1,0}.$$

After considering a finite number of main diagonal blocks, we conclude that $per(\mathcal{A}) > 0$. Therefore, we could not construct a 4-dimensional nonnegative polystochastic tensor of order n with permanent equal to 0. We conclude that the permanent of a 4-dimensional nonnegative polystochastic tensor of order n is positive in this case.

Step III-III: In this case of step III, we put at least two positive numbers (less than 1) on each of the lines $(*, n-1, i, i)$ and $(*, n-2, i, i)$, for each $i = 1, 2, \dots, n-1$.

Step I-III-III: First, we consider the vertical lines $(*, n-1, i, i)$, where $i = 1, 2, \dots, n-1$ and $* = 0, 1, \dots, n-1$. We set positive numbers (less than 1) for the entries with indices

$$(n-2, n-1, i, i), \qquad i = 1, 2, \dots, n-1.$$

Notice that these selected entries lie on the same horizontal line as the entries with indices $(n-2, n-2, i, i)$, and we know that $a_{n-2,n-2,i,i} = 0$. (We consider $(n-2, n-1, i, i)$ because no horizontal line $(n-2, *, i, i)$, $* = 0, 1, \dots, n-1$, has any positive numbers.)

We check the permanent of $\mathcal{A}$, and observe that $per(\mathcal{A}) = 0$.

Now, for other selections of positive entries on the vertical lines $(*, n-1, i, i)$, choose an arbitrary block $(k, n-1, i, i)$, where $k \neq n-2, n-1$. We set positive numbers (less than 1) for the entries of $\mathcal{A}$ with indices $(k, n-1, i, i)$, where $i = 1, 2, \ldots, n-1$.

Notice that we check the permanent of $\mathcal{A}$, and observe that $per(\mathcal{A}) = 0$.

We set positive numbers (less than 1) for the entries of $\mathcal{A}$ with indices

$$(n-2, n-1, i, i), (k, n-1, i, i) \qquad i = 1, 2, \ldots, n-1, \tag{5.6}$$

where $k \in \{0, \ldots, n-3\}$. Thus vertical lines $(*, n-1, i, i)$, where $i = 1, 2, \ldots, n-1$ and $* = 0, 1, \ldots, n-1$, have at least two positive entries.

Step II-III-III: In what follows, we consider the vertical lines $(*, n-2, i, i)$, where $i = 1, \ldots, n-1$. We want to put at least two positive numbers on each vertical line. So, we put positive numbers (less than 1) in the positions

$$(k, n-2, i, i), (m, n-2, i, i) \quad i = 1, 2, \ldots, n-1, \quad m \in \{0, 1, \ldots, n-3\},$$

where k is the nonnegative number that was selected in (5.6) and $m \neq k$ is arbitrary. (Since our goal is to construct a nonnegative polystochastic tensor with permanent equal to 0, we try to do this with the least number of positive entries.)

At the end of this step, we check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Since our goal is to construct a tensor with permanent equal to 0, we set equal to 0 the entries of $\mathcal{A}$ in the $(n-1)$-row block that forms a positive diagonal for $\mathcal{A}$. Thus, we let

$$a_{n-1,n-2,i,i} = a_{n-1,k,i,i} = a_{n-1,m,i,i} = 0, \qquad i = 1, 2, \ldots, n-1.$$

Step III-III-III: In this step, we continue the process used in the proof of step III-III, similar to steps I-II-III, II-II-III, III-II-III, IV-II-III and V-II-III. We will see in this case that we cannot construct a 4-dimensional nonnegative polystochastic tensor of order n with permanent equal to 0.

Step IV-III: In this case of step III, if we want to put at least three positive entries (less than 1) on each of the vertical lines $(*, n-1, i, i)$ and $(*, n-2, i, i)$, where $i = 1, 2, \ldots, n-1$ and $* = 0, 1, \ldots, n-1$, the process of the proof is similar to that of step III-III. (Therefore, we cannot construct a tensor with the aforementioned properties.) Thus, we cannot construct a 4-dimensional nonnegative polystochastic tensor of order n with permanent equal to 0 in all cases. This allows us to conclude that the permanent of each 4-dimensional nonnegative polystochastic tensor of order n that is constructed using the special $n \times (n-1)$ row-Latin rectangle R, mentioned in Example 3.7, is positive. $\square$

Notice that in Step II, we can use an arbitrary row-Latin rectangle R with no transversals and not necessarily isotopic to R, and continue the algorithm by choosing a positive diagonal for the doubly stochastic submatrices of $\mathcal{A}$, and putting positive numbers on each line of $\mathcal{A}$ whose sum of the entries is less than 1, in such a way that the positive entries do not form a positive diagonal for $\mathcal{A}$. Eventually, after these selections, we may obtain a positive diagonal for $\mathcal{A}$.

The following example helps the reader to better understand Theorem 5.1.

Example 5.2. The permanent of a 4-dimensional nonnegative polystochastic tensor $\mathcal{A}$ of order 6 that is constructed using the special $n \times (n-1)$ row-Latin rectangle R, mentioned in Example 3.7, is positive.

Proof. We try to construct a 4-dimensional nonnegative polystochastic tensor of order 6 with permanent equal to 0. (In fact, we would like to show that such a construction is not possible.) We solve this example in three steps. Step III contains four steps, namely, I-III, II-III, III-III and IV-III. Step I-III itself contains two steps: I-I-III and II-I-III. Step II-III has five steps, which are I-II-III, II-II-III, III-II-III, IV-II-III and V-II-III. Step III-III has two steps: I-III-III and II-III-III. In each step, we check the permanent of the tensor under construction by using the algorithm that we presented in previous section.

Step I: According to step I of the proof of Theorem 5.1, we consider a doubly stochastic matrix $B = \mathcal{A}(*_1, *_2, 0, 0)$, where $*_1, *_2 = 0, 1, \ldots, 5$, and we set positive numbers for the following entries. Clearly, these entries form a positive diagonal for the matrix B.

$$a_{0,0,0,0}, a_{1,1,0,0}, \ldots, a_{5,5,0,0}.$$

We denote these positive entries by $+_1$ in Table 3 and Table 4 below.

Step II: Consider doubly stochastic matrices $B_i = \mathcal{A}(i, i, *_1, *_2)$, where $i = 0, 1, \ldots, 5$ and $*_1, *_2 = 0, 1, \ldots, 5$. According to row-Latin rectangle R mentioned in step II of the proof of Theorem 5.1, we set positive values for the following entries which form a positive diagonal for $B_0, B_1, \ldots, B_5$.

The diagonal of B_0 is of the form

$$a_{0,0,0,0}, a_{0,0,1,1}, a_{0,0,2,2}, \ldots, a_{0,0,5,5}.$$

The diagonal of B_1 is of the form

$$a_{1,1,0,0}, a_{1,1,1,1}, a_{1,1,2,2}, \ldots, a_{1,1,5,5}.$$

The diagonal of B_2 is of the form

$$a_{2,2,0,0}, a_{2,2,1,1}, a_{2,2,2,2}, \ldots, a_{2,2,5,5}.$$

The diagonal of B_3 is of the form

$$a_{3,3,0,0}, a_{3,3,1,1}, a_{3,3,2,2}, \ldots, a_{3,3,5,5}.$$

The diagonal of B_4 is of the form

$$a_{4,4,0,0}, a_{4,4,1,5}, a_{4,4,2,1}, a_{4,4,3,2}, \ldots, a_{4,4,5,4}.$$

The diagonal of B_5 is of the form

$$a_{5,5,0,0}, a_{5,5,1,5}, a_{5,5,2,1}, a_{5,5,3,2}, \dots, a_{5,5,5,4}.$$

Assume that for all other indices of the form

$$(i,i,\lambda,\mu), \qquad i=0,1,\dots,5, \quad \lambda,\mu=1,2,\dots,5,$$

the entries of $\mathcal{A}$ are equal to 0. We denote these positive entries by $+_2$, and the 0 entries by 0_2, in Table 3 and Table 4 below.

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A})=0$.

Step III: We know that $a_{5,5,i,i}=a_{4,4,i,i}=0$ for $i=1,2,\dots,5$. According to step III of the proof of Theorem 5.1, in order for $\mathcal{A}$ to remain polystochastic, we consider the vertical lines $(*,5,i,i)$ and $(*,4,i,i)$, where $i=1,2,\dots,5$ and $*=0,1,\dots,5$. Thus, in order for $\mathcal{A}$ to remain polystochastic, we should put a positive number 1 or at least two positive entries (less than 1) on each vertical line $(*,5,i,i)$ and each vertical line $(*,4,i,i)$, where $i=1,2,\dots,n-1$. To do so, we consider the following steps, according to the proof of Theorem 5.1.

Step I-III: We want to set equal to 1 one of the entries on each of the vertical lines $(*,4,i,i)$ and $(*,5,i,i)$, $i=1,2,\dots,5$.

Step I-I-III: We cannot set equal to 1 the entries with indices

$$(*,5,i,i), \quad *=0,\dots,3, \quad i=1,\dots,5,$$

because $a_{*,5,i,i}$, where $*=0,\dots,3$, lie on the same horizontal line as the positive entries $a_{*,*,i,i}$, where $*=0,\dots,3$ and $i=1,\dots,5$. If we set these entries equal to 1, then the summation of horizontal line that contains the entries $a_{*,5,i,i}$ and $0<a_{*,*,i,i}<1$ will be greater than 1, and this contradicts the assumption that $\mathcal{A}$ is polystochastic.

Similarly, we cannot set equal to 1 the entries with indices

$$(*,4,i,i), \quad *=0,\dots,3, \quad i=1,\dots,5,$$

because $a_{*,4,i,i}$, where $*=0,\dots,3$, lie on the same horizontal line as the positive entries $a_{*,*,i,i}$, where $*=0,\dots,3$ and $i=1,\dots,5$.

We denote these entries by $\neq 1$ in Table 3.

Step II-I-III: We want to set equal to 1 one of the entries on each of the vertical lines $(*,4,i,i)$ and $(*,5,i,i)$. We know that $a_{*,5,i,i}$ and $a_{*,4,i,i}$, where $*=0,1,2,3$, cannot be equal to 1 (by step I-I-III of this example), and also $a_{4,4,i,i}=a_{5,5,i,i}=0$, where $i=1,\dots,5$ (by step II of this example). Thus, we have to let $a_{5,4,i,i}=a_{4,5,i,i}=1$ for $i=1,2,\dots,5$.

We denote these entries by 1_{s2c1} in Table 3.

But, this gives a positive diagonal for the nonnegative polystochastic tensor $\mathcal{A}$ of the form

$$a_{0,0,0,0}, a_{1,1,1,1}, a_{2,2,2,2}, a_{3,3,3,3}, a_{4,5,4,4}, a_{5,4,5,5}.$$

Thus, we cannot construct a $6\times 6\times 6\times 6$ nonnegative polystochastic tensor with permanent equal to 0 in this case.

Step II-III: In this step, we set equal to 1 one of the entries on each vertical line $(*, 5, i, i)$, for $* = 0, 1, \ldots, 5$, and we set positive numbers (less than 1) for exactly two entries on each vertical line $(*, 4, i, i)$, where $i = 1, 2, \ldots, 5$.

According to step II-I-III in this example, we let $a_{4,5,i,i} = 1$ for $i = 1, 2, \ldots, 5$. Since the entries with indices $(*, 5, i, i)$ and $(4, 5, i, i)$ are on the same vertical line, $a_{*,5,i,i} = 0$ for $* = 0, \ldots, 3$. Also, since the entries with indices $(4, *, i, i)$ and $(4, 5, i, i)$ are on the same horizontal line, $a_{4,*,i,i} = 0$ for $* = 0, 1, 2, 3$ and $i = 1, 2, \ldots, 5$. We denote the entries equal to 1 by 1_{c2}, and denote the entries equal to 0 by 0_c in Table 4.

In what follows, associated to step II-III of the proof of Theorem 5.1, let $k = 2$ and $m = 3$. Also, set positive numbers (less than 1) for the entries

$$a_{2,4,i,i}, a_{3,4,i,i}, \qquad i = 1, 2, \ldots, 5.$$

We denote these positive entries by $+_{c2}$ in Table 4. Thus, the entries

$$a_{2,2,i,i}, a_{3,3,i,i}, \qquad i = 1, 2, \ldots, 5 \tag{5.7}$$

are positive numbers (less than 1), because the entries mentioned in (5.7) are on the same horizontal line as the entries $0 < a_{2,4,i,i} < 1$ and $0 < a_{3,4,i,i} < 1$, respectively, and the summation of the entries on each horizontal line must be equal to 1. We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Since our goal is to construct a tensor with permanent equal to 0, we set equal to 0 the entries of $\mathcal{A}$ in the last row block that forms a positive diagonal for $\mathcal{A}$. Thus, we set equal to 0 the entries

$$a_{5,2,i,i} = a_{5,3,i,i} = a_{5,4,i,i} = 0, \qquad i = 1, 2, \ldots, 5. \tag{5.8}$$

The reason is that, if we set positive numbers for the entries mentioned in (5.8), then the permanent of $\mathcal{A}$ will not remain equal to 0. We denote these entries by 0_{c2} in Table 4.

It is clear that if we set positive numbers for the entries $a_{5,0,i,i}$ and $a_{5,1,i,i}$, then the permanent of $\mathcal{A}$ remains equal to 0.

Step I-II-III: Now, we consider the horizontal lines

$$(5, *, i, i), \qquad * = 0, 1, \ldots, 5, \quad i = 1, 2, \ldots, 5.$$

We know that in this row block, the entries mentioned in (5.8) and the entries $a_{5,5,i,i}$, $i = 1, 2, \ldots, 5$, are equal to 0. We consider the entries

$$a_{5,0,i,i}, a_{5,1,i,i}, \qquad i = 1, 2, \ldots, 5. \tag{5.9}$$

In order for $\mathcal{A}$ to remain polystochastic on the horizontal lines $(5, *, i, i)$, where $i = 1, 2 \ldots, 5$ and $* = 0, 1, \ldots, 5$, we select two blocks s_1 and s_2 mentioned in (5.9), and set positive numbers (less than 1) for their entries. (By step I-II-III in Theorem 5.1, it is clear that we cannot select one block on the row block $(5, *, i, i)$ and set equal to 1 the entries on this block.) To do so, according to step I-II-III in Theorem 5.1, let $s_1 = 0$ and $s_2 = 1$. Thus, we set positive numbers (less than 1) for the entries with indices

$$(5, 0, i, i), (5, 1, i, i), \qquad i = 1, 2, \ldots, 5.$$

We denote these entries by $+_f$ in Table 4.

Therefore, the entries

$$a_{0,0,i,i}, a_{1,1,i,i}, \qquad i = 1, 2, \ldots, 5$$

of $\mathcal{A}$, which were selected in step II as positive numbers, are less than 1. Since $0 < a_{5,0,i,i} < 1$ and $a_{0,0,i,i}$ are on the same vertical line, and the same is also true for $0 < a_{5,1,i,i} < 1$ and $a_{1,1,i,i}$, the entries $a_{0,0,i,i}$ and $a_{1,1,i,i}$ cannot be equal to 1, and are positive numbers (less than 1). We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step II-II-III: Now, consider the horizontal lines $(0, *, i, i)$, and notice that $s_1 = 0$. We must have at least one more positive number (less than 1) on each horizontal line $(0, *, i, i)$, where $* = 0, 1, \ldots, 5$ and $i = 1, 2, \ldots, 5$. (Since $0 < a_{0,0,i,i} < 1$ and the summation on each horizontal line must be equal to 1.) If we set positive numbers for the entries $a_{0,2,i,i}, a_{0,3,i,i}, a_{0,4,i,i}$, where $i = 1, 2, \ldots, 5$, then $per(\mathcal{A}) > 0$. (We know that $a_{0,5,i,i} = 0$ by step II-III of this example.) Thus, we set equal to 0 the entries

$$a_{0,2,i,i}, a_{0,3,i,i}, a_{0,4,i,i}, \qquad i = 1, 2, \ldots, 5.$$

We denote these 0 entries by 0_a in Table 4.

It is clear that if we set positive numbers for $a_{0,1,i,i}$, where $i = 1, 2, \ldots, 5$, then the permanent of $\mathcal{A}$ also remains 0. (Notice that $s_2 = 1$.) Thus, we set positive numbers (less than 1) for the entries $a_{0,1,i,i}$, $i = 1, 2, \ldots, 5$. These entries are denoted by $+_a$ in Table 4. We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step III-II-III: Now, we consider the horizontal lines $(1, *, i, i)$. (Notice that $s_2 = 1$.) We must have at least one more positive number (less than 1) on each horizontal line $(1, *, i, i)$, where $* = 0, 1, \ldots, 5$ and $i = 1, 2, \ldots, 5$. (Since $0 < a_{1,1,i,i} < 1$, and the summation on each horizontal line must be equal to 1.)

If we set positive numbers for $a_{1,2,i,i}, a_{1,3,i,i}, a_{1,4,i,i}$, where $i = 1, \ldots, 5$, then $per(\mathcal{A}) > 0$. (We know that $a_{1,5,i,i} = 0$.) Since our goal is to construct a tensor with permanent equal to 0, we let

$$a_{1,2,i,i} = a_{1,3,i,i} = a_{1,4,i,i} = 0, \qquad i = 1, 2, \ldots, 5.$$

We denote these 0 entries by 0_e in Table 4.

It is clear that if we set positive numbers for the entries $a_{1,0,i,i}$, where $i = 1, 2, \ldots, 5$, then the permanent of $\mathcal{A}$ remains equal to 0. (Notice that $s_2 = 1$.) Thus, we set 1 for the entries

$$a_{1,0,1,1}, a_{1,0,2,2}, \ldots, a_{1,0,5,5}.$$

We denote these positive entries by 1_d in Table 4. We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step IV-II-III: We know that the entries $a_{2,2,i,i}$ (Remember that $k = 2$.) are positive numbers less than 1 (by step II-III of this example). Now, we consider the vertical lines $(*, 2, i, i)$, where $* = 0, 1, \ldots, 5$ and $i = 1, 2, \ldots, 5$. Hence, we must have at least one more positive number (less than 1) on each vertical line $(*, 2, i, i)$. (We know that $a_{0,2,i,i} = a_{1,2,i,i} = a_{4,2,i,i} = a_{5,2,i,i} = 0$ by step II-III and steps I-II-III, II-II-III and III-II-III.) It is clear that if we set positive numbers (less than 1) for $a_{3,2,i,i}$, then the permanent of $\mathcal{A}$ remains equal to 0. (Notice that $m = 3$.) Thus, in order for $\mathcal{A}$ to remain polystochastic on each vertical line, we set positive numbers (less than 1) for the entries $a_{3,2,i,i}$, $i = 1, 2, \ldots, 5$. We denote these entries by $+_h$ in Table 4. We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

We know that the entries $a_{3,3,i,i}$ (Notice that $m = 3$.) are positive numbers less than 1 (by step II-III of this example). In what follows, we consider the vertical lines $(*, 3, i, i)$, where $* = 0, 1, \ldots, 5$ and $i = 1, 2, \ldots, 5$. Similar to the previous discussion, we set positive numbers (less than 1) for $a_{2,3,i,i}$, $i = 1, 2, \ldots, 5$. (Remember that $k = 2$.) We denote these entries by $+_h$ in Table 4. We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step V-II-III: According to step V-II-III in Theorem 5.1, we know that $a_{2,2,i,i}$ and $a_{3,3,i,i}$ are positive numbers (by step II-III of this example), and that $a_{2,2,\lambda,\mu} = a_{3,3,\lambda,\mu} = 0$, where $\lambda, \mu = 1, 2, \ldots, 5$ and $\lambda \neq \mu$ (by step II of this example). In this step, we consider the vertical lines $(2, 2, i, j)$, where $i = 0, 1, \ldots, 5$ and $j = 1, 2, \ldots, 5$. Since the entries $a_{2,2,i,i}$, $i = 0, 1, \ldots, 5$, are positive numbers (less than 1), in order for $\mathcal{A}$ to remain polystochastic on each vertical line, we have to set positive numbers (less than 1) for the entries

$$a_{2,2,0,1}, a_{2,2,0,2}, \ldots, a_{2,2,0,5}.$$

Similarly, consider the horizontal lines $(2, 2, l, h)$, where $l = 1, 2, \ldots, 5$ and $h = 0, 1, \ldots, 5$. In order for $\mathcal{A}$ to remain polystochastic on each horizontal line, we have to set positive values (less than 1) for the entries

$$a_{2,2,1,0}, a_{2,2,2,0}, \ldots, a_{2,2,5,0}.$$

We denote these positive entries by $+_g$ in Table 4. We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Similar to the discussion above, consider the vertical lines with indices $(3, 3, i, j)$, where $i = 0, 1, \ldots, 5$ and $j = 1, 2, \ldots, 5$, and the horizontal lines with indices $(3, 3, l, h)$, where $l = 1, 2, \ldots, 5$ and $h = 0, 1, \ldots, 5$. In order for $\mathcal{A}$ to remain polystochastic on each of the vertical and horizontal lines, we have to set positive numbers (less than 1) for the entries

$$a_{3,3,0,1}, a_{3,3,0,2}, \ldots, a_{3,3,0,5}$$

and

$$a_{3,3,1,0}, a_{3,3,2,0}, \ldots, a_{3,3,5,0}.$$

We denote these positive entries by $+_g$ in Table 4. But, this gives a positive diagonal for the 4-dimensional nonnegative polystochastic tensor $\mathcal{A}$ of order 6. The conclusion is that we cannot construct a 4-dimensional nonnegative polystochastic tensor $\mathcal{A}$ of order 6 with permanent equal to 0 in this step.

Step III-III: In this case of step III, we put at least two positive entries (less than 1) on each of the vertical lines $(*, 5, i, i)$ and $(*, 4, i, i)$, for each $i = 1, 2, \ldots, 5$ and $* = 0, 1, \ldots, 5$.

Step I-III-III: We set positive numbers (less than 1) for the entries

$$a_{4,5,1,1}, a_{4,5,2,2}, \ldots, a_{4,5,5,5}.$$

(Notice that the entries with indices $(4, 5, i, i)$ and $(4, 4, i, i)$ are on the same horizontal line, and we know that $a_{4,4,i,i} = 0$ for $i = 1, 2, \ldots, 5$. Thus, with our selection, we put at least one positive number (less than 1) on each horizontal line $(4, *, i, i)$, $i = 1, 2, \ldots, 5$.) We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

For the other selection of this row block, we choose $k = 3$ and set positive numbers (less than 1) for the entries

$$a_{3,5,1,1}, a_{3,5,2,2}, \ldots, a_{3,5,5,5}.$$

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step II-III-III: In what follows, we consider the vertical lines $(*, 4, i, i)$, where $i = 1, 2, \ldots, 5$. We must put at least two positive numbers on each vertical line $(*, 4, i, i)$. We choose row blocks $m = 2$ and $k = 3$, where k is the same as what is mentioned in step I-III-III in this example. We put positive numbers (less than 1) in the positions

$$(3, 4, 1, 1), (3, 4, 2, 2), \ldots, (3, 4, 5, 5)$$

and

$$(2, 4, 1, 1), (2, 4, 2, 2), \ldots, (2, 4, 5, 5).$$

Now, we continue the algorithm proposed in the proof similar to steps I-II-III, II-II-III, III-II-III, IV-II-III, V-II-III of this example, and we show that we cannot construct a 4-dimensional nonnegative polystochastic tensor of order 6 with permanent equal to 0 in this case.

Step IV-III: If we want to put at least three positive numbers for the entries on each of the vertical lines $(*, 5, i, i)$ and $(*, 4, i, i)$, where $i = 1, 2, \ldots, 5$ and $* = 0, 1, \ldots, 5$, then the proof progresses similar to the previous cases, and we cannot construct a 4-dimensional nonnegative polystochastic tensor of order 6 with permanent equal to 0 in this step.

Thus, we cannot construct a 4-dimensional nonnegative polystochastic tensor of order 6 with permanent equal to 0 in all steps. This allows us to conclude that the permanent of a 4-dimensional nonnegative polystochastic tensor of order 6 that is constructed using the special $n \times (n - 1)$ row-Latin rectangle R, mentioned in Example 3.7, is positive. □

In what follows, we present two tables to help the readers better understand the algorithm used in Example 5.2. The processes used in the steps I-III and II-III of the proof are illustrated in Table 3 and Table 4, respectively. In the tables, "1" denotes an entry equal to 1, "0" is a 0 entry, and "+" is a positive entry. Dots are used to denote insignificant entries. Indices designate the steps in which the entries are considered.

TABLE 3. The table associated to Step I-III in Example 5.2

$+_1$	. . . . . .	. . . . . .	. . . . . .	$\neq 1$	$\neq 1$ . . . . .
. $+_2$ 0_2 0_2 0_2 0_2	. . . . . .	. . . . . .	. . . . . .	. $\neq 1$	. $\neq 1$ . . . .
. 0_2 $+_2$ 0_2 0_2 0_2	. . . . . .	. . . . . .	. . . . . .	. . $\neq 1$ . . .	. . $\neq 1$ . . .
. 0_2 0_2 $+_2$ 0_2 0_2	. . . . . .	. . . . . .	. . . . . .	. . . $\neq 1$ . .	. . . $\neq 1$ . .
. 0_2 0_2 0_2 $+_2$ 0_2	. . . . . .	. . . . . .	. . . . . .	. . . . $\neq 1$.	. . . . $\neq 1$.
. 0_2 0_2 0_2 0_2 $+_2$	. . . . . .	. . . . . .	. . . . . .	. . . . . $\neq 1$	. . . . . $\neq 1$
. . . . . .	$+_1$	. . . . . .	. . . . . .	$\neq 1$	$\neq 1$ . . . . .
. . . . . .	. $+_2$ 0_2 0_2 0_2 0_2	. . . . . .	. . . . . .	. $\neq 1$	. $\neq 1$ . . . .
. . . . . .	. 0_2 $+_2$ 0_2 0_2 0_2	. . . . . .	. . . . . .	. . $\neq 1$ . . .	. . $\neq 1$ . . .
. . . . . .	. 0_2 0_2 $+_2$ 0_2 0_2	. . . . . .	. . . . . .	. . . $\neq 1$ . .	. . . $\neq 1$ . .
. . . . . .	. 0_2 0_2 0_2 $+_2$ 0_2	. . . . . .	. . . . . .	. . . . $\neq 1$.	. . . . $\neq 1$.
. . . . . .	. 0_2 0_2 0_2 0_2 $+_2$	. . . . . .	. . . . . .	. . . . . $\neq 1$	. . . . . $\neq 1$
. . . . . .	. . . . . .	$+_1$	. . . . . .	$\neq 1$	$\neq 1$ . . . . .
. . . . . .	. . . . . .	. $+_2$ 0_2 0_2 0_2 0_2	. . . . . .	. $\neq 1$	. $\neq 1$ . . . .
. . . . . .	. . . . . .	. 0_2 $+_2$ 0_2 0_2 0_2	. . . . . .	. . $\neq 1$ . . .	. . $\neq 1$ . . .
. . . . . .	. . . . . .	. 0_2 0_2 $+_2$ 0_2 0_2	. . . . . .	. . . $\neq 1$ . .	. . . $\neq 1$ . .
. . . . . .	. . . . . .	. 0_2 0_2 0_2 $+_2$ 0_2	. . . . . .	. . . . $\neq 1$.	. . . . $\neq 1$.
. . . . . .	. . . . . .	. 0_2 0_2 0_2 0_2 $+_2$	. . . . . .	. . . . . $\neq 1$	. . . . . $\neq 1$
. . . . . .	. . . . . .	. . . . . .	$+_1$	$\neq 1$	$\neq 1$ . . . . .
. . . . . .	. . . . . .	. . . . . .	. $+_2$ 0_2 0_2 0_2 0_2	. $\neq 1$	. $\neq 1$ . . . .
. . . . . .	. . . . . .	. . . . . .	. 0_2 $+_2$ 0_2 0_2 0_2	. . $\neq 1$ . . .	. . $\neq 1$ . . .
. . . . . .	. . . . . .	. . . . . .	. 0_2 0_2 $+_2$ 0_2 0_2	. . . $\neq 1$ . .	. . . $\neq 1$ . .
. . . . . .	. . . . . .	. . . . . .	. 0_2 0_2 0_2 $+_2$ 0_2	. . . . $\neq 1$.	. . . . $\neq 1$.
. . . . . .	. . . . . .	. . . . . .	. 0_2 0_2 0_2 0_2 $+_2$	. . . . . $\neq 1$	. . . . . $\neq 1$
. . . . . .	. . . . . .	. . . . . .	. . . . . .	$+_1$	. . . . . .
. . . . . .	. . . . . .	. . . . . .	. . . . . .	. 0_2 0_2 0_2 0_2 $+_2$	. 1_{s2c1} . . . .
. . . . . .	. . . . . .	. . . . . .	. . . . . .	. $+_2$ 0_2 0_2 0_2 0_2	. . 1_{s2c1} . . .
. . . . . .	. . . . . .	. . . . . .	. . . . . .	. 0_2 $+_2$ 0_2 0_2 0_2	. . . 1_{s2c1} . .
. . . . . .	. . . . . .	. . . . . .	. . . . . .	. 0_2 0_2 $+_2$ 0_2 0_2	. . . . 1_{s2c1} .
. . . . . .	. . . . . .	. . . . . .	. . . . . .	. 0_2 0_2 0_2 $+_2$ 0_2	. . . . . 1_{s2c1}
. . . . . .	. . . . . .	. . . . . .	. . . . . .	. . . . . .	$+_1$ . . . . .
. . . . . .	. . . . . .	. . . . . .	. . . . . .	. 1_{s2c1}	. 0_{s2} 0_2 0_2 0_2 $+_2$
. . . . . .	. . . . . .	. . . . . .	. . . . . .	. . 1_{s2c1} . . .	. $+_2$ 0_2 0_2 0_2 0_2
. . . . . .	. . . . . .	. . . . . .	. . . . . .	. . . 1_{s2c1} . .	. 0_2 $+_2$ 0_2 0_2 0_2
. . . . . .	. . . . . .	. . . . . .	. . . . . .	. . . . 1_{s2c1} .	. 0_2 0_2 $+_2$ 0_2 0_2
. . . . . .	. . . . . .	. . . . . .	. . . . . .	. . . . . 1_{s2c1}	. 0_2 0_2 0_2 $+_2$ 0_2

Table 4. The table associated to Step II-III in Example 5.2

$+_1$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	$+_2$	0_2	0_2	0_2	0_2	.	$+_a$	.	.	.	.	.	0_a	.	.	.	.	.	0_a	.	.	.	.	.	0_a	.	.	.	.	.	0_c	.	.	.	.
.	0_2	$+_2$	0_2	0_2	0_2	.	.	$+_a$	.	.	.	.	.	0_a	.	.	.	.	.	0_a	.	.	.	.	.	0_a	.	.	.	.	.	0_c	.	.	.
.	0_2	0_2	$+_2$	0_2	0_2	.	.	.	$+_a$	.	.	.	.	.	0_a	.	.	.	.	.	0_a	.	.	.	.	.	0_a	.	.	.	.	.	0_c	.	.
.	0_2	0_2	0_2	$+_2$	0_2	.	.	.	.	$+_a$	.	.	.	.	.	0_a	.	.	.	.	.	0_a	.	.	.	.	.	0_a	.	.	.	.	.	0_c	.
.	0_2	0_2	0_2	0_2	$+_2$	.	.	.	.	.	$+_a$	.	.	.	.	.	0_a	.	.	.	.	.	0_a	.	.	.	.	.	0_a	.	.	.	.	.	0_c
.	.	.	.	.	.	$+_1$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	1_d	.	.	.	.	.	$+_2$	0_2	0_2	0_2	0_2	.	0_e	.	.	.	.	.	0_e	.	.	.	.	.	0_e	.	.	.	.	.	0_c	.	.	.	.
.	.	1_d	.	.	.	.	0_2	$+_2$	0_2	0_2	0_2	.	.	0_e	.	.	.	.	.	0_e	.	.	.	.	.	0_e	.	.	.	.	.	0_c	.	.	.
.	.	.	1_d	.	.	.	0_2	0_2	$+_2$	0_2	0_2	.	.	.	0_e	.	.	.	.	.	0_e	.	.	.	.	.	0_e	.	.	.	.	.	0_c	.	.
.	.	.	.	1_d	.	.	0_2	0_2	0_2	$+_2$	0_2	.	.	.	.	0_e	.	.	.	.	.	0_e	.	.	.	.	.	0_e	.	.	.	.	.	0_c	.
.	.	.	.	.	1_d	.	0_2	0_2	0_2	0_2	$+_2$	.	.	.	.	.	0_e	.	.	.	.	.	0_e	.	.	.	.	.	0_e	.	.	.	.	.	0_c
.	.	.	.	.	.	.	.	.	.	.	.	$+_1$	$+_g$	$+_g$	$+_g$	$+_g$	$+_g$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	$+_g$	$+_2$	0_2	0_2	0_2	0_2	.	$+_h$	.	.	.	.	.	$+_{c2}$	.	.	.	.	.	0_c	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	$+_g$	0_2	$+_2$	0_2	0_2	0_2	.	.	$+_h$	.	.	.	.	.	$+_{c2}$	.	.	.	.	.	0_c	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	$+_g$	0_2	0_2	$+_2$	0_2	0_2	.	.	.	$+_h$	.	.	.	.	.	$+_{c2}$	.	.	.	.	.	0_c	.	.
.	.	.	.	.	.	.	.	.	.	.	.	$+_g$	0_2	0_2	0_2	$+_2$	0_2	.	.	.	.	$+_h$	.	.	.	.	.	$+_{c2}$	.	.	.	.	.	0_c	.
.	.	.	.	.	.	.	.	.	.	.	.	$+_g$	0_2	0_2	0_2	0_2	$+_2$	.	.	.	.	.	$+_h$	.	.	.	.	.	$+_{c2}$	.	.	.	.	.	0_c
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$+_1$	$+_g$	$+_g$	$+_g$	$+_g$	$+_g$	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	$+_h$	.	.	.	.	$+_g$	$+_2$	0_2	0_2	0_2	0_2	.	$+_{c2}$	.	.	.	.	.	0_c	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	$+_h$	.	.	.	$+_g$	0_2	$+_2$	0_2	0_2	0_2	.	.	$+_{c2}$	.	.	.	.	.	0_c	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$+_h$	.	.	$+_g$	0_2	0_2	$+_2$	0_2	0_2	.	.	.	$+_{c2}$	.	.	.	.	.	0_c	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$+_h$	.	$+_g$	0_2	0_2	0_2	$+_2$	0_2	.	.	.	.	$+_{c2}$	.	.	.	.	.	0_c	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$+_h$	$+_g$	0_2	0_2	0_2	0_2	$+_2$	.	.	.	.	.	$+_{c2}$	.	.	.	.	.	0_c
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$+_1$	.	.	.	.	.	.	.	.	.	.	.
.	0_c	.	.	.	.	.	0_c	.	.	.	.	.	0_c	.	.	.	.	.	0_c	.	.	.	.	.	0_2	0_2	0_2	0_2	$+_2$	.	1_{c2}	.	.	.	.
.	.	0_c	.	.	.	.	.	0_c	.	.	.	.	.	0_c	.	.	.	.	.	0_c	.	.	.	.	$+_2$	0_2	0_2	0_2	0_2	.	.	1_{c2}	.	.	.
.	.	.	0_c	.	.	.	.	.	0_c	.	.	.	.	.	0_c	.	.	.	.	.	0_c	.	.	.	0_2	$+_2$	0_2	0_2	0_2	.	.	.	1_{c2}	.	.
.	.	.	.	0_c	.	.	.	.	.	0_c	.	.	.	.	.	0_c	.	.	.	.	.	0_c	.	.	0_2	0_2	$+_2$	0_2	0_2	.	.	.	.	1_{c2}	.
.	.	.	.	.	0_c	.	.	.	.	.	0_c	.	.	.	.	.	0_c	.	.	.	.	.	0_c	.	0_2	0_2	0_2	$+_2$	0_2	.	.	.	.	.	1_{c2}
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$+_1$	.	.	.	.	.
.	$+_f$	.	.	.	.	.	$+_f$	.	.	.	.	.	0_{c2}	.	.	.	.	.	0_{c2}	.	.	.	.	.	0_{c2}	.	.	.	.	.	0_2	0_2	0_2	0_2	$+_2$
.	.	$+_f$	.	.	.	.	.	$+_f$	.	.	.	.	.	0_{c2}	.	.	.	.	.	0_{c2}	.	.	.	.	.	0_{c2}	.	.	.	.	$+_2$	0_2	0_2	0_2	0_2
.	.	.	$+_f$	.	.	.	.	.	$+_f$	.	.	.	.	.	0_{c2}	.	.	.	.	.	0_{c2}	.	.	.	.	.	0_{c2}	.	.	.	0_2	$+_2$	0_2	0_2	0_2
.	.	.	.	$+_f$	.	.	.	.	.	$+_f$	.	.	.	.	.	0_{c2}	.	.	.	.	.	0_{c2}	.	.	.	.	.	0_{c2}	.	.	0_2	0_2	$+_2$	0_2	0_2
.	.	.	.	.	$+_f$	.	.	.	.	.	$+_f$	.	.	.	.	.	0_{c2}	.	.	.	.	.	0_{c2}	.	.	.	.	.	0_{c2}	.	0_2	0_2	0_2	$+_2$	0_2

In what follows, we use the special row-Latin rectangle R to show that the permanent of an even-dimensional nonnegative polystochastic tensor of order n is positive.

Theorem 5.3. The permanent of an even-dimensional nonnegative polystoc -hastic tensor $\mathcal{A}$ of order n that is constructed using the special $n \times (n-1)$ row-Latin rectangle R, mentioned in Example 3.7, is positive.

Proof. We show that the permanent of each $2s$-dimensional nonnegative polystochastic tensor of order n is positive. We prove this theorem by induction.

Since each 2-dimensional nonnegative polystochastic tensor of order n is a doubly stochastic matrix, the statement of the theorem is true for $d = 2$ (that is, $L = 1$) by [9].

The induction hypothesis: We assume that the statement of the theorem is true for $d = 2s - 2$ (that is, $L = s - 1$): The permanent of each $2s - 2$-dimensional nonnegative polystochastic tensor of order n, where $s \in \mathbb{N}$, is greater than 0.

We must prove that the statement of the theorem is also true for $d = 2s$ (that is, $L = s$).

To see this, we try to construct a $2s$-dimensional nonnegative polystochastic tensor of order n with permanent equal to 0. We will see, in fact, that such a construction is not possible. To construct a tensor with permanent equal to 0, we exhibit the algorithm below step by step. We prove this theorem in three steps, in which step III contains four steps: I-III, II-III, III-III and IV-III. Step I-III itself contains two steps: I-I-III and II-I-III. Step II-III has five steps, which are I-II-III, II-II-III, III-II-III, IV-II-III and V-II-III. Step III-III has three steps, namely, I-III-III, II-III-III and III-III-III. In each step, we check the permanent of the tensor which is under construction.

Step I: Consider the matrix B of the form

$$\mathcal{A}(*_1, *_2, \underbrace{0, 0, \ldots, 0}_{2s-2}), \qquad *_1, *_2 = 0, 1, \ldots, n-1.$$

Since $\mathcal{A}$ is a nonnegative polystochastic tensor, B is a nonnegative doubly stochastic matrix. Thus, B has a positive diagonal by [9]. Without loss of generality, we let $a_{i,i,0,0\ldots,0} > 0$, where $i = 0, 1, \ldots, n-1$. It is clear that these positive entries form a positive diagonal for B.

At the end of this step, we check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step II: Let B_i be composed of the entries of $\mathcal{A}$ which are of the form

$$\mathcal{A}(i, i, *_1, *_2, \ldots, *_{2s-2}), \quad i = 0, 1, \ldots, n-1, \quad *_1, \ldots, *_{2s-2} = 0, 1, \ldots, n-1.$$

Each B_i is a $2s - 2$-dimensional tensor of order n. Since $\mathcal{A}$ is a nonnegative polystochastic tensor, B_i is also $2s-2$-dimensional nonnegative polystochastic tensor of order n. The induction hypothesis allows us to conclude that the permanent of B_i, $i = 0, \ldots, n-1$, is positive. Thus, it has a positive diagonal. To select a positive diagonal for B_i, we consider an $n \times (n-1)$ row-Latin rectangle R such that the number of transversals in R is equal to 0. (Since our goal is to construct a nonnegative polystochastic tensor with permanent equal to 0, and it is clear that a transversal in R gives a positive diagonal for $\mathcal{A}$, we consider a row-Latin rectangle R without any transversals.) We assume that the entries of $\mathcal{A}$ with indices $\{(i, i, \beta_i^j, \gamma_i^j, \beta_i^j, \gamma_i^j, \ldots, \beta_i^j, \gamma_i^j)\}_{j=1}^n$ are positive numbers, and that they form a positive diagonal for B_i containing $a_{i,i,0,0\ldots,0}$, where β_i^j, γ_i^j are determined according to the rectangle R as follows.

> The entry in the $(i+1)$th row and the β_i^jth column of R is γ_i^j, where $i = 0, \ldots, n-1$.

Now, we choose the row-Latin rectangle R of the following form. It is clear that the number of transversals in R is equal to 0 [3].

$$\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & \ldots & \ldots & n-3 & n-2 & n-1 \\
1 & 2 & 3 & 4 & \ldots & \ldots & n-3 & n-2 & n-1 \\
\vdots & \vdots & \vdots & \vdots & \ldots & \ldots & \vdots & \vdots & \vdots \\
1 & 2 & 3 & 4 & \ldots & \ldots & n-3 & n-2 & n-1 \\
n-1 & 1 & 2 & 3 & \ldots & \ldots & n-4 & n-3 & n-2 \\
n-1 & 1 & 2 & 3 & \ldots & \ldots & n-4 & n-3 & n-2.
\end{array}$$

For example, to select a positive diagonal for B_0, we look at the first row of R. In this row, we observe that the entry located in the first column is equal to 1 (that is, $R_{11} = 1$). Thus, we let $a_{0011\ldots1} > 0$. Since $R_{1,n-1} = n-1$, we let $a_{0,0,n-1,n-1\ldots,n-1} > 0$. Also, to select a positive diagonal for B_{n-1}, we look at the last row of R (that is, the nth row of R). Since $R_{n1} = n-1$, we let $a_{n-1,n-1,1,n-1,1,n-1\ldots,1,n-1} > 0$. Also, in this row of R, we observe that $R_{n,n-1} = n-2$. Hence, we let

$$a_{n-1,n-1,n-1,n-2,n-1,n-2\ldots,n-1,n-2} > 0.$$

Therefore, a positive diagonal for B_i, where $i = 0, 1, \ldots, n-1$, is formed by the entries of the tensor $\mathcal{A}$ with indices

$$(i, i, 0, 0\ldots, 0), (i, i, 1, 1\ldots, 1), (i, i, 2, 2, \ldots, 2), \ldots, (i, i, n-1, n-1, \ldots, n-1),$$

for $i = 0, \ldots, n-3$, and

$$(i, i, 0, 0, \ldots, 0), (i, i, 1, n-1, 1, n-1, \ldots, 1, n-1), (i, i, 2, 1, 2, 1\ldots, 2, 1), (i, i, 3, 2, 3, 2, \ldots, 3, 2), \ldots,$$
$$(i, i, n-1, n-2, n-1, n-2, \ldots, n-1, n-2),$$

where $i = n-2, n-1$. So far, the entries of the tensor $\mathcal{A}$ with the following indices are positive numbers. (These entries are considered as positive numbers in steps I and II.)

$$\begin{array}{cccc}
(0,0,\ldots,0) & (0,0,1,\ldots,1) & (0,0,2,\ldots,2) & \ldots \\
(1,1,0,\ldots,0) & (1,1,\ldots,1) & (1,1,2,\ldots,2) & \ldots \\
\vdots & \vdots & \vdots & \ldots \\
(n-3,n-3,0,\ldots,0) & (n-3,n-3,1,\ldots,1) & (n-3,n-3,2,\ldots,2) & \ldots \\
(n-2,n-2,0,\ldots,0) & (n-2,n-2,1,n-1,1,n-1,\ldots,1,n-1) & (n-2,n-2,2,1,2,1,\ldots,2,1) & \ldots \\
(n-1,n-1,0,\ldots,0) & (n-1,n-1,1,n-1,1,n-1,\ldots,1,n-1) & (n-1,n-1,2,1,2,1,\ldots,2,1) & \ldots
\end{array}$$

$$
\begin{array}{ll}
\dots & (0,0,n-1,\dots,n-1),\\
\dots & (1,1,n-1,\dots,n-1),\\
\dots & \vdots\\
\dots & (n-3,n-3,n-1,\dots,n-1),\\
\dots & (n-2,n-2,n-1,n-2,n-1,n-2,\dots,n-1,n-2),\\
\dots & (n-1,n-1,n-1,n-2,n-1,n-2,\dots,n-1,n-2).
\end{array}
$$

Assume that for all other indices of the form

$$(i,i,\lambda_1,\dots,\lambda_{2s-2}),\qquad i=0,1,\dots,n-1,\quad \lambda_1,\dots,\lambda_{2s-2}=1,\dots,n-1,$$

the entries of $\mathcal{A}$ are equal to 0.

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A})=0$.

Step III: We know that $a_{n-1,n-1,i,i\dots,i} = a_{n-2,n-2,i,i\dots,i} = 0$ for $i = 1,2,\dots,n-1$ (by step II). In order for $\mathcal{A}$ to remain polystochastic, we consider the vertical lines $(*,n-1,i,i,\dots,i)$ and $(*,n-2,i,i,\dots,i)$, where $i=1,2,\dots,n-1$. (For each i, put $*=0,1,\dots,n-1$.) Since we want $\mathcal{A}$ to be polystochastic, we should put a positive number 1 or at least two positive entries (less than 1) on each vertical line $(*,n-1,i,i,\dots,i)$ and each vertical line $(*,n-2,i,i,\dots,i)$, where $i=1,2,\dots,n-1$. In what follows, we consider four cases.

Step I-III: We want to set equal to 1 one of the entries on each vertical line $(*,n-1,i,i,\dots,i)$ and each vertical line $(*,n-2,i,i,\dots,i)$, where $i=1,2,\dots,n-1$.

Step I-I-III: We cannot set equal to 1 the entries with indices

$$(*,n-1,i,i,\dots,i),\quad *=0,1,\dots,n-3,\quad i=1,2,\dots,n-1,$$

because $a_{*,n-1,i,i\dots,i}$ lie on the same horizontal line as the positive entries $a_{*,*,i,i\dots,i}$, where $*=0,1,\dots,n-3$ and $i=1,2,\dots,n-1$. For example, if we let $a_{0,n-1,1,\dots,1}=1$, then the summation of the entries in horizontal line that contains the entries $a_{0,n-1,1,1\dots,1}$ and $0<a_{0,0,1,1,\dots,1}<1$ will be greater than 1. But, this contradicts the assumption that $\mathcal{A}$ is polystochastic.

Similarly, we cannot set equal to 1 the entries with indices

$$(*,n-2,i,i,\dots,i),\quad *=0,1,\dots,n-3,\quad i=1,2,\dots,n-1,$$

because $a_{*,n-2,i,\dots,i}$ lie on the same horizontal line as the positive entries $a_{*,*,i,i,\dots,i}$, where $*=0,1,\dots,n-3$ and $i=1,2,\dots,n-1$.

Step II-I-III: According to step I-III, step I-I-III, and the fact that $a_{n-1,n-1,i,i\dots,i}=0$ for $i=1,2,\dots,n-1$, in order for $\mathcal{A}$ to remain polystochastic on the vertical lines $(*,n-1,i,i,\dots,i)$, $i=1,2,\dots,n-1$, we have to let $a_{n-2,n-1,i,i,\dots,i}=1$, where $i=1,2,\dots,n-1$.

According to step I-III, step I-I-III, and the fact that $a_{n-2,n-2,i,i,\dots,i}=0$ for $i=1,2,\dots,n-1$, in order for $\mathcal{A}$ to remain polystochastic on the vertical lines $(*,n-2,i,i,\dots,i)$, where $i=1,2,\dots,n-1$, we have to let $a_{n-1,n-2,i,i,\dots,i}=1$ for $i=1,2,\dots,n-1$.

But, this gives a positive diagonal for the $2s$-dimensional nonnegative polystochastic tensor $\mathcal{A}$ of order n of the form

$$(0,0,\dots,0),(1,1,\dots,1),\dots,(n-3,n-3\dots,n-3),$$
$$(n-2,n-1,n-2,n-2,\dots,n-2),(n-1,n-2,n-1,n-1\dots,n-1).$$

Thus, we cannot construct a $2s$-dimensional nonnegative polystochastic tensor of order n with permanent equal to 0 in this step.

Step II-III: In this case of step III, we set equal to 1 one of the entries on each vertical line $(*,n-1,i,i,\dots,i)$, $i=1,2,\dots,n-1$. (For each i, let $*=0,1,\dots,n-1$.) Also, we put exactly two positive entries (less than 1) on each vertical line $(*,n-2,i,i\dots,i)$, where $i=1,2,\dots,n-1$.

According to step II-I-III, we let $a_{n-2,n-1,i,i,\dots,i}=1$, where $i=1,2,\dots,n-1$. Since the entries with indices $(*,n-1,i,i,\dots,i)$ and $(n-2,n-1,i,i\dots,i)$ are on the same vertical line, $a_{*,n-1,i,i,\dots,i}=0$, where $*=0,1,\dots,n-3$ and $i=1,2,\dots,n-1$. Also, since the entries with indices $(n-2,*,i,i,\dots,i)$ and $(n-2,n-1,i,i,\dots,i)$ are on the same horizontal line, $a_{n-2,*,i,i,\dots,i}=0$, where $*=0,1,\dots,n-3$ and $i=1,2,\dots,n-1$.

Now, we consider the vertical lines $(*,n-2,i,i\dots,i)$, where $*=0,1,\dots,n-1$ and $i=1,\dots,n-1$. We choose arbitrary row blocks k and m, and set positive numbers (less than 1) for the entries of $\mathcal{A}$ with indices

$$(k,n-2,i,i,\dots,i),\quad (m,n-2,i,i,\dots,i),\quad k,m\in\{0,1,\dots,n-3\},\quad k\neq m.$$

Thus, the entries

$$a_{k,k,i,i,\dots,i},a_{m,m,i,i,\dots,i},\qquad i=1,2,\dots,n-1 \tag{5.10}$$

are positive numbers (less than 1), because the entries mentioned in (5.10) are on the same horizontal line as the entries $0<a_{k,n-2,i,i,\dots,i}<1$ and $0<a_{m,n-2,i,i,\dots,i}<1$.

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A})=0$.

Since our goal is to construct a tensor with permanent equal to 0, we set equal to 0 the entries of $\mathcal{A}$ in the last row block that forms a positive diagonal for $\mathcal{A}$. Thus, we let

$$a_{n-1,n-2,i,i\dots,i}=a_{n-1,k,i,i,\dots,i}=a_{n-1,m,i,i,\dots,i}=0,\quad i=1,2,\dots,n-1, \tag{5.11}$$

because, if we set positive numbers for the entries mentioned in (5.11), then $per(\mathcal{A})>0$.

Clearly, if we set positive numbers for the entries with indices

$$(n-1,l,i,i,\dots,i),\qquad l=0,1,\dots,n-3,\quad l\neq k,m,\quad i=1,2,\dots,n-1, \tag{5.12}$$

then the permanent of $\mathcal{A}$ also remains 0.

Step I-II-III: We consider the horizontal lines

$$(n-1,*,i,i,\dots,i),\qquad *=0,1,\dots,n-1,\quad i=1,2,\dots,n-1.$$

Since we want $\mathcal{A}$ to be polystochastic, each horizontal line $(n-1,*,i,i,\dots,i)$, where $*=0,1,\dots,n-1$, must have exactly one entry equal to 1 or at least

two positive entries (less than 1). We choose these positions to put the positive entries among the positions mentioned in (5.12), because we want the permanent of $\mathcal{A}$ to remain equal to 0.
If we set equal to 1 one of the entries on each horizontal line

$$(n-1,*,i,i,\ldots,i),$$

where $*=0,1,\ldots,n-1$, for example, if we let

$$a_{n-1,s,i,i,\ldots,i}=1, \qquad s\in\{0,1,\ldots,n-3\}, \qquad s\neq k,m,$$

then this allows us to conclude that $a_{s,s,i,i,\ldots,i}=0$. (Since $a_{s,s,i,i,\ldots,i}$ and $a_{n-1,s,i,i,\ldots,i}$ are on the same vertical line, and the summation of the entries on the vertical lines containing $a_{s,s,i,i,\ldots,i}$ and $a_{n-1,s,i,i,\ldots,i}$ will be greater than 1.) But, this contradicts $a_{s,s,i,i,\ldots,i}>0$, which is true by step II.

Therefore, we have to put at least two positive entries on each horizontal line $(n-1,*,i,i,\ldots,i)$, in the positions mentioned in (5.12). Without loss of generality, we choose column blocks s_1 and s_2 of l, and set positive numbers (less than 1) for the entries of $\mathcal{A}$ with indices

$$(n-1,s_1,i,i,\ldots,i),(n-1,s_2,i,i,\ldots,i), \qquad s_1,s_2\in\{0,1,\ldots,n-3\},\quad s_1,s_2\neq k,m,$$

where $i=1,2,\ldots,n-1$. In what follows, in relation (5.12), the column blocks different from $s_1,s_2,k,m,n-2,n-1$ in the last row block are indexed by $s_3,s_4,\ldots,s_{\max v}$ in the form

$$\mathcal{A}(n-1,s_v,i,\ldots,i), \qquad s_v\neq s_1,s_2,k,m,n-2,n-1,$$

where $s_3,s_4,\ldots,s_{\max v}$ are shown by s_v. We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A})=0$.

Since $0<a_{n-1,s_1,i,i,\ldots,i}<1$ and $a_{s_1,s_1,i,i,\ldots,i}$ are on the same vertical line, and the same is also true for $0<a_{n-1,s_2,i,i,\ldots,i}<1$ and $a_{s_2,s_2,i,i,\ldots,i}$, the entries $a_{s_1,s_1,i,i,\ldots,i}$ and $a_{s_2,s_2,i,i,\ldots,i}$ cannot be equal to 1 and are positive numbers (less than 1). Therefore, each of the horizontal lines $(s_1,*,i,i,\ldots,i)$ and $(s_2,*,i,i,\ldots,i)$, where $*=0,1,\ldots,n-1$ and $i=1,2,\ldots,n-1$, must have at least one more positive entry (less than 1).

Step II-II-III: We consider the horizontal lines $(s_1,*,i,i,\ldots,i)$, where $*=0,1,\ldots,n-1$ and $i=1,2,\ldots,n-1$. It is clear that the permanent of $\mathcal{A}$ remains 0 if we set positive numbers (less than 1) for the entries with indices
(5.13)

$$(s_1,s_v,i,i,\ldots,i), \quad s_v=s_2,s_3,s_4,\ldots,s_{\max v}, \quad s_v\neq s_1,k,m,n-2,n-1.$$

Next, we put exactly one positive number (less than 1) on each horizontal line $(s_1,*,i,i,\ldots,i)$. (This position is selected from the indices mentioned in (5.13).) For example, we put a positive number (less than 1) in the position $(s_1,j_1,i,i,\ldots,i)$, where $j_1\in\{s_2,s_3,\ldots,s_{\max v}\}$. This implies that the entries $a_{j_1,j_1,i,i,\ldots,i}$, for $i=1,2,\ldots,n-1$, are positive numbers (less than 1). Now, we set equal to 0 the entries with indices $(s_1,*,i,i,\ldots,i)$,

where $* \neq s_1, s_2, \ldots, s_{\max v}$, because, if we consider positive numbers for these entries, then the permanent of $\mathcal{A}$ does not remain equal to 0.
At the end of this step, we check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step III-II-III: Now, we consider the horizontal lines $(s_2, *, i, i, \ldots, i)$, where $* = 0, 1, \ldots, n-1$ and $i = 1, 2, \ldots, n-1$. It is clear that the permanent of $\mathcal{A}$ remains 0 if we set positive numbers (less than 1) for the entries of $\mathcal{A}$ with indices

(5.14)
$$(s_2, s_v, i, i, \ldots, i), \quad s_v = s_1, s_3, s_4, \ldots, s_{\max v}, \quad s_v \neq s_2, k, m, n-2, n-1.$$

Next, we put exactly one positive number (less than 1) on each horizontal line $(s_2, *, i, i, \ldots, i)$. (This position is selected from the indices mentioned in (5.14).) For example, we put a positive number (less than 1) in the position $(s_2, j_2, i, i, \ldots, i)$. This implies that the entries $a_{j_2, j_2, i, i, \ldots, i}$, for $i = 1, 2, \ldots, n-1$, are positive numbers (less than 1). Now, we set equal to 0 the entries with indices $(s_2, *, i, i, \ldots, i)$, where $* \neq s_1, s_2, \ldots, s_{\max v}$, because, if we set positive numbers for these entries, then the permanent of $\mathcal{A}$ does not remain equal to 0.

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step IV-II-III: By step II-III, we know that $0 < a_{k,k,i,i,\ldots,i} < 1$, where k is mentioned in (5.10). Since we want $\mathcal{A}$ to be a polystochastic tensor, we must put at least one more positive entry on each vertical line $(*, k, i, i, \ldots, i)$, where $* = 0, 1, \ldots, n-1$ and $i = 1, 2, \ldots, n-1$. It is clear that if we set positive numbers for the entries with indices $(*, k, i, i, \ldots, i)$, where $* = 0, 1, \ldots, n-3, * \neq m$, then the permanent of $\mathcal{A}$ is not 0. (We remember that $a_{n-1,k,i,i,\ldots,i} = a_{n-2,k,i,i,\ldots,i} = 0$, by step II-III.) Thus, we let $a_{*,k,i,i,\ldots,i} = 0$, where $* = 0, 1, \ldots, n-3, * \neq m$. Also, if we set positive numbers for $a_{m,k,i,i,\ldots,i}$, where k is mentioned in step II-III and $i = 1, 2, \ldots, n-1$, then the permanent of $\mathcal{A}$ remains 0. Therefore, we put positive numbers (less than 1) in positions $(m, k, i, i, \ldots, i)$, where $i = 1, 2, \ldots, n-1$.
We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Now, we know that $0 < a_{m,m,i,i,\ldots,i} < 1$ (by step II-III), where m is mentioned in (5.10). Since we want $\mathcal{A}$ to be a polystochastic tensor, we must put at least one more positive entry on each vertical line $(*, m, i, i, \ldots, i)$, where $i = 1, 2, \ldots, n-1$ and $* = 0, 1, \ldots, n-1$. It is clear that if we set positive numbers for the entries with indices $(*, m, i, i, \ldots, i)$, where $* = 0, 1, \ldots, n-3, * \neq k$, then the permanent of $\mathcal{A}$ is not 0. (We remember that $a_{n-1,m,i,i,\ldots,i} = a_{n-2,m,i,i,\ldots,i} = 0$, by step II-III.) Thus, we let $a_{*,m,i,i,\ldots,i} = 0$, where $* = 0, 1, \ldots, n-3, * \neq k$. If we set positive numbers for the entries $a_{k,m,i,i,\ldots,i}$, where k is mentioned in step II-III and $i = 1, 2, \ldots, n-1$, then the permanent of $\mathcal{A}$ remains equal to 0. Hence, we put positive numbers (less than 1) in positions $(m, k, i, \ldots, i)$, where $i = 1, 2, \ldots, n-1$.
We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step V-II-III: Now, we consider the block $(k, k, *_1, *_2, \dots, *_{2s-2})$, where $*_1, \dots, *_{2s-2} = 0, 1, \dots, n-1$ and k is mentioned in step II-III. Since we want $\mathcal{A}$ to be polystochastic, the summation of the entries on each of the vertical and horizontal lines of this block must be equal to 1. We know that the entries $a_{k,k,i,i,\dots,i}$, where $i = 1, 2, \dots, n-1$, are positive numbers (less than 1) (by (5.10)). Hence, in order for $\mathcal{A}$ to remain polystochastic, we must put at least one more positive number (less than 1) on each vertical line and each horizontal line of this block. For example, consider the vertical line $(k, k, *, 1, 1, \dots, 1)$, where $* = 0, 1, \dots, n-1$. Since $0 < a_{k,k,i,i,\dots,i} < 1$ and $a_{k,k,l,1,1,\dots,1} = 0$, where $l \neq 0, 1$ (by step II), in order for $\mathcal{A}$ to remain polystochastic, we have to set a positive number (less than 1) for $a_{k,k,0,1,1,\dots,1}$. In what follows, consider the vertical lines with indices $(k, k, *, z, z, \dots, z)$ for $z = 2, 3, \dots, n-1$. Similarly, for these vertical lines, we set positive numbers (less than 1) for the entries $a_{k,k,0,z,z,\dots,z}$, where $z = 2, 3, \dots, n-1$.

Similarly, consider the horizontal line $(k, k, 1, 1, \dots, 1, *)$, where $* = 0, 1, \dots, n-1$. Since $0 < a_{k,k,1,1,\dots,1} < 1$ and $a_{k,k,1,1,\dots,1,l} = 0$, where $l \neq 0, 1$ (by step II), in order for $\mathcal{A}$ to remain polystochastic, we have to set a positive number (less than 1) for $a_{k,k,1,1,\dots,1,0}$. In what follows, consider the horizontal lines $(k, k, z, z, \dots, z, *)$ for $z = 2, 3, \dots, n-1$ and $* = 0, 1, \dots, n-1$. Similarly, for these vertical lines, we set positive numbers (less than 1) for the entries $a_{k,k,z,z,\dots,z,0}$, where $z = 2, 3, \dots, n-1$. Therefore, the following entries are positive and less than 1.

$$a_{k,k,0,1,1,\dots,1}, a_{k,k,0,2,2,\dots,2}, \cdots, a_{k,k,0,n-1,n-1\dots,n-1}$$
$$a_{k,k,1,1,\dots,1,0}, a_{k,k,2,2,\dots,2,0}, \cdots, a_{k,k,n-1,n-1,\dots,n-1,0}.$$

In what follows, we consider the block with index $(m, m, *_1, \dots, *_{2s-2})$, where $*_1, \dots, *_{2s-2} = 0, 1, \dots, n-1$ and m is mentioned in step II-III, because we know that the positive entries on this block are less than 1. For the vertical and horizontal lines in the block $(m, m, *_1, *_2 \dots, *_{2s-2})$, similar to what we did for the block $(k, k, *_1, *_2 \dots, *_{2s-2})$, we set positive numbers (less than 1) for the entries

$$a_{m,m,0,1,1,\dots,1}, a_{m,m,0,2,2,\dots,2}, \cdots, a_{m,m,0,n-1,n-1,\dots,n-1}$$
$$a_{m,m,1,1,\dots,1,0}, a_{m,m,2,2,\dots,2,0}, \cdots, a_{m,m,n-1,n-1,\dots,n-1,0}.$$

Now, we check the permanent of $\mathcal{A}$. If $per(\mathcal{A}) > 0$, then the proof of step II-III is complete. If $per(A) = 0$, we consider the block $(s_1, s_1, *_1, \dots, *_{2s-2})$, where $*_1, \dots, *_{2s-2} = 0, 1, \dots, n-1$ and s_1 is mentioned in step I-II-III, because we know that the positive entries on this block are less than 1. We set positive values less than 1 for the following entries, similar to $a_{k,k,0,*,*\dots,*}$ and $a_{k,k,*,*,\dots,*,0}$, where $* = 1, 2, \dots, n-1$.

$$a_{s_1,s_1,0,1,1,\dots,1}, a_{s_1,s_1,0,2,2,\dots,2}, \cdots, a_{s_1,s_1,0,n-1,n-1,\dots,n-1}$$
$$a_{s_1,s_1,1,1,\dots,1,0}, a_{s_1,s_1,2,2,\dots,2,0}, \cdots, a_{s_1,s_1,n-1,n-1,\dots,n-1,0}.$$

Now, we check the permanent of $\mathcal{A}$. If $per(\mathcal{A}) > 0$, then the proof of step II-III is complete. Otherwise, we consider the block $(s_2, s_2, *_1, \dots, *_{2s-2})$, and set positive numbers (less than 1) for the entries

$$a_{s_2,s_2,0,1,1,\dots,1},\, a_{s_2,s_2,0,2,2,\dots,2},\, \cdots,\, a_{s_2,s_2,0,n-1,n-1,\dots,n-1}$$
$$a_{s_2,s_2,1,1,\dots,1,0},\, a_{s_2,s_2,2,2,\dots,2,0},\, \cdots,\, a_{s_2,s_2,n-1,n-1,\dots,n-1,0}.$$

After considering a finite number of main diagonal blocks, we conclude that $per(\mathcal{A}) > 0$. Thus, we could not construct a $2s$-dimensional nonnegative polystochastic tensor of order n with permanent equal to 0. This allows us to conclude that the permanent of a $2s$-dimensional nonnegative polystochastic tensor of order n is positive in this step.

Step III-III: In what follows, we continue step III and put at least two positive numbers (less than 1) on each of the vertical lines $(*, n-1, i, i, \dots, i)$ and $(*, n-2, i, i, \dots, i)$, for each $i = 1, 2, \dots, n-1$.

Step I-III-III: First, we consider the vertical lines $(*, n-1, i, i, \dots, i)$, where $i = 1, 2, \dots, n-1$ and $* = 0, 1, \dots, n-1$. We set positive numbers (less than 1) for the entries with indices

$$(n-2, n-1, i, i, \dots, i), \qquad i = 1, 2, \dots, n-1.$$

Notice that these selected entries lie on the same horizontal line as the entries with indices $(n-2, n-2, i, i, \dots, i)$. Also, we know that $a_{n-2,n-2,i,i,\dots,i} = 0$. (We consider $(n-2, n-1, i, i, \dots, i)$, because each horizontal line $(n-2, *, i, i, \dots, i)$, where $* = 0, 1, \dots, n-1$, does not have any positive numbers.)

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Now, for other selection of positive entries on the vertical lines $(*, n-1, i, i, \dots, i)$, where $i = 1, 2, \dots, n-1$, choose an arbitrary block $(k, n-1, i, i, \dots, i)$, where $k \neq n-2, n-1$. Now, we set positive numbers (less than 1) for the entries of $\mathcal{A}$ with indices $(k, n-1, i, i, \dots, i)$, where $i = 1, 2, \dots, n-1$.

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

So far, we set positive numbers (less than 1) for the entries of $\mathcal{A}$ with indices

$$(n-2, n-1, i, i, \dots, i), (k, n-1, i, i, \dots, i) \qquad i = 1, 2, \dots, n-1, \tag{5.15}$$

where $k \in \{0, \dots, n-3\}$. Thus, the vertical lines $(*, n-1, i, i, \dots, i)$, where $i = 1, 2, \dots, n-1$ and $* = 0, 1, \dots, n-1$, have at least two positive entries.

Step II-III-III: In this step, we consider the vertical lines

$$(*, n-2, i, i, \dots, i),$$

where $i = 1, \dots, n-1$. We want to put at least two positive numbers on each vertical line. So, we put positive numbers in the positions

$$(k, n-2, i, i, \dots, i), (m, n-2, i, i, \dots, i),$$

where $i = 1, 2, \dots, n-1$, and $m \in \{0, 1, \dots, n-3\}$, where k is the nonnegative number, selected in (5.15), and $m \neq k$ is arbitrary. (Since our goal is to construct a nonnegative polystochastic tensor with permanent equal to 0, we try to do this with the least number of positive entries.)

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Now, since our goal is to construct a tensor with permanent 0, we set equal to 0 the entries of $\mathcal{A}$ in the $(n-1)$-row block that forms a positive diagonal for $\mathcal{A}$. Thus, we let

$$a_{n-1,n-2,i,i,\dots,i} = a_{n-1,k,i,i,\dots,i} = a_{n-1,m,i,i,\dots,i} = 0, \qquad i = 1, 2, \dots, n-1.$$

Step III-III-III: In this step, we continue the process of the proof of step III-III, similar to steps I-II-III, II-II-III, III-II-III, IV-II-III and V-II-III. We will see in this step that, we cannot construct a $2s$-dimensional nonnegative polystochastic tensor of order n with permanent equal to 0.

Step IV-III: In this case of step III, if we want to put at least three positive entries (less than 1) on each of the vertical lines $(*, n-1, i, i, \dots, i)$ and $(*, n-2, i, i, \dots, i)$, where $i = 1, 2, \dots, n-1$ and $* = 0, 1, \dots, n-1$, then the proof is similar to step III-III. (Therefore, we cannot construct a tensor with the aforementioned properties.) Hence, we cannot construct a $2s$-dimensional nonnegative polystochastic tensor of order n with permanent equal to 0 in all steps. This allows us to conclude that the permanent of an even-dimensional nonnegative polystochastic tensor of order n that is constructed using the special $n \times (n-1)$ row-Latin rectangle R, mentioned in Example 3.7, is positive. □

Notice that in Step II, we can use an arbitrary row-Latin rectangle R with no transversals and not necessarily isotopic to R, and continue the algorithm by choosing a positive diagonal for 2-dimensional plane and $2s-2$-dimensional plane subtensors of $\mathcal{A}$, and putting positive numbers on each line of $\mathcal{A}$ whose sum of the entries is less than 1, in such a way that the positive entries do not form a positive diagonal for $\mathcal{A}$. Eventually, after these selections, we may obtain a positive diagonal for $\mathcal{A}$.

Corollary 5.4. *The permanent of each even-dimensional polystochastic nonnegative tensor of order* 4 *is positive.*

Proof. We know that the row-Latin rectangle R is the only $n \times (n-1)$ row-Latin rectangle with no transversals for $n = 4$ [14]. Thus, the desired result follows from Theorem 5.3. □

6. Conclusion

In this paper, we proposed an algorithm for computing the permanent of a tensor, which was based on a recursion relation. Next, we proved the positivity of the permanent of a 4-dimensional polystochastic $(0,1)$-tensor of order n that was constructed using a special $n \times (n-1)$ row-Latin rectangle R with no transversals. Also, we established the positivity of the permanent of an even-dimensional polystochastic $(0,1)$-tensor of order n that was constructed using the row-Latin rectangle R. Although we proved these theorems by using the special $n \times (n-1)$ row-Latin rectangle R with no transversals, we presented an algorithm to study these theorems by using each $n \times (n-1)$ row-Latin rectangle R' with no transversals and not necessarily isotopic to R. In the special case $n = 4$, we proved the positivity of the permanent of an even-dimensional polystochastic $(0, 1)$-tensor of order 4 (answering Wanless' conjecture for odd-dimensional Latin hypercubes of order 4). Moreover, we applied our main theorem to the theory of hypergraphs; we proved the positivity of the number of perfect matchings of the bipartite hypergraph associated to an even-dimensional polystochastic $(0, 1)$-tensor of order 4. Furthermore, we proved the positivity of the permanent of a 4-dimensional nonnegative polystochastic tensor of order n that was constructed using the special $n \times (n - 1)$ row-Latin rectangle R. Then, we established the positivity of the permanent of an even-dimensional nonnegative polystochastic tensor of order n that was constructed using the row-Latin rectangle R.

As an idea for further research, interested readers can investigate other ways to simplify the computation of the permanents of tensors. Also, they can try to present a complete proof for the conjecture proposed by Wanless in [16].

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