Contributions to Discrete Mathematics

Volume 18, Number 1, Pages 66–89
ISSN 1715-0868

INTRIGUING SETS OF STRONGLY REGULAR GRAPHS AND THEIR RELATED STRUCTURES

DEAN CRNKOVIĆ, FRANCESCO PAVESE, AND ANDREA ŠVOB

ABSTRACT. In this paper we outline a technique for constructing directed strongly regular graphs by using strongly regular graphs having a "nice" family of intriguing sets. Further, we investigate such a construction method for rank three strongly regular graphs having at most 45 vertices. Finally, several examples of intriguing sets of polar spaces are provided.

1. INTRODUCTION

A *finite incidence structure* consists of a finite set $\mathcal{V}$, called points, a set $\mathcal{B}$ of subsets of $\mathcal{V}$, called blocks, and the incidence relation $\in$ (containment) between points and blocks. An incident point-block pair is called a *flag*, and a nonincident point-block pair is called an *antiflag*. A *tactical configuration* with parameters (v, b, k, r) is a finite incidence structure $(\mathcal{V}, \mathcal{B})$ with $|\mathcal{V}| = v$, $|\mathcal{B}| = b$ such that every block contains k points and every point belongs to exactly r blocks. A *partial geometric design* [10] or a $1\frac{1}{2}$*-design* [56] with parameters $(v, b, k, r; \alpha, \beta)$ is a tactical configuration $(\mathcal{V}, \mathcal{B})$ with parameters (v, b, k, r) such that for every point $x \in \mathcal{V}$ and every block $B \in \mathcal{B}$, the number of flags (y, C) such that $y \in B \setminus \{x\}$, $x \in C \neq B$ equals α or β, for $x \notin B$ or $x \in B$ respectively. A *special partially balanced incomplete block design (SPBIBD)* [11] with parameters $(v, b, k, r, \lambda_1, \lambda_2)$ of type (α_1, α_2), with $v, b, r, k \geq 2$, $\lambda_1, \lambda_2, \alpha_1, \alpha_2 \geq 0$, $\lambda_1 \neq \lambda_2$ and $r < b$, is a tactical configuration with parameters (v, b, k, r) such that

(i) Two distinct points are either in exactly λ_1 (when they are λ_1-associated) or in exactly λ_2 common blocks (when they are λ_2-associated).

Received by the editors September 19, 2021, and in revised form November 5, 2021.
2000 *Mathematics Subject Classification.* 05C20; 51A50; 05B25.
Key words and phrases. (directed) strongly regular graph; intriguing set; finite classical polar space.
The authors would like to thank Morgan Rodgers for pointing out Construction 5.3.1.
A. Švob was supported by Croatian Science Foundation under the project 5713.

(ii) A point x is λ_1-associated to exactly α_1 points of a block B if $x \in B$, and to α_2 points of B if $x \notin B$.

A SPBIBD is called *quasi-symmetric* if any two distinct blocks have either μ_1 or μ_2, $\mu_1 \neq \mu_2$, points in common. A *strongly regular graph (SRG)* Γ with parameters (v, k, λ, μ) is a (connected, simple, undirected, and loopless) k-regular graph with v vertices such that any two adjacent vertices have λ common neighbours and any two nonadjacent vertices have μ common neighbours. If Γ is a strongly regular graph, then $V(\Gamma)$ will denote the set of its vertices. A subset $\mathcal{S}$ of vertices in a strongly regular graph is said to be *intriguing* if the number of neighbours in $\mathcal{S}$ of a vertex x only takes two values, according as $x \in \mathcal{S}$ or $x \in V(\Gamma) \setminus \mathcal{S}$. An intriguing set $\mathcal{S}$ is said to be *proper* if $0 < |\mathcal{S}| < v$. A *directed strongly regular graph* [34] with parameters (v, k, t, λ, μ) is a directed graph on v vertices without loops such that

(i) every vertex has in-degree and out-degree k,
(ii) every vertex x has t out-neighbours that are also in-neighbours of x,
(iii) the number of directed paths of length 2 from a vertex x to another vertex y is λ if there is an edge from x to y, and is μ if there is no edge from x to y.

Let G be a group of permutations acting on a set Ω. The *rank* of the action is the number of orbits of the subgroup G_x fixing $x \in \Omega$ on Ω. The orbits of G on $\Omega \times \Omega$ are called *orbitals* and they are symmetric if for all $x, y \in \Omega$ the pairs (x, y) and (y, x) belong to the same orbital. Let G be transitive of rank three. Then its orbitals, say $I = \{(x, x) \mid x \in \Omega\}, R, S$, are symmetric if and only if G has even order. In this case (Ω, R) and (Ω, S) form a pair of complementary strongly regular graphs, called *rank three strongly regular graph*. In particular, they are connected if and only if G is primitive and the group G acts transitively on ordered pairs of adjacent vertices and on ordered pairs of non-adjacent vertices of each of these graphs. See [47], [48], [65].

Recently, it has been shown that directed strongly regular graphs can be constructed from partial geometric designs [14]. Moreover, a partial geometric design with parameters $(v, b, k, r; \alpha, \beta)$ gives rise to two distinct DSRGs having parameters:

$$(b(v-k), r(v-k), kr-\alpha, kr-(k+r-1+\beta), kr-\alpha),$$
$$(vr, rk-1, \beta+r+k-2, \beta+r+k-3, \alpha).$$

We will consider *proper* partial geometric design, i.e., the design for which $\alpha > 0$, $3 \leq k \leq v-3$ and $3 \leq r \leq b-3$ (see [56]).

In this paper we show that a strongly regular graph having a "nice" family of intriguing sets gives rise to SPBIBD (section 3). See section 2 for the basic properties and the definition of intriguing sets. Since SPBIBDs form a particular class of partial geometric designs (see Lemma 3.1), a technique for constructing directed strongly regular graphs arises in this way. In section 4 we investigate such a construction method for rank three strongly regular graphs on at most 45 vertices. Finally, several examples of intriguing sets of polar spaces are provided in section 5.

2. Preliminaries

In this section we recall some basic facts regarding strongly regular graphs, intriguing sets, special partially balanced incomplete block designs and quasi-symmetric special partially balanced incomplete block designs. For a more comprehensive treatment of these topics we refer the reader to [9, 12].

2.1. **Strongly regular graphs.** Let Γ be a strongly regular graph with parameters (v, k, λ, μ). Let A be the adjacency matrix of Γ. The matrix A satisfies the equation $A^2 = kI + \lambda A + \mu(J - I - A)$, where I denotes the identity matrix of order v and J the all-ones matrix of order v. On the other hand, if A is a $v \times v$ matrix and there exist non-negative integers k, λ, μ such that

$$A^2 = kI + \lambda A + \mu(J - I - A) = (\lambda - \mu)A + (k - \mu)I + \mu J,$$

then A can be seen as the adjacency matrix of a strongly regular graph. The matrix A has three distinct eigenvalues: $\theta_0 > \theta_1 > \theta_2$, where

$$\begin{aligned}
\theta_0 &= k,\\
\theta_1 &= \left(\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right)/2,\\
\theta_2 &= \left(\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right)/2.
\end{aligned}$$

The matrices $A_0 := I, A_1 := A, A_2 := J - I - A$ are symmetric and they pairwise commute. Moreover, $A_i A_j = \sum_{k=0}^{2} p_{ij}^k A_k$ where

$$\begin{aligned}
p_{0j}^k &= \delta_{j,k},\\
p_{11}^0 &= k,\\
p_{11}^1 &= \lambda,\\
p_{11}^2 &= \mu,\\
p_{12}^0 &= 0,\\
p_{12}^1 &= k - \lambda - 1,\\
p_{12}^2 &= k - \mu,\\
p_{22}^0 &= v - k - 1,\\
p_{22}^1 &= v - 2k + \lambda,\\
p_{22}^2 &= v - 2k + \mu - 2.
\end{aligned}$$

Since the matrices A_0, A_1, A_2 are linearly independent, they generate a commutative 3-dimensional algebra $\mathcal{A}$ consisting of real symmetric matrices, called *Bose-Mesner algebra* of Γ. Also, $\mathcal{A}$ admits a basis $\{E_0, E_1, E_2\}$, of so called *minimal idempotents*, where $E_iE_j = \delta_{i,j}E_i$ and $E_0 + E_1 + E_2 = I$. Here

$$E_0 = \frac{1}{v}J,$$
$$E_1 = \frac{1}{\theta_1 - \theta_2}\left(A - \theta_2 I - \frac{k-\theta_2}{v}J\right),$$
$$E_2 = \frac{1}{\theta_2 - \theta_1}\left(A - \theta_1 I - \frac{k-\theta_1}{v}J\right).$$

A subset $\mathcal{I}$ of vertices of Γ, $0 < |\mathcal{I}| < v$, is said to be *intriguing* with parameters (h_1, h_2) if there exist constants h_1 and h_2 such that every vertex of $\mathcal{I}$ is adjacent to precisely h_1 vertices of $\mathcal{I}$ and every vertex of $V(\Gamma) \setminus \mathcal{I}$ is adjacent to precisely h_2 vertices of $\mathcal{I}$. This concept has been introduced by Delsarte [32] in the more general framework of association schemes and investigated in different contexts by several authors [1, 2, 3, 15, 17, 20, 36, 57]. If $\mathcal{I}$ is intriguing with parameters (h_1, h_2), then $(h_1 - h_2 - k)\boldsymbol{j}_{\mathcal{I}} + h_2\boldsymbol{j}$ is an eigenvector of the adjacency matrix A with the eigenvalue $h_1 - h_2$. Here and in the sequel $\boldsymbol{j}$ denotes the $v \times 1$ all ones vector, $\mathbf{0}$ the $v \times 1$ all zeros vector and $\boldsymbol{j}_{\mathcal{I}}$ the $v \times 1$ characteristic vector of $\mathcal{I}$. Hence, either $h_1 - h_2$ is θ_1 and $\mathcal{I}$ is said to be *a positive intriguing set* or $h_1 - h_2$ is θ_2 and $\mathcal{I}$ is said to be *a negative intriguing set*. For an intriguing set $\mathcal{I}$, we have that $|\mathcal{I}| = \frac{h_2 v}{k-\theta_i}$, where i equals 1 or 2 according as $\mathcal{I}$ is positive or negative, respectively. Note that the complement of an intriguing set is an intriguing set of the same type; the union of two disjoint intriguing sets of the same type is an intriguing set of the same type; if A and B are intriguing sets of the same type and $A \subseteq B$, then $B \setminus A$ is an intriguing set of the same type. Moreover, if Γ^c denotes the complement of Γ and $\mathcal{I}$ is a (positive or negative) intriguing set of Γ, then $\mathcal{I}$ is a (negative or positive) intriguing set of Γ^c. As a consequence we have the following.

Proposition 2.1. *A self-complementary strongly regular graph has a positive intriguing set of size x if and only if it has a negative intriguing set of size x.*

An equivalent definition of an intriguing set is the following:

Definition 2.2. *$\mathcal{I}$ is a positive intriguing set of Γ if $E_2\boldsymbol{j}_{\mathcal{I}} = \mathbf{0}$, and $\mathcal{I}$ is a negative intriguing set of Γ if $E_1\boldsymbol{j}_{\mathcal{I}} = \mathbf{0}$.*

Since both h_1, h_2 are non-negative integers, the definition of an intriguing set does not make sense if Γ is a conference graph with non-integral eigenvalues.

2.2. **Finite classical polar spaces.** Let q be a prime power and let $\mathrm{PG}(r, q)$ be the r-dimensional finite projective space over the finite field $\mathrm{GF}(q)$. We will use the term *n-space* to denote an n-dimensional projective subspace of $\mathrm{PG}(r,q)$. Let $\mathcal{P}_r$ be one of the following nondegenerate polar spaces of $\mathrm{PG}(r,q)$:

$$\mathcal{H}(r,q^2),\ \mathcal{Q}^-(r,q)\ (r \text{ odd}),\ \mathcal{Q}^+(r,q)\ (r \text{ odd}),\ \mathcal{Q}(r,q)\ (r \text{ even}).$$

Associated with $\mathcal{P}_r$, there is a polarity $\perp$ of $\mathrm{PG}(r,q)$, which is nondegenerate except when $\mathcal{P}_r = \mathcal{Q}(r,q)$ and q is even. In particular, the polarity $\perp$ is symplectic if $\mathcal{P}_r = \mathcal{W}(r,q)$ or $\mathcal{P}_r \in \{\mathcal{Q}^+(r,q), \mathcal{Q}^-(r,q)\}$ with q even, orthogonal if $\mathcal{P}_r \in \{\mathcal{Q}(r,q), \mathcal{Q}^+(r,q), \mathcal{Q}^-(r,q)\}$ with q odd, and Hermitian if $\mathcal{P}_r = \mathcal{H}(r,q^2)$. If $\mathcal{P}_r = \mathcal{Q}(r,q)$ with q even, then $\perp$ is degenerate, indeed $N^\perp = \mathrm{PG}(r,q)$ if N is the nucleus of $\mathcal{Q}(r,q)$ and $P^\perp$ is a hyperplane of $\mathrm{PG}(r,q)$ for any other point P of $\mathrm{PG}(r,q)$. A *generator* of $\mathcal{P}_r$ is a projective space of maximal dimension contained in $\mathcal{P}_r$ and the union of pairwise disjoint generators is a *partial spread* of $\mathcal{P}_r$. More background information on the properties of the finite classical polar spaces can be found in [49, 50, 51].

Let Γ be the point graph of $\mathcal{P}_r$. A subset $\mathcal{I}$ of points of $\mathcal{P}_r$ is called *intriguing* if the corresponding set of vertices of Γ is an *intriguing set* of Γ. An *m-ovoid* $\mathcal{O}$ of $\mathcal{P}_r$ is a subset of points of $\mathcal{P}_r$ such that every generator of $\mathcal{P}_r$ meets $\mathcal{O}$ in exactly m points [63]. A subset $\mathcal{T}$ of points of $\mathcal{P}_r$ is said to be *i-tight* if the average number of points of $\mathcal{T}$ collinear with a given point of $\mathcal{T}$ attains a maximum possible value [33, 60, 61]. Tight sets and m-ovoids are intriguing sets of $\mathcal{P}_r$. Viceversa, a positive intriguing set of $\mathcal{P}_r$ is an i-tight set, whereas a negative intriguing set of $\mathcal{P}_r$ is an m-ovoid [2, Theorem 6], [3, Theorem 4.1]. The points covered by a partial spreads of size x form an x-tight set of $\mathcal{P}_r$. For more results and constructions of intriguing sets of finite polar spaces see [4, 31, 24, 26, 27, 30, 28, 29, 6, 7, 16, 39, 40, 41, 42, 44, 43, 51, 52, 53, 54, 55, 59].

2.3. **SPBIBDs.** Let $\mathcal{D}$ be a SPBIBD with parameters $(v,b,k,r,\lambda_1,\lambda_2)$ of type (α_1,α_2). Let $\Gamma_{\mathcal{D}}$ be the graph having as vertices the points of $\mathcal{D}$, where two distinct vertices are adjacent whenever the corresponding points of $\mathcal{D}$ are λ_1-associated. The graph $\Gamma_{\mathcal{D}}$ is strongly regular. Moreover, if $\mathcal{D}$ is quasi-symmetric, then its block graph is strongly regular. These facts are stated implicitly in [11, p. 3–4] and [11, p. 10] and a proof is given here for completeness.

Lemma 2.3. *The graph $\Gamma_{\mathcal{D}}$ is strongly regular.*

Proof. Let N be $v \times b$ the incidence matrix of $\mathcal{D}$ and let P be the $v \times v$ adjacency matrix of $\Gamma_{\mathcal{D}}$. Then

(2.1)
$$NN^t = (r-\lambda_2)I_v + (\lambda_1-\lambda_2)P + \lambda_2 J_{v,v} = (\lambda_1-\lambda_2)\left(P - \frac{\lambda_2 - r}{\lambda_1-\lambda_2} I_v\right) + \lambda_2 J_{v,v}$$

and

$$PN = (\alpha_1 - \alpha_2)N + \alpha_2 J_{v,b}. \tag{2.2}$$

It follows that on one hand

$$PNN^t = P(NN^t) = (\lambda_1-\lambda_2)P\left(P - \frac{\lambda_2 - r}{\lambda_1-\lambda_2}I_v\right) + \lambda_2 PJ_{v,v}. \tag{2.3}$$

On the other hand

$$\begin{aligned} PNN^t &= (PN)N^t \\ &= \left((\alpha_1-\alpha_2)N + \alpha_2 J_{v,b}\right)N^t \\ &= (\alpha_1-\alpha_2)NN^t + \alpha_2 J_{v,b}N^t \\ &= (\alpha_1-\alpha_2)\left[(\lambda_1-\lambda_2)\left(P - \frac{\lambda_2-r}{\lambda_1-\lambda_2}I_v\right) + \lambda_2 J_{v,v}\right] \\ &\quad + \alpha_2 r J_{v,v}. \end{aligned} \tag{2.4}$$

Taking into account (2.1), (2.2), we have that

$$\begin{aligned} krJ_{v,v} &= k(NJ_{b,v}) \\ &= N(N^tJ_{v,v}) \\ &= (NN^t)J_{v,v} \\ &= (r-\lambda_2)J_{v,v} + (\lambda_1-\lambda_2)PJ_{v,v} + \lambda_2 J_{v,v}^2, \end{aligned}$$

and hence

$$PJ_{v,v} = \frac{kr - r + \lambda_2 - \lambda_2 v}{\lambda_1 - \lambda_2}J_{v,v}. \tag{2.5}$$

Therefore, from (2.3), (2.4) and (2.5), we obtain

$$\begin{aligned} (\lambda_1-\lambda_2)P^2 \;=\; & \left(\lambda_2 - r + (\alpha_1-\alpha_2)(\lambda_1-\lambda_2)\right)P - (\alpha_1-\alpha_2)(\lambda_2-r)I_v \\ & + \left((\alpha_1-\alpha_2)\lambda_2 + \alpha_2 r - \lambda_2\frac{kr - r + \lambda_2 - \lambda_2 v}{\lambda_1-\lambda_2}\right)J_{v,v}. \end{aligned}$$

□

Lemma 2.4. *The block graph of a quasi-symmetric SPBIBD is strongly regular.*

Proof. Assume that two distinct blocks of $\mathcal{D}$ have either μ_1 or μ_2 points in common, $\mu_1 < \mu_2$. Let $\Gamma'_{\mathcal{D}}$ be the graph having as vertices the blocks of $\mathcal{D}$, where two distinct vertices are adjacent whenever the corresponding blocks of $\mathcal{D}$ have μ_1 points in common. Let N be the $v \times b$ incidence matrix of $\mathcal{D}$ and let A be the $b \times b$ adjacency matrix of $\Gamma'_{\mathcal{D}}$. Then

$$N^tN = (k-\mu_2)I_b + (\mu_1-\mu_2)A + \mu_2 J_{b,b} = (\mu_1-\mu_2)\left(A - \frac{\mu_2-k}{\mu_1-\mu_2}I_b\right) + \mu_2 J_{b,b}.$$

As a consequence we have that

$$\begin{aligned}
N^tNN^t &= (N^tN)N^t \\
&= (\mu_1-\mu_2)\left(AN^t - \frac{\mu_2-k}{\mu_1-\mu_2}N^t\right) + \mu_2 J_{b,b}N^t \\
&= (\mu_1-\mu_2)\left(AN^t - \frac{\mu_2-k}{\mu_1-\mu_2}N^t\right) + \mu_2 r J_{b,v} \\
&= (NN^tN)^t \\
&= \left((\lambda_1-\lambda_2)\left(PN - \frac{\lambda_2-r}{\lambda_1-\lambda_2}N\right) + \lambda_2 J_{v,v}N\right)^t \\
&= \left((\lambda_1-\lambda_2)(\alpha_1-\alpha_2)-\lambda_2+r\right)N^t + \left((\lambda_1-\lambda_2)\alpha_2+\lambda_2 k\right)J_{b,v},
\end{aligned}$$

and

$$\begin{aligned}
N^tNJ_{b,b} &= (N^tN)J_{b,b} \\
&= (k-\mu_2)J_{b,b} + (\mu_1-\mu_2)AJ_{b,b} + \mu_2 bJ_{b,b} \\
&= N^t(NJ_{v,b}) \\
&= rkJ_{b,b}.
\end{aligned}$$

Therefore,

$$(\mu_1-\mu_2)AJ_{b,b} = (kr-k+\mu_2-\mu_2 b)J_{b,b}. \tag{2.6}$$

$$\begin{aligned}
(\mu_1-\mu_2)AN^t = {}&(\mu_2-k+r-\lambda_2+(\lambda_1-\lambda_2)(\alpha_1-\alpha_2))N^t \\
&+ \left(\lambda_2 k-\mu_2 r+(\lambda_1-\lambda_2)\alpha_2\right)J_{b,v}.
\end{aligned} \tag{2.7}$$

Taking into account (2.6) and (2.7), it follows that

$$\begin{aligned}
AN^tN &= A(N^tN) \\
&= (\mu_1-\mu_2)\left(A^2 - \frac{\mu_2-k}{\mu_1-\mu_2}A\right) + \mu_2 AJ_{b,b} \\
&= (\mu_1-\mu_2)\left(A^2 - \frac{\mu_2-k}{\mu_1-\mu_2}A\right) + \frac{\mu_2(kr-k+\mu_2-\mu_2 b)}{\mu_1-\mu_2}J_{b,b} \\
&= (AN^t)N \\
&= \frac{\mu_2-k+r-\lambda_2+(\lambda_1-\lambda_2)(\alpha_1-\alpha_2)}{\mu_1-\mu_2}N^tN \\
&\quad + \frac{\lambda_2 k-\mu_2 r+(\lambda_1-\lambda_2)\alpha_2}{\mu_1-\mu_2}J_{b,v}N \\
&= \left[\frac{\mu_2-k+r-\lambda_2+(\lambda_1-\lambda_2)(\alpha_1-\alpha_2)}{\mu_1-\mu_2}\right. \\
&\qquad \left.\times\left((\mu_1-\mu_2)\left(A-\frac{\mu_2-k}{\mu_1-\mu_2}I_b\right)+\mu_2 J_{b,b}\right)\right] \\
&\quad + \frac{\left(\lambda_2 k-\mu_2 r+(\lambda_1-\lambda_2)\alpha_2\right)k}{\mu_1-\mu_2}J_{b,b}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
(\mu_1-\mu_2)A^2 = {} & (2(\mu_2-k)+(\alpha_1-\alpha_2)(\lambda_1-\lambda_2)-\lambda_2+r)\,A \\
& - \frac{(\mu_2-k)\,((\alpha_1-\alpha_2)(\lambda_1-\lambda_2)-\lambda_2+r+\mu_2-k)}{\mu_1-\mu_2} I_b \\
& + \frac{(\lambda_1-\lambda_2)(\alpha_1\mu_2-\alpha_2\mu_2+\alpha_2 k)}{\mu_1-\mu_2} J_{b,b} \\
& + \frac{\mu_2(r-\lambda_2)+\lambda_2 k^2-2\mu_2 kr+\mu_2^2 b}{\mu_1-\mu_2} J_{b,b}.
\end{aligned}
$$

□

3. Intriguing sets and partial geometric designs

For the convenience of the reader we remark that SPBIBDs form a particular class of partial geometric designs.

Lemma 3.1. *A SPBIBD with parameters $(v, b, k, r, \lambda_1, \lambda_2)$ of type (α_1, α_2) is a partial geometric design with parameters*

$$(v, b, k, r; \alpha_2(\lambda_1-\lambda_2)+k\lambda_2, \alpha_1(\lambda_1-\lambda_2)+(k-1)(\lambda_2-1)).$$

Proof. Let x be a point and B a block. We count the number N of flags (y, C) such that $x \in C, y \in B$, with $y \neq x$ and $C \neq B$. Assume first that $x \notin B$. Let $y \in B$ such that there are exactly λ_1 blocks containing both x and y, then y can be chosen in α_2 ways. The remaining $k-\alpha_2$ elements of B are λ_2-associated with x. Hence $N = \lambda_1\alpha_2+(k-\alpha_2)\lambda_2 = k\lambda_2+\alpha_2(\lambda_1-\lambda_2)$.

Assume that $x \in B$. Let $y \in B$ such that there are exactly $\lambda_1 - 1$ blocks distinct from B and containing both x, y; then y can be chosen in α_1 ways. The remaining $k - \alpha_1 - 1$ elements of B are λ_2-associated with x. Then $N = (\lambda_1 - 1)\alpha_1 + (k - \alpha_1 - 1)(\lambda_2 - 1) = \alpha_1(\lambda_1 - \lambda_2) + (k - 1)(\lambda_2 - 1)$. □

The converse situation has been investigated in [8]. See also [64].

Theorem 3.2. *Let Γ be a strongly regular graph and let $\mathcal{F}$ be a family of subsets of $V(\Gamma)$ such that*

1) *all elements of $\mathcal{F}$ have that same number z of elements, $0 < z < |V(\Gamma)|$;*
2) *there exist constants λ_i, $0 \le i \le 2$, such that $\forall x, y \in V(\Gamma)$, $d(x, y) = i$, then $\lambda_i = |\{\mathcal{I} \in \mathcal{F} \mid \{x, y\} \subset \mathcal{I}\}|$.*

Then $(V(\Gamma), \mathcal{F})$ is a SPBIBD with parameter $(|V(\Gamma)|, |\mathcal{F}|, z, \lambda_0, \lambda_1, \lambda_2)$ of type

$$\left(\theta_i + \frac{k-\theta_i}{|V(\Gamma)|}z, \frac{k-\theta_i}{|V(\Gamma)|}z\right),$$

if and only if $\mathcal{F}$ consists of intriguing sets of Γ with parameters

$$\left(\theta_i + \frac{k-\theta_i}{|V(\Gamma)|}z, \frac{k-\theta_i}{|V(\Gamma)|}z\right).$$

Proof. Firstly, observe that $(V(\Gamma), \mathcal{F})$ is a tactical configuration with parameters $(|V(\Gamma)|, |\mathcal{F}|, z, \lambda_0)$. Assume that $(V(\Gamma), \mathcal{F})$ is a SPBIBD; then two distinct vertices x, y are adjacent in Γ if and only if they are λ_1-associated. Let $x \in V(\Gamma)$ and $B \in \mathcal{F}$. Then x is adjacent to either

$$\theta_i + \frac{k-\theta_i}{|V(\Gamma)|}z, \text{ or } \frac{k-\theta_i}{V(\Gamma)}z,$$

vertices of B, for $x \in B$ or $x \notin B$, respectively. Hence B is an intriguing set of Γ. Viceversa, assume that $\mathcal{F}$ consists of intriguing sets of Γ. From 2), we have that through two distinct elements x, y of $V(\Gamma)$ there pass either λ_1 or λ_2 blocks of $\mathcal{F}$ according as x, y are adjacent or not in Γ. Let $x \in V(\Gamma)$ and $B \in \mathcal{F}$. Since B is an intriguing set of Γ, we have that x is λ_1-associated to exactly

$$\theta_i + \frac{k-\theta_i}{|V(\Gamma)|}z,$$

points of B if $x \in B$, and to

$$\frac{k-\theta_i}{|V(\Gamma)|}z,$$

points of B if $x \notin B$. □

Remark: Note that, taking into account [20, Proposition A.2], if Γ is a connected regular graph of diameter 2 with $s+1 \geq 3$ eigenvalues and $\mathcal{F}$ is a family of intriguing sets of Γ of fixed index satisfying 1), 2) of Theorem 3.2, then Γ is strongly regular.

Proposition 3.4. *Let Γ be a strongly regular graph admitting a rank three automorphism group G and let $\mathcal{I} \neq V(\Gamma)$ be a nonempty subset of vertices of Γ. Then $(V(\Gamma), \mathcal{I}^G)$ is a SPBIBD with parameters $(|V(\Gamma)|, b, k, r, r_1, r_2)$ of type $(\theta_i + h_2, h_2)$, with $b = |G|/|G_{\mathcal{I}}|$, $k = |\mathcal{I}|$, if and only if $\mathcal{I}$ is an intriguing set of Γ with parameters $(\theta_i + h_2, h_2)$.*

Proof. The group G has three orbits on $V(\Gamma) \times V(\Gamma)$, namely I, R, S, where $x, y \in V(\Gamma)$, $x \neq y$, are adjacent if and only if $(x, y) \in R$. Let $\mathcal{I} \neq V(\Gamma)$ be a nonempty subset of vertices of Γ, hence $0 < |\mathcal{I}| = k < |V(\Gamma)|$, and let $b = |G|/|G_{\mathcal{I}}|$. Then each of the incidence structures $(I, \mathcal{I}^G)$, $(R, \mathcal{I}^G)$ and $(S, \mathcal{I}^G)$ is a tactical configuration. Therefore, through a vertex of Γ there pass a constant number of elements of $\mathcal{I}^G$, say r, and through two distinct vertices x, y of Γ there pass either r_1 or r_2 elements of $\mathcal{I}^G$, according as x is adjacent to y or not. The result follows from Theorem 3.2. □

As a consequence, the next result is immediately obtained.

Corollary 3.5. *Let $\mathcal{P}_r$ be a nondegenerate polar space of $\mathrm{PG}(r, q)$ and let G be the subgroup of either $\mathrm{PSL}(r+1, q)$ or $\mathrm{PGL}(r+1, q)$ or $\mathrm{P\Gamma L}(r+1, q)$ fixing $\mathcal{P}_r$. If $\mathcal{I}$ is a nontrivial intriguing set of $\mathcal{P}_r$, then the incidence structure whose points are the points of $\mathcal{P}_r$ and whose blocks are the elements of $\mathcal{I}^G$ is a SPBIBD.*

Corollary 3.5 provides motivation to construct intriguing sets in polar space, a task that will be discussed further in section 5.

4. Intriguing sets in small rank three strongly regular graphs

In what follows, by using GAP list of primitive groups [46], we consider a primitive rank three group G of even order and the strongly regular graph Γ obtained from one of its orbitals. Of course $G \leq \mathrm{Aut}(\Gamma)$. If Γ has at most 40 vertices, we completely classify its intriguing sets and compute the corresponding DSRGs via Proposition 3.4. Moreover, some partial results are obtained for Γ having 45 vertices. Most of them have a large number of vertices. We omit the known DSRGs whose parameters are included in Tables [13]. Besides the conference graphs with nonintegral eigenvalues, we exclude the Petersen graph, the Clebsch graph and the Hoffman–Singleton graph since they have been considered in [1]. For more information on some families of strongly regular we refer the reader to [12, section 9.9.1].

The Paley graph SRG$(9, 4, 1, 2)$

There are two rank three groups: $3^2 : 4 \leq 3^2 : D(8) = \mathrm{Aut}(\Gamma)$. The eigenvalues of Γ are 1 and -2 and Γ has one positive and one negative intriguing set both of size 3 and both stabilized by a subgroup of $\mathrm{Aut}(\Gamma)$ of order 12.

The point graph of $\mathcal{Q}(4, 2)$ SRG$(15, 6, 1, 3)$

There are two rank three groups: $A_6 \leq S_6 = \mathrm{Aut}(\Gamma) \simeq \mathrm{PGO}(5, 2)$ and Γ has eigenvalues 1 and -3. There is one example of tight set of size 3 corresponding to a line of $\mathcal{Q}(4, 2)$ and two tight sets of size 6 corresponding to either the complement of a $\mathcal{Q}^+(3, 2)$ or to two disjoint lines. In the latter case there arise a DSRG$(540, 216, 96, 72, 96)$ and a DSRG$(360, 143, 71, 70, 48)$. There is only one negative intriguing set of size 5, being the ovoid $\mathcal{Q}^-(3, 2)$.

The point graph of $\mathcal{Q}^+(3, 3)$ SRG$(16, 6, 2, 2)$

There are four rank three groups: $(A_4 \times A_4) : 2$, $2^4.S_3 \times S_3$, $2^4.3^2 : 4$, $(S_4 \times S_4) : 2 = \mathrm{Aut}(\Gamma) \simeq \mathrm{PGO}^+(4, 3)$ and the eigenvalues of Γ are 2 and -2. There is a tight set of size 4 (that is a line of $\mathcal{Q}^+(3, 3)$) stabilized by a subgroup of G of order 144 and a tight set of size 8 (a pair of disjoint lines of $\mathcal{Q}^+(3, 3)$) fixed by a subgroup of G of order 96. Regarding m-ovoids of $\mathcal{Q}^+(3, 3)$, there is a unique class of ovoids, being the conic sections and two distinct examples of 2-ovoids: one of which is a pair of disjoint conics admitting a subgroup of G of order 16 and there is one more stabilized by a subgroup of G of order 64. The related DSRGs have parameters $(144, 36, 10, 6, 10)$, $(144, 71, 39, 38, 32)$, $(144, 72, 40, 32, 40)$ and $(288, 72, 20, 12, 20)$.

The triangular graph $T(7)$ SRG$(21, 10, 3, 6)$

In this case there are two rank three groups: $A_7 \leq S_7 = \mathrm{Aut}(\Gamma)$ and $\theta_1 = 1, \theta_2 = -4$. There is one example of negative intriguing set of size 6 left invariant by S_6, which is a coclique. Regarding positive intriguing set there are two examples of size 7, admitting an automorphism group of size 14 and 48, respectively. Moreover, the DSRGs associated with the SP-BIBDs have parameters $(5040, 1680, 600, 480, 600)$, $(2520, 839, 359, 358, 240)$, $(1470, 490, 175, 140, 175)$, $(735, 244, 104, 103, 70)$.

The point graph of $\mathcal{Q}^+(3,4)$ SRG$(25,8,3,2)$

There are six rank three groups: $5^2 : 8 : 2$, $5^2 : O^+(2,5)$, $(A_5 \times A_5) : 2$, $(A_5 \times A_5) : 4$, $(A_5 \times A_5) : 2^2 \simeq \mathrm{P\Gamma O}^+(4,4)$ and $(S_5 \times S_5) : 2 = \mathrm{Aut}(\Gamma)$. The eigenvalues of Γ are 3 and -2. There is a tight set of size 5 (that is a line of $\mathcal{Q}^+(3,4)$) stabilized by a subgroup of $\mathrm{Aut}(\Gamma)$ of order 2880 and a tight set of size 10 (a pair of disjoint lines of $\mathcal{Q}^+(3,4)$) fixed by a subgroup of $\mathrm{Aut}(\Gamma)$ of order 1440. Regarding m-ovoids of $\mathcal{Q}^+(3,4)$, all the ovoids are $\mathrm{Aut}(\Gamma)$-equivalent, nevertheless they fall into two sets under the action of $\mathrm{P\Gamma O}^+(4,4)$: the conic sections and the elliptic quadric $\mathcal{Q}^-(3,2)$ (which coincide with the twisted cubic in this case). There are two distinct examples of 2-ovoids: one of which admits a group of order 48 and consists of a pair of conics having in common two points of a line ℓ together with $\ell^\perp \cap \mathcal{Q}^+(3,4)$. The other example is obtained from two disjoint ovoids and is left invariant by a group of order 20. These example corresponds to 26 DSRGs; those on less than 10^3 vertices have parameters

$$(400, 159, 72, 71, 58), (200, 79, 40, 39, 26), (400, 80, 17, 12, 17),$$
$$(600, 119, 47, 46, 18), (600, 240, 102, 87, 102), (200, 40, 9, 4, 9),$$
$$(500, 99, 39, 38, 15), (300, 59, 23, 22, 9), (300, 120, 54, 39, 54).$$

The Paley graph SRG$(25,12,5,6)$

There are three rank three groups: $5^2 : Q(12)$, $5^2 : 12$, $3^2 : D(8) = \mathrm{Aut}(\Gamma)$. The eigenvalues of Γ are 2 and -3, and Γ has one positive and one negative intriguing set of size 5, both stabilized by a subgroup of $\mathrm{Aut}(\Gamma)$ of order 40. There are also two positive and two negative intriguing sets of size 10, invariant under by a subgroup of $\mathrm{Aut}(\Gamma)$ of order 6 and 20, respectively. The corresponding DSRGs have parameters

$$(300, 60, 13, 8, 13), (1500, 600, 260, 210, 260), (1000, 399, 189, 188, 140),$$
$$(450, 180, 78, 63, 78), (300, 119, 56, 55, 42).$$

The point graph of $\mathcal{Q}^-(5,2)$ SRG$(27,10,1,5)$

There are two rank three groups: $\mathrm{P}\Omega^-(6,2) \leq \mathrm{PGO}^-(6,2) = \mathrm{Aut}(\Gamma)$ and the eigenvalues of Γ are $\theta_1 = 1, \theta_2 = -5$. There are no examples of negative intriguing set (or m-ovoids), indeed this would correspond to a regular system of order m of $\mathcal{H}(3,4)$ [49], [62]. There is one example of a tight set of size 3 corresponding to a line of $\mathcal{Q}^-(5,2)$ and two tight sets of size 6 corresponding to either two disjoint lines or the 6 points of a $\mathcal{Q}(4,2) \setminus \mathcal{Q}^+(3,2)$. There are four examples of tight sets of size 9: the points of a $\mathcal{Q}^+(3,2)$; the union of three pairwise disjoint lines of $\mathcal{Q}^-(5,2)$ generating the whole $\mathrm{PG}(5,2)$; the union of three pairwise disjoint lines of $\mathcal{Q}^-(5,2)$ generating a four-space and having a common transversal. The last example can be described by using Construction 5.0.1. The related DSRGs have parameters

$$(1080, 120, 14, 8, 14), (135, 14, 6, 5, 1), (15120, 3360, 784, 616, 784),$$
$$(4320, 959, 343, 342, 176), (7560, 1680, 392, 308, 392), (2160, 479, 171, 170, 88),$$
$$(2160, 720, 252, 216, 252), (1080, 359, 143, 142, 108),$$
$$(58320, 19440, 6804, 5832, 6804), (29160, 9719, 3887, 3886, 2916),$$
$$(51840, 17280, 6048, 5184, 6048), (25920, 8639, 3455, 3454, 2592),$$
$$(38880, 12960, 3888, 3264, 3888), (19440, 6479, 3215, 3214, 2592).$$

The graph $\mathrm{NO}^+(6,2)$ $\mathrm{SRG}(28,15,6,10)$

There are two rank three groups:

$$\mathrm{P}\Omega^+(6,2) \simeq A_8 \leq S_8 = \mathrm{Aut}(\Gamma) \simeq \mathrm{PGO}^+(6,2),$$

and $\theta_1 = 1, \theta_2 = -5$. Concerning positive intriguing sets there is one example of size 4, that is an affine plane disjoint from $\mathcal{Q}^+(5,2)$ such that its line at infinity is a line of $\mathcal{Q}^+(5,2)$, fixed by a group of order 384, three of size 8 stabilized by a group of order 128, 60 and 16, respectively, and six examples of size 12. One of these consists of the points of $\mathcal{Q}^+(5,2)$ on a tangent hyperplane. The remaining are left invariant by a group of order 4, 12, 16, 16, 48, respectively. There arise 22 distinct DSRGs; those on less than 10^4 vertices have parameters

$$(2520, 360, 54, 36, 54), (420, 59, 23, 22, 6), (6300, 1800, 540, 450, 540),$$
$$(2520, 719, 269, 268, 180), (5376, 1535, 575, 574, 384),$$
$$(6720, 2880, 1296, 1152, 1296), (5040, 2159, 1007, 1006, 864),$$
$$(560, 240, 108, 96, 108), (420, 179, 83, 82, 72).$$

There is one negative intriguing set of size 7 left invariant by S_7, which is a coclique. The DSRGs associated have parameters $(168, 42, 12, 6, 12)$ and $(56, 13, 7, 6, 2)$.

The point graph of $\mathcal{Q}^+(5,2)$ $\mathrm{SRG}(35,18,9,9)$

There are two rank three groups:

$$\mathrm{P}\Omega^+(6,2) \simeq A_8 \leq S_8 = \mathrm{Aut}(\Gamma) \simeq \mathrm{PGO}^+(6,2).$$

The eigenvalues of Γ are 3 and -3. The positive intriguing sets are determined in [21]; we have one example of a 1-tight set, a plane of $\mathcal{Q}^+(5,2)$, and one example of 2-tight sets, i.e., the union of two disjoint planes. The corresponding DSRGs have parameters

$$(840, 168, 36, 24, 36), (210, 41, 17, 16, 6), (420, 84, 18, 12, 18),$$
$$(2520, 1008, 432, 360, 432), (1680, 671, 311, 310, 240).$$

There is a unique ovoid, that is the elliptic quadric $\mathcal{Q}^-(3, q)$, two types of 2-ovoids: the points of $\mathcal{Q}(4, q) \setminus \mathcal{Q}^-(3, 2)$ which admits a group of order 240 or two disjoint elliptic quadrics $\mathcal{Q}^-(3, 2)$, left invariant by a group of order 48. Finally, there are five 3-ovoids: two of them are a disjoint union of elliptic quadrics, and admit a group of order 12 or 48, respectively; a third example was pointed out by D. Glynn [45] and it is stabilized by a group of order 60; a fourth example is a $\mathcal{Q}(4, 2)$ embedded in $\mathcal{Q}^+(5, 2)$ and the last example is left invariant by a group of order 48 and can be obtained from Construction 5.0.1. The related DSRGs have parameters

$$(1680, 240, 36, 24, 36), (280, 39, 15, 14, 4), (21000, 6000, 1800, 1500, 1800),$$
$$(420, 179, 83, 82, 72), (8400, 2399, 899, 898, 600), (4200, 1200, 360, 300, 360),$$
$$(1680, 479, 179, 178, 120), (8400, 3600, 1620, 1440, 1620),$$
$$(6300, 2699, 1259, 1258, 1080), (67200, 28800, 12960, 11520, 12960),$$
$$(50400, 21599, 10079, 10078, 8640), (13440, 5760, 2592, 2304, 2592),$$
$$(10080, 4319, 2015, 2014, 1728), (16800, 7200, 3240, 2880, 3240),$$
$$(12600, 5399, 2519, 2518, 2160), (560, 240, 108, 96, 108).$$

The point graph of $\mathcal{Q}^+(3, 5)$ SRG$(36, 10, 4, 2)$

There are eight rank three groups: $(A_5 \times A_5) : 2$, $(A_5 \times A_5).4$, $((A_5 \times A_5) : 2)2$, $(S_5 \times S_5) : 2 \simeq \mathrm{PGO}^+(4, 5)$, $(A_6 \times A_6) : 2$, $(A_6 \times A_6) : 2^2$, $(A_6 \times A_6) : 4$, $(S_6 \times S_6) : 2 = \mathrm{Aut}(\Gamma)$ and $\theta_1 = 4$, $\theta_2 = -2$. Regarding the tight sets, there are either one, two, or three pairwise disjoint lines. The corresponding DSRGs have parameters

$$(360, 60, 11, 5, 11), (360, 119, 55, 54, 32), (360, 179, 98, 97, 81),$$
$$(360, 180, 99, 81, 99), (720, 240, 88, 64, 88),$$
$$(720, 359, 197, 196, 162), (720, 360, 198, 162, 198).$$

Under the action of $\mathrm{Aut}(\Gamma)$ there is one ovoid stabilized by a group of order 1440, four types of 2-ovoids, fixed by a group of order 24, 64, 144, 768, respectively, and six examples of 3-ovoids, admitting a group of order 8, 12, 24, 48, 64, 5184, respectively. Note that there are m-ovoids that are equivalent under the action of $\mathrm{Aut}(\Gamma)$, although they are not $\mathrm{PGO}^+(4, 5)$-equivalent. For instance, under the action of $\mathrm{PGO}^+(4, 5)$, there are two types of ovoids: the conic sections and the ovoids that span the whole $\mathrm{PG}(3, 5)$, see also [23, Proposition 2.10]. Hence, these two types of ovoids are not $\mathrm{PGO}^+(4, 5)$-equivalent, whereas they are $\mathrm{Aut}(\Gamma)$-equivalent. Varying G in one of the eight rank three groups listed above, there arise 86 distinct DSRGs. If $G = \mathrm{Aut}(\Gamma)$, the related DSRGs on less than 10^5 vertices have parameters

$$(21600, 3600, 624, 480, 624), (4320, 719, 239, 238, 96),$$
$$(86400, 28799, 11135, 11134, 8832), (32400, 10800, 3744, 3312, 3744),$$
$$(16200, 5399, 2087, 2086, 1656), (3600, 1800, 936, 864, 936),$$
$$(3600, 1799, 935, 934, 864).$$

$\boxed{\mathrm{SRG}(36, 14, 4, 6)}$

In this case $G = \mathrm{P\Gamma U}(3, 9) = \mathrm{Aut}(\Gamma)$ and $\theta_1 = 2, \theta_2 = -4$. Concerning positive intriguing sets, there is one example of size 6 left invariant by a group of order 96, four types of size 12 fixed by a group of order 6, 16, 24 and 192, respectively, and eight examples of size 18, four of which are fixed by a group of order 6, two by a group of order 12 and the remaining two by a group of order 24 and 216, respectively. The corresponding DSRGs have parameters

$$(3780, 630, 110, 80, 110), (756, 125, 45, 44, 16), (1512, 504, 176, 152, 176),$$
$$(24192, 8063, 3199, 3198, 2432), (18144, 6048, 2112, 1824, 2112),$$
$$(36288, 18144, 9504, 8640, 9504), (36288, 18143, 9503, 9502, 8640),$$
$$(18144, 9072, 4752, 4320, 4752), (18144, 9071, 4751, 4750, 4320),$$
$$(12096, 4032, 1408, 1216, 1408), (9072, 3023, 1199, 1198, 912),$$
$$(6048, 2015, 799, 798, 608), (48384, 16128, 5632, 4864, 5632),$$
$$(9072, 4536, 2376, 2160, 2376), (9072, 4535, 2375, 2374, 2160),$$
$$(1008, 504, 264, 240, 264), (1008, 503, 263, 262, 240), (756, 251, 99, 98, 76).$$

As for positive intriguing sets, there is one example of size 12 admitting an automorphism group of order 192, and one example of size 18 fixed by a group of order 108. The related DSRGs have parameters

$$(1512, 504, 180, 144, 180), (756, 251, 107, 106, 72), (2016, 1008, 540, 468, 540),$$
$$(2016, 1007, 539, 538, 468).$$

$\boxed{\textit{The triangular graph } T(9)\ \mathrm{SRG}(36, 14, 7, 4)}$

There are three rank three groups: $\mathrm{P\Gamma L}(2, 8) \leq A_9 \leq S_9 = \mathrm{Aut}(\Gamma)$ and $\theta_1 = 5, \theta_2 = -2$. There is only one positive intriguing set, which is of size 8 and left invariant by S_8. The DSRG obtained has parameters $(252, 56, 14, 7, 14)$. Regarding negative intriguing sets there are four examples of size 9 left invariant by a group of order 18, 72, 80, 1296, respectively, and sixteen examples of size 18, three of which are stabilized by an involution, two by a group of order 4, three by a group of order 8, two by a group of order 12, two by a group of order 16, two by a group of order 18 and the remaining two by a group of order 32 and 72, respectively. There are 52 distinct corresponding

DSRGs. Many of them have a quite large number of vertices; those on less than 10^3 vertices have parameters $(756, 189, 49, 42, 49)$, $(756, 188, 62, 61, 42)$ and $(252, 62, 20, 19, 14)$.

The graph $\mathrm{NO}^-(6,2)$ $\mathrm{SRG}(36,15,6,6)$

There are two rank three groups: $\mathrm{P}\Omega^-(6,2) \le \mathrm{PGO}^-(6,2) = \mathrm{Aut}(\Gamma)$ and the eigenvalues of Γ are $\theta_1 = 3, \theta_2 = -3$. There are two examples of positive intriguing sets of size 9 and 18, stabilized by a group of order 1296 and 216, respectively. The corresponding DSRGs have parameters

$$(1080, 270, 72, 54, 72), (360, 89, 35, 34, 18), (4320, 2160, 1152, 1008, 1152),$$
$$(4320, 2159, 1151, 1150, 1008).$$

As for negative intriguing sets there is one example of size 8 admitting a group of order 384, two examples of size 12 one of which is $\ell^\perp \setminus \ell$, where ℓ is a line of $\mathcal{Q}^-(5,2)$. The remaining one is fixed by a group of order 36. There are three examples of size 16, one of these consists of the points off $\mathcal{Q}^-(5,2)$ not on a tangent hyperplane; the others are left invariant by a group of order 20, 48, respectively. The related DSRGs have parameters

$$(51840, 23040, 10752, 9600, 10752), (41472, 18431, 8831, 8830, 7680),$$
$$(21600, 9600, 4480, 4000, 4480), (17280, 7679, 3679, 3678, 3200),$$
$$(540, 240, 112, 100, 112), (432, 191, 91, 90, 80).$$

The point graph of $\mathcal{W}(3,3)$ $\mathrm{SRG}(40,12,2,4)$

There are two rank three groups: $\mathrm{PSp}(4,3) \le \mathrm{PGSp}(4,3) = \mathrm{Aut}(\Gamma)$ and $\theta_1 = 2$, $\theta_2 = -4$. As for tight sets, there is one example of size 4, a line of $\mathcal{W}(3,3)$ and two examples of size 8: a pair of disjoint lines of $\mathcal{W}(3,3)$ and the set $\ell \cup \ell^\perp$, where ℓ is a line of $\mathrm{PG}(3,3)$, that is not a line of $\mathcal{W}(3,3)$. There are four 3-tight sets. One of them consists of three pairwise disjoint lines of $\mathcal{W}(3,3)$ such that the opposite of the regulus determined by them has no lines of $\mathcal{W}(3,3)$. The second one consists of three pairwise disjoint lines of $\mathcal{W}(3,3)$ such that the opposite of the regulus determined by them has two lines of $\mathcal{W}(3,3)$. The third example is $r \cup \ell \cup \ell^\perp$, where r is a line of $\mathcal{W}(3,q)$, ℓ is a line of $\mathrm{PG}(3,3)$, that is not a line of $\mathcal{W}(3,3)$ and $|r \cap \ell| = 0$. The fourth example can be described by using Construction 5.1.1 and it is left invariant by a group of order 48. There are seven 4-tight sets. Two of these 4-tight sets are reguli consisting of lines of $\mathcal{W}(3,3)$. Two further examples are four pairwise disjoint lines of $\mathcal{W}(3,3)$ not forming a regulus having a line of $\mathcal{W}(3,3)$ as a transversal line (stabilized by a group of order 24), or not (fixed by a group of order 8). Two further examples arise by gluing two generators of $\mathcal{W}(3,3)$ that are disjoint from ℓ, to $\ell \cup \ell^\perp$, where ℓ is not a generator of $\mathcal{W}(3,3)$, and are left invariant by a group of order 16 or 12. Another example admits a group of order 12. There are nine types of 5-tight sets. Five of them are five pairwise disjoint lines and are left invariant by a group of order 12, 16, 16, 20 or 240, respectively. One more example arises from Construction 5.3.1 and admits a group of order 240. The remaining three examples are left invariant by groups of order 4, 12, and 24. There arise 38 DSRGs. Those with less than 10^4 vertices have parameters

$$(4320, 1727, 755, 754, 648), (8640, 3455, 1511, 1510, 1296), (160, 15, 6, 5, 1),$$
$$(4320, 2159, 1124, 1123, 1035), (4320, 2160, 1125, 1035, 1125),$$
$$(4320, 863, 287, 286, 144), (6480, 2592, 1080, 972, 1080),$$
$$(1440, 144, 15, 9, 15), (360, 71, 23, 22, 12), (1440, 288, 60, 48, 60).$$

Regarding m-ovoids, the symplectic polar space $\mathcal{W}(3, 3)$ has a unique 2-ovoid [3, Theorem 5.1], which admits a group of order 120. The related DSRGs have parameters

$$(8640, 4320, 2304, 2016, 2304), (8640, 4319, 2303, 2302, 2016).$$

$\boxed{\textit{The point graph of } \mathcal{Q}(4,3) \text{ SRG}(40,12,2,4)}$

There are two rank three groups: $\mathrm{P}\Omega(5, 3) \leq \mathrm{PGO}(5, 3) = \mathrm{Aut}(\Gamma)$ and $\theta_1 = 2$, $\theta_2 = -4$. Concerning tight sets, there is one example of size 4 (a line) and one example of size 8 (a pair of disjoint lines). The 3-tight sets are of three types. One consists of three pairwise disjoint lines spanning a three-space stabilized by a group of order 144. The second one consists of three pairwise disjoint lines spanning the whole four-space fixed by a group of order 18. The third example is left invariant by a group of order 36 and can be described as follows: $(\mathcal{Q}^+(3, q) \setminus (\ell_1 \cup \ell_2)) \cup (\ell \setminus \{P\})$, where P is a point of $\mathcal{Q}^+(3, 3) \subset \mathcal{Q}(4, 3)$, ℓ_1, ℓ_2 are the lines of $\mathcal{Q}^+(3, 3)$ through P and ℓ is a line of $\mathcal{Q}(4, 3)$ meeting $\mathcal{Q}^+(3, 3)$ exactly in P. It is easily seen that such a set is a tight set being the union of 4 pairwise skew lines minus a transversal. Of course it generalizes for $q > 3$ as well. There are five 4-tight sets, and four pairwise disjoint lines spanning a three-space, i.e., the points of a $\mathcal{Q}^+(3, 3)$ embedded in $\mathcal{Q}(4, 3)$. Further, four pairwise disjoint lines, three of them span a three-space, admitting a group of order 18. The third example consists of four pairwise disjoint lines, no three in a three-space, left invariant by a group of order 8. A fourth example can be described by means of Construction 5.0.1 and it is left invariant by a group of order 72. A fifth example admits a group of order 6. Finally, there are five types of 5-tight sets. Two of them are five pairwise disjoint lines and these are left invariant by a group of order 6 or 10, respectively. The other examples admit either a group of order 6, or a group of order 10 fixing a $\mathcal{Q}^-(3, 3)$, or a group of order 24 fixing a $\mathcal{Q}^+(3, 3)$. There arise 26 distinct DSRGs. Those having less than 10^4 vertices have parameters

$$(4320, 863, 287, 286, 144), (720, 287, 125, 124, 108), (160, 15, 6, 5, 1),$$
$$(1080, 432, 180, 162, 180), (1440, 144, 15, 9, 15), (4320, 1295, 476, 475, 351).$$

As for m-ovoids, the parabolic quadric $\mathcal{Q}(4,3)$ possesses a unique ovoid, the elliptic quadric $\mathcal{Q}^-(3,3)$, and a unique 2-ovoid, which is obtained by intersecting $\mathcal{Q}(4,3)$ with the unique 2-ovoids of $\mathcal{Q}^-(5,3)$ and admits a group of order 160. The related DSRGs have parameters

$$(1080,270,72,54,72),(360,89,35,34,18),(6480,3240,1728,1512,1728),$$
$$(6480,3239,1727,1726,1512),(3240,1620,864,756,864),$$
$$(3240,1619,863,862,756).$$

$\boxed{\textit{The point graph of } \mathcal{H}(3,4) \text{ SRG}(45,12,3,3)}$

There are two rank three groups: $\mathrm{PGU}(4,4) \leq \mathrm{P\Gamma U}(4,4) = \mathrm{Aut}(\Gamma)$ and $\theta_1 = 3$, $\theta_2 = -3$. Regarding positive intriguing sets, we have one example of 1-tight set, a line of $\mathcal{H}(3,4)$, and one example of 2-tight sets, i.e., the union of two disjoint lines. Examples of size 15 arise either from a $\mathcal{W}(3,2)$ embedded in $\mathcal{H}(3,4)$ or by the union of three pairwise disjoint lines of $\mathcal{H}(3,4)$. As for 4-tight sets, we have either four pairwise disjoint lines or the complement of the two non-equivalent sets of five pairwise disjoint generators of $\mathcal{H}(3,4)$. The related DSRGs have parameters

$$(1080,120,14,8,14),(135,14,6,5,1),(7560,1680,392,308,392),$$
$$(2160,479,171,170,88),(21600,7200,2520,2160,2520),$$
$$(10800,3599,1439,1438,1080),(1080,360,126,108,126),$$
$$(540,179,71,70,54),(5400,2400,1120,1000,1120),$$
$$(4320,1919,919,918,800),(10800,4800,2240,2000,2240),$$
$$(8640,3839,1839,1838,1600),(27000,12000,5600,5000,5600),$$
$$(21600,9599,4599,4598,4000).$$

As for m-ovoids there are two classes of ovoids, a nondegenerate plane section and an ovoid spanning the whole 3-space admitting a group of order 324, whereas from [19] there are six types of 2-ovoids. Some of the related DSRGs have parameters

$$(5760,1152,240,192,240),(1440,287,95,94,48),(1440,288,60,48,60),$$
$$(360,71,23,22,12),(116640,46655,20411,20410,17496),$$
$$(174960,69984,29160,26244,29160),(38880,15552,6480,5832,6480),$$
$$(25920,10367,4535,4534,3888).$$

$\boxed{\text{The triangular graph } T(10) \text{ SRG}(45,16,8,4)}$

There are two rank three groups: $A_{10} \leq S_{10} = \mathrm{Aut}(\Gamma)$ and $\theta_1 = 6$, $\theta_2 = -2$. There is one example of a positive intriguing set of size 9 admitting S_9 whose associated DSRGs has parameters $(360,72,16,8,16)$.

The negative intriguing sets of the graph are the following: a coclique of size 5 fixed by a group of order 3840; five distinct examples of size 10 left invariant by a group of order 20, 84, 96, 200, 576, respectively; 21 intriguing sets of size 15 admitting a subgroup of order 2^2, 4^4, 6^2, 8^3, 12, 16^2, 20, 32, 48, 32, 120, 288, 1728. Some of the related DSRGs have parameters

$(1323000, 294000, 67200, 58800, 67200), (378000, 83999, 25199, 25198, 16800),$
$(1512000, 336000, 76800, 67200, 76800), (432000, 95999, 28799, 28798, 19200),$
$(6350400, 1411200, 322560, 282240, 322560), (63000, 13999, 4199, 4198, 2800),$
$(1814400, 403199, 120959, 120958, 80640), (181440, 40319, 12095, 12094, 8064),$
$(220500, 49000, 11200, 9800, 11200), (635040, 141120, 32256, 28224, 32256).$

4.1. **Quasi-symmetric SPBIBDs.** According to Lemma 2.4 a quasi -symmetric SPBIBD yields a strongly regular graph. Note that the lines of a generalized quadrangle are blocks of quasi-symmetric SPBIBDs and the obtained SRG is the point graph of its dual generalized quadrangle. More interestingly, the nondegenerate hyperplane sections of a parabolic quadric $\mathcal{Q}(2n, q)$ or of a Hermitian polar space $\mathcal{H}(n, q^2)$ are intriguing sets and form the blocks of quasi-symmetric SPBIBDs. The SRGs that arise are the graphs $\mathrm{NO}^{\pm}(2n+1, q)$ or $\mathrm{NU}(n, q^2)$. Similarly, in $\mathrm{NO}^{\pm}(2n, 2)$ or $\mathrm{NU}(n, q^2)$ the nonisotropic points on tangent hyperplanes form an intriguing set. These are blocks of quasi-symmetric SPBIBDs. The related SRGs are the point graphs of the corresponding polar spaces. In what follows we provide some interesting SPBIBDs arising from Proposition 3.4 that are quasi–symmetric and compute the parameters of the associated strongly regular block graph.

(1) In $\mathrm{NO}^{-}(6, 2)$, there are 40 positive intriguing sets of size 9 and two of them are either disjoint or meet in 3 points. There arises a SRG with parameters $(40, 12, 2, 4)$ that is the point graph of $\mathcal{Q}(4, 3)$. There are 45 negative intriguing sets of size 12 fixed by a group of order 1152, any two of them have 3 or 6 points in common. The corresponding SRG has parameters $(45, 32, 22, 24)$ that is the complement of the point graph of $\mathcal{H}(3, 4)$.
(2) In $\mathrm{NU}(3, 25)$, there is a negative intriguing set $\mathcal{I}$ of size 105, fixed by the group A_7. If $\mathcal{Z} = \mathcal{I}^{\mathrm{PSU}(3,5)}$, then $|\mathcal{Z}| = 50$. Since two distinct members of $\mathcal{Z}$ meet in either 15 or 45 points, there arises a SRG with parameters $(50, 7, 0, 1)$, i.e., the Hoffman–Singleton graph.
(3) Let $\mathcal{Q}^{-}(5, 3)$ be a nondegenerate elliptic quadric of $\mathrm{PG}(5, 3)$. Up to isomorphism, there is a unique 2-ovoid (being a negative intriguing set of the point graph of $\mathcal{Q}^{-}(5, 3)$) of $\mathcal{Q}^{-}(5, 3)$ admitting the group $\mathrm{PSL}(4, 3)$ as an automorphism group. If $\mathcal{Z} = \mathcal{I}^{\mathrm{P}\Omega^{-}(6,3)}$, then $|\mathcal{Z}| = 162$. Since two distinct members of $\mathcal{Z}$ meet in either 20 or 32 points, there arises the unique SRG with parameters $(162, 56, 10, 24)$.
(4) Let $\mathcal{H}(5, 4)$ be a nondegenerate Hermitian variety of $\mathrm{PG}(5, 4)$. A *hyperoval* of $\mathcal{H}(5, 4)$ is a set of points of $\mathcal{H}(5, 4)$ such that every line of $\mathcal{H}(5, 4)$ meets in either 0 or 2 points. There exists a hyperoval $\mathcal{I}$ of $\mathcal{H}(5, 4)$ of size 126 [58], [25]. Moreover, $\mathcal{I}$ is the unique hyperoval

of $\mathcal{H}(5,4)$ of size 126, up to isomorphism. The stabilizer of $\mathcal{I}$ in $\mathrm{PSU}(6,2)$ is a group S of order 6531840 containing $\mathrm{PSU}(4,3)$. The group S has two orbits on the points of $\mathcal{H}(5,4)$, hence $\mathcal{I}$ is an intriguing set of $\mathcal{H}(5,4)$. Particularly, $\mathcal{I}$ is a positive intriguing set (tight set) of the point graph of $\mathcal{H}(5,4)$ with $h_1 = 45$ and $h_2 = 30$. Let $\mathcal{Z} = \mathcal{I}^{\mathrm{PSU}(6,2)}$. Since two distinct members of $\mathcal{Z}$ meet in either 18 or 30 points, there arises a SRG with parameters $(1408, 567, 246, 216)$. The SRG can be described as a rank three graph, obtained from the group $\mathrm{PSU}(6,2)$.

5. Intriguing sets of polar spaces: some constructions

In this section we present some constructions of intriguing sets of finite classical polar spaces. We say that an intriguing set $\mathcal{I}$ of $\mathcal{P}_r$ is *classical* if $\mathcal{I} = \mathcal{P}_r \cap \Sigma$, for some subspace Σ not contained in $\mathcal{P}_r$. Then it is easily seen ([2, Lemma 7], [18]), that Σ is either an $(r-1)$-space or an $(r-2)$-space of $\mathrm{PG}(r,q)$ such that $\mathcal{P}_r \cap \Sigma = \tilde{\mathcal{P}}_{r-1}$ or $\mathcal{P}_r \cap \Sigma = \tilde{\mathcal{P}}_{r-2}$, where

(5.1)

$\mathcal{P}_r$	$\tilde{\mathcal{P}}_{r-1}$	$\tilde{\mathcal{P}}_{r-2}$
$\mathcal{H}(r,q^2)$	$\mathcal{H}(r-1,q^2)$	
$\mathcal{Q}^-(r,q)$	$\mathcal{Q}(r-1,q)$	$\mathcal{Q}^+(r-2,q)$
$\mathcal{Q}^+(r,q)$	$\mathcal{Q}(r-1,q)$	$\mathcal{Q}^-(r-2,q)$
$\mathcal{Q}(r,q)$	$\mathcal{Q}^+(r-1,q)$	
$\mathcal{Q}(r,q)$	$\mathcal{Q}^-(r-1,q)$	

First we show that by perturbating a classical intriguing set of $\mathcal{P}_r$, a nonclassical intriguing set can be obtained. Then some tight sets of $\mathcal{W}(3,q)$ are described.

Construction. *Let $\mathcal{P}_r$ be a polar space of $\mathrm{PG}(r,q)$, $r \geq 4$, and let Σ be an $(r-1)$-space or an $(r-2)$-space of $\mathrm{PG}(r,q)$ such that $\mathcal{P}_r \cap \Sigma = \tilde{\mathcal{P}}_{r-1}$ or $\mathcal{P}_r \cap \Sigma = \tilde{\mathcal{P}}_{r-2}$ as in (5.1). Let σ be an s-space of $\mathcal{P}_r$ contained in Σ such that $\sigma^\perp \cap \mathcal{P}_r \cap \Sigma \neq \sigma$. Then $\sigma^\perp \cap \mathcal{P}_r$ is a cone having σ as the vertex and $\mathcal{P}_{r-2s-2}$ as the base, and $\sigma^\perp \cap \mathcal{P}_r \cap \Sigma$ is a cone, say $\mathcal{C}$, having σ as the vertex and as the base the polar space $\tilde{\mathcal{P}}_{r-2s-3} \subset \mathcal{P}_{r-2s-2}$ or the polar space $\tilde{\mathcal{P}}_{r-2s-4} \subset \mathcal{P}_{r-2s-2}$. Let $\tilde{\mathcal{P}}'_{r-2s-3}$ or $\tilde{\mathcal{P}}'_{r-2s-4}$ be a polar space embedded in $\mathcal{P}_{r-2s-2}$ distinct from $\tilde{\mathcal{P}}_{r-2s-3}$ or $\tilde{\mathcal{P}}_{r-2s-4}$ and of the same type as $\tilde{\mathcal{P}}_{r-2s-3}$ or $\tilde{\mathcal{P}}_{r-2s-4}$, respectively. Let $\mathcal{C}'$ be the cone having σ as the vertex and $\tilde{\mathcal{P}}'_{r-2s-3}$ or $\tilde{\mathcal{P}}'_{r-2s-4}$ as the base . Set $\mathcal{X} = ((\mathcal{P}_r \cap \Sigma) \setminus \mathcal{C}) \cup \mathcal{C}'$.*

Proposition 5.1. *The set $\mathcal{X}$ is a nonclassical intriguing set of $\mathcal{P}_r$ of the same type as $\mathcal{P}_r \cap \Sigma$.*

Proof. Let P be a point of $\mathcal{P}_r$. Assume first that $P \notin \sigma^\perp$. Then $\sigma \cap P^\perp$ is an $(s-1)$-space and $P^\perp \cap \sigma^\perp \cap \mathcal{P}_r$ is a cone having $\sigma \cap P^\perp$ as the vertex and $\mathcal{P}_{r-2s-2}$ as the base. Hence, $|P^\perp \cap \mathcal{C}| = |P^\perp \cap \mathcal{C}'|$.

Assume now that $P \in \sigma^\perp$. If $P \in \sigma$, then $\sigma^\perp \subset P^\perp$ and $|P^\perp \cap \mathcal{C}| = |P^\perp \cap \mathcal{C}'|$. If $P \notin \sigma$, then we may suppose w.l.o.g. that it belongs to the base of the cone $\sigma^\perp \cap \mathcal{P}_r$, i.e., $P \in \mathcal{P}_{r-2s-2}$. Note that both $\tilde{\mathcal{P}}_{r-2s-3}$ and $\tilde{\mathcal{P}}'_{r-2s-3}$ or $\tilde{\mathcal{P}}_{r-2s-4}$ and $\tilde{\mathcal{P}}'_{r-2s-4}$ are intriguing sets of $\tilde{\mathcal{P}}_{r-2s-2}$. If

$$P \in \mathcal{P}_{r-2s-2} \setminus ((\tilde{\mathcal{P}}_{r-2s-3} \cup \tilde{\mathcal{P}}'_{r-2s-3}) \setminus (\tilde{\mathcal{P}}_{r-2s-3} \cap \tilde{\mathcal{P}}'_{r-2s-3})),$$

or

$$P \in \mathcal{P}_{r-2s-2} \setminus ((\tilde{\mathcal{P}}_{r-2s-4} \cup \tilde{\mathcal{P}}'_{r-2s-4}) \setminus (\tilde{\mathcal{P}}_{r-2s-4} \cap \tilde{\mathcal{P}}'_{r-2s-4})),$$

then $|P^\perp \cap \tilde{\mathcal{P}}_{r-2s-3}| = |P^\perp \cap \tilde{\mathcal{P}}'_{r-2s-3}|$ or $|P^\perp \cap \tilde{\mathcal{P}}_{r-2s-4}| = |P^\perp \cap \tilde{\mathcal{P}}'_{r-2s-4}|$. Hence, $|P^\perp \cap \mathcal{C}| = |P^\perp \cap \mathcal{C}'|$. If $R \in \tilde{\mathcal{P}}_{r-2s-3} \setminus \tilde{\mathcal{P}}'_{r-2s-3}$ and $Q \in \tilde{\mathcal{P}}'_{r-2s-3} \setminus \tilde{\mathcal{P}}_{r-2s-3}$, then $|R^\perp \cap \tilde{\mathcal{P}}_{r-2s-3}| = |Q^\perp \cap \tilde{\mathcal{P}}'_{r-2s-3}|$ and $|R^\perp \cap \tilde{\mathcal{P}}'_{r-2s-3}| = |Q^\perp \cap \tilde{\mathcal{P}}_{r-2s-3}|$. Similarly, if $R \in \tilde{\mathcal{P}}_{r-2s-4} \setminus \tilde{\mathcal{P}}'_{r-2s-4}$ and $Q \in \tilde{\mathcal{P}}'_{r-2s-4} \setminus \tilde{\mathcal{P}}_{r-2s-4}$, then $|R^\perp \cap \tilde{\mathcal{P}}_{r-2s-4}| = |Q^\perp \cap \tilde{\mathcal{P}}'_{r-2s-4}|$ and $|R^\perp \cap \tilde{\mathcal{P}}'_{r-2s-4}| = |Q^\perp \cap \tilde{\mathcal{P}}_{r-2s-4}|$. Therefore, $|R^\perp \cap \mathcal{C}| = |Q^\perp \cap \mathcal{C}'|$ and $|R^\perp \cap \mathcal{C}'| = |Q^\perp \cap \mathcal{C}|$. The result follows from the fact that $\tilde{\mathcal{P}}_{r-1}$ or $\tilde{\mathcal{P}}_{r-2}$ is an intriguing set of $\mathcal{P}_r$.

Finally, notice that $\mathcal{X}$ is not contained in Σ, hence, it is not classical. □

5.1. **Tight sets of $\mathcal{W}(3, q)$.**

Construction. *Assume that q is odd. Let $\mathcal{W}(3, q)$ be a symplectic polar space of* $\mathrm{PG}(3, q)$ *and let $\mathcal{Q}^+(3, q)$ be a hyperbolic quadric with reguli $\mathcal{R}_1$ and $\mathcal{R}_2$ such that all the lines of $\mathcal{R}_1$ and two of the lines of $\mathcal{R}_2$, say ℓ_1 and ℓ_2, are lines of $\mathcal{W}(3, q)$. Let K be the group of projectivities of* $\mathrm{PG}(3, q)$ *isomorphic to* $\mathrm{PSL}(2, q) \times \mathrm{PSL}(2, q)$ *fixing $\mathcal{Q}^+(3, q)$. Then K has two orbits of size $(q^3 - q)/2$ on points of* $\mathrm{PG}(3, q) \setminus \mathcal{Q}^+(3, q)$. *Let $\mathcal{X}$ be one of these two K-orbits.*

Proposition 5.2. *The set $\mathcal{X}$ is a $(q^2 - q)/2$-tight set of $\mathcal{W}(3, q)$.*

Proof. Let P be a point of $\mathrm{PG}(3, q)$. Assume first that $P \in \mathcal{X}$. Then $P^\perp$ meets ℓ_i at the point P_i, $i = 1, 2$, and the lines PP_i are lines of $\mathcal{W}(3, q)$ that are tangent to $\mathcal{Q}^+(3, q)$. Hence, the plane $P^\perp$ meets $\mathcal{Q}^+(3, q)$ in a nondegenerate conic $\mathcal{C}$. Moreover, $\mathcal{X} \cap P^\perp$ are the points of $P^\perp$ that are external to $\mathcal{C}$. Hence, $|P^\perp \cap \mathcal{X}| = (q^2 + q)/2$. Assume now that $P \notin \mathcal{X}$. If $P \in \mathcal{Q}^+(3, q)$, then $P^\perp$ is a plane tangent to $\mathcal{Q}^+(3, q)$ at a point P'. Note that $P = P'$ if and only if $P \in \ell_1 \cup \ell_2$. Among the $q-1$ lines through P' that are tangent to $\mathcal{Q}^+(3, q)$ there are $(q-1)/2$ lines containing q points of $\mathcal{X}$ and $(q-1)/2$ lines containing no points of $\mathcal{X}$. Hence, $|P^\perp \cap \mathcal{X}| = (q^2 - q)/2$. If $P \notin \mathcal{Q}^+(3, q)$, then again $P^\perp$ meets $\mathcal{Q}^+(3, q)$ in a nondegenerate conic $\mathcal{C}$. In this case $\mathcal{X} \cap P^\perp$ consists of the points of $P^\perp$ that are internal to $\mathcal{C}$. Therefore, $|P^\perp \cap \mathcal{X}| = (q^2 - q)/2$. □

Construction. *Assume that $q \equiv 1 \pmod 3$. Let $\mathcal{C}$ be a twisted cubic of* $\mathrm{PG}(3, q)$ *and let $\mathcal{W}(3, q)$ be the symplectic polar space whose polarity $\perp$ maps the points of $\mathcal{C}$ to their osculating planes and interchanges the chords and axes of $\mathcal{C}$, see* [49, Theorem 21.1.2]. *The union of the $q + 1$ tangents to $\mathcal{C}$,* *the $q(q+1)$ unisecant lines in the osculating planes and q^3-q lines external to $\mathcal{C}$ not lying in osculating planes is the set of generators of $\mathcal{W}(3,q)$. With the same notation used in* [49, Corollary 5, Lemma 21.1.11, Corollary]*, let $\mathcal{M}_4$ be the set of points lying on the imaginary chords of $\mathcal{C}$.*

Proposition 5.3. *The set $\mathcal{M}_4$ is a $(q^2-q)/2$-tight set of $\mathcal{W}(3,q)$.*

Proof. The points of $\mathcal{W}(3,q)$ are partitioned into five sets, namely $\mathcal{M}_1$, points of $\mathcal{C}$, $\mathcal{M}_2$, points off $\mathcal{C}$ on a tangent, $\mathcal{M}_3 \cup \mathcal{M}_5$, points off $\mathcal{C}$ on a real chord, $\mathcal{M}_4$ points on an imaginary chord. Similarly, the planes are partitioned into the five sets $\mathcal{N}_i$, $1 \le i \le 5$, and the polarity $\perp$ maps points of $\mathcal{M}_i$ to planes of $\mathcal{N}_i$, $1 \le i \le 5$, [49, Corollary 4, Corollary 5, Lemma 21.1.11, Corollary]. Moreover, a point off $\mathcal{C}$ lies on exactly one real chord, tangent or imaginary chord of $\mathcal{C}$, [49, Theorem 21.1.9]. This means that $\mathcal{M}_4$ consists of the points on $q(q-1)/2$ pairwise skew lines having no point in common with $\mathcal{C}$. Thus $|\mathcal{M}_4| = (q^3-q)/2$. Assume first that P is a point of $\mathcal{W}(3,q)$ not in $\mathcal{M}_4$. If $P \in \mathcal{M}_1 \cup \mathcal{M}_2$, then $P^\perp$ contains a tangent, say t; hence, $P^\perp$ cannot contain an imaginary chord, otherwise it would meet the line t at a point not on $\mathcal{C}$, a contradiction. Therefore, $|P^\perp \cap \mathcal{M}_4| = q(q-1)/2$. Analogously, if $P \in \mathcal{M}_3$, then $|P^\perp \cap \mathcal{C}| = 3$ and $P^\perp$ contains three real chords; therefore, $P^\perp$ cannot contain an imaginary chord, otherwise it would meet a real chord at a point off $\mathcal{C}$, a contradiction. It follows that $|P^\perp \cap \mathcal{M}_4| = q(q-1)/2$. If $P \in \mathcal{M}_5$, then $|P^\perp \cap \mathcal{C}| = 0$. In this case $P^\perp$ cannot contain a tangent or a real chord and hence, $|P^\perp \cap \mathcal{M}_2| = q+1$, $|P^\perp \cap (\mathcal{M}_3 \cup \mathcal{M}_5)| = q(q-1)/2$. It turns out that $|P^\perp \cap \mathcal{M}_4| = q(q-1)/2$. Assume now that $P \in \mathcal{M}_4$. We have seen so far that no plane of $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3 \cup \mathcal{N}_5$ contains an imaginary chord. Hence, $P^\perp \in \mathcal{N}_4$ has to contain exactly one imaginary chord and therefore, $|P^\perp \cap \mathcal{M}_4| = q+1+q(q-1)/2-1 = q(q+1)/2$. □

Construction. *Assume that q is odd. Let $\mathcal{W}(3,q)$ be a symplectic polar space of* $\mathrm{PG}(3,q)$*. There is a partition of the points of* $\mathrm{PG}(3,q)$ *into $q+1$ elliptic quadrics* [35]*. These elliptic quadrics can be paired in such a way they give rise to $(q+1)/2$ pairwise disjoint 2-ovoids of $\mathcal{W}(3,q)$* [3, Corollary 5.2]*, say $\{\mathcal{O}_1, \mathcal{O}'_1\}, \dots, \{\mathcal{O}_{(q+1)/2}, \mathcal{O}'_{(q+1)/2}\}$. Let $\mathcal{X}$ be the set obtained by selecting one elliptic quadric for each of the $(q+1)/2$ pairs and taking their union.*

Proposition 5.4. *The set $\mathcal{X}$ is a $(q^2+1)/2$-tight set of $\mathcal{W}(3,q)$.*

Proof. By construction $|\mathcal{X}| = (q+1)(q^2+1)/2$. Let P be a point of $\mathcal{O}_i$. Note that among the $q+1$ lines of $\mathcal{W}(3,q)$ through P there is exactly one that is tangent to $\mathcal{O}_i$ and the remaining q are secant to $\mathcal{O}_i$, see also [3]. Hence, $|P^\perp \cap \mathcal{O}_i| = q+1$ and $|P^\perp \cap \mathcal{O}'_i| = 1$, since $\mathcal{O}_i \cup \mathcal{O}'_i$ is a 2-ovoid of $\mathcal{W}(3,q)$. Moreover, a plane of $\mathrm{PG}(3,q)$ is tangent to exactly one elliptic quadric of the partition and it is secant to the remaining q. This means that $|P^\perp \cap \mathcal{O}_j| = |P^\perp \cap \mathcal{O}'_j| = q+1$, if $i \ne j$.

If R is a point of $\mathcal{X}$, then we may assume that $R \in \mathcal{O}_i$. Hence, $\mathcal{O}_i \subset \mathcal{X}$ and $|\mathcal{O}'_i \cap \mathcal{X}| = 0$. Thus $|R^\perp \cap \mathcal{X}| = (q-1)(q+1)/2 + q + 1 = (q^2+1)/2 + q$.

If $R \notin \mathcal{X}$, then we may assume that $R \in \mathcal{O}'_i$ and

$$|R^\perp \cap \mathcal{X}| = \frac{(q-1)(q+1)}{2} + 1 = \frac{(q^2+1)}{2},$$

as required. □

References

1. J. Bamberg, F. De Clerck, and N. Durante, *Intriguing sets in partial quadrangles*, J. Combin. Des. **19** (2011), 217–245.
2. J. Bamberg, S. Kelly, M. Law, and T. Penttila, *Tight sets and m–ovoids of finite polar spaces*, J. Combin. Theory Ser. A **114** (2007), 1293–1314.
3. J. Bamberg, M. Law, and T. Penttila, *Tight sets and m–ovoids of generalised quadrangles*, Combinatorica **29** (2009), 1–17.
4. J. Bamberg, M. Lee, K. Momihara, and Q. Xiang, *A new infinite family of hemisystems of the Hermitian surface*, Combinatorica **38** (2018), no. 1, 43–66.
5. J. De Beule, J. Demeyer, K. Metsch, and M. Rodgers, *A new family of tight sets in $\mathcal{Q}^+(5, q)$*, Des. Codes Cryptogr. **78** (2016), no. 3, 655–678.
6. J. De Beule, P. Govaerts, A. Hallez, and L. Storme, *Tight sets, weighted m–covers, weighted m–ovoids, and minihypers*, Des. Codes Cryptogr. **50** (2009), 187–201.
7. J. De Beule and K. Metsch, *On the smallest non–trivial tight sets in Hermitian polar spaces*, Electron. J. Combin. **24** (2017), no. 1, Paper No. 1.62, 13 pp.
8. R. C. Bose, W. G. Bridges, and M. S. Shrikhande, *Partially geometric designs and two–class partially balanced designs*, Discrete Math. **21** (1978), 97–101.
9. R. C. Bose and T. Shimamoto, *Classification and analysis of partially balanced incomplete block designs with two associate classes*, J. Amer. Statist. Assoc. **47** (1952), 151–184.
10. R. C Bose, S. S. Shrikhande, and N. M. Singhi, *Edge regular multigraphs and partial geometric designs*, Proc. Internat. Colloq. Combin. Theory **17** (1976), 49–81.
11. W. G. Bridges and M. S. Shrikhande, *Special partially balanced incomplete block designs and associated graphs*, Discrete Math. **9** (1974), 1–18.
12. A. E. Brouwer and W. H. Haemers, *Spectra of graphs*, second ed., Universitext, Springer, New York, 2012.
13. A. E. Brouwer and S. A. Hobart, *Parameters of directed strongly regular graphs*, `https://homepages.cwi.nl/~aeb/math/dsrg/dsrg.html`.
14. A. E. Brouwer, O. Olmez, and S. Y. Song, *Directed strongly regular graphs from $1\frac{1}{2}$–designs*, European J. Combin. **33** (2012), 1174–1177.
15. B. De Bruyn, *Tight sets of points in the half–spin geometry related to $\mathcal{Q}^+(9, q)$*, Linear Algebra Appl. **424** (2007), no. 2-3, 480–491.
16. ———, *A characterization of m–ovoids and i–tight sets of polar spaces*, Adv. Geom. **8** (2008), no. 3, 367–375.
17. ———, *Intriguing sets of points of $\mathcal{Q}(2n, 2) \setminus \mathcal{Q}^+(2n-1, 2)$*, Graphs Combin. **28** (2012), no. 6, 791–805.
18. ———, *On some 2–tight sets of polar spaces*, Ars Combin. **133** (2017), 115–131.
19. B. De Bruyn and M. Gao, *The homogeneous pseudo-embeddings and hyperovals of the generalized quadrangle $\mathcal{H}(3, 4)$*, Linear Algebra Appl. **593** (2020), 90–115.
20. B. De Bruyn and H. Suzuki, *Intriguing sets of vertices of regular graphs*, Graphs Combin. **26** (2010), 629–646.
21. P. J. Cameron and R. A. Liebler, *Tactical decompositions and orbits of projective groups*, Linear Algebra Appl. **46** (1982), 91–102.
22. Z. Chai, R. Q. Feng, and L. W. Zeng, *Constructions of $1\frac{1}{2}$–designs from symplectic geometry over finite fields*, Acta Math. Sin. **31** (2015), 1367–1378.

23. A. Cossidente, G. Marino, and F. Pavese, *The covering radius of* $\mathrm{PGL}(3,q)$, Discrete Math. **342** (2019), no. 3, 664–670.
24. A. Cossidente and F. Pavese, *Intriguing sets of* $\mathcal{W}(5,q)$*, q even*, J. Combin. Theory Ser. A **127** (2014), 303–313.
25. ———, *Sets of even type on* $\mathcal{H}(5,q^2)$*, q even*, Discrete Appl. Math. **181** (2015), 280–282.
26. ———, *Intriguing sets of quadrics in* $\mathrm{PG}(5,q)$, Adv. Geom. **17** (2017), no. 3, 339–345.
27. ———, *On intriguing sets of finite symplectic spaces*, Des. Codes Cryptogr. **86** (2018), no. 5, 1161–1174.
28. ———, *Cameron–liebler line classes of* $\mathrm{PG}(3,q)$ *admitting* $\mathrm{PGL}(2,q)$, J. Combin. Theory Ser. A **167** (2019), 104–120.
29. ———, *New Cameron–Liebler line classes with parameter* $\frac{q^2+1}{2}$, J. Algebraic Combin. **49** (2019), no. 2, 193–208.
30. ———, *On line covers of finite projective and polar spaces*, Des. Codes Cryptogr. **87** (2019), no. 9, 1985–2002.
31. A. Cossidente and T. Penttila, *Hemisystems on the Hermitian surface*, J. London Math. Soc. **72** (2005), no. 3, 731–741.
32. P. Delsarte, *An algebraic approach to the association schemes of coding theory*, Philips Res. Rep. Suppl. (1973), no. 10, 97 pp.
33. K. W. Drudge, *Extremal sets in projective and polar spaces*, Ph.D. thesis, The University of Western Ontario (Canada), 1998.
34. A. Duval, *A directed graph version of strongly regular graphs*, J. Combin. Theory, Ser. A **47** (1988), 71–100.
35. G. L. Ebert, *Partitioning projective geometries into caps*, Canad. J. Math. **37** (1985), no. 6, 1163–1175.
36. J. Eisfeld, *On the common nature of spreads and pencils in* $\mathrm{PG}(d,q)$, Discrete Math. **189** (1998), 95–104.
37. R. Q. Feng, L. W. Zeng, and Y. Zhang, *Constructions of* $1\frac{1}{2}$*–designs from unitary geometry over finite fields*, Algebra Colloq. **24** (2017), 381–392.
38. R. Q. Feng, M. Zhao, and L. W. Zeng, *Constructions of* $1\frac{1}{2}$*–designs from orthogonal geometry over finite fields*, Discrete Math. **339** (2016), 382–390.
39. T. Feng, K. Momihara, and Q. Xiang, *Cameron–Liebler line classes with parameter* $x = \frac{q^2-1}{2}$, J. Combin. Theory Ser. A **133** (2015), 307–338.
40. ———, *A family of* m*–ovoids of parabolic quadrics*, J. Combin. Theory Ser. A **140** (2016), 97–111.
41. T. Feng and R. Tao, *An infinite family of* m*–ovoids of* $\mathcal{Q}(4,q)$, Finite Fields Appl. **63** (2020), 101644, 16 pp.
42. T. Feng, Y. Wang, and Q. Xiang, *On* m*–ovoids of symplectic polar spaces*, J. Combin. Theory Ser. A **175** (2020), 105279, 14 pp.
43. A. L. Gavrilyuk, I. Matkin, and T. Penttila, *Derivation of Cameron–Liebler line classes*, Des. Codes Cryptogr. **86** (2018), no. 1, 231–236.
44. A. L. Gavrilyuk and K. Metsch, *A modular equality for Cameron–Liebler line classes*, J. Combin. Theory Ser. A **127** (2014), 224–242.
45. D. G. Glynn, *On a set of lines of* $\mathrm{PG}(3,q)$ *corresponding to a maximal cap contained in the klein quadric of* $\mathrm{PG}(5,q)$, Geom. Dedicata **26** (1988), no. 3, 273–280.
46. The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.11.0*, `https://www.gap-system.org`, 2020.
47. M. D. Hestenes and D. G. Higman, *Rank* 3 *groups and strongly regular graphs*, Computers in algebra and number theory, Proc. SIAM-AMS Sympos. Appl. Math., New York, 1970, vol. IV, Amer. Math. Soc., Providence, R.I., 1971, pp. 141–159.
48. D. G. Higman, *Finite permutation groups of rank* 3, Math. Z. **86** (1964), 145–156.

49. J. W. P. Hirschfeld, *Finite projective spaces of three dimensions*, first ed., Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1985.
50. ———, *Projective Geometries over Finite Fields*, second ed., Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1998.
51. J. W. P. Hirschfeld and J. A. Thas, *General Galois Geometries*, second ed., Springer Monographs in Mathematics, Springer, London, 2016.
52. S. Kelly, *Constructions of intriguing sets of polar spaces from field reduction and derivation*, Des. Codes Cryptogr. **43** (2007), no. 1, 1–8.
53. G. Korchmáros, G. P. Nagy, and P. Speziali, *Hemisystems of the Hermitian surface*, J. Combin. Theory Ser. A **165** (2019), 408–439.
54. K. Metsch, *Small tight sets in finite elliptic, parabolic and Hermitian polar spaces*, Combinatorica **36** (2016), no. 6, 725–744.
55. K. Metsch and D. Werner, *On the smallest non–trivial tight sets in Hermitian polar spaces $\mathcal{H}(d, q^2)$, d even*, Discrete Math. **342** (2019), no. 5, 1336–1342.
56. A. Neumaier, *$t\frac{1}{2}$–designs*, J. Combin. Theory, Ser. A **28** (1980), 226–248.
57. ———, *Regular sets and quasisymmetric 2–designs*, Combinatorial theory (Schloss Rauischholzhausen), Lecture Notes in Math., vol. 969, Springer, Berlin-New York, 1982, pp. 258–275.
58. D. Pasechnik, *Extending polar spaces of rank at least 3*, J. Combin. Theory Ser. A **72** (1995), 232–242.
59. F. Pavese, *Geometric constructions of two–character sets*, Discrete Math. **338** (2015), no. 3, 202–208.
60. S. E. Payne, *Tight pointsets in finite generalized quadrangles*, Congr. Numer. **60** (1987), 243–260.
61. M. J. Rodgers, *On some new examples of Cameron–Liebler line classes*, Ph.D. thesis, University of Colorado, Denver, 2012.
62. B. Segre, *Forme e geometrie Hermitiane, con particolare riguardo al caso finito*, Ann. Mat. Pura Appl. **70** (1965), no. 4, 1–201.
63. J. A. Thas, *Interesting pointsets in generalized quadrangles and partial geometries*, Linear Algebra Appl. **114/115** (1989), 103–131.
64. E. R. van Dam and E. Spence, *Combinatorial designs with two singular values. ii. partial geometric designs*, Linear Algebra Appl. **396** (2005), 303–316.
65. H. Wielandt, *Finite permutation groups*, AcademicPress, New York, 1964.

Department of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia
E-mail address: `deanc@math.uniri.hr`

Dipartimento di Meccanica Matematica e Management, Politecnico di Bari, Via Orabona 4, I-70125 Bari, Italy
E-mail address: `francesco.pavese@poliba.it`

Department of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia
E-mail address: `asvob@math.uniri.hr`