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PRIME 2-STRUCTURES

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To Roland Fraïssé (1920-2008)

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Preface

My purpose is to present the main results on prime 2-structures. We consider primality in terms of the usual modular decomposition. We are mainly interested in the downward hereditary properties of primality. The first six sections are devoted to finite prime 2-structures whereas the last four sections are devoted to infinite prime 2-structures. The main focus is to establish results from the literature proven for graphs, digraphs, binary relational structures, etc., in the setting of 2-structures.

In section 1, we provide the definition of a 2-structure. The 2-structures are the suitable generalizations of usual structures in graph theory, such as graphs and digraphs, to study the modular decomposition. In a 2-structure, the link between two vertices is not an edge or an arc, but a type of links, that is, an equivalence class of ordered pairs of distinct vertices. In this manner, a 2-structure is defined as an equivalence relation on the set of ordered pairs of distinct vertices. This equivalence relation is sufficient to define the notion of a module.

In section 2, we define different types of connectedness for 2-structures. They generalize known connectedness for graphs and tournaments. We examine the components which are generated by these different types of connectedness. This examination leads us to introduce the notions of a module, a modular cut, and a strong module. These three notions induce three different types of primality. We study these three types of primality, and we conclude with Gallai's decomposition theorem.

In section 3, we examine the prime 2-substructures in a prime 2-structure. First, we prove that every vertex is covered by prime 2-substructures of size 3, 4, or 5. Second, we introduce the outside partition associated with a prime 2-substructure. The outside partition allows us to build from a prime 2-substructure a new prime 2-substructure by adding two vertices. The first downward hereditary property of primality follows: A prime 2-structure admits prime 2-substructures obtained by removing one or two vertices.

In section 4, we characterize the critical 2-structures, that is, the prime 2-structures with the property that all the 2-substructures obtained by removing one vertex are decomposable. We introduce the primality graph associated with every prime 2-structures. Its edges are the unordered pairs whose removal provides a prime 2-substrucure. We examine the neighbourhoods of the primality graph of a critical graph. We deduce that the primality graph of a critical graph is a path, a cycle of odd length or a path of odd length together with one isolated vertex. For each of these four types, we characterize the corresponding critical 2-structures. The characterization of critical 2-structures constitutes an important step in the study of prime 2-structures.

In section 5, we demonstrate the Schmerl–Trotter theorem: a prime 2-structure, with at least seven vertices, admits an unordered pair whose removal provides a prime 2-substructure. In other words, the primality graph of a prime graph, with at least seven vertices, is nonempty. The Schmerl–Trotter theorem is the first substantial theorem in the study of prime 2-structures. It is an important downward hereditary property of primality. We prove also different refinements of the Schmerl–Trotter theorem.

In section 6, we characterize the prime 2-structures that are minimal for a singleton or an unordered pair. Precisely, a prime 2-structure is minimal for a vertex subset if every proper induced 2-substructure with at least three vertices containing this vertex subset is not prime. We mainly characterize the prime 2-structures with at least six vertices that are minimal for an unordered pair. This characterization allows us to provide a concise proof of the Schmerl–Trotter theorem.

Section 7 is devoted to the following compactness theorem on infinite prime 2-structures. An infinite 2-structure is prime if and only if every finite vertex subset is contained in a finite vertex subset which induces a prime 2-substructure.

Section 8 is the analogue of section 4 for infinite 2-structures. Precisely, we characterize the infinite prime 2-structures, all the 2-substructures of which are obtained by removing one vertex are decomposable, and which admit at least a prime 2-substructure obtained by removing finitely many vertices.

In section 9, we characterize finite or infinite partially critical 2-structures. A prime 2-structure is partially critical whenever the removal of every vertex outside a given proper and prime 2-substructure provides a decomposable 2-substructure. As in section 3, we associate with the prime 2-substructure an outside partition. We also associate with it an outside graph which plays an important role in our characterization.

Finally, in section 10, we provide a downward hereditary property of primality in the case of infinite 2-structures. Precisely, we prove that an infinite prime 2-structure admits a proper vertex subset equipotent to the vertex set which induces a prime 2-substructure.

Contents

1. 2-STRUCTURES

A *2-structure* [14] σ consists of a *vertex set* $V(\sigma)$, and of an equivalence relation $\equiv_\sigma$ defined on $(V(\sigma) \times V(\sigma)) \setminus \{(v, v) : v \in V(\sigma)\}$. The cardinality of $V(\sigma)$ is denoted by $v(\sigma)$. A *vertex subset* of σ is a subset of $V(\sigma)$. The set of the equivalence classes of $\equiv_\sigma$ is denoted by $E(\sigma)$. Given a 2-structure σ, if $E(\sigma)$ admits a unique element e, then σ is said to be *constant* or *e-constant*.

2-structure *vertex set* *vertex subset* *constant* *e-constant*

Warning. Unless indicated to the contrary, we consider 2-structures to be finite.

Notation 1.1. Let σ be a 2-structure. Given distinct $v, w \in V(\sigma)$, the equivalence class of $\equiv_\sigma$ to which (v, w) belongs is denoted by $(v, w)_\sigma$. Moreover, set

$$[v, w]_\sigma = ((v, w)_\sigma, (w, v)_\sigma),$$

and

$$\prec v, w \succ_\sigma = \{(v, w)_\sigma, (w, v)_\sigma\}.$$

Let σ be a 2-structure. With each $W \subseteq V(\sigma)$ associate the *2-substructure* $\sigma[W]$ of σ induced by W defined on $V(\sigma[W]) = W$ such that

2-substructure

$$\equiv_{\sigma[W]} \; = \; (\equiv_\sigma)_{\restriction (W \times W) \setminus \{(w,w) : w \in W\}}.$$

Given $W \subseteq V(\sigma)$, $\sigma[V(\sigma) \setminus W]$ is denoted by $\sigma - W$, and by $\sigma - w$ when $W = \{w\}$.

We use the next notation.

Notation 1.2. Let S be a set. Given $W \subseteq S \times S$, set $W^\star = \{(v, w) : (w, v) \in W\}$.

We associate with a 2-structure σ its *dual* $\sigma^\star$ defined on $V(\sigma^\star) = V(\sigma)$ as follows. Given $x, y, v, w \in V(\sigma^\star)$, with $x \neq y$ and $v \neq w$, $(x, y) \equiv_{\sigma^\star} (v, w)$ if $(y, x) \equiv_\sigma (w, v)$. Hence $E(\sigma^\star) = \{e^\star : e \in E(\sigma)\}$. A 2-structure σ is *reversible* if $\sigma = \sigma^\star$. Hence, a 2-structure σ is reversible if and only if for each $e \in E(\sigma)$, $e^\star \in E(\sigma)$. Let σ be a reversible 2-structure. For each $e \in E(\sigma)$, we have $e^\star \in E(\sigma)$, so $e = e^\star$ or $e \cap e^\star = \varnothing$. A 2-structure σ is *symmetric* if for each $e \in E(\sigma)$, $e = e^\star$. On the other hand, it is *asymmetric* if for each $e \in E(\sigma)$, $e \cap e^\star = \varnothing$ [1.1].

dual *reversible* *symmetric* *asymmetric*

1.1. **Isomorphism.** Given 2-structures σ and τ, an *isomorphism* from σ onto τ is a bijection from $V(\sigma)$ onto $V(\tau)$ satisfying for $x, y, v, w \in V(\sigma)$, with $x \neq y$ and $v \neq w$, $(x, y) \equiv_\sigma (v, w)$ if and only if $(f(x), f(y)) \equiv_\tau$

isomorphism

[1.1] In general, a reversible 2-structure is neither symmetric nor asymmetric.

$(f(v), f(w))$. Therefore, given a bijection $f : V(\sigma) \longrightarrow V(\tau)$, f is an isomorphism from σ onto τ if and only if f induces a bijection $\underline{f} : E(\sigma) \longrightarrow E(\tau)$ satisfying for any $v, w \in V(\sigma)$, with $v \neq w$, we have

$$(f(v), f(w))_\tau = \underline{f}((v, w)_\sigma).$$

isomorphic

Two 2-structures are *isomorphic* if there exists an isomorphism from one onto the other.

automorphism
identity function

Let σ be a 2-structure. An *automorphism* of σ is an isomorphism from σ onto itself. For example, the *identity function* $\mathrm{Id}_{V(\sigma)} : V(\sigma) \longrightarrow V(\sigma)$, defined by $\mathrm{Id}_{V(\sigma)}(v) = v$ for every $v \in V(\sigma)$, is an automorphism of σ. The family of the automorphisms of σ, endowed with composition, is the *automorphism group* of σ. It is denoted by $\mathrm{Aut}(\sigma)$. A 2-structure σ is *rigid* if $\mathrm{Aut}(\sigma) = \{\mathrm{Id}_{V(\sigma)}\}$. On the other hand, it is *vertex-transitive* if for any $v, w \in V(\sigma)$, there is $f \in \mathrm{Aut}(\sigma)$ such that $f(v) = w$.

automorphism group
rigid
vertex-transitive
embedding

Lastly, given 2-structures σ and τ, σ *embeds* into τ if σ is isomorphic to a 2-substructure of τ.

graph
vertex set
edge set
complete graph
empty graph
complement

1.2. **Graphs.** A (simple) *graph* G is defined by a *vertex set* $V(G)$ and an *edge set* $E(G)$, where an edge of G is an unordered pair of distinct vertices of G. Such a graph is denoted by $(V(G), E(G))$. For instance, given a nonempty set S, $K_S = (S, \binom{S}{2})$ is the *complete* graph on S whereas $(S, \varnothing)$ is the *empty* graph. With each graph G we associate its *complement* $\overline{G} = (V(G), \binom{V(G)}{2} \setminus E(G))$.

multipartite
bipartite
path
length

A graph G is *multipartite* with a partition P of $V(G)$ if the subgraph $G[X]$ of G induced by X is empty for each $X \in P$. It is *bipartite* when $|P| = 2$. Given $n \geq 2$, the *path* P_n is the graph defined on $V(P_n) = \{0, \dots, n-1\}$ as follows. Given $v, w \in \{0, \dots, n-1\}$, with $v \neq w$, $\{v, w\} \in E(P_n)$ if $|v - w| = 1$ (see Figure 1.1). The *length* of the path P_n is $n - 1$.

FIGURE 1.1. The path P_n

cycle

Given $n \geq 3$, the *cycle* C_n is the graph defined on $V(C_n) = \{0, \dots, n-1\}$ obtained from P_n by adding the edge $\{0, n-1\}$. The length of C_n is n.

neighbour
neighbourhood
degree
connected
component
isolated

Let G be a graph. Given a vertex v of G, a *neighbour* of v is a vertex w of G such that $\{v, w\} \in E(G)$. The *neighbourhood* of v is the set $N_G(v)$ of its neighbours, and $d_G(v) = |N_G(v)|$ is its *degree*. Given a nonempty subset X of $V(G)$, $G[X]$ is *connected* if for any $x, y \in X$, with $x \neq y$, there are elements $x_0, \dots, x_n$ of X such that $x_0 = x$, $x_n = y$, and $\{x_m, x_{m+1}\} \in E(G)$ for every $0 \leq m \leq n - 1$. Given a nonempty subset X of $V(G)$, $G[X]$ is a *component* of G if $G[X]$ is connected, and for any $x \in X$ and $v \in V(G) \setminus X$, $\{x, v\} \notin E(G)$. A vertex v of a graph G is *isolated* if $G[\{v\}]$ is a component of G.

Let G and H be graphs such that $V(G) \cap V(H) = \varnothing$. The *disjoint union* of G and H is the graph $G \oplus H = (V(G) \cup V(H), E(G) \cup E(H))$. If $V(G) \cap V(H) \neq \varnothing$, then we can define $G \oplus H$ up to isomorphism by considering a graph H' such that $H \simeq H'$, and $V(G) \cap V(H') = \varnothing$. *disjoint union*

A graph G is identified with the symmetric 2-structure $\sigma(G)$ defined on $V(\sigma(G)) = V(G)$ as follows. Given $x, y, v, w \in V(\sigma(G))$, with $x \neq y$ and $v \neq w$,

$$(x,y) \equiv_{\sigma(G)} (v,w) \text{ if } \begin{cases} \{x,y\}, \{v,w\} \in E(G) \\ \text{or} \\ \{x,y\}, \{v,w\} \notin E(G). \end{cases}$$

Given a graph G, observe that $\sigma(G) = \sigma(\overline{G})$. A graph is *self-complementary* if it is isomorphic to its complement. Consider a self-complementary graph G. Since $\sigma(G) = \sigma(\overline{G})$, an isomorphism from G onto $\overline{G}$ is an automorphism of $\sigma(G)$. *self-complementary*

1.3. **Digraphs.** A *digraph* D is defined by a *vertex set* $V(D)$ and an *arc set* $A(D)$, where an arc of D is an ordered pair of distinct vertices of D. Such a digraph is denoted by $(V(D), A(D))$. With each digraph D we associate its *dual* $D^\star$ defined on $V(D^\star) = V(D)$ as follows. Given $v, w \in V(D^\star)$, with $v \neq w$, $(v,w) \in A(D^\star)$ if $(w,v) \in A(D)$. Given a vertex v of a digraph D, the *in-neighbourhood* of v is the set $N_D^-(v) = \{w \in V(D) : (w,v) \in A(D)\}$, and its *out-neighbourhood* is the set $N_D^+(v) = \{w \in V(D) : (v,w) \in A(D)\}$. *digraph* *vertex set* *arc set* *dual* *in-neighbourhood* *out-neighbourhood*

A digraph D is identified with the 2-structure $\sigma(D)$ defined on $V(\sigma(D)) = V(D)$ as follows. Given $x, y, v, w \in V(\sigma(D))$, with $x \neq y$ and $v \neq w$,

$$(x,y) \equiv_{\sigma(D)} (v,w) \text{ if } \begin{cases} (x,y), (v,w) \in A(D) \\ \text{or} \\ (x,y), (v,w) \notin A(D). \end{cases}$$

A digraph D is a *tournament* if for any $v, w \in V(D)$, with $v \neq w$, $|A(D) \cap \{(v,w), (w,v)\}| = 1$. It is a *transitive digraph* provided that for any $u, v, w \in V(D)$, if $(u,v) \in A(D)$ and $(v,w) \in A(D)$, then $(u,w) \in A(D)$. A transitive digraph is also called a (strict) *partial order*. With each partial order O, we associate its *comparability graph* $\text{Comp}(O)$ defined on $V(\text{Comp}(O)) = V(O)$ as follows. For any $v, w \in V(\text{Comp}(O))$, with $v \neq w$, $\{v,w\} \in E(\text{Comp}(O))$ if $(v,w) \in A(O)$ or $(w,v) \in A(O)$. A *linear order* is a transitive tournament. Given a nonempty set S of integers, the usual linear order on S is denoted by L_S. Given $m \geq 1$, $L_{\{0,\ldots,m-1\}}$ is also denoted by L_m. Given $n \geq 1$, we consider the tournament T_{2n+1} defined on $V(T_{2n+1}) = \{0, \ldots, 2n\}$ by *tournament* *transitive digraph* *partial order* *comparability graph* *linear order*

$$\begin{cases} T_{2n+1} - (2n) = L_{2n}, \\ (2n, 2m) \in A(T_{2n+1}) \text{ for } 0 \leq m \leq n-1, \\ \text{and} \\ (2m+1, 2n) \in A(T_{2n+1}) \text{ for } 0 \leq m \leq n-1 \text{ (see Figure 1.2).} \end{cases}$$

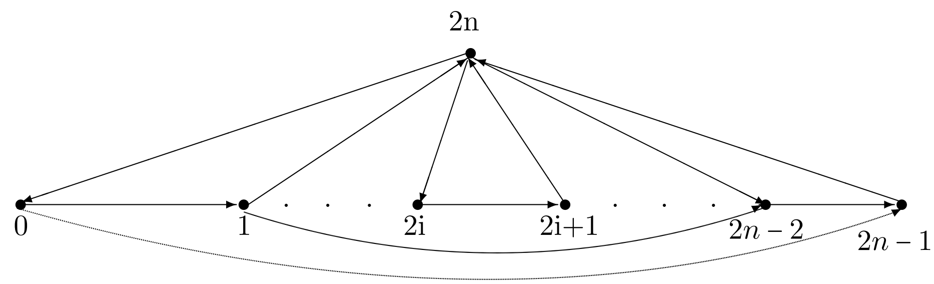

FIGURE 1.2. The tournament T_{2n+1}

linear, e-linear

Consider a reversible 2-structure σ. Given $e \in E(\sigma)$, the 2-structure σ is *linear* or *e-linear* if $(V(\sigma), e)$ is a linear order.

Remark 1.3. Consider an e-linear 2-structure σ, where $e \in E(\sigma)$. We have $(V(\sigma), e)$ is a linear order. Clearly, $(V(\sigma), e^\star)$ is a linear order as well. Since σ is reversible, $e^\star \in E(\sigma)$. Thus, σ is $(e^\star)$-linear, and $E(\sigma) = \{e, e^\star\}$. Moreover, we have $\sigma = \sigma((V(\sigma), e)) = \sigma((V(\sigma), e^\star))$.

2. Connectedness and modules

We use the following notation.

Notation 2.1. Let σ be a 2-structure. For $W, W' \subseteq V(\sigma)$, with $W \cap W' = \varnothing$, $W \longleftrightarrow_\sigma W'$ signifies that $(v, v') \equiv_\sigma (w, w')$ and $(v', v) \equiv_\sigma (w', w)$ for any $v, w \in W$ and $v', w' \in W'$. The negation is denoted by $W \not\longleftrightarrow_\sigma W'$. Given $v \in V(\sigma)$ and $W \subseteq V(\sigma) \setminus \{v\}$, $\{v\} \longleftrightarrow_\sigma W$ is also denoted by $v \longleftrightarrow_\sigma W$. The negation is denoted by $v \not\longleftrightarrow_\sigma W$.

Given $W, W' \subseteq V(\sigma)$ such that $W \longleftrightarrow_\sigma W'$, $(W, W')_\sigma$ denotes the equivalence class $(w, w')_\sigma$ of (w, w'), where $w \in W$ and $w' \in W'$. Furthermore, set

$$[W, W']_\sigma = ((W, W')_\sigma, (W', W)_\sigma).$$

Lastly, given $v \in V(\sigma)$ and $W \subseteq V(\sigma) \setminus \{v\}$ such that $\{v\} \longleftrightarrow_\sigma W$, $(\{v\}, W)_\sigma$ is also denoted by $(v, W)_\sigma$, $(W, \{v\})_\sigma$ is denoted by $(W, v)_\sigma$, and $[\{v\}, W]_\sigma$ is denoted by $[v, W]_\sigma$.

2.1. **Different types of connectedness.** Let σ be a 2-structure. With each $(e, f) \in E(\sigma) \times E(\sigma)$, we associate a type of connectedness. Given $(e, f) \in E(\sigma) \times E(\sigma)$, we require that if σ is not connected in terms of the type associated with (e, f), then the ordered pairs of vertices that are not in the same component, belong to e or f.

Given a 2-structure σ, consider $e, f \in E(\sigma)$. We define on $V(\sigma)$ the equivalence relation $\approx_{(e,f)}$ in the following way. Given $v, w \in V(\sigma)$, $v \approx_{(e,f)} w$ if $v = w$ or $v \neq w$ and there exist sequences $v_0, \ldots, v_m$ and $w_0, \ldots, w_n$ of vertices of σ satisfying

- $v_0 = v$ and $v_m = w$;
- for $0 \leq i \leq m - 1$, $[v_i, v_{i+1}]_\sigma \neq (e, f)$;
- $w_0 = w$ and $w_n = v$;
- for $0 \leq j \leq n - 1$, $[w_j, w_{j+1}]_\sigma \neq (e, f)$.

Note that we do not need the second sequence $w_0, \ldots, w_p$ when $e = f$. Moreover, for $0 \leq i \leq m - 1$, $[v_{i+1}, v_i]_\sigma \neq (f, e)$, and for $0 \leq j \leq n - 1$, $[w_{j+1}, w_j]_\sigma \neq (f, e)$. By considering the sequences $v = w_n, \ldots, w_0 = w$ and $w = v_m, \ldots, v_0 = v$, we obtain $v \approx_{(f,e)} w$. Consequently, for any $e, f \in E(\sigma)$ and $v, w \in V(\sigma)$, we have $v \approx_{(e,f)} w$ if and only if $w \approx_{(f,e)} v$.

Definition 2.2. Let σ be a 2-structure. Consider $e, f \in E(\sigma)$. The equivalence classes of $\approx_{(e,f)}$ are called the *$\{e, f\}$-components* of σ. The family of the $\{e, f\}$-components of σ is denoted by $\mathcal{C}_{\{e,f\}}(\sigma)$. Lastly, we say that the 2-structure σ is *$\{e, f\}$-connected* if it admits a unique $\{e, f\}$-component. Moreover, the 2-structure σ is *connected* if σ is $\{e, f\}$-connected for all $e, f \in E(\sigma)$.

$\{e, f\}$-components
$\{e, f\}$-connected
connected

Remark 2.3. First, consider a graph G. Set

$$\begin{cases} e_1 = \{(v,w) : \{v,w\} \in E(G)\} \\ \text{and} \\ e_0 = \{(v,w) : \{v,w\} \notin E(G)\}. \end{cases}$$

We have

$$E(\sigma(G)) = \{e_0, e_1\}.$$

The $\{e_0\}$-components of $\sigma(G)$ are exactly the components of G, whereas the $\{e_1\}$-components of $\sigma(G)$ are exactly the components of $\overline{G}$. Since $\sigma(G)$ is symmetric, $\sigma(G)$ is $\{e_0, e_1\}$-connected.

Second, consider a tournament T. We have

$$E(\sigma(T)) = \{A(T), A(T)^\star\}.$$

The $\{A(T), A(T)^\star\}$-components of $\sigma(T)$ are exactly the strongly connected components of T. Since $\sigma(T)$ is asymmetric, $\sigma(T)$ is $\{A(T)\}$-connected, and $\{A(T)^\star\}$-connected.

The following lemma is established in [23] for binary structures, that is, labeled 2-structures [14].

Lemma 2.4. *Given a 2-structure σ, consider $e, f \in E(\sigma)$. Let X be an $\{e,f\}$-component of σ. For each $v \in V(\sigma) \smallsetminus X$, we have $v \longleftrightarrow_\sigma X$. Precisely, for each $y \in V(\sigma) \smallsetminus X$, we have $[v, X]_\sigma = (e,f)$ or (f,e).*

Proof. Let $v \in V(\sigma) \smallsetminus X$. Consider $x \in X$. Since $v \not\approx_{(e,f)} x$, we have

$$[x, v]_\sigma = (e,f) \text{ or } (f,e). \tag{2.1}$$

For a contradiction, suppose that there exist $x, y \in X$ such that $[x,v]_\sigma \neq [y,v]_\sigma$. It follows from (2.1) that $e \neq f$. Morever, by interchanging x and y if necessary, we can assume that $[x,v]_\sigma = (e,f)$ and $[y,v]_\sigma = (f,e)$. Hence $[v,x]_\sigma \neq (e,f)$ and $[y,v]_\sigma \neq (e,f)$. Since $x \approx_{(e,f)} y$, there exists a sequence $x_0, \ldots, x_m$ satisfying

- $x_0 = x$ and $x_m = y$;
- for $0 \leq i \leq m-1$, $[x_i, x_{i+1}]_\sigma \neq (e,f)$.

By considering the sequences $x_0, \ldots, x_m, v$ and v, x, we obtain $x \approx_{(e,f)} v$, which contradicts $v \notin X$. Therefore, $[x,v]_\sigma = [y,v]_\sigma$ for any $x, y \in X$. It follows from (2.1) that $[v, X]_\sigma = (e,f)$ or (f,e). □

2.2. **Modules and quotient.** Given Lemma 2.4, we introduce the following definition. Given a 2-structure σ, a subset M of $V(\sigma)$ is a *module*[2.1] of

module

[2.1] This notion of a module generalizes the usual notion of module for a graph [34]. One also uses *homogeneous set* [11, 28] for graphs. For a partial order, Gallai [18] uses *closed set* (*geschlossen Menge* in German), and Kelly [26] uses *autonomous set*. For a linear order L, the notion of a module of $\sigma(L)$ coincide with the classical notion of an interval of L. For relations and multirelations [17], Fraïssé introduced the notion of an *interval* [16]. It is also used for digraphs [22, 33] . The notion of clan was introduced by Ehrenfeucht and Rozenberg for 2-structures [13].

σ if for each $v \in V(\sigma) \setminus M$, we have $v \longleftrightarrow_\sigma M$. The classical properties of modules follow.

Proposition 2.5. *Let σ be a 2-structure.*

(M1) *$\varnothing$, $V(\sigma)$, and $\{v\}$, where $v \in V(\sigma)$, are modules of σ.*
(M2) *Given $W \subseteq V(\sigma)$, if M is a module of σ, then $M \cap W$ is a module of $\sigma[W]$.*
(M3) *Let M be a module of σ. For every $N \subseteq M$, N is a module of $\sigma[M]$ if and only if N is a module of σ.*
(M4) *For any modules M and N of σ, $M \cap N$ is a module of σ.*
(M5) *Given modules M and N of σ, if $M \cap N \neq \varnothing$, then $M \cup N$ is a module of σ.*
(M6) *Given modules M and N of σ, if $M \setminus N \neq \varnothing$, then $N \setminus M$ is a module of σ.*
(M7) *Given modules M and N of σ, if $M \cap N = \varnothing$, then $M \longleftrightarrow_\sigma N$.*

Proof. It is easy to verify that the first assertion holds. For the second one, consider a subset W of $V(\sigma)$, and a module M of σ. Let $v \in W \setminus M$. Clearly, $v \in V(\sigma) \setminus M$. Since M is a module of σ, we have $v \longleftrightarrow_\sigma M$, so $v \longleftrightarrow_\sigma M \cap W$.

For the third assertion, consider a module M of σ and a subset N of M. By the preceding assertion, if N is a module of σ, then $M \cap N = N$ is a module of $\sigma[M]$. Conversely, suppose that N is a module of $\sigma[M]$, and consider $v \in V(\sigma) \setminus N$. We have $v \in V(\sigma) \setminus M$ or $v \in M \setminus N$. In the first instance, since M is a module of σ, $v \longleftrightarrow_\sigma M$, and hence $v \longleftrightarrow_\sigma N$. In the second instance, $v \longleftrightarrow_\sigma N$ because N is a module of $\sigma[M]$.

Now, let M and N be modules of σ.

To verify that $M \cap N$ is a module of σ, consider $v \in V(B) \setminus (M \cap N)$. We have $v \in (V(\sigma) \setminus M) \cup (V(\sigma) \setminus N)$. By interchanging M and N if necessary, assume that $v \in V(\sigma) \setminus M$. As M is a module of σ, $v \longleftrightarrow_\sigma M$, and hence $v \longleftrightarrow_\sigma M \cap N$.

To show that $M \cup N$ is a module of σ, suppose that there exists $x \in M \cap N$. Let $v \in V(\sigma) \setminus (M \cup N)$. Since M is a module of σ, $x \in M$ and $v \in V(\sigma) \setminus M$, we have $[v, M]_\sigma = [v, x]_\sigma$. Similarly, we have $[v, N]_\sigma = [v, x]_\sigma$. It follows that $[v, M \cup N]_\sigma = [v, x]_\sigma$. Thus $v \longleftrightarrow_\sigma M \cup N$.

Lastly, to prove that $N \setminus M$ is a module of σ, suppose that there exists $x \in M \setminus N$. Let $v \in V(\sigma) \setminus (N \setminus M)$. Clearly, $v \in (V(\sigma) \setminus N) \cup (M \cap N)$. First, suppose that $v \in V(\sigma) \setminus N$. Since N is a module of σ, $v \longleftrightarrow_\sigma N$, so $v \longleftrightarrow_\sigma N \setminus M$. Second, suppose that $v \in M \cap N$. Consider $u, u' \in N \setminus M$. We have to verify that $v \longleftrightarrow_\sigma \{u, u'\}$. Since M is a module of σ, $x, v \in M$ and $u \in V(\sigma) \setminus M$, we have $[v, u]_\sigma = [x, u]_\sigma$. Similarly, $[v, u']_\sigma = [x, u']_\sigma$. Moreover, we have $[x, u]_\sigma = [x, u']_\sigma$ because N is a module of σ with $u, u' \in N$ and $x \in V(\sigma) \setminus N$. It follows that $[v, u]_\sigma = [v, u']_\sigma$, so $v \longleftrightarrow_\sigma \{u, u'\}$.

Finally, let M and N be nonempty modules of σ such that $M \cap N = \varnothing$. Consider $x \in M$ and $y \in N$. For any $v \in M$ and $w \in N$, we have $[v, w]_\sigma = [x, w]_\sigma$ because M is a module of σ with $x, v \in M$ and $w \in V(\sigma) \setminus M$. Furthermore, $[x, w]_\sigma = [x, y]_\sigma$ because N is a module of σ with $y, w \in N$ and

$x \in V(\sigma) \setminus N$. Therefore, $[v, w]_\sigma = [x, y]_\sigma$ for any $v \in M$ and $w \in N$. Thus $M \longleftrightarrow_\sigma N$. □

trivial module
indecomposable
decomposable
prime

Let σ be a 2-structure. Following assertion (M1) of Proposition 2.5, the modules $\varnothing$, $V(\sigma)$, and $\{v\}$, where $v \in V(\sigma)$, are called *trivial modules*. A 2-structure is *indecomposable* if all its modules are trivial [2.2]. Otherwise, it is *decomposable*. Observe that a 2-structure, with at most two vertices, is indecomposable. This leads us to the following notion. A 2-structure σ is *prime* if σ is indecomposable, with $v(\sigma) \geq 3$.

For instance, if σ is a constant 2-structure, then all the subsets of $V(\sigma)$ are modules of σ. Hence, a constant 2-structure σ is decomposable if $v(\sigma) \geq 3$. The same holds for linear 2-structures. Instead, consider a linear 2-structure σ such that $v(\sigma) \geq 3$. By Remark 1.3, there exists a linear order L such that $\sigma = \sigma(L)$. As above mentioned, the intervals of L are modules of σ. By denoting by v and w the first two vertices of L, we obtain that $\{v, w\}$ is an interval of L. Thus, $\{v, w\}$ is a nontrivial module of σ, so σ is decomposable.

Fact 2.6. *For $n \geq 4$, the path P_n (see Figure 1.1) is prime.*

Proof. Let M be a module of P_n with $|M| \geq 2$. We have to show that $M = \{0, \dots, n-1\}$. Consider $p, q \in M$ such that $0 < p < q$. Since $\{p-1, p\} \in E(P_n)$ and $\{p-1, q\} \notin E(P_n)$, we have $p - 1 \in M$. In the same manner, if $0 < p - 1$, then $p - 2 \in M$. It follows that $\{0, \dots, p\} \subseteq M$. Similarly $\{q, \dots, n-1\} \subseteq M$. Therefore

$$\{0, \dots, p\} \cup \{q, \dots, n-1\} \subseteq M.$$

Now, consider $p, q \in M$ such that $p < q$ and $M \cap \{p, \dots, q\} = \{p, q\}$. Suppose for a contradiction that $p < q - 1$. Since $\{p, p+1\} \in E(P_n)$ and $p + 1 \notin M$, we have $\{x, p+1\} \in E(P_n)$ for every $x \in M$. Therefore $M \subseteq \{p, p+2\}$. Since $\{0, \dots, p\} \cup \{q, \dots, n-1\} \subseteq M$, we obtain $p = 0$, $q = n - 1 = 2$. Since $n \geq 4$, we have $p = q - 1$. Thus $M = \{0, \dots, n-1\}$. □

Fact 2.7. *For $n \geq 1$, the tournament T_{2n+1} (see Figure 1.2) is prime.*

Proof. Consider a module M of T_{2n+1} such that $|M| \geq 2$. We have to show that $M = \{0, \dots, 2n\}$. By Proposition 2.5, $M \cap \{0, \dots, 2n-1\}$ is a module of $T_{2n+1}[\{0, \dots, 2n-1\}] = L_{2n}$. Since $M \cap \{0, \dots, 2n-1\} \neq \varnothing$, there exist $p, q \in \{0, \dots, 2n-1\}$ such that $p \leq q$ and $M \cap \{0, \dots, 2n-1\} = \{p, \dots, q\}$. If $p = q$, then $2n \in X$ because $|M| \geq 2$. If $p < q$, then $2n \in X$ because $2n \not\longleftrightarrow_\sigma \{2m, 2m+1\}$ for $0 \leq m \leq n-1$. Thus $2n \in M$. Since $(2n, 0) \in A(T_{2n+1})$ and $(0, r) \in A(T_{2n+1})$ for $1 \leq r \leq 2n-1$, we have $0 \in M$. Since $(2n-1, 2n) \in A(T_{2n+1})$ and $(r, 2n-1) \in A(T_{2n+1})$ for $0 \leq r \leq 2n-2$, we have $2n - 1 \in M$. Consequently, $p = 0$, $q = 2n - 1$ and $M = \{0, \dots, 2n\}$. □

modular partition

Let σ be a 2-structure. For any $e, f \in E(\sigma)$, the $\{e, f\}$-components of σ are modules of σ by Lemma 2.4. Hence, the family $\mathcal{C}_{\{e,f\}}(\sigma)$ (see Definition 2.2) realizes a partition of $V(\sigma)$ in modules of σ. Generally, we introduce the following definition. A partition P of $V(\sigma)$ is a *modular partition* of σ if all

[2.2] Ehrenfeucht et al. [13, 14] use *primitive* instead of *indecomposable*.

the blocks of P are modules of σ. Given a modular partition P of σ, it follows from assertion (M7) of Proposition 2.5 that for distinct $X, Y \in P$, we have $X \longleftrightarrow_\sigma Y$. Hence, the blocks of P can be considered as the vertices of a new 2-structure defined in the following manner. With each modular partition P of σ, we associate the *quotient* σ/P of σ by P defined on $V(\sigma/P) = P$ as follows. Given $X, X', Y, Y' \in V(\sigma/P)$, with $X \neq X'$ and $Y \neq Y'$, *quotient*

$$(X, X') \equiv_{(\sigma/P)} (Y, Y') \text{ if } (x, x') \equiv_\sigma (y, y'),$$

where $x \in X$, $x' \in X'$, $y \in Y$, and $y' \in Y'$.

Let σ be a 2-structure. Given $e, f \in E(\sigma)$, $\mathcal{C}_{\{e,f\}}(\sigma)$ is a modular partition of σ as mentioned above. We characterize the quotient $\sigma/\mathcal{C}_{\{e,f\}}(\sigma)$ as follows.

Proposition 2.8 (Ille [23])**.** *Let σ be a 2-structure. For every $e \in E(\sigma)$, $\sigma/\mathcal{C}_{\{e\}}(\sigma)$ is constant. Moreover, for distinct $e, f \in E(\sigma)$, $\sigma/\mathcal{C}_{\{e,f\}}(\sigma)$ is linear.*

Proof. To begin, consider $e \in E(\sigma)$. Given distinct $X, Y \in \mathcal{C}_{\{e\}}(\sigma)$, it follows from Lemma 2.4 that

$$(X, Y)_\sigma = e. \tag{2.2}$$

Consider $X, X', Y, Y' \in \mathcal{C}_{\{e\}}(\sigma)$, with $X \neq X'$ and $Y \neq Y'$. Let $x \in X$, $x' \in X'$, $y \in Y$, and $y' \in Y'$. It follows from (2.2) that $(x, x')_\sigma = e$ and $(y, y')_\sigma = e$, so $(x, x') \equiv_\sigma (y, y')$. By the definition of quotient, we have $(X, X') \equiv_{(\sigma/\mathcal{C}_{\{e\}}(\sigma))} (Y, Y')$. Hence $\sigma/\mathcal{C}_{\{e\}}(\sigma)$ is constant.

Now, consider distinct $e, f \in E(\sigma)$. Given distinct $X, Y \in \mathcal{C}_{\{e,f\}}(\sigma)$, it follows from Lemma 2.4 and assertion (M7) of Proposition 2.5 that

$$[X, Y]_\sigma = (e, f) \text{ or } (f, e). \tag{2.3}$$

Set

$$\begin{cases} e/\mathcal{C}_{\{e,f\}}(\sigma) = \{(X, Y) \in \mathcal{C}_{\{e,f\}}(\sigma) \times \mathcal{C}_{\{e,f\}}(\sigma) : X \neq Y, (X, Y)_\sigma = e\} \\ \text{and} \\ f/\mathcal{C}_{\{e,f\}}(\sigma) = \{(X, Y) \in \mathcal{C}_{\{e,f\}}(\sigma) \times \mathcal{C}_{\{e,f\}}(\sigma) : X \neq Y, (X, Y)_\sigma = f\}. \end{cases}$$

We prove that

$$E(\sigma/\mathcal{C}_{\{e,f\}}(\sigma)) = \{e/\mathcal{C}_{\{e,f\}}(\sigma), f/\mathcal{C}_{\{e,f\}}(\sigma)\}. \tag{2.4}$$

Consider $X, X', Y, Y' \in \mathcal{C}_{\{e,f\}}(\sigma)$, with $X \neq X'$ and $Y \neq Y'$. Let $x \in X$, $x' \in X'$, $y \in Y$, and $y' \in Y'$. First, suppose that $(X, X') \equiv_{(\sigma/\mathcal{C}_{\{e\}}(\sigma))} (Y, Y')$. By the definition of quotient, we have $(x, x') \equiv_\sigma (y, y')$, so $(x, x')_\sigma = (y, y')_\sigma$. By (2.3), $(x, x')_\sigma, (y, y')_\sigma \in \{e, f\}$. Thus, either $(x, x')_\sigma = (y, y')_\sigma = e$ or $(x, x')_\sigma = (y, y')_\sigma = f$. In the first instance, we obtain $(X, X')_\sigma = (Y, Y')_\sigma = e$, and hence $(X, X'), (Y, Y') \in e/\mathcal{C}_{\{e,f\}}(\sigma)$. In the second one, we have $(X, X'), (Y, Y') \in f/\mathcal{C}_{\{e,f\}}(\sigma)$. Second, suppose that $(X, X'), (Y, Y') \in e/\mathcal{C}_{\{e,f\}}(\sigma)$. We have $(X, X')_\sigma = (Y, Y')_\sigma = e$. Thus $(x, x')_\sigma = (y, y')_\sigma = e$, so

$(x,x') \equiv_\sigma (y,y')$. By the definition of quotient, we have $(X,X') \equiv_{(\sigma/\mathcal{C}_{\{e\}}(\sigma))} (Y,Y')$. Similarly, we have

$$(X,X') \equiv_{(\sigma/\mathcal{C}_{\{e\}}(\sigma))} (Y,Y')$$

when $(X,X'),(Y,Y') \in f/\mathcal{C}_{\{e,f\}}(\sigma)$. Consequently (2.4) holds.

We continue by showing that

$$(e/\mathcal{C}_{\{e,f\}}(\sigma))^\star = f/\mathcal{C}_{\{e,f\}}(\sigma). \tag{2.5}$$

Consider distinct $X,Y \in \mathcal{C}_{\{e,f\}}(\sigma)$. Suppose that $(X,Y) \in (e/\mathcal{C}_{\{e,f\}}(\sigma))^\star$. We have $(Y,X) \in e/\mathcal{C}_{\{e,f\}}(\sigma)$, so $(Y,X)_\sigma = e$. By (2.2), $[X,Y]_\sigma = (e,f)$ or (f,e). Since $(Y,X)_\sigma = e$, we obtain $(X,Y)_\sigma = f$, so $(X,Y) \in f/\mathcal{C}_{\{e,f\}}(\sigma)$. Conversely, suppose that $(X,Y) \in f/\mathcal{C}_{\{e,f\}}(\sigma)$. We have $(X,Y)_\sigma = f$. By (2.2), $[X,Y]_\sigma = (e,f)$ or (f,e). Hence $(Y,X)_\sigma = e$, so $(Y,X) \in e/\mathcal{C}_{\{e,f\}}(\sigma)$, that is, $(X,Y) \in (e/\mathcal{C}_{\{e,f\}}(\sigma))^\star$. Consequently (2.5) holds.

To conclude, we have to prove that $(\mathcal{C}_{\{e,f\}}(\sigma), e/\mathcal{C}_{\{e,f\}}(\sigma))$ is a linear order. It follows from (2.4) and (2.5) that $(\mathcal{C}_{\{e,f\}}(\sigma), e/\mathcal{C}_{\{e,f\}}(\sigma))$ is a tournament. Thus, we have to verify that $(\mathcal{C}_{\{e,f\}}(\sigma), e/\mathcal{C}_{\{e,f\}}(\sigma))$ is transitive. Consider $X,Y,Z \in \mathcal{C}_{\{e,f\}}(\sigma)$ such that $(X,Y),(Y,Z) \in e/\mathcal{C}_{\{e,f\}}(\sigma)$. Since $(X,Y),(Y,Z) \in e/\mathcal{C}_{\{e,f\}}(\sigma)$, we have $(X,Y)_\sigma = (Y,Z)_\sigma = e$. It follows from (2.3) that $[X,Y]_\sigma = [Y,Z]_\sigma = (e,f)$. Since $e \neq f$, we have $X \neq Z$. Let $x \in X$, $y \in Y$, and $z \in Z$. We obtain $[z,y]_\sigma = [y,x]_\sigma = (f,e)$. Since $e \neq f$, we have $[z,y]_\sigma \neq (e,f)$ and $[y,x]_\sigma \neq (e,f)$. Since $X \neq Z$, we have $x \not\preccurlyeq_{(e,f)} z$. It follows that $[x,z]_\sigma = (e,f)$. By (2.3), $[X,Z]_\sigma = (e,f)$. Consequently, $(\mathcal{C}_{\{e,f\}}(\sigma), e/\mathcal{C}_{\{e,f\}}(\sigma))$ is transitive. □

Notation 2.9. Given a 2-structure σ, consider a partition P of $V(\sigma)$. With $W \subseteq V(\sigma)$, we associate the set W/P of the blocks X of P such that $X \cap W \neq \varnothing$. Moreover, with $Q \subseteq P$, we associate the union $\cup Q$ of the elements of Q.

In the following result, we compare the modules of a 2-structure with those of its quotients.

Lemma 2.10. *Given a 2-structure σ, consider a modular partition P of σ.*
1. *If M is a module of σ, then M/P is a module of σ/P.*
2. *If Q is a module of σ/P, then $\cup Q$ a module of σ.*

Proof. First, we consider a module M of σ. Consider $X \in P \setminus (M/P)$, and $Y,Z \in M/P$. Let $x \in X$. Since $Y,Z \in M/P$, there exist $y,z \in M$ such that $y \in Y \cap M$ and $z \in Z \cap M$. Since M is a module of σ, we have $(x,y) \equiv_\sigma (x,z)$ and $(y,x) \equiv_\sigma (z,x)$. By the definition of quotient, $(X,Y) \equiv_{(\sigma/P)} (X,Z)$ and $(Y,X) \equiv_{(\sigma/P)} (Z,X)$. Thus, M/P is a module of σ/P.

Second, let Q be a module of σ/P. Consider $v \in V(\sigma) \setminus (\cup Q)$, and $y,z \in (\cup Q)$. Since $v \in V(\sigma) \setminus (\cup Q)$, there exist $X \in P \setminus Q$ such that $v \in X$. Furthermore, since $y,z \in (\cup Q)$, there exist $Y,Z \in Q$ such that $y \in Y$ and $z \in Z$. Since Q is a module of σ/P, $Y,Z \in Q$ and $X \in P \setminus Q$, we have $(X,Y) \equiv_{\sigma/P} (X,Z)$ and $(Y,X) \equiv_{\sigma/P} (Z,X)$. It follows from the definition of quotient that $(v,y) \equiv_\sigma (v,z)$ and $(y,v) \equiv_\sigma (z,v)$. Therefore, $\cup Q$ is a module of σ. □

2.3. **Modular cuts.** Given a 2-structure σ, we continue the examination of the properties of the $\{e,f\}$-components of σ, where $e,f \in E(\sigma)$. The next result is a consequence of Proposition 2.8.

Corollary 2.11 (Ille [23])**.** *Given a 2-structure σ, consider $e,f \in E(\sigma)$. If σ is not $\{e,f\}$-connected, then there exists $X \in \mathcal{C}_{\{e,f\}}(\sigma)$ such that $[X, V(\sigma) \setminus X]_\sigma = (e,f)$.*

Proof. If $e = f$, then it follows from Proposition 2.8 that $[X, V(\sigma) \setminus X]_\sigma = (e,e)$ for every $X \in \mathcal{C}_{\{e\}}(\sigma)$. Suppose that $e \neq f$. By Proposition 2.8 and Remark 1.3, there exists a linear order L defined on $V(L) = \mathcal{C}_{\{e,f\}}(\sigma)$ such that $\sigma/\mathcal{C}_{\{e,f\}}(\sigma) = \sigma(L)$. The least vertex Y of L satisfies $[Y, V(\sigma) \setminus Y]_\sigma = (e,f)$ or (f,e). Similarly, the greatest vertex Z of L satisfies $[Z, V(\sigma) \setminus Z]_\sigma = (e,f)$ or (f,e). Since $[Y,Z]_\sigma \neq [Z,Y]_\sigma$, we have $[Y, V(\sigma) \setminus Y]_\sigma \neq [Z, V(\sigma) \setminus Z]_\sigma$. Therefore, there exists $X \in \{Y,Z\}$ such that $[X, V(\sigma) \setminus X]_\sigma = (e,f)$. □

In Corollary 2.11, observe that X and $V(\sigma) \setminus X$ are modules of σ. This leads us to the following definition. Given a 2-structure σ, a subset X of $V(\sigma)$ is a *modular cut*[2.3] of σ if X and $V(\sigma) \setminus X$ are modules of σ. For instance, $\varnothing$ and $V(\sigma)$ are modular cuts of σ, called *trivial modular cuts.* A 2-structure is *uncuttable* if all its modular cuts are trivial, otherwise it is *cuttable*[2.4]. The following characterization of uncuttable 2-structures follows from assertion (M7) of Proposition 2.5 and from Corollary 2.11.

modular cut
trivial modular cut
uncuttable
cuttable

Proposition 2.12 (Ille [23])**.** *A 2-structure is uncuttable if and only if it is connected.*

Proof. Let σ be a 2-structure. To begin, suppose that σ is not connected. There exist $e,f \in E(\sigma)$ such that σ is not $\{e,f\}$-connected. By Corollary 2.11, σ admits a nontrivial modular cut among its $\{e,f\}$-components. Hence σ is cuttable.

Conversely, suppose that σ is cuttable, and consider a nontrivial modular cut X of σ. Since X is a nontrivial modular cut of σ, X and $V(\sigma) \setminus X$ are nonempty modules of σ. It follows from assertion (M7) of Proposition 2.5 that there exist $e,f \in E(\sigma)$ such that $[X, V(\sigma) \setminus X]_\sigma = (e,f)$. Consequently, there is no sequence $x_0, \ldots, x_n$ satisfying $x_0 \in X$, $x_n \in V(\sigma) \setminus X$, and $[x_m, x_{m+1}]_\sigma \neq (e,f)$ for $m \in \{0, \ldots, n-1\}$. Thus σ is not $\{e,f\}$-connected, so σ is not connected. □

[2.3] Also called *cut* in [8] for digraphs.

[2.4] Ehrenfeucht, Harju, and Rozenberg [14] say that a 2-structure has the *2-block* property if it is cuttable.

2.4. **Strong modules and Gallai's decomposition.** Let σ be a 2-structure. If σ is prime, then $\{V(\sigma)\}$ and $\{\{v\} : v \in V(\sigma)\}$ are the only modular partitions of σ. On the other hand, if σ is constant, then every partition of $V(\sigma)$ is a modular partition of σ. Hence, in order to obtain a successful modular decomposition process, we have to associate intrinsically a modular partition with each 2-structure and to characterize the corresponding quotient. Furthermore, for the efficiency of the process, we require that if we repeat the process a second time, we would get an isomorphic quotient. For instance, consider a binary structure σ, and suppose that σ is not $\{e\}$-connected, where $e \in E(\sigma)$. First, given Lemma 2.4, we can associate with σ the modular partition $\mathcal{C}_{\{e\}}(\sigma)$. By Proposition 2.8, the corresponding quotient $\sigma/\mathcal{C}_{\{e\}}(\sigma)$ is constant. Set

$$\tau = \sigma/\mathcal{C}_{\{e\}}(\sigma),$$

and

$$\varepsilon = (\mathcal{C}_{\{e\}}(\sigma) \times \mathcal{C}_{\{e\}}(\sigma)) \smallsetminus \{(X, X) : X \in \mathcal{C}_{\{e\}}(\sigma)\}.$$

Since τ is constant, we have $E(\tau) = \{\varepsilon\}$. Moreover, $|\mathcal{C}_{\{e\}}(\sigma)| \geq 2$ because σ is not $\{e\}$-connected. Thus τ is not $\{\varepsilon\}$-connected. Second, associate with τ the family $\mathcal{C}_{\{\varepsilon\}}(\tau)$ of its $\{\varepsilon\}$-components. Since $E(\tau) = \{\varepsilon\}$, the $\{\varepsilon\}$-components of τ are reduced to singletons. Therefore, the quotient of $\tau/\mathcal{C}_{\{\varepsilon\}}(\tau)$ is isomorphic to τ. To proceed for any 2-structure σ, we return to the examination of the properties of the $\{e, f\}$-components of σ, where $e, f \in E(\sigma)$.

Lemma 2.13 (Ille [23]). *Given a 2-structure σ, consider an $\{e, f\}$-component X of σ, where $e, f \in E(\sigma)$. For every module M of σ, if $X \cap M \neq \varnothing$, then $X \subseteq M$ or $M \subseteq X$.*

Proof. Let M be a module of σ such that $X \cap M \neq \varnothing$ and $X \smallsetminus M \neq \varnothing$. We have to show that $M \subseteq X$. Consider $x \in X \smallsetminus M$ and $y \in X \cap M$. Since X is an $\{e, f\}$-component of σ containing x and y, there exist sequences $x = x_0, \dots, x_p = y$ and $y = y_0, \dots, y_q = x$ of elements of X such that for $0 \leq m \leq p-1$, $[x_m, x_{m+1}]_\sigma \neq (e, f)$, and for $0 \leq m \leq q-1$, $[y_m, y_{m+1}]_\sigma \neq (e, f)$. Since $x_0 \notin M$ and $x_p \in M$, there exists $m \in \{0, \dots, p-1\}$ such that $x_m \in X \smallsetminus M$ and $x_{m+1} \in X \cap M$. Similarly, since $y_0 \in M$ and $y_q \notin M$, there exists $n \in \{0, \dots, q-1\}$ such that $y_n \in X \cap M$ and $y_{n+1} \in X \smallsetminus M$. Now, let $v \in M$. Since M is a module of σ, $x_{m+1}, v \in M$ and $x_m \notin M$, we have $[x_m, x_{m+1}]_\sigma = [x_m, v]_\sigma$. Hence $[x_m, v]_\sigma \neq (e, f)$. Since $y_n, v \in M$ and $y_{n+1} \notin M$, $[y_n, y_{n+1}]_\sigma = [v, y_{n+1}]_\sigma$. Thus $[v, y_{n+1}]_\sigma \neq (e, f)$. By considering the sequences $x = x_0, \dots, x_m, v$ and $v, y_{n+1}, \dots, y_q = x$, we obtain $x \approx_{(e,f)} v$. It follows that $v \in X$. Therefore $M \subseteq X$. □

strong module

This result leads us to introduce the following definition. Given a 2-structure σ, a subset M of $V(\sigma)$ is a *strong module* [2.5] of σ provided that

[2.5] Also called *prime module* in [14] for 2-structures, and *strong interval* for digraphs.

M is a module of σ, and for every module N of σ, we have

$$\text{if } M \cap N \neq \varnothing, \text{ then } M \subseteq N \text{ or } N \subseteq M.$$

Given $e, f \in E(\sigma)$, it follows from Lemma 2.13 that each $\{e, f\}$-component of σ is a strong module of σ. As for modules, $\varnothing$, $V(\sigma)$ and $\{v\}$, $v \in V(\sigma)$, are strong modules of σ, called *trivial strong modules*. A 2-structure is *primitive* [2.6] if all its strong modules are trivial. Three types of primitive 2-structures occur. *primitive*

Lemma 2.14. *Given a 2-structure σ, if σ is prime, constant, or linear, then σ is primitive.*

Proof. If σ is indecomposable, then all its modules are trivial, and hence all its strong modules are also. Therefore, if σ is prime, then σ is primitive.

Now, suppose that σ is constant or linear. Recall that a 2-structure with at most 2 vertices is indecomposable. Hence, suppose also that $v(\sigma) \geq 3$. To show that σ is primitive, it suffices to verify that every nontrivial module M of σ is not strong, that is, there exists a module N of σ such that $M \cap N \neq \varnothing$, $M \setminus N \neq \varnothing$ and $N \setminus M \neq \varnothing$.

Suppose that σ is constant. As previously observed, any subset of $V(\sigma)$ is a module of σ. Consider distinct $x, y \in M$, and $v \in V(\sigma) \setminus M$. The module $\{x, v\}$ of σ satisfies $x \in M \cap \{x, v\}$, $y \in M \setminus \{x, v\}$ and $v \in \{x, v\} \setminus M$.

Lastly, suppose that σ is linear. By Remark 1.3, there exists a linear order L defined on $V(L) = V(\sigma)$ such that $\sigma = \sigma(L)$. Recall that the modules of σ are exactly the intervals of L. Hence, M is a nontrivial interval of L. Up to isomorphism, we can assume that $L = L_n$, where $n \geq 3$. Since M is a nontrivial interval of L, $M = [p, q]$, where $0 \leq p < q \leq n-1$ and $(p, q) \neq (0, n-1)$. Observe that $\sigma = \sigma(L^\star)$ as well. Thus, by considering $L^\star$ instead of L if necessary, we can assume that $p \geq 1$. To conclude, it suffices to consider for N the interval $[0, p]$ of L. □

The analogue of Lemma 2.10 for strong modules follows.

Lemma 2.15. *Given a 2-structure σ, consider a modular partition P of σ.*

(1) *If M is a strong module of σ, then M/P is a strong module of σ/P.*
(2) *Suppose that all the blocks of P are strong modules of σ. If Q is a strong module of σ/P, then $\cup Q$ is a strong module of σ.*

Proof. First, let M be a strong module of σ. By Lemma 2.10, M/P is a module of σ/P. Consider a module Q of σ/P such that $Q \cap (M/P) \neq \varnothing$. If $|M/P| = 1$, that is, if there is $X \in P$ such that $M \subseteq X$, then $M/P = \{X\}$, so $X \in Q$ and $M/P \subseteq Q$. Hence suppose that $|M/P| \geq 2$. For each $X \in M/P$, we have $M \cap X \neq \varnothing$ and $M \setminus X \neq \varnothing$. Since M is a strong module of σ, $X \subseteq M$. Consequently $M = \cup(M/P)$. By Lemma 2.10, $\cup Q$ is a module of σ. Clearly $M \cap (\cup Q) \neq \varnothing$ because $Q \cap (M/P) \neq \varnothing$. Since M is a strong module

[2.6] Also called *special* in [14] .

of σ, we have $M \subseteq \cup Q$ or $\cup Q \subseteq M$. Since $M = \cup(M/P)$, we obtain $M/P \subseteq Q$ or $Q \subseteq M/P$. Consequently, M/P is a strong module of σ/P.

Second, let Q be a strong module of σ/P. Consider a module M of σ such that there exists $x \in (\cup Q) \cap M$. Denote by X the block of P containing x. Clearly $X \in Q \cap (M/P)$. By Lemma 2.10, M/P is a module of σ/P. If $|M/P| = 1$, then $M \subseteq X \subseteq \cup Q$. Thus suppose that $|M/P| \geq 2$. Consider $Y \in M/P$. Since $|M/P| \geq 2$, we have $Y \cap M \neq \varnothing$ and $M \setminus Y \neq \varnothing$. Moreover, Y is a strong module of σ because $Y \in P$. It follows that $Y \subseteq M$. Therefore $M = \cup(M/P)$. Since Q is a strong module of σ/P and $X \in Q \cap (M/P)$, we have $Q \subseteq M/P$ or $M/P \subseteq Q$. It follows that $\cup Q \subseteq M$ or $M \subseteq \cup Q$. Consequently, $\cup Q$ is a strong module of σ. □

In the second assertion of Lemma 2.15, the hypothesis that all the blocks of P are strong modules of σ is necessary. Indeed, for each $X \in P$, $\{X\}$ is a strong module of σ/P, so we must have $\cup\{X\} = X$ is a strong module of σ.

The following property of the $\{e, f\}$-components of a 2-structure σ, where $e, f \in E(\sigma)$, completes our examination.

Lemma 2.16 (Ille [23])**.** *Let σ be a 2-structure σ. Consider an $\{e, f\}$-component X of σ, where $e, f \in E(\sigma)$. For every strong module M of σ, if $X \subseteq M$, then $M = X$ or $M = V(\sigma)$.*

Proof. Let M be a strong module of σ such that $X \subsetneq M \subseteq V(\sigma)$. We have to show that $M = V(\sigma)$. It follows from Lemma 2.4 that $\mathcal{C}_{\{e,f\}}(\sigma)$ is a modular partition of σ. Furthermore, each block of $\mathcal{C}_{\{e,f\}}(\sigma)$ is a strong module of σ by Lemma 2.13. Since M is a strong module of σ, it follows from Lemma 2.15 that $M/\mathcal{C}_{\{e,f\}}(\sigma)$ is a strong module of $\sigma/\mathcal{C}_{\{e,f\}}(\sigma)$. By Proposition 2.8, $\sigma/\mathcal{C}_{\{e,f\}}(\sigma)$ is constant or linear. Thus, $\sigma/\mathcal{C}_{\{e,f\}}(\sigma)$ is primitive by Lemma 2.14. Therefore, $M/\mathcal{C}_{\{e,f\}}(\sigma)$ is a trivial strong module of $\sigma/\mathcal{C}_{\{e,f\}}(\sigma)$. Since $X \subsetneq M$, $|M/\mathcal{C}_{\{e,f\}}(\sigma)| \geq 2$, so $M/\mathcal{C}_{\{e,f\}}(\sigma) = \mathcal{C}_{\{e,f\}}(\sigma)$. Lastly, consider $Y \in \mathcal{C}_{\{e,f\}}(\sigma)$. Since $M/\mathcal{C}_{\{e,f\}}(\sigma) = \mathcal{C}_{\{e,f\}}(\sigma)$, we have $Y \cap M \neq \varnothing$ and $M \setminus Y \neq \varnothing$. Since M is a strong module of σ, we obtain $Y \subseteq M$. It follows that $M = V(\sigma)$. □

Notation 2.17. Let σ be a 2-structure. Suppose that σ is not $\{e, f\}$-connected, where $e, f \in E(\sigma)$. It follows from Lemma 2.16 that $\mathcal{C}_{\{e,f\}}(\sigma)$ is the set of the strong modules of σ that are maximal under inclusion among the proper strong modules of σ. In a general way, we associate with each 2-structure σ the set $\Pi(\sigma)$ of the strong modules of σ that are maximal under inclusion among the proper strong modules of σ. (Note that $\Pi(\sigma)$ can be empty when σ is infinite.)

Proposition 2.18. *Let σ be a 2-structure such that $v(\sigma) \geq 2$. The set $\Pi(\sigma)$ constitutes a modular partition of σ*[2.7]*, and the quotient $\sigma/\Pi(\sigma)$ is primitive.*

Proof. To begin, consider $X, Y \in \Pi(\sigma)$ such that $X \cap Y \neq \varnothing$. Since X is a strong module of σ, we have $X \subseteq Y$ or $Y \subseteq X$. It follows from the

[2.7] Recall that we consider finite 2-structures.

maximality of X and Y that $X = Y$. Moreover, consider $v \in V(\sigma)$. As previously mentioned, $\{v\}$ is a strong module of σ. Denote by $\mathcal{S}_v$ the set of the proper strong modules of σ containing v. Since $v(\sigma) \geq 2$, $\{v\} \in \mathcal{S}_v$. Let $M, N \in \mathcal{S}_v$. We have $v \in M \cap N$. Since M is a strong module of σ, we obtain $M \subseteq N$ or $N \subseteq M$. Therefore, $(\mathcal{S}_v, \subsetneq)$ is a linear order. Since $V(\sigma)$ is finite, $(\mathcal{S}_v, \subsetneq)$ admits a greatest element M_v. Clearly, $M_v \in \Pi(\sigma)$. Consequently, we have $\cup\Pi(\sigma) = V(\sigma)$. It follows that $\Pi(\sigma)$ is a modular partition of σ.

Now, we prove that $\sigma/\Pi(\sigma)$ is primitive. Consider a strong module Q of $\sigma/\Pi(\sigma)$ such that $|Q| \geq 2$. We have to show that $Q = \Pi(\sigma)$. Since all the blocks of $\Pi(\sigma)$ are strong modules of σ, it follows from Lemma 2.15 that $\cup Q$ is a strong module of σ. Given $X \in Q$, we have $X \subsetneq (\cup Q)$ because $|Q| \geq 2$. By the maximality of X, $\cup Q = V(\sigma)$, and hence $Q = \Pi(\sigma)$. □

The characterization of primitive 2-structures is an easy consequence of Lemma 2.14, and of the following two propositions.

Proposition 2.19. *Given a primitive 2-structure σ such that $v(\sigma) \geq 3$, σ is prime if and only if σ is uncuttable.*

Proof. To begin, suppose that σ is cuttable, and consider a nontrivial modular cut X of σ. Since $v(\sigma) \geq 3$, X or $V(\sigma) \setminus X$ are nontrivial modules of σ. Therefore σ is decomposable.

Conversely, suppose that σ is decomposable. Hence σ admits nontrivial modules. Consider a module M of σ that is maximal under inclusion among the nontrivial modules of σ. Since σ is primitive, M is not a strong module of σ. Thus there exists a module N of σ such that $M \cap N \neq \varnothing$, $M \setminus N \neq \varnothing$, and $N \setminus M \neq \varnothing$. By assertion (M5) of Proposition 2.5, $M \cup N$ is a module of σ because $M \cap N \neq \varnothing$. Since $N \setminus M \neq \varnothing$, we have $M \subsetneq M \cup N$. By the maximality of M, we obtain $M \cup N = V(\sigma)$. Thus $N \setminus M = V(\sigma) \setminus M$. By assertion (M6) of Proposition 2.5, $N \setminus M = V(\sigma) \setminus M$ is a module of σ because $M \setminus N \neq \varnothing$. Consequently, M is a nontrivial modular cut of σ. Therefore σ is cuttable. □

Proposition 2.20. *Given a 2-structure σ, σ is primitive and cuttable if and only if σ is constant or linear, with $v(\sigma) \geq 2$.*

Proof. Suppose that σ is constant or linear, with $v(\sigma) \geq 2$. By Lemma 2.14, σ is primitive. Since σ is constant or linear, it follows from Corollary 2.11 that there exists $v \in V(\sigma)$ such that $\{v\}$ is a modular cut of σ. Since $v(\sigma) \geq 2$, $\{v\}$ is a nontrivial modular cut of σ, so σ is cuttable.

Conversely, suppose that σ is primitive and cuttable. By Proposition 2.12, there exist $e, f \in E(\sigma)$ such that σ is not $\{e, f\}$-connected. Furthermore, by Lemma 2.13, each $\{e, f\}$-component of σ is a strong module of σ. Since σ is primitive and not $\{e, f\}$-connected, each $\{e, f\}$-component of σ is reduced to a singleton. Consequently, the function $V(\sigma) \longrightarrow \mathcal{C}_{\{e,f\}}(\sigma)$, defined by $v \mapsto \{v\}$ for every $v \in V(\sigma)$, realizes an isomorphism from σ onto $\sigma/\mathcal{C}_{\{e,f\}}(\sigma)$. It follows from Proposition 2.8 that σ is constant or linear. Note that $v(\sigma) \geq 2$ because σ is cuttable. □

There is another approach to establish the forward direction of Proposition 2.20. It reveals the importance of the notion of a modular cut in the study of nonconnected 2-structures.

Second proof of the forward direction of Proposition 2.20. Let σ be a cuttable and primitive 2-structure. We consider a maximal set $\mathcal{S}$ under inclusion among the sets of modular cuts of σ that are linearly ordered by inclusion. By the maximality of $\mathcal{S}$, we have $\varnothing, V(\sigma) \in \mathcal{S}$, and $\mathcal{S} \setminus \{\varnothing, V(\sigma)\} \neq \varnothing$ because σ is cuttable. We denote the elements of $\mathcal{S}$ by $X_0, \dots, X_n$, where $n \geq 2$, in such a way that

$$\varnothing = X_0 \subsetneq \cdots \subsetneq X_n = V(\sigma).$$

Let $m \in \{0, \dots, n-1\}$. We show that

$$X_{m+1} \setminus X_m \text{ is a strong module of } \sigma.$$

Since $V(\sigma) \setminus X_m$ is a module of σ, $X_{m+1} \cap (V(\sigma) \setminus X_m) = X_{m+1} \setminus X_m$ is a module of σ by assertion (M4) of Proposition 2.5. Now, consider a module M of σ such that $M \cap (X_{m+1} \setminus X_m) \neq \varnothing$, and $M \setminus (X_{m+1} \setminus X_m) \neq \varnothing$. We have to verify that $(X_{m+1} \setminus X_m) \subseteq M$. Since $M \setminus (X_{m+1} \setminus X_m) \neq \varnothing$, we have $M \cap X_m \neq \varnothing$ or $M \cap (V(\sigma) \setminus X_{m+1}) \neq \varnothing$. The set $\{V(\sigma) \setminus X_p : 0 \leq p \leq n\}$ is also maximal under inclusion among the sets of modular cuts of σ that are linearly ordered by inclusion. So, by interchanging $\mathcal{S}$ and $\{V(\sigma) \setminus X_p : 0 \leq p \leq n\}$ if necessary, we can assume that

$$M \cap X_m \neq \varnothing.$$

We verify that

$$X_m \cup (M \cap X_{m+1}) \text{ is a modular cut of } \sigma.$$

Since $X_m \cap (M \cap X_{m+1}) = M \cap X_m$, $X_m \cup (M \cap X_{m+1})$ is a module of σ by assertion (M5) of Proposition 2.5. Clearly, $V(\sigma) \setminus (X_m \cup (M \cap X_{m+1})) = (V(\sigma) \setminus X_m) \setminus (M \cap X_{m+1})$. Since $(M \cap X_{m+1}) \setminus (V(\sigma) \setminus X_m) = M \cap X_m$, it follows from assertion (M6) of Proposition 2.5 that $(V(\sigma) \setminus X_m) \setminus (M \cap X_{m+1}) = V(\sigma) \setminus (X_m \cup (M \cap X_{m+1}))$ is a module of σ. Therefore, $X_m \cup (M \cap X_{m+1})$ is a modular cut of σ.

Since $X_m \cup (M \cap X_{m+1}) = X_m \cup (M \cap (X_{m+1} \setminus X_m))$, we have

$$X_m \subsetneq X_m \cup (M \cap X_{m+1}) \subseteq X_{m+1}.$$

It follows from the maximality of $\mathcal{S}$ that $X_m \cup (M \cap X_{m+1}) = X_{m+1}$ or, equivalently, $(X_{m+1} \setminus X_m) \subseteq M$.

Consequently, $X_{m+1} \setminus X_m$ is a strong module of σ for every $m \in \{0, \dots, n-1\}$. Since σ is primitive, $|X_{m+1} \setminus X_m| = 1$ for every $0 \leq m \leq n-1$. Denote by x_{m+1} the unique element of $X_{m+1} \setminus X_m$. We have $X_m = \{x_1, \dots, x_m\}$ for each $0 < m \leq n$. In particular, $V(\sigma) = X_n = \{x_1, \dots, x_n\}$. Consider $p, q \in \{1, \dots, n\}$ such that $p < q$. Since $X_p = \{x_1, \dots, x_p\}$ is a module of σ, we obtain $[x_p, x_q]_\sigma = [x_1, x_q]_\sigma$. Since $V(\sigma) \setminus X_1 = \{x_2, \dots, x_n\}$ is a module of σ, $[x_1, x_q]_\sigma = [x_1, x_2]_\sigma$. Thus

$$[x_p, x_q]_\sigma = [x_1, x_2]_\sigma \text{ for any } p, q \in \{1, \dots, n\} \text{ such that } p < q.$$

It follows that σ is constant if $(x_1, x_2)_\sigma = (x_2, x_1)_\sigma$, and σ is linear if $(x_1, x_2)_\sigma \neq (x_2, x_1)_\sigma$. □

The characterization of primitive 2-structures follows.

Theorem 2.21 (Ille [23][2.8]). *Given a 2-structure σ, σ is primitive if and only if σ is prime, constant, or linear.*

Proof. By Lemma 2.14, if σ is prime, constant, or linear, then σ is primitive. Conversely, we verify that if σ is primitive and decomposable, then σ is constant or linear. Hence, suppose that σ is primitive and decomposable. Obviously, $v(\sigma) \geq 3$ because σ is decomposable. It follows from Proposition 2.19 that σ is cuttable, and it suffices to apply Proposition 2.20. □

The next result, called Gallai's decomposition theorem, is a direct consequence of Proposition 2.18 and Theorem 2.21.

Theorem 2.22 (Gallai [18, 28][2.9]). *Given a 2-structure σ, with $v(\sigma) \geq 2$, the quotient $\sigma/\Pi(\sigma)$ is prime, constant, or linear.*

Remark 2.23. Chein, Habib, and Maurer [9] adopted a different approach to establish Theorem 2.22 for *partitive hypergraphs*, which constitutes a nice generalization of Theorem 2.22. We transcribe it in terms of symmetric 2-structures. (The set of the modules of a symmetric 2-structure is a partitive hypergraph.) Given a symmetric 2-structure σ, define a partial order $\mathbb{O}$ on the set of the modular partitions of σ as follows. Given distinct modular partitions P and Q of σ, $(P,Q) \in A(\mathbb{O})$ if for every $X \in P$, there exists $Y \in Q$ such that $X \subseteq Y$. Clearly, $\{\{v\} : v \in V(\sigma)\}$ is the least vertex of $\mathbb{O}$, and $\{V(\sigma)\}$ is the greatest one. Furthermore, with modular partitions P and Q of σ associate their *join* $P \vee Q$, and their *meet* $P \wedge Q$ defined as follows. First, given distinct $v, w \in V(\sigma)$, v and w belong to the same block of $P \vee Q$ if there exist $X_0, \dots, X_n \in P \cup Q$ satisfying $v \in X_0$, $w \in X_n$, and for $0 \leq i \leq n-1$ (when $n \geq 1$), $X_i \cap X_{i+1} \neq \varnothing$. Second, given distinct $v, w \in V(\sigma)$, v and w belong to the same block of $P \wedge Q$ if there exist $X \in P$ and $Y \in Q$ such that $v, w \in X \cap Y$. Clearly, $P \vee Q$ and $P \wedge Q$ are modular partitions of σ. Therefore, $\mathbb{O}$ is a *lattice*, that is, for any modular partitions P, Q, and R of σ, we have: if $(P,R), (Q,R) \in A(\mathbb{O})$, then $((P \vee Q), R) \in A(\mathbb{O})$, and if $(R,P), (R,Q) \in A(\mathbb{O})$, then $(R, (P \wedge Q)) \in A(\mathbb{O})$. The maximal vertices of $\mathbb{O} - \{V(\sigma)\}$ are called the *coatoms* of $\mathbb{O}$. Lastly, Chein, Habib, and Maurer observed that $\Pi(\sigma)$ is the meet of all the coatoms of $\mathbb{O}$.

join *meet* *lattice* *coatoms*

[2.8] Ehrenfeucht, Harju and Rozenberg [14, Theorem 5.3] established a more general result. They associate with a decomposable and primitive 2-structure σ a graph Γ defined on $V(\Gamma) = V(\sigma)$ as follows. Given distinct $v, w \in V(\Gamma)$, $vw \in E(\Gamma)$ if $\{v, w\}$ is a module of σ. Then, they proved that either Γ is complete or Γ is a path. In the first instance, σ is constant whereas σ is linear in the second one.

[2.9] Gallai [18] demonstrated this theorem for graphs; Boussaïri, Ille, Lopez, Thomassé [8, Theorem 5] for digraphs; Ehrenfeucht, Harju, and Rozenberg [14, Theorem 5.5] for 2-structures; Ille [23, Theorem 2] for binary structures.

We specify Theorem 2.22 as follows.

Theorem 2.24. *Given a 2-structure σ, with $v(\sigma) \geq 2$, the assertions below hold.*

(1) *There exists $e \in E(\sigma)$ such that σ is not $\{e\}$-connected if and only if $\Pi(\sigma) = \mathcal{C}_{\{e\}}(\sigma)$ and $\sigma/\Pi(\sigma)$ is constant.*
(2) *There exist distinct $e, f \in E(\sigma)$ such that σ is not $\{e, f\}$-connected if and only if $\Pi(\sigma) = \mathcal{C}_{\{e,f\}}(\sigma)$ and $\sigma/\Pi(\sigma)$ is linear.*
(3) *The 2-structure σ is connected if and only if $\sigma/\Pi(\sigma)$ is prime.*

Proof. To begin, suppose that there exist $e, f \in E(\sigma)$ such that σ is not $\{e, f\}$-connected. It follows from Lemma 2.4, Lemma 2.13, and Lemma 2.16 that $\Pi(\sigma) = \mathcal{C}_{\{e,f\}}(\sigma)$. By Proposition 2.8, $\sigma/\Pi(\sigma)$ is constant if $e = f$, and $\sigma/\Pi(\sigma)$ is linear if $e \neq f$. Conversely, suppose that $\sigma/\Pi(\sigma)$ is constant or linear. There exists $X \in \Pi(\sigma)$ such that $\{X\}$ is a modular cut of $\sigma/\Pi(\sigma)$. By Lemma 2.10, X is a modular cut of σ. It follows from assertion (M7) of Proposition 2.5 that there exist $e, f \in E(\sigma)$ such that $[X, V(\sigma) \setminus X]_\sigma = (e, f)$. Therefore, σ is not $\{e, f\}$-connected.

Lastly, suppose that σ is $\{e, f\}$-connected for any $e, f \in E(\sigma)$. By Proposition 2.12, σ is uncuttable. It follows that $|\Pi(\sigma)| \geq 3$. For every modular cut Q of $\sigma/\Pi(\sigma)$, $\cup Q$ is a modular cut of σ by Lemma 2.10. Thus $\sigma/\Pi(\sigma)$ is uncuttable as well. Moreover, by Proposition 2.18, $\sigma/\Pi(\sigma)$ is primitive. Since $|\Pi(\sigma)| \geq 3$, it follows from Proposition 2.19 that $\sigma/\Pi(\sigma)$ is prime. Conversely, suppose that $\sigma/\Pi(\sigma)$ is prime. Since $|\Pi(\sigma)| \geq 3$, $\sigma/\Pi(\sigma)$ is neither constant nor linear. It follows from the first two assertions that σ is connected. □

Notation 2.25. Let σ be a nonconnected 2-structure. It follows from Theorem 2.24 that there exists a unique subset $\nu(\sigma)$ of $E(\sigma)$ such that $|\nu(\sigma)| = 1$ or 2, and σ is not $\nu(\sigma)$-connected. The $\nu(\sigma)$-components of σ are simply called the *components* of σ, and $\mathcal{C}_{\nu(\sigma)}(\sigma)$ is denoted by $\mathcal{C}(\sigma)$.

component

Finally, the last assertion of Theorem 2.24 is developed as follows.

Theorem 2.26. *Given a 2-structure σ, with $v(\sigma) \geq 2$, the following assertions are equivalent*

(1) *σ is connected;*
(2) *σ is uncuttable;*
(3) *$\sigma/\Pi(\sigma)$ is prime;*
(4) *There exists a modular partition P of σ such that σ/P is prime;*
(5) *$|\Pi(\sigma)| \geq 3$ and $\Pi(\sigma)$ is the set of the maximal modules of σ under inclusion among the proper modules of σ.*

Proof. We denote by $\mathcal{M}$ the set of the maximal modules of σ under inclusion among the proper modules of σ. Hence, (5) is restated as follows

$$|\Pi(\sigma)| \geq 3 \text{ and } \Pi(\sigma) = \mathcal{M}.$$

By Proposition 2.12, the first two assertions are equivalent. Hence, it follows from the last assertion of Theorem 2.24 that the first three assertions are equivalent.

Clearly, (3) implies (4). Now, we show that (4) implies (5). Suppose that there exists a modular partition P of σ such that σ/P is prime. First, we prove that for every module M of σ,

$$\text{if } |M/P| \geq 2, \text{ then } M/P = P. \tag{2.6}$$

Let M be a module of σ such that $|M/P| \geq 2$. By Lemma 2.10, M/P is a module of σ/P. Thus $M/P = P$ because σ/P is prime. Therefore (2.6) holds. Second, we prove that for every module M of σ,

$$\text{if } |M/P| \geq 2, \text{ then } M = V(\sigma). \tag{2.7}$$

Let M be a module of σ such that $|M/P| \geq 2$. By (2.6), $M/P = P$. For a contradiction, suppose that $M \neq V(\sigma)$, and consider $X \in P$ such that $X \setminus M \neq \varnothing$. By assertion (M6) of Proposition 2.5, $M \setminus X$ is a module of σ. We have $(M \setminus X)/P = P \setminus \{X\}$. Since $|P| \geq 3$, we obtain $|(M \setminus X)/P| \geq 2$ and $(M \setminus X)/P \neq P$, which contradicts (2.6). It follows that $X \subseteq M$ for every $X \in P$, so $M = V(\sigma)$. Therefore (2.7) holds. Third, we prove that $P = \mathcal{M}$. Given $X \in P$, consider a module M of σ such that $X \subsetneq M$. Since $X \subsetneq M$, $|M/P| \geq 2$. By (2.7), $M = V(\sigma)$. Thus $P \subseteq \mathcal{M}$. Conversely, consider $Y \in \mathcal{M}$. Since $Y \neq V(\sigma)$, it follows from (2.7) that there exists $X \in P$ such that $Y \subseteq X$. By the maximality of Y, $Y = X$, so $Y \in P$. Therefore $\mathcal{M} \subseteq P$. It follows that $P = \mathcal{M}$. Fourth, we verify that the blocks of P are strong modules of σ. Given $X \in P$, consider a module M of σ such that $X \cap M \neq \varnothing$ and $M \setminus X \neq \varnothing$. We have $|M/P| \geq 2$. By (2.7), $M = V(\sigma)$. Hence $X \subseteq M$. It follows that X is a strong module of σ. Since $P = \mathcal{M}$, $X \in \mathcal{M}$. It follows that $X \in \Pi(\sigma)$. Therefore $P \subseteq \Pi(\sigma)$. Since P and $\Pi(\sigma)$ are partitions of $V(\sigma)$, we obtain $P = \Pi(\sigma)$. Consequently

$$P = \Pi(\sigma) = \mathcal{M}.$$

Note that $|\Pi(\sigma)| \geq 3$ because $\Pi(\sigma) = P$ and σ/P is prime. It follows that (4) implies (5).

Lastly, we show that (5) implies (3). Hence suppose that $|\Pi(\sigma)| \geq 3$ and $\Pi(\sigma) = \mathcal{M}$. Since $|\Pi(\sigma)| \geq 3$, we have to show that $\sigma/\Pi(\sigma)$ is indecomposable. Let Q be a module of $\sigma/\Pi(\sigma)$ such that $|Q| \geq 2$. We have to verify that $Q = \Pi(\sigma)$. By Lemma 2.10, $\cup Q$ is a module of σ. Consider $X \in Q$. Since $\Pi(\sigma) = \mathcal{M}$, we have $X \in \mathcal{M}$. Since $|Q| \geq 2$, we obtain $X \subsetneq \cup Q$. It follows from the maximality of X that $\cup Q = V(\sigma)$. Hence $Q = \Pi(\sigma)$. □

3. Prime 2-substructures of a prime 2-structure: the first results

Notation 3.1. Let σ be a 2-structure. For $n \in \{3, \ldots, v(\sigma) - 1\}$, we denote by $\mathscr{P}_n(\sigma)$ the set of $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime and $|X| = n$. Furthermore, we denote by $\mathscr{R}_n(\sigma)$ the union of the elements of $\mathscr{P}_n(\sigma)$.

Question 3.2. Let σ be a prime 2-structure. A natural question is:

$$\text{For which } n \in \{3, \ldots, v(\sigma) - 1\}, \text{ is } \mathscr{P}_n(\sigma) \neq \varnothing?$$

Obviously, we can refine the question as follows. Given $v \in V(\sigma)$, for which $n \in \{3, \ldots, v(\sigma) - 1\}$, do we have $v \in \mathscr{R}_n(\sigma)$?

For instance, given $n \geq 2$, consider the graph B_{2n+1} defined on $V(B_{2n+1}) = \{0, \ldots, 2n\}$ by

$$\begin{aligned} E(B_{2n+1}) = & \{\{i, j\} : i, j \in \{0, \ldots, n-1\}, i \neq j\} \\ & \cup \{\{i, i+n\} : i \in \{0, \ldots, n-1\}\} \\ & \cup \{\{i, 2n\} : i \in \{0, \ldots, n-1\}\} \text{ (see Figure 3.1).} \end{aligned}$$

the bull

The graph B_5 is called the *bull*.

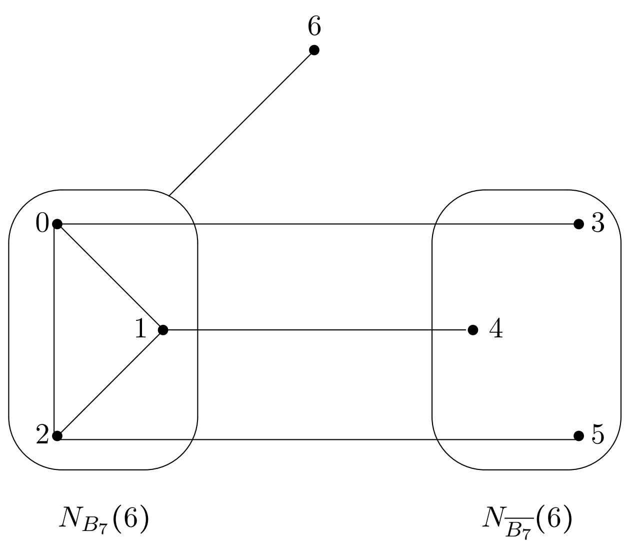

Figure 3.1. The graph B_7

Claim 3.3. *$B_{2n+1} - 2n$ is prime.*

Proof. Consider a module M of $B_{2n+1} - 2n$ such that $|M| \geq 2$. We have to show that $M = \{0, \ldots, 2n-1\}$. For a contradiction, suppose that $M \cap \{0, \ldots, n-1\} = \varnothing$. Hence $M \subseteq \{n, \ldots, 2n-1\}$. Since $|M| \geq 2$, there exist distinct $i, j \in \{0, \ldots, n-1\}$ such that $i+n, j+n \in M$, which is impossible because $i \notin M$ with $\{i, i+n\} \in E(B_{2n+1})$ and $\{i, j+n\} \notin E(B_{2n+1})$. It follows that $M \cap \{0, \ldots, n-1\} \neq \varnothing$. Similarly, $M \cap \{n, \ldots, 2n-1\} \neq \varnothing$.

We prove that for every $i \in \{0, \ldots, n-1\}$,

$$i \in M \Longrightarrow i+n \in M. \tag{3.1}$$

Indeed, consider $i \in M \cap \{0, \ldots, n-1\}$. Since $M \cap \{n, \ldots, 2n-1\} \neq \varnothing$, there exists $j \in \{0, \ldots, n-1\}$ such that $j+n \in M \cap \{n, \ldots, 2n-1\}$. If $i = j$, then $i+n \in M$. Suppose that $i \neq j$. We have $\{i, i+n\} \in E(B_{2n+1})$ and $\{j+n, i+n\} \notin E(B_{2n+1})$. Since M is a module of $B_{2n+1} - 2n$ such that $i, j+n \in M$, we obtain $i+n \in M$. Hence (3.1) holds. Now, we prove that for every $i \in \{0, \ldots, n-1\}$,

$$i \in M \iff i+n \in M. \tag{3.2}$$

Since (3.1) holds, consider $i \in \{0, \ldots, n-1\}$ such that $i+n \in M$. Since $M \cap \{0, \ldots, n-1\} \neq \varnothing$, there exists $j \in \{0, \ldots, n-1\} \cap M$. If $j = i$, then $i \in M$. Suppose that $i \neq j$. Since (3.1) holds, $j+n \in M$. We have $\{i, i+n\} \in E(B_{2n+1})$ and $\{i, j+n\} \notin E(B_{2n+1})$. Since M is a module of $B_{2n+1} - 2n$ such that $i+n, j+n \in M$, we obtain $i \in M$. Hence (3.2) holds.

Lastly, since (3.2) holds, there exists $i \in \{0, \ldots, n-1\}$ such that $i, i+n \in M$. For each $j \in \{0, \ldots, n-1\} \setminus \{i\}$, we have $\{i, j\} \in E(B_{2n+1})$ and $\{i+n, j\} \notin E(B_{2n+1})$. Since M is a module of $B_{2n+1} - 2n$ such that $i, i+n \in M$, we obtain $j \in M$. Therefore $\{0, \ldots, n-1\} \subseteq M$. It follows from (3.2) that $M = \{0, \ldots, 2n-1\}$. □

Claim 3.4. *B_{2n+1} is prime.*

Proof. Consider a module M of B_{2n+1} such that $|M| \geq 2$. We have to show that $M = \{0, \ldots, 2n\}$. For a contradiction, suppose that $2n \notin M$. By assertion (M2) of Proposition 2.5, M is a module of $B_{2n+1} - 2n$. Since $|M| \geq 2$, it follows from Claim 3.3 that $M = \{0, \ldots, 2n-1\}$, which is impossible because $\{0, 2n\} \in E(B_{2n+1})$ and $\{n, 2n\} \notin E(B_{2n+1})$. Thus $2n \in M$.

We prove that $|M \setminus \{2n\}| \geq 2$. We have $M \setminus \{2n\} \neq \varnothing$ because $|M| \geq 2$. First, suppose that there exists $i \in M \cap \{0, \ldots, n-1\}$. Since $\{i, i+n\} \in E(B_{2n+1})$ and $\{i+n, 2n\} \notin E(B_{2n+1})$, we have $i+n \in M$. Second, suppose that there exists $i \in \{0, \ldots, n-1\}$ such that $i+n \in M$. Consider $j \in \{0, \ldots, n-1\} \setminus \{i\}$. Since $\{j, 2n\} \in E(B_{2n+1})$ and $\{j, i+n\} \notin E(B_{2n+1})$, we have $j \in M$. It follows that $|M \setminus \{2n\}| \geq 2$.

By assertion (M2) of Proposition 2.5, $M \setminus \{2n\}$ is a module of $B_{2n+1} - 2n$. Moreover, $B_{2n+1} - 2n$ is prime by Claim 3.3. Since $|M \setminus \{2n\}| \geq 2$, we obtain $M \setminus \{2n\} = \{0, \ldots, 2n-1\}$. Thus $M = \{0, \ldots, 2n\}$ because $2n \in M$. □

Claim 3.5. *We have $2n \in \mathscr{R}_5(B_{2n+1}) \setminus (\mathscr{R}_3(B_{2n+1}) \cup \mathscr{R}_4(B_{2n+1}))$.*

Proof. It follows from Claim 3.4 that the 2-substructure $B_{2n+1}[\{0, 1, n, n+1, 2n\}]$ is prime because it is isomorphic to B_5. Thus $2n \in \mathscr{R}_5(B_{2n+1})$.

Now, consider $X \subseteq V(B_{2n+1})$ such that $|X| = 3$ or 4, and $2n \in X$. We have to prove that $B_{2n+1}[X]$ is decomposable.

First, suppose that for all $i \in X \cap \{0, \dots, n-1\}$ and all $j \in X \cap \{n, \dots, 2n-1\}$, we have $j - i \neq n$. We have $(X \cap \{0, \dots, n-1\}) \cup \{2n\}$ and $X \cap \{n, \dots, 2n-1\}$ are modules of $B_{2n+1}[X]$. If $(X \cap \{0, \dots, n-1\}) \cup \{2n\}$ is a trivial module of $B_{2n+1}[X]$, then $X \subseteq \{n, \dots, 2n\}$, and hence $X \cap \{n, \dots, 2n-1\}$ is a nontrivial module of $B_{2n+1}[X]$. Therefore, $(X \cap \{0, \dots, n-1\}) \cup \{2n\}$ or $X \cap \{n, \dots, 2n-1\}$ are nontrivial modules of $B_{2n+1}[X]$.

Second, suppose that there exists $i \in \{0, \dots, n-1\}$ such that $i, i+n \in X$. If $X \cap \{0, \dots, n-1\} = \{i\}$, then $\{i+n, 2n\}$ is a nontrivial module of $B_{2n+1}[X]$. Otherwise, if there exists $j \in \{0, \dots, n-1\} \setminus \{i\}$ such that $X = \{i, j, i+n, 2n\}$, then $\{j, 2n\}$ is a nontrivial module of $B_{2n+1}[X]$. □

In Corollary 3.9, we establish that such a vertex $2n$ is unique.

3.1. **Sumner's theorem.**

Remark 3.6. Let σ be a prime 2-structure. Consider

$$v \notin \mathscr{R}_3(\sigma) \cup \mathscr{R}_4(\sigma). \tag{3.3}$$

Let $X \subseteq V(\sigma)$ such that $v \in X$ and $|X| = 3$ or 4. We verify that $\sigma[X]$ is not connected. Otherwise, $\sigma[X]$ is connected, and it follows from Theorem 2.26 that $\sigma[X]/\Pi(\sigma[X])$ is prime. Consider a subset X' of X such that $v \in X'$ and $|X' \cap Y| = 1$ for each $Y \in \Pi(\sigma[X])$. The function $f : \Pi(\sigma[X]) \longrightarrow X'$, satisfying $X' \cap Y = \{f(Y)\}$ for each $Y \in \Pi(\sigma[X])$, realizes an isomorphism from $\sigma[X]/\Pi(\sigma[X])$ onto $\sigma[X']$. Thus, $\sigma[X']$ is prime with $v \in X'$ and $|X'| = 3$ or 4, which contradicts (3.3).

Notation 3.7. Consider a 2-structure σ. Let $v \in V(\sigma)$. For $e, f \in E(\sigma)$, set

$$N_\sigma^{(e,f)}(v) = \{w \in V(\sigma) \setminus \{v\} : [v, w]_\sigma = (e, f)\}.$$

Proposition 3.8 (Ille [23][3.1]). *Given a prime 2-structure σ, consider $v \notin \mathscr{R}_3(\sigma) \cup \mathscr{R}_4(\sigma)$.*

(1) *For each $e \in E(\sigma)$, $\sigma[N_\sigma^{(e,e)}(v)]$ is e-constant.*
(2) *For distinct $e, f \in E(\sigma)$, $\sigma[N_\sigma^{(e,f)}(v)]$ is e-linear (and f-linear).*

Proof. Consider $e, f \in E(\sigma)$ such that $N_\sigma^{(e,f)}(v) \neq \varnothing$. We prove that each $\{e, f\}$-component C of $\sigma[N_\sigma^{(e,f)}(v)]$ is a module of σ. Consider $x \in V(\sigma) \setminus C$. We have to verify that $x \longleftrightarrow_\sigma C$. Since $C \subseteq N_\sigma^{(e,f)}(v)$, we have $[v, C]_\sigma = (e, f)$. Hence, suppose that $x \neq v$. Moreover, since C is a module of $\sigma[N_\sigma^{(e,f)}(v)]$ by Lemma 2.4, suppose that $x \notin N_\sigma^{(e,f)}(v)$. Thus suppose that

$$x \notin N_\sigma^{(e,f)}(v) \cup \{v\}.$$

Let $\gamma \in C$. Since $\gamma \in N_\sigma^{(e,f)}(v)$ and $x \notin N_\sigma^{(e,f)}(v)$, $\{\gamma, x\}$ is not a module of $\sigma[\{v, x, \gamma\}]$. Since $\sigma[\{v, x, \gamma\}]$ is decomposable, $\{v, x\}$ or $\{v, \gamma\}$ are modules of $\sigma[\{v, x, \gamma\}]$. First, suppose that $\{v, \gamma\}$ is a module of $\sigma[\{v, x, \gamma\}]$. We

[3.1] Cournier and Ille [12] established this proposition for digraphs. Ille [23] proved this proposition for binary structures, that is, labeled 2-structures [14].

obtain $x \longleftrightarrow_\sigma \{v,\gamma\}$. Thus $[x,\gamma]_\sigma = [x,v]_\sigma$, so $\gamma \in N_\sigma^{[x,v]_\sigma}(x)$. Second, suppose that $\{v,x\}$ is a module of $\sigma[\{v,x,\gamma\}]$. We obtain $\gamma \longleftrightarrow_\sigma \{v,x\}$. Hence $[x,\gamma]_\sigma = [v,\gamma]_\sigma$. Since $C \subseteq N_\sigma^{(e,f)}(v)$, we obtain $\gamma \in N_\sigma^{(e,f)}(x)$. Therefore

$$C \subseteq N_\sigma^{(e,f)}(x) \cup N_\sigma^{[x,v]_\sigma}(x). \tag{3.4}$$

Suppose that $\prec v,x \succ_\sigma = \{e,f\}$ (see Notation 1.1). We obtain $e \neq f$ and $[v,x]_\sigma = (f,e)$ because $x \notin N_\sigma^{(e,f)}(v)$. It follows from (3.4) that $[x,C]_\sigma = (e,f)$. Now, suppose that $\prec v,x \succ_\sigma \neq \{e,f\}$. We obtain

$$(C \cap N_\sigma^{[x,v]_\sigma}(x)) \cap (C \cap N_\sigma^{(e,f)}(x)) = \varnothing.$$

Consider $\gamma \in C \cap N_\sigma^{[x,v]_\sigma}(x)$ and $\delta \in C \cap N_\sigma^{(e,f)}(x)$. By Remark 3.6, $\sigma[\{v,x,\gamma,\delta\}]$ is not connected. Since $\prec v,\gamma \succ_\sigma = \prec v,\delta \succ_\sigma = \prec x,\delta \succ_\sigma = \{e,f\}$, we obtain

$$\nu(\sigma[\{v,x,\gamma,\delta\}]) = \{e,f\} \text{ (see Notation 2.25).}$$

Furthermore, since $\prec v,x \succ_\sigma \neq \{e,f\}$ and $\prec x,\gamma \succ_\sigma = \prec v,x \succ_\sigma$, we have

$$\{v,x,\gamma\} \in \mathcal{C}(\sigma[\{v,x,\gamma,\delta\}]).$$

By Lemma 2.4, $\delta \longleftrightarrow_\sigma \{v,x,\gamma\}$, and hence $[\gamma,\delta]_\sigma = [v,\delta]_\sigma = (e,f)$. Consequently, if $C \cap N_\sigma^{[x,v]_\sigma}(x) \neq \varnothing$ and $C \cap N_\sigma^{(e,f)}(x) \neq \varnothing$, then $[C \cap N_\sigma^{[x,v]_\sigma}(x), C \cap N_\sigma^{(e,f)}(x)]_\sigma = (e,f)$. Since $\sigma[C]$ is $\{e,f\}$-connected and $C \subseteq N_\sigma^{(e,f)}(x) \cup N_\sigma^{[x,v]_\sigma}(x)$ by (3.4), we have $C \subseteq N_\sigma^{[x,v]_\sigma}(x)$ or $C \subseteq N_\sigma^{(e,f)}(x)$. In both instances, we obtain $x \longleftrightarrow_\sigma C$.

Consequently, the $\{e,f\}$-components of $\sigma[N_\sigma^{(e,f)}(v)]$ are modules of σ. Since σ is prime, they are reduced to singletons. Thus, the function from $N_\sigma^{(e,f)}(v)$ onto $\mathcal{C}_{\{e,f\}}(\sigma[N_\sigma^{(e,f)}(v))$, defined by

$$u \mapsto \{u\} \text{ for every } u \in N_\sigma^{(e,f)}(v),$$

is an isomorphism from $\sigma[N_\sigma^{(e,f)}(v)]$ onto

$$\sigma[N_\sigma^{(e,f)}(v)]/\mathcal{C}_{\{e,f\}}(\sigma[N_\sigma^{(e,f)}(v)]).$$

It follows from Lemma 2.4 and Proposition 2.8 that $\sigma[N_\sigma^{(e,f)}(v)]$ is e-constant if $e = f$, and $\sigma[N_\sigma^{(e,f)}(v)]$ is e-linear if $e \neq f$. □

Corollary 3.9 (Ille [23][3.2]). *Given a prime 2-structure σ, we have*

$$|V(\sigma) \smallsetminus (\mathscr{R}_3(\sigma) \cup \mathscr{R}_4(\sigma))| \leq 1.$$

Proof. For a contradiction, suppose that there exist distinct $x,y \in V(\sigma) \smallsetminus (\mathscr{R}_3(\sigma) \cup \mathscr{R}_4(\sigma))$. We prove that σ is decomposable. This is the case if $\{x,y\}$ is a module of σ because $v(\sigma) \geq 3$. Hence, suppose that $\{x,y\}$ is not a module of σ, and consider $v \in V(\sigma) \smallsetminus \{x,y\}$ such that $v \not\longleftrightarrow_\sigma \{x,y\}$. Since $\sigma[\{x,y,v\}]$ is decomposable, $\{x,v\}$ or $\{y,v\}$ are modules of $\sigma[\{x,y,v\}]$.

[3.2]Cournier and Ille [12] proved this corollary for digraphs. Ille [23] proved this proposition for binary structures, that is, labeled 2-structures [14]

Suppose that $\{x,v\}$ is a module of $\sigma[\{x,y,v\}]$. Thus $x,v\in N_\sigma^{[y,x]_\sigma}(y)$. By Proposition 3.8, $\prec x,v\succ_\sigma=\prec x,y\succ_\sigma$. It follows that

$$\prec x,v\succ_\sigma=\prec y,v\succ_\sigma=\prec x,y\succ_\sigma\ . \tag{3.5}$$

Since $v\not\longleftrightarrow_\sigma\{x,y\}$ and $\prec x,v\succ_\sigma=\prec y,v\succ_\sigma$, we obtain $|\prec x,v\succ_\sigma|=2$ and $[x,v]_\sigma=[v,y]_\sigma$. Since $\{x,v\}$ is a module of $\sigma[\{x,y,v\}]$, we have

$$[x,v]_\sigma=[v,y]_\sigma=[x,y]_\sigma. \tag{3.6}$$

We obtain also that (3.6) is satisfied when $\{y,v\}$ is a module of $\sigma[\{x,y,v\}]$. Consequently, by setting

$$W=\{v\in V(\sigma)\smallsetminus\{x,y\}:v\not\longleftrightarrow_\sigma\{x,y\}\},$$

we obtain that $W\cup\{x\}$ is a module of $\sigma[W\cup\{x,y\}]$. We show that $W\cup\{x\}$ is a module of σ. By assertion (M3) of Proposition 2.5, it suffices to verify that $W\cup\{x,y\}$ is a module of σ. Consider $v,w\in V(\sigma)\smallsetminus\{x,y\}$ such that $v\not\longleftrightarrow_\sigma\{x,y\}$ (i.e. $v\in W$) and $w\longleftrightarrow_\sigma\{x,y\}$. We prove that

$$w\longleftrightarrow_\sigma\{x,y,v\}. \tag{3.7}$$

By Remark 3.3, $B[\{x,y,u,w\}]$ is not connected. We distinguish the following two cases.

CASE 1: $\prec x,y\succ_\sigma\neq\nu(\sigma[\{x,y,v,w\}])$ (see Notation 2.25).

Since $\prec x,v\succ_\sigma=\prec y,v\succ_\sigma=\prec x,y\succ_\sigma$ by (3.5), it follows from Lemma 2.4 and Proposition 2.8 that $\{x,y,v\}\in\mathcal{C}(\sigma[\{x,y,v,w\}])$. By Lemma 2.4, $\{x,y,v\}$ is a module of $\sigma[\{x,y,v,w\}]$, and hence $w\longleftrightarrow_\sigma\{x,y,v\}$.

CASE 2: $\prec x,y\succ_\sigma=\nu(\sigma[\{x,y,v,w\}])$ (see Notation 2.25).

Since $w\longleftrightarrow_\sigma\{x,y\}$, we have $\prec x,w\succ_\sigma=\prec y,w\succ_\sigma$. For a contradiction, suppose that

$$\prec x,w\succ_\sigma\neq\nu(\sigma[\{x,y,v,w\}]).$$

By Lemma 2.4 and Proposition 2.8, $\{x,y,w\}\in\mathcal{C}(\sigma[\{x,y,v,w\}])$. By Lemma 2.4, $\{x,y,w\}$ is a module of $B[\{x,y,v,w\}]$, which contradicts $v\not\longleftrightarrow_\sigma\{x,y\}$. It follows that $\prec x,w\succ_\sigma=\nu(\sigma[\{x,y,v,w\}])$. Since $w\longleftrightarrow_\sigma\{x,y\}$, $[w,\{x,y\}]_\sigma=[x,y]_\sigma$ or $[y,x]_\sigma$. Suppose that

$$[w,\{x,y\}]_\sigma=[x,y]_\sigma.$$

Since $|\prec x,y\succ_\sigma|=2$ and $[v,x]_\sigma=[y,x]_\sigma$ by (3.6), $\{v,w\}$ is not a module of $\sigma[\{x,v,w\}]$. Since $\sigma[\{x,u,w\}]$ is decomposable, we have $\{x,v\}$ is a module of $\sigma[\{x,v,w\}]$ and $[w,v]_\sigma=[w,x]_\sigma=[x,y]_\sigma$ or $\{x,w\}$ is a module of $\sigma[\{x,v,w\}]$ and $[w,v]_\sigma=[x,v]_\sigma$. Since $[x,v]_\sigma=[x,y]_\sigma$ by (3.6), we obtain

$$[w,\{x,y,v\}]_\sigma=[x,y]_\sigma$$

in both instances. When $[w,\{x,y\}]_\sigma=[y,x]_\sigma$, we obtain

$$[w,\{x,y,v\}]_\sigma=[y,x]_\sigma$$

by considering $\sigma[\{y,u,w\}]$ instead of $\sigma[\{x,u,w\}]$.

In both cases, we obtain that (3.7) holds. It follows that $W \cup \{x, y\}$ is a module of σ. By assertion (M3) of Proposition 2.5, $W \cup \{x\}$ is a module of σ. Hence σ is decomposable. Consequently $|V(\sigma) \smallsetminus (\mathscr{R}_3(\sigma) \cup \mathscr{R}_4(\sigma))| \leq 1$. □

Sumner's theorem is an immediate consequence of Corollary 3.9.

Theorem 3.10 (Sumner [35][3.3]). *Given a prime 2-structure σ, we have*

$$\mathscr{P}_3(\sigma) \cup \mathscr{P}_4(\sigma) \neq \varnothing.$$

Sumner's theorem is improved as follows.

Theorem 3.11 (Cournier, Ille [12][3.4]). *Given a prime 2-structure σ, we have*

$$V(\sigma) = \mathscr{R}_3(\sigma) \cup \mathscr{R}_4(\sigma) \cup \mathscr{R}_5(\sigma).$$

Proof. We prove that

$$V(\sigma) \smallsetminus (\mathscr{R}_3(\sigma) \cup \mathscr{R}_4(\sigma)) \subseteq \mathscr{R}_5(\sigma).$$

Hence, consider $v \in V(\sigma) \smallsetminus (\mathscr{R}_3(\sigma) \cup \mathscr{R}_4(\sigma))$. By Corollary 3.9, $V(\sigma) \smallsetminus (\mathscr{R}_3(\sigma) \cup \mathscr{R}_4(\sigma)) = \{v\}$. Thus, by considering an element of $V(\sigma) \smallsetminus \{v\}$, we obtain $X \in \mathscr{P}_3(\sigma) \cup \mathscr{P}_4(\sigma)$ such that $X \subseteq V(\sigma) \smallsetminus \{v\}$. We prove that $\sigma[X \cup \{v\}]$ is prime. Otherwise, $\sigma[X \cup \{v\}]$ admits a nontrivial module M. By assertion (M2) of Proposition 2.5, $M \cap X$ is a module of $\sigma[X]$. Since $|M| \geq 2$, $M \cap X \neq \varnothing$. Since $\sigma[X]$ is prime, we obtain $|M \cap X| = 1$ or $M \cap X = X$. In the first instance, there is $y \in X$ such that $M = \{y, v\}$. Since $\{y, v\}$ is a module of $\sigma[X \cup \{v\}]$, the function $X \longrightarrow (X \smallsetminus \{y\}) \cup \{v\}$, defined by $y \mapsto v$ and $z \mapsto z$ for each $z \in X \smallsetminus \{y\}$, is an isomorphism from $\sigma[X]$ onto $\sigma[(X \smallsetminus \{y\}) \cup \{v\}]$. Thus $\sigma[(X \smallsetminus \{u\}) \cup \{v\}]$ is prime, which contradicts $v \in V(\sigma) \smallsetminus (\mathscr{R}_3(\sigma) \cup \mathscr{R}_4(\sigma))$. In the second instance, $v \longleftrightarrow_\sigma X$. Hence there exist $e, f \in E(\sigma)$ such that $X \subseteq N_\sigma^{(e,f)}(v)$. By Proposition 3.8, $\sigma[N_\sigma^{(e,f)}(v)]$ is constant or linear. Therefore, $\sigma[X]$ is constant or linear as well, which contradicts the fact that $\sigma[X]$ is prime. Consequently $\sigma[X \cup \{v\}]$ is prime. □

3.2. **The Ehrenfeucht–Rozenberg theorem.** We continue examining the existence of prime 2-substructures of cardinality greater than 5 in a prime 2-structure (see Question 3.2).

Notation 3.12. Given a 2-structure σ, suppose that there exists $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. By Theorem 3.10, such a subset X exists if σ is prime with $v(\sigma) \geq 5$. The discussion on $M \cap X$ in the proof of Theorem 3.11, where M is a module of $\sigma[X \cup \{v\}]$, leads us to consider the following subsets of $V(\sigma) \smallsetminus X$

- $\text{Ext}_\sigma(X)$ denotes the set of $v \in V(\sigma) \smallsetminus X$ such that $\sigma[X \cup \{v\}]$ is prime;

[3.3]Sumner [35] demonstrated this theorem for graphs.

[3.4]Cournier and Ille [12] proved this theorem for digraphs, and Ille [23] for binary structures by using the same proof.

- $\langle X\rangle_\sigma$ denotes the set of $v \in V(\sigma) \setminus X$ such that X is a module of $\sigma[X \cup \{v\}]$;
- For each $y \in X$, $X_\sigma(y)$ denotes the set of $v \in V(\sigma) \setminus X$ such that $\{y, v\}$ is a module of $\sigma[X \cup \{v\}]$.

Furthermore, $p_{(\sigma,X)}$ denotes the set $\{\text{Ext}_\sigma(X), \langle X\rangle_\sigma\} \cup \{X_\sigma(y) : y \in X\}$.

Lemma 3.13. *Given a 2-structure σ, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. The set $p_{(\sigma,X)}$ constitutes a partition of $V(\sigma) \setminus X$.*

Proof. To begin, we verify that the union of the elements of $p_{(\sigma,X)}$ equals $V(\sigma) \setminus X$. Let $v \in V(\sigma) \setminus X$. If $\sigma[X \cup \{v\}]$ is prime, then $v \in \text{Ext}_\sigma(X)$. Suppose that $\sigma[X \cup \{v\}]$ is decomposable. Hence, $\sigma[X \cup \{v\}]$ admits a nontrivial module M. By assertion (M2) of Proposition 2.5, $M \cap X$ is a module of $\sigma[X]$. Since M is a nontrivial module of $\sigma[X \cup \{v\}]$, we have $|M| \geq 2$, so $M \cap X \neq \varnothing$. Since $\sigma[X]$ is prime, we obtain $|M| = 1$ or $M \cap X = X$. In the first instance, there exists $y \in X$ such that $M \cap X = \{y\}$. Since $|M| \geq 2$, we obtain $M = \{y, v\}$, and hence $v \in X_\sigma(y)$. In the second instance, we obtain $M = X$ because $M \neq X \cup \{v\}$. It follows that $v \in \langle X\rangle_\sigma$.

Now, we show that the elements of $p_{(\sigma,X)}$ are pairwise disjoint. By definition of the elements of $p_{(\sigma,X)}$, we have $\text{Ext}_\sigma(X) \cap \langle X\rangle_\sigma = \varnothing$, and $\text{Ext}_\sigma(X) \cap X_\sigma(y) = \varnothing$ for every $y \in X$.

Let $y \in X$. Suppose for a contradiction that there exists $v \in X_\sigma(y) \cap \langle X\rangle_\sigma$. We obtain that $\{y, v\}$ and X are modules of $\sigma[X \cup \{v\}]$. By assertion (M6) of Proposition 2.5, $X \setminus \{y, v\} = X \setminus \{y\}$ is a module of $\sigma[X \cup \{v\}]$ because $v \in \{y, v\} \setminus X$. By assertion (M2) of Proposition 2.5, $X \setminus \{y\}$ is a module of $\sigma[X]$, which contradicts the fact that $\sigma[X]$ is prime. It follows that $X_\sigma(y) \cap \langle X\rangle_\sigma = \varnothing$.

Lastly, consider distinct $y, z \in X$. Suppose for a contradiction that there is $v \in X_\sigma(y) \cap X_\sigma(z)$. We obtain that $\{y, v\}$ and $\{z, v\}$ are modules of $\sigma[X \cup \{v\}]$. By assertion (M5) of Proposition 2.5, $\{y, v\} \cup \{z, v\} = \{y, z, v\}$ is a module of $\sigma[X \cup \{x\}]$ because $v \in \{y, v\} \cap \{z, v\}$. By assertion (M2) of Proposition 2.5, $X \cap \{y, z, v\} = \{y, z\}$ is a module of $\sigma[X]$, which contradicts the fact that $\sigma[X]$ is prime. □

Lemma 3.13 justifies the following definition.

Definition 3.14. Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. By Lemma 3.13, $p_{(\sigma,X)}$ is a partition of $V(\sigma) \setminus X$. It is called the *outside partition* induced by σ and X.

outside partition

Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. We study the modules of $\sigma[X \cup \{v, w\}]$, where $v, w \in V(\sigma) \setminus X$. We begin with two remarks.

Remark 3.15. Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime.

- For every $v \in \langle X\rangle_\sigma$, X is a module of $\sigma[X \cup \{v\}]$, that is, $v \longleftrightarrow_\sigma X$. Thus, X is a module of $\sigma[X \cup \langle X\rangle_\sigma]$, and X is a module of $\sigma[X \cup \{v,w\}]$ for $v,w \in \langle X\rangle_\sigma$.
- Let $y \in X$. For $z \in X \smallsetminus \{y\}$ and $v \in X_\sigma(y)$, we have $z \longleftrightarrow_\sigma \{y,v\}$ because $\{y,v\}$ is a module of $\sigma[X \cup \{v\}]$. Therefore $z \longleftrightarrow_\sigma \{y\} \cup X_\sigma(y)$. Consequently $\{y\} \cup X_\sigma(y)$ is a module of $\sigma[X \cup X_\sigma(y)]$, and $\{y,v,w\}$ is a module of $\sigma[X \cup \{v,w\}]$ for $v,w \in X_\sigma(y)$.

Remark 3.16. Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that σ admits a nontrivial module M. By assertion (M2) of Proposition 2.5, $X \cap M$ is a module of $\sigma[X]$. Since $\sigma[X]$ is prime, we obtain $X \cap M = \varnothing$, $|X \cap M| = 1$ or $X \cap M = X$. We consider the three cases below.

Case 1: $X \cap M = \varnothing$.

We prove that there exists $B \in p_{(\sigma,X)}$ such that $M \subseteq B$. For distinct $v,w \in M$, we have $(X \cup \{v,w\}) \cap M = \{v,w\}$ is a module of $\sigma[X \cup \{v,w\}]$ by assertion (M2) of Proposition 2.5. Therefore, the function $f : X \cup \{v\} \longrightarrow X \cup \{w\}$, defined by $v \mapsto w$ and $y \mapsto y$ for every $y \in X$, is an isomorphism from $\sigma[X \cup \{v\}]$ onto $\sigma[X \cup \{w\}]$. Consequently, if $v \in \mathrm{Ext}_\sigma(X)$, that is, $\sigma[X \cup \{v\}]$ is prime, then $\sigma[X \cup \{w\}]$ is prime too, so $w \in \mathrm{Ext}_\sigma(X)$. Furthermore, if $v \in \langle X\rangle_\sigma$, that is, if X is a module of $\sigma[X \cup \{v\}]$, then $f(X) = X$ is a module of $\sigma[X \cup \{w\}]$, so $w \in \langle X\rangle_\sigma$. Lastly, given $y \in X$, if $v \in X(y)$, that is, $\{y,v\}$ is a module of $\sigma[X \cup \{v\}]$, then $f(\{y,v\}) = \{y,w\}$ is a module of $\sigma[X \cup \{w\}]$, so $w \in X_\sigma(y)$. Therefore, v and w belong to the same block of $p_{(\sigma,X)}$.

Case 2: There is $y \in X$ such that $X \cap M = \{y\}$.

We verify that $M \smallsetminus \{y\} \neq \varnothing$ and $M \smallsetminus \{y\} \subseteq X_\sigma(y)$. We have $M \smallsetminus \{y\} \neq \varnothing$ because $|M| \geq 2$. For each $v \in M \smallsetminus \{y\}$, it follows from assertion (M2) of Proposition 2.5 that $(X \cup \{v\}) \cap M = \{y,v\}$ is a module of $\sigma[X \cup \{v\}]$ or, equivalently, $v \in X_\sigma(y)$.

Case 3: $X \subseteq M$.

Since M is a nontrivial module of σ, we have $M \subsetneq V(\sigma)$. Moreover, $(V(\sigma) \smallsetminus M) \subseteq \langle X\rangle_\sigma$. Indeed, for each $v \in V(\sigma) \smallsetminus M$, it follows from assertion (M2) of Proposition 2.5 that $(X \cup \{v\}) \cap M = X$ is a module of $\sigma[X \cup \{v\}]$ or, equivalently, $v \in \langle X\rangle_\sigma$.

Lemma 3.17. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. The following statements hold.*

(P1) *For $v \in \langle X\rangle_\sigma$ and $w \in X_\sigma(y)$, where $y \in X$, if $\sigma[X \cup \{v,w\}]$ is decomposable, then $X \cup \{w\}$ and $\{y,w\}$ are the only nontrivial modules of $\sigma[X \cup \{v,w\}]$.*

(P2) *For $v \in \langle X\rangle_\sigma$ and $w \in \mathrm{Ext}_\sigma(X)$, if $\sigma[X \cup \{v,w\}]$ is decomposable, then $X \cup \{w\}$ is the unique nontrivial module of $\sigma[X \cup \{v,w\}]$.*

(P3) *Given distinct $y,z \in X$, for $v \in X_\sigma(y)$ and $w \in X_\sigma(z)$, if $\sigma[X \cup \{v,w\}]$ is decomposable, then $\{y,v\}$ and $\{z,w\}$ are the only nontrivial modules of $\sigma[X \cup \{v,w\}]$.*

(P4) *Given* $y \in X$*, for* $v \in X_\sigma(y)$ *and* $w \in \mathrm{Ext}_\sigma(X)$*, if* $\sigma[X \cup \{v,w\}]$ *is decomposable, then* $\{y,v\}$ *is the unique nontrivial module of* $\sigma[X \cup \{v,w\}]$.

(P5) *Given distinct* $v,w \in \mathrm{Ext}_\sigma(X)$*, if* $\sigma[X \cup \{v,w\}]$ *is decomposable, then* $\{v,w\}$ *is the unique nontrivial module of* $\sigma[X \cup \{x,y\}]$.

Proof. For statements (P1),. . . ,(P5) above, consider $v,w \in V(\sigma) \setminus X$ such that $v \neq w$. Suppose that $\sigma[X \cup \{v,w\}]$ admits a nontrivial module M. By assertion (M2) of Proposition 2.5, $X \cap M$ is a module of $\sigma[X]$. Since $\sigma[X]$ is prime, we obtain $|X \cap M| \leq 1$ or $X \cap M = X$. Observe that in statements (P1), (P2), (P3), and (P4) above, v and w do not belong to the same block of $p_{(\sigma,X)}$. By Remark 3.16, we have $X \cap M \neq \varnothing$. Hence, we have $|X \cap M| = 1$ or $X \cap M = X$ in statements (P1), (P2), (P3), and (P4) above.

For statement (P1), suppose that $v \in \langle X \rangle_\sigma$ and $w \in X_\sigma(y)$, where $y \in X$. As above observed, $X \subseteq M$ or there is $z \in X$ such that $X \cap M = \{z\}$. First, suppose that $X \subseteq M$. Since $w \in X_\sigma(y)$, $w \notin \langle X \rangle_\sigma$ by Lemma 3.13. It follows from Remark 3.16 that $w \in M$. Since $M \neq X \cup \{v,w\}$, we obtain $M = X \cup \{w\}$. Thus $v \longleftrightarrow_\sigma X \cup \{w\}$, so $v \longleftrightarrow_\sigma \{y,w\}$. Since $w \in X_\sigma(y)$, that is, $\{y,w\}$ is a module of $\sigma[X \cup \{w\}]$, we obtain that $\{y,w\}$ is a module of $\sigma[X \cup \{v,w\}]$. Second, suppose that $X \cap M = \{z\}$. By Lemma 3.13, $v \notin X_\sigma(z)$. It follows from Remark 3.16 that $v \notin M$. Therefore $M = \{z,w\}$ because $|M| \geq 2$. We obtain $w \in X_\sigma(z)$. By Lemma 3.13, $y = z$. Since $\{y,w\}$ is a module of $\sigma[X \cup \{v,w\}]$, we have $v \longleftrightarrow_\sigma \{y,w\}$. We have also $v \longleftrightarrow_\sigma X$ because $v \in \langle X \rangle_\sigma$. Thus $v \longleftrightarrow_\sigma X \cup \{w\}$, and hence $X \cup \{w\}$ is a module of $\sigma[X \cup \{v,w\}]$.

For statement (P2), suppose that $v \in \langle X \rangle_\sigma$ and $w \in \mathrm{Ext}_\sigma(X)$. We verify that $|X \cap M| \geq 2$. Otherwise, there exits $y \in X$ such that $X \cap M = \{y\}$. By Remark 3.16, $M \setminus \{y\} \neq \varnothing$ and $M \setminus \{y\} \subseteq X_\sigma(y)$, which contradicts Lemma 3.13. Therefore, $|X \cap M| \geq 2$, and hence $X \subseteq M$. Since $w \notin \langle X \rangle_\sigma$, we obtain $w \in M$ by Remark 3.16. Hence $M = X \cup \{w\}$ because $M \subsetneq X\{v,w\}$.

For statement (P3), suppose that $v \in X_\sigma(y)$ and $w \in X_\sigma(z)$, where $y,z \in X$ and $y \neq z$. Suppose for a contradiction that $X \subseteq M$. By Remark 3.16, $\{v,w\} \setminus M \neq \varnothing$ and $\{v,w\} \setminus M \subseteq \langle X \rangle_\sigma$, which contradicts Lemma 3.13. Consequently, $X \setminus M \neq \varnothing$, and hence there exists $t \in X$ such that $X \cap M = \{t\}$. By Remark 3.16, $M \setminus \{t\} \neq \varnothing$ and $M \setminus \{t\} \subseteq X_\sigma(t)$. It follows from Lemma 3.13 that $t \in \{y,z\}$. By interchanging y and z, and hence v and w if necessary, we can assume that $y = t$. As previously, $M \setminus \{y\} \neq \varnothing$ and $M \setminus \{y\} \subseteq X_\sigma(y)$. By Lemma 3.13, $w \notin X_\sigma(y)$, and hence $w \notin M$. Since $|M| \geq 2$, we obtain $M = \{y,v\}$. It remains to show that $\{z,w\}$ is a module of $\sigma[X \cup \{x,y\}]$ as well. Since $\{z,w\}$ is a module of $\sigma[X \cup \{w\}]$, it suffices to verify that $v \longleftrightarrow_\sigma \{z,w\}$. We have $[z,y]_\sigma = [z,v]_\sigma$ and $[w,y]_\sigma = [w,v]_\sigma$ because $\{y,v\}$ is a module of $\sigma[X \cup \{v,w\}]$. Furthermore, we have $[z,y]_\sigma = [w,y]_\sigma$ because $\{z,w\}$ is a module of $\sigma[X \cup \{w\}]$. Therefore $[z,v]_\sigma = [w,v]_\sigma$.

For statement (P4), suppose that $v \in X_\sigma(y)$, where $y \in X$, and $w \in \mathrm{Ext}_\sigma(X)$. Suppose for a contradiction that $X \subseteq M$. By Remark 3.16,

$\{v,w\} \setminus M \neq \emptyset$ and $\{v,w\} \setminus M \subseteq \langle X \rangle_\sigma$, which contradicts Lemma 3.13. Consequently, $X \setminus M \neq \emptyset$, and hence there exists $z \in X$ such that $X \cap M = \{z\}$. By Remark 3.16, $M \setminus \{z\} \neq \emptyset$ and $M \setminus \{z\} \subseteq X_\sigma(z)$. By Lemma 3.13, $w \notin X_\sigma(v)$, so $w \notin M$. Since $|M| \geq 2$, we obtain $M = \{z,v\}$. By Lemma 3.13, we have $z = y$.

For statement (P5), suppose that $v,w \in \mathrm{Ext}_\sigma(X)$. First, suppose that $X \subseteq M$. By Remark 3.16, $\{v,w\} \setminus M \neq \emptyset$ and $\{v,w\} \setminus M \subseteq \langle X \rangle_\sigma$. It follows from Lemma 3.13 that $X \setminus M \neq \emptyset$. Second, suppose that there exists $y \in X$ such that $X \cap M = \{y\}$. By Remark 3.16, $M \setminus \{y\} \neq \emptyset$ and $M \setminus \{y\} \subseteq X_\sigma(y)$. It follows from Lemma 3.13 that $|X \cap M| \neq 1$. Consequently, we have $X \setminus M \neq \emptyset$, and $|X \cap M| \neq 1$. By Remark 3.16, $X \cap M = \emptyset$, and hence $M = \{v,w\}$. □

The following result is a direct consequence of Lemma 3.17.

Corollary 3.18. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. The following two assertions hold.*

(Q1) *For $v \in \langle X \rangle_\sigma$ and $w \in V(\sigma) \setminus (X \cup \langle X \rangle_\sigma)$, if $\sigma[X \cup \{v,w\}]$ is decomposable, then $X \cup \{w\}$ is a nontrivial module of $\sigma[X \cup \{v,w\}]$.*

(Q2) *Given $y \in X$, for $v \in X_\sigma(y)$ and $w \in V(\sigma) \setminus (X \cup X_\sigma(y))$, if $\sigma[X \cup \{v,w\}]$ is decomposable, then $\{y,v\}$ is a nontrivial module of $\sigma[X \cup \{v,w\}]$.*

At present, we are ready to establish the Ehrenfeucht–Rozenberg theorem.

Theorem 3.19 (Ehrenfeucht and Rozenberg [13])**.** *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $|V(\sigma) \setminus X| \geq 2$ and $\sigma[X]$ is primitive. If σ is prime, then there exist distinct $v,w \in V(\sigma) \setminus X$ such that $\sigma[X \cup \{v,w\}]$ is primitive. More precisely, if σ is prime, then the following two statements hold.*

(1) *If $\langle X \rangle_\sigma \neq \emptyset$, then there exist $v \in \langle X \rangle_\sigma$ and $w \in V(\sigma) \setminus (X \cup \langle X \rangle_\sigma)$ such that $\sigma[X \cup \{v,w\}]$ is prime.*

(2) *For each $y \in X$, if $X_\sigma(y) \neq \emptyset$, then there exist $v \in X_\sigma(y)$ and $w \in V(\sigma) \setminus (X \cup X_\sigma(y))$ such that $\sigma[X \cup \{v,w\}]$ is prime.*

Proof. First, suppose that $\langle X \rangle_\sigma \neq \emptyset$. Since σ is prime, $V(\sigma) \setminus \langle X \rangle_\sigma$ is not a module of σ. Thus, there exists $v \in \langle X \rangle_\sigma$ such that $v \not\longleftrightarrow_\sigma V(\sigma) \setminus \langle X \rangle_\sigma$. But $v \longleftrightarrow_\sigma X$ because $v \in \langle X \rangle_\sigma$. It follows that there exists $w \in V(\sigma) \setminus (X \cup \langle X \rangle_\sigma)$ such that $v \not\longleftrightarrow_\sigma X \cup \{w\}$. It follows from assertion (Q1) of Corollary 3.18 that $\sigma[X \cup \{v,w\}]$ is prime.

Second, consider $y \in X$ such that $X_\sigma(y) \neq \emptyset$. Since σ is prime, $\{y\} \cup X_\sigma(y)$ is not a module of σ. By Remark 3.15, $\{y\} \cup X_\sigma(y)$ is a module of $\sigma[X \cup X_\sigma(y)]$. Consequently, there exists $w \in V(\sigma) \setminus (X \cup X_\sigma(y))$ such that $w \not\longleftrightarrow_\sigma \{y\} \cup X_\sigma(y)$. Observe that for $u \in V(\sigma) \setminus (\{y\} \cup X_\sigma(y))$, we have $u \longleftrightarrow_\sigma \{y\} \cup X_\sigma(y)$ if and only if $u \longleftrightarrow_\sigma \{y,v\}$ for every $v \in X_\sigma(y)$. It follows that there is $v \in X_\sigma(y)$ such that $w \not\longleftrightarrow_\sigma \{y,v\}$. Therefore, $\{y,v\}$ is not a module of $\sigma[X \cup \{v,w\}]$. It follows from assertion (Q2) of Corollary 3.18 that $\sigma[X \cup \{v,w\}]$ is prime.

Finally, suppose that $\langle X\rangle_\sigma = \varnothing$, and $X_\sigma(y) = \varnothing$ for each $y \in X$. By Lemma 3.13, we have $V(\sigma)\smallsetminus X = \mathrm{Ext}_\sigma(X)$. Since σ is prime, $V(\sigma)\smallsetminus X$ is not a module of σ. Therefore, there exist $y \in X$ and distinct $v, w \in V(\sigma)\smallsetminus X$ such that $y \not\leftrightarrow_\sigma \{v, w\}$. We obtain that $\{v, w\}$ is not a module of $\sigma[X \cup \{v, w\}]$. Since $v, w \in \mathrm{Ext}_\sigma(X)$, it follows from statement (P5) of Lemma 3.17 that $\sigma[X \cup \{v, w\}]$ is prime. □

The next result, called the *parity property*, follows by applying Theorem 3.19 several times. It also provides an upward hereditary property of primality.

Corollary 3.20 (Ehrenfeucht and Rozenberg [13]). *Given a prime 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. For each $n \geq 0$ such that $|V(\sigma) \smallsetminus X| \geq 2n$, there exists $Y \subseteq V(\sigma) \smallsetminus X$ such that $|Y| = 2n$ and $\sigma[X \cup Y]$ is prime.*

The next result is a simple consequence of Corollary 3.20.

Corollary 3.21 (Ehrenfeucht and Rozenberg [13]). *Given a prime 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. There exist $v, w \in V(\sigma) \smallsetminus X$ such that $\sigma - \{v, w\}$ is prime.*

Proof. It suffices to apply Corollary 3.20 with $n = \lceil \frac{v(\sigma)-|X|}{2} \rceil - 1$. □

The first downward hereditary property of primality ends the section. It is an immediate consequence of Theorem 3.10 and Corollary 3.21. The second downward hereditary property of primality is the Schmerl–Trotter theorem (see Theorem 5.3).

Proposition 3.22 (Ehrenfeucht and Rozenberg [13]). *Given a prime 2-structure σ, with $v(\sigma) \geq 5$, there exist $v, w \in V(\sigma)$ such that $\sigma - \{v, w\}$ is prime.*

Proof. By Theorem 3.10, there exists $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime, and $|X| = 3$ or 4. To conclude, it suffices to apply Corollary 3.21. □

4. Critical 2-structures

Given $n \geq 2$, the tournament T_{2n+1} (see Figure 1.2) is prime by Fact 2.7. Moreover, we have

- $T_{2n+1} - 2n = L_{2n}$ and, for instance, $\{0,1\}$ is a nontrivial module of $T_{2n+1} - 2n$;
- $\{2,\dots,2n\}$ is a nontrivial module of $T_{2n+1} - 0$;
- $\{0,\dots,2n-3\}$ is a nontrivial module of $T_{2n+1} - (2n-1)$;
- for $1 \leq p \leq 2n-2$, $\{p-1,p+1\}$ is a nontrivial module of $T_{2n+1} - p$.

This leads us to the following definition.

Definition 4.1. Given a prime 2-structure σ, a vertex v of σ is *critical* (in terms of primality) if $\sigma - v$ is decomposable. The set of the noncritical vertices of σ is called the *support* of σ, it is denoted by $\mathscr{S}(\sigma)$. Generally, a proper subset W of $V(\sigma)$ is *critical* if $\sigma - W$ is decomposable. A primitive 2-structure is *critical* if all its vertices are critical.

critical

support

From the example above, given $n \geq 2$, the tournament T_{2n+1} is critical. Since critical 2-structures exist, the only attempt to improve Proposition 3.22 is to answer the following question positively.

Question 4.2. Let σ be a prime 2-structure. If $v(\sigma)$ is large enough, then does there exist $Z \subsetneq V(\sigma)$ such that $\sigma[Z]$ is prime and $|V(\sigma) \setminus Z| = 2$?

The second downward hereditary property of primality, that is, the Schmerl–Trotter theorem (see Theorem 5.3) answered Question 4.2 positively. Before providing such an answer, Schmerl and Trotter [33] characterized the critical partial orders, graphs, tournaments, etc.. Bonizzoni [4] independently characterized the critical 2-structures. To describe the structure of the critical digraphs, Boubabbous and Ille [7] study the components of the primality graph[4.1] associated with every prime 2-structure. The primality graph was introduced by Ille [20] as below. It plays a decisive role in the structural study of the prime 2-structures.

4.1. **The primality graph.**

Definition 4.3. Given a prime 2-structure σ, the *primality graph* $\mathbb{P}(\sigma)$ of σ is the graph defined on $V(\mathbb{P}(\sigma)) = V(\sigma)$, the edges of which are the noncritical unordered pairs. Therefore, given $v, w \in V(\sigma)$, with $v \neq w$, we have

primality graph

$$\{v,w\} \in E(\mathbb{P}(\sigma)) \text{ if and only if } \sigma - \{v,w\} \text{ is prime.}$$

To begin, given a prime 2-structure σ, we examine the neighbourhood $N_{\mathbb{P}(\sigma)}(v)$ of a critical vertex v of σ.

Lemma 4.4 (Ille [20])**.** *Let σ be a prime 2-structure with $v(\sigma) \geq 5$. For every $v \in V(\sigma) \setminus \mathscr{S}(\sigma)$, we have $d_{\mathbb{P}(\sigma)}(v) \leq 2$. Moreover, we have*

[4.1] The same approach is adopted in [6] to characterize the critical infinite digraphs.

(1) *if* $d_{\mathbb{P}(\sigma)}(v) = 1$*, then* $V(\sigma) \smallsetminus (N_{\mathbb{P}(\sigma)}(v) \cup \{v\})$ *is the unique nontrivial module of* $\sigma - v$*;*
(2) *if* $d_{\mathbb{P}(\sigma)}(v) = 2$*, then* $N_{\mathbb{P}(\sigma)}(v)$ *is the unique nontrivial module of* $\sigma - v$*.*

Proof. To begin, we prove that $d_{\mathbb{P}(\sigma)}(v) \leq 2$ for each $v \in V(\sigma) \smallsetminus \mathscr{S}(\sigma)$. Consider $v \in V(\sigma) \smallsetminus \mathscr{S}(\sigma)$ such that $N_{\mathbb{P}(\sigma)}(v) \neq \varnothing$. Let $w \in N_{\mathbb{P}(\sigma)}(v)$. Set

$$X = V(\sigma) \smallsetminus \{v, w\}.$$

Hence, $\sigma[X]$ is prime. Since $v \notin \mathscr{S}(\sigma)$, $\sigma - v$ is decomposable. Thus,

$$w \notin \mathrm{Ext}_\sigma(X) \text{ (see Notation 3.12).} \tag{4.1}$$

By Lemma 3.13, $w \in \langle X \rangle_\sigma$ or $w \in X_\sigma(y)$, where $y \in X$. Therefore, we distinguish the following two cases.

CASE 1: $w \in \langle X \rangle_\sigma$.

For every $y \in X$, $X \smallsetminus \{y\}$ is a nontrivial module of $\sigma - \{v, y\}$. Therefore $y \notin N_{\mathbb{P}(\sigma)}(v)$. Consequently,

$$\text{if there exists } w \in N_{\mathbb{P}(\sigma)}(v) \cap \langle X \rangle_\sigma \text{ , then } N_{\mathbb{P}(\sigma)}(v) = \{w\}. \tag{4.2}$$

CASE 2: There exists $y \in X$ such that $w \in X_\sigma(y)$.

For every $z \in X \smallsetminus \{y\}$, $\{y, w\}$ is a nontrivial module of $\sigma - \{v, z\}$. Consequently, $z \notin N_{\mathbb{P}(\sigma)}(v)$, and hence $N_{\mathbb{P}(\sigma)}(v) \subseteq \{y, w\}$. Since $\{y, w\}$ is a module of $\sigma[X \cup \{w\}]$, the function $X \longrightarrow (X \smallsetminus \{y\}) \cup \{w\}$, defined by $y \mapsto w$ and $z \mapsto z$ for every $z \in X \smallsetminus \{y\}$, is an isomorphism from $\sigma - \{v, w\}$ onto $\sigma - \{v, y\}$. It follows that $y \in N_{\mathbb{P}(\sigma)}(v)$. Consequently, given $y \in X$,

$$\text{if there exists } w \in N_{\mathbb{P}(\sigma)}(v) \cap X_\sigma(y) \text{, then } N_{\mathbb{P}(\sigma)}(v) = \{y, w\}. \tag{4.3}$$

It follows from both cases above that $d_{\mathbb{P}(\sigma)}(v) \leq 2$.

Now, consider $v \in V(\sigma) \smallsetminus \mathscr{S}(\sigma)$ such that $d_{\mathbb{P}(\sigma)}(v) = 1$. Denote by w the unique neighbour of v in $\mathbb{P}(\sigma)$. Set $X = V(\sigma) \smallsetminus \{v, w\}$. It follows from (4.3) that $w \notin X_\sigma(y)$ for every $y \in X$. Moreover, $w \notin \mathrm{Ext}_\sigma(X)$ by (4.1). By Lemma 3.13, $w \in \langle X \rangle_\sigma$, and $V(\sigma) \smallsetminus \{v, w\}$ is the only nontrivial module of $\sigma - v$.

Lastly, consider $v \in V(\sigma) \smallsetminus \mathscr{S}(\sigma)$ such that $d_{\mathbb{P}(\sigma)}(v) = 2$. Denote by w and w' the neighbours of v in $\mathbb{P}(\sigma)$. Set $X = V(\sigma) \smallsetminus \{v, w\}$. It follows from (4.2) that $w \notin \langle X \rangle_\sigma$. Moreover, $w \notin \mathrm{Ext}_\sigma(X)$ by (4.1). By Lemma 3.13, there exists $y \in X$ such that $w \in X_\sigma(y)$. By (4.3), $N_{\mathbb{P}(\sigma)}(v) = \{y, w\}$. Therefore, $w' = y$, so $w \in X_\sigma(w')$. It follows from Lemma 3.13 that $\{w, w'\}$ is the only nontrivial module of $\sigma - v$. □

Given a prime 2-structure σ, consider a component C of $\mathbb{P}(\sigma)$ such that $v(C) \geq 2$ and $V(C) \subseteq V(\sigma) \smallsetminus \mathscr{S}(\sigma)$. It follows from Lemma 4.4 that C is a cycle or a path.

Proposition 4.5 (Boudabbous and Ille [7]). *Let* σ *be a prime 2-structure with* $v(\sigma) \geq 5$*. For every component* C *of* $\mathbb{P}(\sigma)$ *such that* $v(C) \geq 2$ *and* $V(C) \subseteq V(\sigma) \smallsetminus \mathscr{S}(\sigma)$*, the following statements hold.*

(1) *If C is a cycle, then its length is odd and $V(C) = V(\sigma)$;*
(2) *If C is a path of odd length, then $|V(\sigma) \setminus V(C)| \le 1$;*
(3) *If C is a path of even length, then $V(C) = V(\sigma)$.*

Proof. We denote the vertices of C by $0, \dots, v(C) - 1$ in such a way that $C = C_{v(C)}$ or $P_{v(C)}$.

First, suppose that $v(C) \ge 3$ and $C = C_{v(C)}$. For a contradiction, suppose that $v(C)$ is even. Hence, $v(C) = 2n$, where $n \ge 2$. We show that $\{1, 2n-1\}$ is a nontrivial module of σ. Since $N_{\mathbb{P}(\sigma)}(0) = \{1, 2n-1\}$, $\{1, 2n-1\}$ is a module of $\sigma - 0$ by Lemma 4.4. To show that $\{1, 2n-1\}$ is a nontrivial module of σ, it suffices to verify that $[0,1]_\sigma = [0, 2n-1]_\sigma$. For $m \in \{1, \dots, n-1\}$, we have $N_{\mathbb{P}(\sigma)}(2m) = \{2m-1, 2m+1\}$. By Lemma 4.4, $\{2m-1, 2m+1\}$ is a module of $\sigma - 2m$. In particular, we obtain $[0, 2m-1]_\sigma = [0, 2m+1]_\sigma$. Therefore, we have $[0,1]_\sigma = [0,3]_\sigma = \dots = [0, 2n-1]_\sigma$. Consequently, $\{1, 2n-1\}$ is a nontrivial module of σ, which contradicts the fact that σ is prime. It follows that $v(C)$ is odd. Hence $v(C) = 2n+1$, where $n \ge 1$. For a contradiction, suppose that $V(C) \subsetneq V(\sigma)$. We show that $V(C)$ is a module of σ. Consider $v \in V(\sigma) \setminus V(C)$. For $m \in \{0, \dots, n-1\}$, we have $N_{\mathbb{P}(\sigma)}(2m+1) = \{2m, 2m+2\}$. By Lemma 4.4, $\{2m, 2m+2\}$ is a module of $\sigma - (2m+1)$. We obtain $[v,0]_\sigma = [v,2]_\sigma = \dots = [v,2n]_\sigma$, so $v \longleftrightarrow_\sigma \{0, 2, \dots, 2n\}$. Similarly, since for $m \in \{1, \dots, n-1\}$, $N_{\mathbb{P}(\sigma)}(2m) = \{2m-1, 2m+1\}$, we have $v \longleftrightarrow_\sigma \{1, 3, \dots, 2n-1\}$. Since $N_{\mathbb{P}(\sigma)}(0) = \{1, 2n\}$, $[v, 2n]_\sigma = [v, 1]_\sigma$. It follows that $v \longleftrightarrow_\sigma V(C)$. Consequently, $V(C)$ is a nontrivial module of σ, which contradicts the fact that σ is prime. Therefore $V(C) = V(\sigma)$.

Second, suppose that $v(C) \ge 2$, $v(C)$ is even, and $C = P_{v(C)}$. Hence $v(C) = 2n$, where $n \ge 1$. For a contradiction, suppose that $n = 1$. We obtain $N_{\mathbb{P}(\sigma)}(0) = \{1\}$ and $N_{\mathbb{P}(\sigma)}(1) = \{0\}$. By Lemma 4.4, $V(\sigma) \setminus \{0,1\}$ is a module of $\sigma - 0$ and $\sigma - 1$. Thus $V(\sigma) \setminus \{0,1\}$ is a nontrivial module of σ. Therefore $n \ge 2$. We show that $V(\sigma) \setminus V(C)$ is a module of σ. Consider $v \in V(\sigma) \setminus V(C)$. Since $N_{\mathbb{P}(\sigma)}(2m+1) = \{2m, 2m+2\}$ for $m \in \{0, \dots, n-2\}$, we have $v \longleftrightarrow_\sigma \{0, 2, \dots, 2n-2\}$. Moreover, since $N_{\mathbb{P}(\sigma)}(2n-1) = \{2n-2\}$, $[v, 2n-2]_\sigma = [1, 2n-2]_\sigma$. It follows that for any $v, w \in V(\sigma) \setminus V(C)$ and $m \in \{0, \dots, n-1\}$, $[v, 2m]_\sigma = [w, 2m]_\sigma$. Similarly, for $v \in V(\sigma) \setminus V(C)$, we have $v \longleftrightarrow_\sigma \{1, 3, \dots, 2n-1\}$ because $N_{\mathbb{P}(\sigma)}(2m) = \{2m-1, 2m+1\}$ for $m \in \{1, \dots, n-1\}$. Now, since $N_{\mathbb{P}(\sigma)}(0) = \{1\}$, $[v,1]_\sigma = [2n-2, 1]_\sigma$. Consequently, for any $v, w \in V(\sigma) \setminus V(C)$ and $m \in \{0, \dots, n-1\}$, $[v, 2m+1]_\sigma = [w, 2m+1]_\sigma$. Consequently, $V(\sigma) \setminus V(C)$ is a module of σ. Since σ is prime, we obtain $|V(\sigma) \setminus V(C)| \le 1$.

Lastly, suppose that $v(C)$ is odd. Hence $v(C) = 2n+1$, where $n \ge 1$. For a contradiction, suppose that $V(C) \subsetneq V(\sigma)$. We show that $V(\sigma) \setminus \{1\}$ is a nontrivial module of σ. Since $N_{\mathbb{P}(\sigma)}(0) = \{1\}$, $V(\sigma) \setminus \{0,1\}$ is a module of $\sigma - 0$ by Lemma 4.4. Let $v \in V(\sigma) \setminus V(C)$. It suffices to verify that $[1, v]_\sigma = [1, 0]_\sigma$. We distinguish the following two cases.

Case 1: $n = 1$.

We have $N_{\mathbb{P}(\sigma)}(2) = \{1\}$. By Lemma 4.4, $V(\sigma) \smallsetminus \{1,2\}$ is a module of $\sigma - 2$. In particular, we obtain $[1,v]_\sigma = [1,0]_\sigma$.

CASE 2: $n \geq 2$.

Since for $m \in \{1,\dots,n-1\}$, $N_{\mathbb{P}(\sigma)}(2m) = \{2m-1, 2m+1\}$, we obtain $v \longleftrightarrow_\sigma \{1,3,\dots,2n-1\}$ and $0 \longleftrightarrow_\sigma \{1,3,\dots,2n-1\}$. Therefore, $[1,v]_\sigma = [2n-1,v]_\sigma$ and $[1,0]_\sigma = [2n-1,0]_\sigma$. Furthermore, since $N_{\mathbb{P}(\sigma)}(2n) = \{2n-1\}$, $[2n-1,v]_\sigma = [2n-1,0]_\sigma$. It follows that $[1,v]_\sigma = [1,0]_\sigma$.

In both cases above, $V(\sigma) \smallsetminus \{1\}$ is a nontrivial module of σ, which contradicts the fact that σ is prime. Consequently, $V(C) = V(\sigma)$. □

Corollary 4.6 (Boudabbous and Ille [7]). *For every critical 2-structure σ, with $v(\sigma) \geq 5$, there exists $n \geq 2$ such that $\mathbb{P}(\sigma)$ is isomorphic to $P_{2n} \oplus K_{\{2n\}}$, P_{2n+1}, C_{2n+1}, or P_{2n}. (In the last instance, $n \geq 3$.)*

Proof. By Proposition 3.22, there exist $v, w \in V(\sigma)$ such that $\sigma - \{v,w\}$ is prime. Since σ is critical, we have $v \neq w$, and hence $\{v,w\} \in E(\mathbb{P}(\sigma))$. Consider the component C of $\mathbb{P}(\sigma)$ containing v and w. As observed before stating Proposition 4.5, it follows from Lemma 4.4 that C is a cycle or a path. To begin, suppose that C is a cycle. It follows from Proposition 4.5 that there exists $n \geq 2$ such that $\mathbb{P}(\sigma) \simeq C_{2n+1}$. Similarly, if there exists $n \geq 2$ such that C is a path of length $2n$, then $V(C) = V(\sigma)$, and hence $\mathbb{P}(\sigma) \simeq P_{2n+1}$. Lastly, suppose that there is $n \geq 2$ such that C is a path of length $2n-1$. By Proposition 4.5, $|V(\sigma) \smallsetminus V(C)| \leq 1$. Obviously, if $V(C) = V(\sigma)$, then $n \geq 3$ and $\mathbb{P}(\sigma) \simeq P_{2n}$. Suppose that $|V(\sigma) \smallsetminus C| = 1$. The single element of $V(\sigma) \smallsetminus V(C)$ is an isolated vertex of $\mathbb{P}(\sigma)$ because C is a component of $\mathbb{P}(\sigma)$. Therefore, $\mathbb{P}(\sigma) \simeq P_{2n} \oplus K_{\{2n\}}$. □

We end the section with some specific results on critical 2-structures. The first one follows from Corollary 3.20.

Corollary 4.7. *Let σ be a critical 2-structure σ such that $v(\sigma) \geq 5$. Let $X \subsetneq V(\sigma)$.*

1. *If $\sigma[X]$ is prime, then $v(\sigma) - |X|$ is even.*
2. *Moreover, if $\sigma[X]$ is prime and $|X| \geq 4$, then $\sigma[X]$ is critical.*

Proof. Let $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. For a contradiction, suppose that $v(\sigma) - |X| = 2n+1$, where $n \geq 0$. By Corollary 3.20, there exists $Y \subseteq V(\sigma) \smallsetminus X$ such that $|Y| = 2n$ and $\sigma[X \cup Y]$ is prime. We have $|V(\sigma) \smallsetminus (X \cup Y)| = 1$. By denoting by v the unique element of $V(\sigma) \smallsetminus (X \cup Y)$, we obtain $\sigma - v$ is prime, which contradicts the fact that σ is critical. Consequently, $v(\sigma) - |X|$ is even.

Now, suppose that $|X| \geq 4$. For each $x \in X$, we have $v(\sigma) - |X \smallsetminus \{x\}|$ is odd. It follows from the first assertion that $\sigma[X \smallsetminus \{x\}]$ is decomposable. Consequently, σ is critical. □

The second result follows from Lemma 4.4 and Corollary 4.6.

Corollary 4.8. *Let σ be a critical 2-structure σ such that $v(\sigma) \geq 5$. For $e, f \in E(\mathbb{P}(\sigma))$, we have $\sigma - e \simeq \sigma - f$.*

Proof. Consider distinct $e, f \in E(\mathbb{P}(\sigma))$. It follows from Corollary 4.6 that e and f are contained in the same component of $\mathbb{P}(\sigma)$. Consequently, there exist distinct vertices $v_0, \ldots, v_p$ of σ satisfying

- $\{v_0, v_1\} = e$;
- $p \geq 2$, and $\{v_{p-1}, v_p\} = f$;
- for $i \in \{0, \ldots, p-1\}$, $\{v_i, v_{i+1}\} \in E(\mathbb{P}(\sigma))$.

Let $i \in \{0, \ldots, p-1\}$. We have $v_{i-1}, v_{i+1} \in N_{\mathbb{P}(\sigma)}(v_i)$. Since $v_i \notin \mathscr{S}(\sigma)$, it follows from Lemma 4.4 that $N_{\mathbb{P}(\sigma)}(v_i) = \{v_{i-1}, v_{i+1}\}$, and $\{v_{i-1}, v_{i+1}\}$ is a module of $\sigma - v_i$. Thus, $\sigma - \{v_{i-1}, v_i\} \simeq \sigma - \{v_i, v_{i+1}\}$. It follows that $\sigma - \{v_0, v_1\} \simeq \sigma - \{v_{p-1}, v_p\}$, that is, $\sigma - e \simeq \sigma - f$. □

The third result is an easy consequence of Corollaries 4.7 and 4.8.

Corollary 4.9. *Let σ be a critical 2-structure σ such that $v(\sigma) \geq 6$. Let $X, Y \subsetneq V(\sigma)$ such that $|X| = |Y|$ and $|X| \geq 4$. If $\sigma[X]$ and $\sigma[Y]$ are prime, then $\sigma[X] \simeq \sigma[Y]$.*

Proof. By Corollary 3.21, there exist $x, x' \in V(\sigma) \smallsetminus X$ such that $\sigma - \{x, x'\}$ is prime. Since σ is critical, we have $x \neq x'$. Thus, $v(\sigma) - |X| \geq 2$. We proceed by induction on $v(\sigma) - |X| \geq 2$. If $v(\sigma) - |X| = 2$, then it suffices to apply Corollary 4.8. Hence, suppose that $v(\sigma) - |X| \geq 3$. Similarly, there exist distinct $y, y' \in V(\sigma) \smallsetminus Y$ such that $\sigma - \{y, y'\}$ is prime. By Corollary 4.8, $\sigma - \{x, x'\} \simeq \sigma - \{y, y'\}$. Therefore, there exists $Y' \subsetneq V(\sigma) \smallsetminus \{x, x'\}$ such that $\sigma[Y'] \simeq \sigma[Y]$. Since $\sigma - \{x, x'\}$ is prime and $v(\sigma) \geq 6$, $\sigma - \{x, x'\}$ is critical by Corollary 4.7. To conclude, it suffices to apply the induction hypothesis to $\sigma - \{x, x'\}$ with $\sigma[X]$ and $\sigma[Y']$. □

Lastly, we obtain the following result.

Corollary 4.10. *Let σ be a critical 2-structure σ such that $v(\sigma) \geq 6$. Consider $X, Y \subsetneq V(\sigma)$ such that $\sigma[X]$ and $\sigma[Y]$ are prime. If $4 \leq |X| \leq |Y|$, then $\sigma[X]$ embeds into $\sigma[Y]$.*

Proof. By Corollary 4.9, $\sigma[X] \simeq \sigma[Y]$ if $|X| = |Y|$. Hence, suppose that $|X| < |Y|$. By Corollary 4.7, there exist $m > n \geq 0$ such that $v(\sigma) - |X| = 2m$ and $v(\sigma) - |Y| = 2n$. By applying Theorem 3.19 $(m - n)$ times from $\sigma[X]$, we obtain $X \subsetneq X' \subseteq V(\sigma)$ such that $\sigma[X']$ is prime, and $|X'| = |Y|$. By Corollary 4.9, we have $\sigma[X'] \simeq \sigma[Y]$. It follows that $\sigma[X]$ embeds into $\sigma[Y]$. □

4.2. **The characterization of critical 2-structures.** Given a critical 2-structure σ, it follows from Corollary 4.6 that σ has four possible types according to whether $\mathbb{P}(\sigma)$ is isomorphic to P_{2n}, $P_{2n} \oplus K_{\{2n\}}$, P_{2n+1}, or C_{2n+1}. The following remark is very useful in the characterization of critical 2-structures of a given type.

Remark 4.11. Consider a set S. We denote by $\Delta(S)$ the set of all 2-structures defined on S. We consider the partial order $<_S$ defined on $\Delta(S)$ as follows. Given $\sigma, \tau \in \Delta(S)$, $\sigma <_S \tau$ if ($\sigma \neq \tau$ and) for every $e \in E(\sigma)$, there exists $f \in E(\tau)$ such that $e \subseteq f$. Consider $\sigma, \tau \in \Delta(S)$. As in Remark 2.23, we define their meet $\sigma \wedge \tau$ and their join $\sigma \vee \tau$ as follows. Given $x, y, v, w \in V(\sigma)$, with $x \neq y$ and $v \neq w$, $(x,y) \equiv_{\sigma\wedge\tau} (v,w)$ if $(x,y) \equiv_\sigma (v,w)$ and $(x,y) \equiv_\tau (v,w)$. Hence

$$E(\sigma \wedge \tau) = \{e \cap f : e \in E(\sigma), f \in E(\tau), e \cap f \neq \varnothing\}.$$

Given $x, y, v, w \in V(\sigma)$, with $x \neq y$ and $v \neq w$, $(x,y) \equiv_{\sigma\vee\tau} (v,w)$ if there exists a sequence $(e_0, \dots, e_n)$ of elements of $E(\sigma) \cup E(\tau)$ such that $(x,y) \in e_0$, $(v,w) \in e_n$, and (when $n \geq 1$,) $e_i \cap e_{i+1} \neq \varnothing$ for $0 \leq i \leq n-1$. Hence, $\Delta(S)$ ordered by $<_S$ is a lattice.

Since it is easy to verify that the next fact holds, we omit its proof.

Fact 4.12. *Given $\sigma, \tau \in \Delta(S)$, the following statements hold.*

(1) *If $\sigma <_S \tau$, then all the modules of σ are modules of τ.*
(2) *The modules of $\sigma \wedge \tau$ are exactly the modules of both σ and τ.*

We obtain the following consequences.

Fact 4.13. *Given $\sigma, \tau \in \Delta(S)$ such that $\sigma <_S \tau$, the next statements hold.*

(1) *If τ is prime, then σ is prime too.*
(2) *For each $n \in \{3, \dots, |S| - 1\}$, we have*

$$\mathscr{P}_n(\tau) \subseteq \mathscr{P}_n(\sigma) \quad \text{(see Notation 3.1).}$$

In particular, when τ is prime, we have $\mathscr{S}(\tau) \subseteq \mathscr{S}(\sigma)$ and $\mathbb{P}(\tau) \subseteq \mathbb{P}(\sigma)$.

It follows from the first statement of Fact 4.13 that the set of the prime 2-structures defined on S is an ideal of the lattice $(\Delta(S), <_S)$. We end the remark with the following consequence of Lemma 4.4, Fact 4.12, and Fact 4.13.

Fact 4.14. *Consider $\sigma, \tau \in \Delta(S)$. Suppose that σ and τ are critical. Suppose also that $\mathbb{P}(\sigma) = \mathbb{P}(\tau)$. Lastly, suppose that $\mathbb{P}(\sigma)$ does not have isolated vertices. Under these assumptions, we obtain that $\sigma \wedge \tau$ is critical, and*

$$\mathbb{P}(\sigma \wedge \tau) = \mathbb{P}(\sigma).$$

Proof. To begin, we verify that $\sigma \wedge \tau$ is prime. We have $\sigma \wedge \tau \leq_S \sigma$ and σ is prime. By the first statement of Fact 4.13, $\sigma \wedge \tau$ is prime.

Now, we show that $\sigma \wedge \tau$ is critical, and $\mathbb{P}(\sigma \wedge \tau) = \mathbb{P}(\sigma)$. Let $v \in V(\sigma)$. Since $\mathbb{P}(\sigma)$ does not have isolated vertices, $d_{\mathbb{P}(\sigma)}(v) \neq 0$. Since v is a critical vertex of σ, it follows from Lemma 4.4 that $d_{\mathbb{P}(\sigma)}(v) = 1$ or 2. We distinguish the following two cases.

Case 1: $d_{\mathbb{P}(\sigma)}(v) = 2$.

Since $\mathbb{P}(\sigma) = \mathbb{P}(\tau)$, we have $N_{\mathbb{P}(\sigma)}(v) = N_{\mathbb{P}(\tau)}(v)$. Furthermore, since v is a critical vertex of σ and τ, it follows from Lemma 4.4 that $N_{\mathbb{P}(\sigma)}(v)$ is a nontrivial module of $\sigma - v$ and $\tau - v$. Note that $(\sigma - v) \wedge (\tau - v) = (\sigma \wedge \tau) - v$. Therefore, it follows from the second statement of Fact 4.12 that $N_{\mathbb{P}(\sigma)}(v)$

is a nontrivial module of $(\sigma \wedge \tau) - v$. Thus, v is a critical vertex of $\sigma \wedge \tau$, and

$$N_{\mathbb{P}(\sigma\wedge\tau)}(v) \subseteq N_{\mathbb{P}(\sigma)}(v).$$

Lastly, it follows from the second statement of Fact 4.13 that

$$N_{\mathbb{P}(\sigma)}(v) \subseteq N_{\mathbb{P}(\sigma\wedge\tau)}(v).$$

Consequently, we obtain $N_{\mathbb{P}(\sigma\wedge\tau)}(v) = N_{\mathbb{P}(\sigma)}(v)$.

CASE 2: $d_{\mathbb{P}(\sigma)}(v) = 1$.

Since $\mathbb{P}(\sigma) = \mathbb{P}(\tau)$, we have $N_{\mathbb{P}(\sigma)}(v) = N_{\mathbb{P}(\tau)}(v)$. Furthermore, since v is a critical vertex of σ and τ, it follows from Lemma 4.4 that $V(\sigma) \smallsetminus (N_{\mathbb{P}(\sigma)}(v) \cup \{v\})$ is a nontrivial module of $\sigma - v$ and $\tau - v$. We conclude as in the preceding case.

It follows from both cases above that $\sigma \wedge \tau$ is critical, and $\mathbb{P}(\sigma \wedge \tau) = \mathbb{P}(\sigma)$. □

4.2.1. *The type P_{2n}.*

Proposition 4.15. *Given $n \geq 3$, consider a 2-structure τ defined on $V(\tau) = \{0, \ldots, 2n-1\}$. The following two statements are equivalent*

(1) *τ is critical and $\mathbb{P}(\tau) = P_{2n}$;*
(2) *$\prec 0,1 \succ_\tau \neq \prec 0,2 \succ_\tau$ (see Notation 1.1), and for $p, q \in \{0, \ldots, 2n-1\}$ such that $p < q$, we have*

$$[p,q]_\tau = \begin{cases} [0,1]_\tau & \text{if } p \text{ is even and } q \text{ is odd,} \\ [0,2]_\tau & \text{otherwise.} \end{cases} \tag{4.4}$$

Proof. To begin, suppose that τ is critical and $\mathbb{P}(\tau) = P_{2n}$. First, we show that (4.4) holds. Consider $p, q \in \{0, \ldots, 2n-1\}$ such that $p < q$. We prove that there exist $p' \in \{0, 1\}$ and $q' \in \{2n-2, 2n-1\}$ such that

$$p' \equiv p \bmod 2,\ q' \equiv q \bmod 2, \text{ and } [p,q]_\tau = [p',q']_\tau. \tag{4.5}$$

For instance, suppose that $p \geq 2$. Since $\mathbb{P}(\tau) = P_{2n}$, we have $N_{\mathbb{P}(\tau)}(p-1) = \{p-2, p\}$. By Lemma 4.4, $\{p-2, p\}$ is a module of $\tau - (p-1)$. In particular, we obtain $[p,q]_\tau = [p-2,q]_\tau$. By iteration, we obtain $p' \in \{0,1\}$ such that

$$p' \equiv p \bmod 2 \text{ and } [p,q]_\tau = [p',q]_\tau.$$

Similarly, we obtain $q' \in \{2n-2, 2n-1\}$ such that $q' \equiv q \bmod 2$ and $[p',q]_\tau = [p',q']_\tau$. Therefore, (4.5) holds. It follows from (4.5) that for any $p', q' \in \{0, \ldots, 2n-1\}$,

$$\text{if } p' < q',\ p' \equiv p \bmod 2 \text{ and } q' \equiv q \bmod 2, \text{ then } [p,q]_\tau = [p',q']_\tau. \tag{4.6}$$

We distinguish the following four cases, where $p, q \in \{0, \ldots, 2n-1\}$ such that $p < q$.

CASE 1: p and q are even.

By (4.6), $[p,q]_\tau = [0,2]_\tau$.

CASE 2: p and q are odd.

By (4.6), $[p,q]_\tau = [1,2n-1]_\tau$. Since $\mathbb{P}(\tau) = P_{2n}$, we have $N_{\mathbb{P}(\tau)}(0) = \{1\}$. By Lemma 4.4, $\{2,\dots,2n-1\}$ is a module of $\tau - 0$. In particular, we obtain $[1,2n-1]_\tau = [1,2n-2]_\tau$. Moreover, we have $N_{\mathbb{P}(\tau)}(2n-1) = \{2n-2\}$. By Lemma 4.4, $\{0,\dots,2n-3\}$ is a module of $\tau - (2n-1)$. In particular, we obtain $[1,2n-2]_\tau = [0,2n-2]_\tau$. By (4.6), $[0,2n-2]_\tau = [0,2]_\tau$. Consequently, we obtain $[p,q]_\tau = [0,2]_\tau$.

Case 3: p is even and q is odd.

By (4.6), $[p,q]_\tau = [0,1]_\tau$.

Case 4: p is odd and q is even.

By (4.6), $[p,q]_\tau = [1,2]_\tau$. Since $N_{\mathbb{P}(\tau)}(0) = \{1\}$, we have $\{2,\dots,2n-1\}$ is a module of $\tau - 0$. In particular, we obtain $[1,2]_\tau = [1,2n-2]_\tau$. Since $N_{\mathbb{P}(\tau)}(2n-1) = \{2n-2\}$, we have $\{0,\dots,2n-3\}$ is a module of $\tau - (2n-1)$. In particular, we obtain $[1,2n-2]_\tau = [0,2n-2]_\tau$. By (4.6), $[0,2n-2]_\tau = [0,2]_\tau$. Consequently, we obtain $[p,q]_\tau = [0,2]_\tau$.

It follows from the four cases above that (4.4) holds.

Second, we show that $< 0,1 >_\tau \neq < 0,2 >_\tau$. Since τ is prime, τ is neither constant nor linear. It follows from (4.4) that $[0,1]_\tau \neq [0,2]_\tau$. Furthermore, since $N_{\mathbb{P}(\tau)}(1) = \{0,2\}$, it follows from Lemma 4.4 that $\{0,2\}$ is a module of $\tau - 1$. Since τ is prime, $\{0,2\}$ is not a module of τ. Therefore, we have $[0,1]_\tau \neq [2,1]_\tau$. Since $[2,1]_\tau = [2,0]_\tau$ by (4.4), we obtain $[0,1]_\tau \neq [2,0]_\tau$. Consequently, we obtain $[0,1]_\tau \neq [0,2]_\tau$ and $[0,1]_\tau \neq [2,0]_\tau$. It follows that $<0,1>_\tau \neq <0,2>_\tau$.

Conversely, suppose that $< 0,1 >_\tau \neq < 0,2 >_\tau$ and (4.4) holds. Since $<0,1>_\tau \neq <0,2>_\tau$, we have

$$\begin{cases} [0,1]_\tau \neq [0,2]_\tau \\ \text{and} \\ [0,1]_\tau \neq [2,0]_\tau. \end{cases} \tag{4.7}$$

We prove by induction on $k \in \{2,\dots,n\}$ that

$$\tau[\{0,\dots,2k-1\}] \text{ is prime.} \tag{4.8}$$

To begin, we verify that $\tau[\{0,1,2,3\}]$ is prime. By (4.4), we have $[0,2]_\tau = [1,2]_\tau = [1,3]_\tau$ and $[0,1]_\tau = [0,3]_\tau = [2,3]_\tau$. Using (4.7), it is easy to verify that for each $W \subseteq \{0,1,2,3\}$, with $|W| = 2$ or 3, W is not a module of $\tau[\{0,1,2,3\}]$. Therefore, $\tau[\{0,1,2,3\}]$ is prime. Now, suppose that $\tau[\{0,\dots,2k-1\}]$ is prime, where $k \in \{2,\dots,n-1\}$. Set

$$X = \{0,\dots,2k-1\}.$$

By (4.4), we have $[y,2k]_\tau = [0,2]_\tau$ for every $y \in X$. Thus, $2k \in \langle X\rangle_\tau$. Furthermore, it follows from (4.4) that for every $y \in X \setminus \{2k-1\}$, $[y,2k-1]_\tau = [y,2k+1]_\tau$. Therefore, $2k+1 \in X_\tau(2k-1)$. Lastly, by (4.4), we have $[2k-1,2k]_\tau = [0,2]_\tau$ and $[2k+1,2k]_\tau = [1,0]_\tau$. By (4.7), $\{2k-1,2k+1\}$ is not a module of $\tau[X \cup \{2k,2k+1\}]$. It follows from statement (P1) of Lemma 3.17 that $\tau[X \cup \{2k,2k+1\}] = \tau[\{0,\dots,2k+1\}]$ is prime. Consequently, τ is prime.

Now, we verify that τ is critical. We consider the following cases.

Case 1: $p \in \{2, \dots, 2n-1\}$.

We have $[1, p]_\tau = [0, 2]_\tau$. Hence, $\{2, \dots, 2n-1\}$ is a module of $\tau - 0$.

Case 2: $p \in \{0, \dots, 2n-3\}$.

We have $[p, 2n-2]_\tau = [0, 2]_\tau$. Thus, $\{0, \dots, 2n-3\}$ is a module of $\tau - (2n-1)$.

Case 3: $p \in \{1, \dots, 2n-2\}$.

Consider $v \in V(\tau) \smallsetminus \{p-1, p, p+1\}$. Since $p - 1 \equiv p + 1 \mod 2$, it follows from (4.4) that $[v, p-1]_\tau = [v, p+1]_\tau$. Therefore, $\{p-1, p+1\}$ is a module of $\tau - p$.

It follows that τ is critical.

Lastly, we have to prove that $\mathbb{P}(\tau) = P_{2n}$. Let $p \in \{0, \dots, 2n-2\}$. The function $\{0, \dots, 2n-1\} \smallsetminus \{p, p+1\} \longrightarrow \{0, \dots, 2n-3\}$, defined by $q \mapsto q$ if $q \leq p-1$ and $q \mapsto q-2$ if $q \geq p+2$, is an isomorphism from $\tau - \{p, p+1\}$ onto $\tau[\{0, \dots, 2n-3\}]$. It follows from (4.8) that $\tau[\{0, \dots, 2n-3\}]$ is prime. Hence $\tau - \{p, p+1\}$ is prime too, so $\{p, p+1\} \in E(\mathbb{P}(\tau))$. It follows that

$$E(P_{2n}) \subseteq E(\mathbb{P}(\tau)). \tag{4.9}$$

Thus, since τ is critical, it follows from Lemma 4.4 that

$$\text{for } p \in \{1, \dots, 2n-2\},\ N_{\mathbb{P}(\tau)}(p) = \{p-1, p+1\}. \tag{4.10}$$

By (4.9), $1 \in N_{\mathbb{P}(\tau)}(0)$. As previously seen, $\{2, \dots, 2n-1\}$ is a nontrivial module of $\tau - 0$. Since $|\{2, \dots, 2n-1\}| \geq 4$, it follows from Lemma 4.4 that $d_{\mathbb{P}(\tau)}(0) = 1$. Therefore,

$$N_{\mathbb{P}(\tau)}(0) = \{1\}. \tag{4.11}$$

By (4.9), $2n-2 \in N_{\mathbb{P}(\tau)}(2n-1)$. Furthermore, it follows from (4.10) that $N_{\mathbb{P}(\tau)}(2n-1) \cap \{1, \dots, 2n-3\} = \varnothing$. Finally, by (4.11), $0 \notin N_{\mathbb{P}(\tau)}(2n-1)$. Thus,

$$N_{\mathbb{P}(\tau)}(2n-1) = \{2n-2\}.$$

Consequently, we obtain that $\mathbb{P}(\tau) = P_{2n}$. □

By using Proposition 4.15, we construct critical graphs, digraphs or 2-structures of type P_{2n} that allow us to characterize the critical 2-structures of type P_{2n}. We use the following notation.

Notation 4.16. Let $n \geq 2$. Recall that $A(L_n)$ is the set of ordered pairs (p, q), where $0 \leq p < q \leq n-1$. Given $i, j \in \{0, 1\}$, set

$$A(L_n)_{(i,j)} = \{(p, q) \in A(L_n) : p \equiv i \bmod 2, q \equiv j \bmod 2\}.$$

Let τ be a 2-structure defined on $V(\tau) = \{0, \dots, 2n-1\}$, where $n \geq 3$. Suppose that $\prec 0, 1 \succ_\tau \neq \prec 0, 2 \succ_\tau$. Suppose also that τ satisfies (4.4). By Proposition 4.15, τ is critical and $\mathbb{P}(\tau) = P_{2n}$. We distinguish the following cases.

Case 1: $(0, 2)_\tau = (2, 0)_\tau$.

Subcase a: $(0, 1)_\tau = (1, 0)_\tau$.

Since $<0,1>_\tau \neq <0,2>_\tau$, $(0,1)_\tau \neq (0,2)_\tau$. Therefore, we obtain

$$\begin{aligned} E(\tau) = \{&A(L_{2n})_{(0,1)} \cup (A(L_{2n})_{(0,1)})^\star, \text{ (see Notation 1.2)} \\ &(A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)}) \\ &\cup (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)})^\star\}. \end{aligned} \tag{4.12}$$

half-graph

In fact, τ is the 2-structure associated with a graph (see the end of subsection 1.2). Given $m \geq 1$, we consider the *half-graph* H_{2m} defined on $V(H_{2n}) = \{0,\ldots,2m-1\}$ as follows (see Figure 4.1). For $p,q \in \{0,\ldots,2m-1\}$, with $p \neq q$, $\{p,q\} \in E(H_{2m})$ if there exist $0 \leq i \leq j \leq m-1$ such that $\{p,q\} = \{2i, 2j+1\}$. It follows from (4.12) that

$$\tau = \sigma(H_{2n}).$$

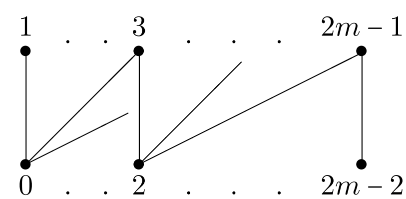

FIGURE 4.1. The half-graph H_{2m}

Subcase b: $(0,1)_\tau \neq (1,0)_\tau$.

We distinguish the following three subcases.

(1) Suppose that $(1,0)_\tau = (0,2)_\tau$. We obtain

$$\begin{aligned} E(\tau) = \{&A(L_{2n})_{(0,1)}, \\ &(A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)}) \\ &\cup (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)})^\star \\ &\cup (A(L_{2n})_{(0,1)})^\star\}. \end{aligned} \tag{4.13}$$

In fact, τ is the 2-structure associated with a partial order (see Subsection 1.3). Given $m \geq 2$, we consider the partial order Q_{2m} defined on $V(Q_{2m}) = \{0,\ldots,2m-1\}$ as follows (see Figure 4.2). For $p,q \in \{0,\ldots,2m-1\}$, with $p \neq q$, $(p,q) \in A(Q_{2m})$ if there exist $0 \leq i \leq j \leq m-1$ such that $(p,q) = (2i, 2j+1)$.

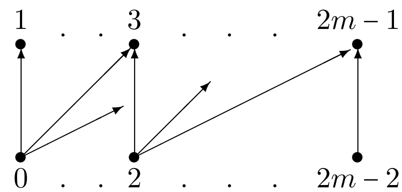

FIGURE 4.2. The partial order Q_{2m}

Observe that $\mathrm{Comp}(Q_{2m}) = H_{2m}$ [4.2] . It follows from (4.13) that

$$\tau = \sigma(Q_{2n}).$$

(2) Suppose that $(0,1)_\tau = (0,2)_\tau$. We obtain

$$\begin{aligned} E(\tau) = \{&(A(L_{2n})_{(0,1)})^\star, \\ &(A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)}) \\ &\cup (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)})^\star \\ &\cup A(L_{2n})_{(0,1)}\}. \end{aligned} \tag{4.14}$$

It follows from (4.14) that

$$\tau = \sigma((Q_{2n})^\star).$$

(3) Suppose that $<0,1>_\tau \cap <0,2>_\tau = \varnothing$. We obtain

$$\begin{aligned} E(\tau) = \{&A(L_{2n})_{(0,1)}, (A(L_{2n})_{(0,1)})^\star, \\ &(A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)}) \\ &\cup (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)})^\star\}. \end{aligned} \tag{4.15}$$

It follows from (4.13), (4.14), and (4.15) that

$$\tau = \sigma(Q_{2n}) \wedge \sigma((Q_{2n})^\star) \quad \text{(see Notation 4.11)}.$$

Recall that by Proposition 4.15, $\sigma(Q_{2n}) \wedge \sigma((Q_{2n})^\star)$ is critical and $\mathbb{P}(\sigma(Q_{2n}) \wedge \sigma((Q_{2n})^\star)) = P_{2n}$. Note that we obtain the same by using Fact 4.14 because $\sigma(Q_{2n})$ and $\sigma((Q_{2n})^\star)$ are critical, and $\mathbb{P}(\sigma(Q_{2n})) = \mathbb{P}(\sigma((Q_{2n})^\star)) = P_{2n}$. Note also that $\sigma(Q_{2n}) \wedge \sigma((Q_{2n})^\star) = \sigma(H_{2n}) \wedge \sigma(Q_{2n}) = \sigma(H_{2n}) \wedge \sigma((Q_{2n})^\star)$.

[4.2] The following theorem is due to Gallai [18, 28] .

Theorem 4.17 (Gallai [18, 28]). *A partial order is prime if and only if its comparability graph is prime.*

Let $m \geq 3$. As showed above, $\sigma(H_{2n})$, and hence H_{2n} are critical. Moreover, $\mathbb{P}(H_{2n}) = P_{2n}$. It follows from Theorem 4.17 that Q_{2n} is critical, and $\mathbb{P}(Q_{2n}) = P_{2n}$.

CASE 2: $(0,2)_\tau \neq (2,0)_\tau$.

Subcase a: $(0,1)_\tau = (1,0)_\tau$.

We distinguish the following three subcases.

(1) Suppose that $(0,1)_\tau = (2,0)_\tau$. We obtain

$$
\begin{aligned}
E(\tau) = \{&(A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)}),\\
&A(L_{2n})_{(0,1)} \cup (A(L_{2n})_{(0,1)})^\star\\
&\cup (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)})^\star\}.
\end{aligned} \tag{4.16}
$$

In fact, τ is the 2-structure associated with a partial order. Given $m \geq 2$, we consider the partial order R_{2m} defined on $V(R_{2m}) = \{0, \ldots, 2m-1\}$ as follows (see Figure 4.3). For $p, q \in \{0, \ldots, 2m-1\}$, with $p \neq q$, $(p,q) \in A(R_{2n})$ if $p < q$ and either p is odd or q is even. Equivalently, R_{2m} is obtained from the linear order L_{2m} by removing the arcs $(2i, 2j+1)$ for $0 \leq i \leq j \leq m-1$.

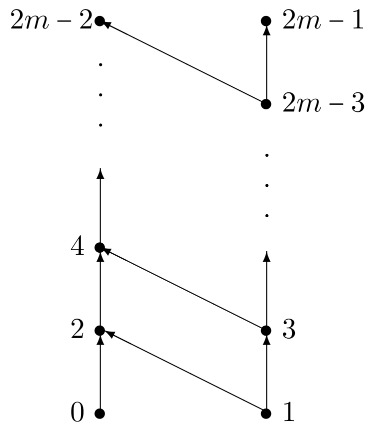

FIGURE 4.3. The partial order R_{2m}

Observe that $\mathrm{Comp}(R_{2m}) = \overline{H_{2m}}$. It follows from (4.16) that

$$\tau = \sigma(R_{2n}).$$

(2) Suppose that $(0,1)_\tau = (0,2)_\tau$. We obtain

$$
\begin{aligned}
E(\tau) = \{&A(L_{2n})_{(0,1)} \cup (A(L_{2n})_{(0,1)})^\star\\
&\cup (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)}),\\
&(A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)})^\star\}.
\end{aligned} \tag{4.17}
$$

It follows from (4.17) that

$$\tau = \sigma((R_{2n})^\star).$$

(3) Suppose that $\prec 0,1\succ_\tau \cap \prec 0,2\succ_\tau = \varnothing$. We obtain

$$E(\tau) = \{A(L_{2n})_{(0,1)} \cup (A(L_{2n})_{(0,1)})^\star, \quad (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)}), \quad (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)})^\star\}. \tag{4.18}$$

It follows from (4.16), (4.17) and (4.18) that

$$\tau = \sigma(R_{2n}) \wedge \sigma((R_{2n})^\star).$$

Note that

$$\sigma(R_{2n}) \wedge \sigma((R_{2n})^\star) = \sigma(H_{2n}) \wedge \sigma(R_{2n}) = \sigma(H_{2n}) \wedge \sigma((R_{2n})^\star).$$

Subcase b: $(0,1)_\tau \neq (1,0)_\tau$.

We distinguish the following five subcases.

(1) Suppose that $(0,1)_\tau = (0,2)_\tau$. Since $\prec 0,1\succ_\tau \neq \prec 0,2\succ_\tau$, we have $(1,0)_\tau \neq (2,0)_\tau$. We obtain

$$E(\tau) = \{A(L_{2n})_{(0,1)} \cup (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)}), \quad (A(L_{2n})_{(0,1)})^\star, \quad (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)})^\star\}. \tag{4.19}$$

It follows from (4.19) that

$$\tau = \sigma((Q_{2n})^\star) \wedge \sigma((R_{2n})^\star).$$

(2) Suppose that $(1,0)_\tau = (0,2)_\tau$. Since $\prec 0,1\succ_\tau \neq \prec 0,2\succ_\tau$, we have $(0,1)_\tau \neq (2,0)_\tau$. We obtain

$$E(\tau) = \{(A(L_{2n})_{(0,1)})^\star \cup (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)}), \quad A(L_{2n})_{(0,1)}, \quad (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)})^\star\}. \tag{4.20}$$

It follows from (4.20) that

$$\tau = \sigma(Q_{2n}) \wedge \sigma((R_{2n})^\star).$$

(3) Suppose that $(0,1)_\tau = (2,0)_\tau$. Since $\prec 0,1\succ_\tau \neq \prec 0,2\succ_\tau$, we have $(1,0)_\tau \neq (0,2)_\tau$. We obtain

$$E(\tau) = \{A(L_{2n})_{(0,1)} \cup (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)})^\star, \quad (A(L_{2n})_{(0,1)})^\star, \quad (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)})\}. \tag{4.21}$$

It follows from (4.21) that

$$\tau = \sigma((Q_{2n})^\star) \wedge \sigma(R_{2n}).$$

(4) Suppose that $(1,0)_\tau = (2,0)_\tau$. Since $<0,1>_\tau \neq <0,2>_\tau$, we have $(0,1)_\tau \neq (0,2)_\tau$. We obtain

$$\begin{aligned} E(\tau) = \{ & (A(L_{2n})_{(0,1)})^\star \\ & \cup (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)})^\star, \\ & A(L_{2n})_{(0,1)}, \\ & (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)})\}. \end{aligned} \tag{4.22}$$

It follows from (4.22) that

$$\tau = \sigma(Q_{2n}) \wedge \sigma(R_{2n}).$$

(5) Suppose that $<0,1>_\tau \cap <0,2>_\tau = \varnothing$. We obtain

$$\begin{aligned} E(\tau) = \{ & A(L_{2n})_{(0,1)}, \\ & (A(L_{2n})_{(0,1)})^\star \\ & (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)}), \\ & (A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,0)} \cup A(L_{2n})_{(1,1)})^\star\}. \end{aligned} \tag{4.23}$$

It follows from (4.23) that

$$\tau = \sigma(Q_{2n}) \wedge \sigma((Q_{2n})^\star) \wedge \sigma(R_{2n}).$$

Remark 4.18. We showed previously that $\sigma(H_{2n})$, $\sigma(Q_{2n})$, $\sigma((Q_{2n})^\star)$, $\sigma(R_{2n})$, and $\sigma((R_{2n})^\star)$ are critical. Furthermore, their primality graph equals P_{2n}. We also obtained that some of their meets are also critical, and admit P_{2n} as primality graph. Observe that, by Fact (4.14), all their meets are also critical, and admit P_{2n} as a primality graph.

We summarize the previous examination in the next theorem.

Theorem 4.19 (Boudabbous and Ille[4.3] [7])**.** *Consider a 2-structure τ defined on $V(\tau) = \{0, \dots, 2n-1\}$, where $n \geq 3$. The following two statements are equivalent*

- *τ is critical, and $\mathbb{P}(\tau) = P_{2n}$;*
- *$\tau = \sigma(H_{2n})$, $\sigma(Q_{2n})$, $\sigma((Q_{2n})^\star)$, $\sigma(R_{2n})$, $\sigma((R_{2n})^\star)$, $\sigma(Q_{2n}) \wedge \sigma((Q_{2n})^\star)$, $\sigma(R_{2n}) \wedge \sigma((R_{2n})^\star)$, $\sigma(Q_{2n}) \wedge \sigma(R_{2n})$, $\sigma(Q_{2n}) \wedge \sigma((R_{2n})^\star)$, $\sigma((Q_{2n})^\star) \wedge \sigma(R_{2n})$, $\sigma((Q_{2n})^\star) \wedge \sigma((R_{2n})^\star)$, or $\sigma(Q_{2n}) \wedge \sigma((Q_{2n})^\star) \wedge \sigma(R_{2n})$.*

The following result is an immediate consequence of Theorem 4.19.

Corollary 4.20. *Consider a reversible 2-structure τ defined on $V(\tau) = \{0, \dots, 2n-1\}$, where $n \geq 3$. The following two statements are equivalent*

- *τ is critical, and $\mathbb{P}(\tau) = P_{2n}$;*
- *$\tau = \sigma(H_{2n})$, $\sigma(Q_{2n}) \wedge \sigma((Q_{2n})^\star)$, $\sigma(R_{2n}) \wedge \sigma((R_{2n})^\star)$, or $\sigma(Q_{2n}) \wedge \sigma((Q_{2n})^\star) \wedge \sigma(R_{2n})$.*

[4.3] Boudabbous and Ille [7] proved this theorem for digraphs.

The next remark completes subsubsection 4.2.1. We use the following notation.

Notation 4.21. Given $n \geq 2$, π_n denotes the permutation of $\{0, \ldots, n-1\}$ which exchanges i and $(n-1)-i$ for $i \in \{0, \ldots, n-1\}$.

Remark 4.22. Given $n \geq 3$, consider a critical 2-structure τ defined on $V(\tau) = \{0, \ldots, 2n-1\}$, and such that $\mathbb{P}(\tau) = P_{2n}$. Set

$$\begin{aligned}\mathcal{E}(\tau) = \{&A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,1)} \cup A(L_{2n})_{(1,0)},\\ &(A(L_{2n})_{(0,0)} \cup A(L_{2n})_{(1,1)} \cup A(L_{2n})_{(1,0)})^\star,\\ &A(L_{2n})_{(0,1)}, A(L_{2n})^\star_{(0,1)}\}.\end{aligned}$$

We obtain

$$\pi_{2n}(e) = e^\star \text{ for each } e \in \mathcal{E}(\tau). \tag{4.24}$$

Consider $e \in E(\tau)$. By Proposition 4.15, there exists $\mathcal{B}_e \subsetneq \mathcal{E}(\tau)$ such that

$$e = \bigcup_{f \in \mathcal{B}_e} f.$$

Thus, we obtain

$$\begin{aligned}\pi_{2n}(e) &= \bigcup_{f \in \mathcal{B}_e} \pi_{2n}(f)\\ &= \bigcup_{f \in \mathcal{B}_e} f^\star \quad \text{(by (4.24))}\\ &= e^\star.\end{aligned}$$

Consequently, π_{2n} is an isomorphism from τ onto $\tau^\star$. If τ is reversible, then $\tau = \tau^\star$, and hence π_{2n} is an automorphism of τ.

4.2.2. *The type $P_{2n} \oplus K_{\{2n\}}$.*

Proposition 4.23. *Given $n \geq 2$, consider a 2-structure τ defined on $V(\tau) = \{0, \ldots, 2n\}$. The following two statements are equivalent*

(1) *τ is critical, and $\mathbb{P}(\tau) = P_{2n} \oplus K_{\{2n\}}$;*
(2) *$(0,1)_\tau \neq (1,0)_\tau$, and for $p, q \in \{0, \ldots, 2n-1\}$, we have*

$$\text{if } p < q, \text{ then } [p,q]_\tau = [0,1]_\tau; \tag{4.25}$$

moreover, for $i \in \{0, \ldots, n-1\}$, we have

$$[2i, 2n]_\tau = [1,0]_\tau \text{ and } [2i+1, 2n]_\tau = [0,1]_\tau. \tag{4.26}$$

Proof. To begin, suppose that τ is critical, and $\mathbb{P}(\tau) = P_{2n} \oplus K_{\{2n\}}$. In a similar way as in the proof of Proposition 4.15, we verify that (4.4) holds. For a contradiction, suppose that $\prec 0,1 \succ_\tau \neq \prec 0,2 \succ_\tau$. If $n = 2$, then it is not difficult to verify that $\tau - 4$ is prime. Furthermore, if $n \geq 3$, then it follows from Proposition 4.15 that $\tau - (2n)$ is critical, and hence prime. Since $\tau - (2n)$ is decomposable, we obtain

$$\prec 0,1 \succ_\tau = \prec 0,2 \succ_\tau . \tag{4.27}$$

Since $\mathbb{P}(\tau) = P_{2n} \oplus K_{\{2n\}}$, we have $N_{\mathbb{P}(\tau)}(0) = \{1\}$. By Lemma 4.4, $\{2, \ldots, 2n\}$ is a module of $\tau - 0$. Since τ is primitive, $\{0\} \cup \{2, \ldots, 2n\}$ is not a module of τ, so $[1,0]_\tau \neq [1,2]_\tau$. Since (4.4) holds, we have $[1,2]_\tau = [0,2]_\tau$. Therefore, we obtain $[1,0]_\tau \neq [0,2]_\tau$. It follows from (4.27) that $(0,1)_\tau \neq (1,0)_\tau$ and $[0,1]_\tau = [0,2]_\tau$. Since (4.4) holds and $[0,1]_\tau = [0,2]_\tau$, (4.25) holds.

Lastly, we show that (4.26) holds. As previously seen, $\{2, \ldots, 2n\}$ is a module of $\tau - 0$. Hence we have $[1,2n]_\tau = [1,2]_\tau$. Since (4.25) holds, we have $[1,2]_\tau = [0,1]_\tau$. We obtain $[1,2n]_\tau = [0,1]_\tau$. Let $i \in \{0, \ldots, n-2\}$. Since $\mathbb{P}(\tau) = P_{2n} \oplus K_{\{2n\}}$, we have $N_{\mathbb{P}(\tau)}(2i+2) = \{2i+1, 2i+3\}$. By Lemma 4.4, $\{2i+1, 2i+3\}$ is a module of $\tau - (2i)$. In particular, we have $[2i+1, 2n]_\tau = [2i+3, 2n]_\tau$. It follows that

$$[0,1]_\tau = [1,2n]_\tau = [3,2n]_\tau = \cdots = [2n-1,2n]_\tau.$$

Since $\mathbb{P}(\tau) = P_{2n} \oplus K_{\{2n\}}$, we have $N_{\mathbb{P}(\tau)}(2n-1) = \{2n-2\}$. By Lemma 4.4, $\{0, \ldots, 2n-3\} \cup \{2n\}$ is a module of $\tau - (2n-1)$. In particular, we have $[2n-2, 2n]_\tau = [2n-2, 0]_\tau$. Since (4.25) holds, we have $[2n-2, 0]_\tau = [1,0]_\tau$. We obtain $[2n-2, 2n]_\tau = [1,0]_\tau$. Let $i \in \{1, \ldots, n-1\}$. Since $\mathbb{P}(\tau) = P_{2n} \oplus K_{\{2n\}}$, we have $N_{\mathbb{P}(\tau)}(2i-1) = \{2i-2, 2i\}$. By Lemma 4.4, $\{2i-2, 2i\}$ is a module of $\tau - (2i-1)$. In particular, we have $[2i-2, 2n]_\tau = [2i, 2n]_\tau$. It follows that

$$[1,0]_\tau = [2n-2,2n]_\tau = [2n-4,2n]_\tau = \cdots = [0,2n]_\tau.$$

Consequently, (4.26) holds.

Conversely, suppose that (4.25) and (4.26) hold. Moreover, suppose that $(0,1)_\tau \neq (1,0)_\tau$. It follows that $\tau = \sigma(T_{2n+1})$ (see Figure 1.2). By Fact 2.7, τ is prime. We continue with the following observation

$$\text{for every } p \in \{0, \ldots, 2n-2\},\ \tau - \{p, p+1\} \text{ is prime.} \tag{4.28}$$

Indeed, consider $p \in \{0, \ldots, 2n-2\}$. Recall that $\tau - \{p, p+1\} = \sigma(T_{2n+1}) - \{p, p+1\}$. The bijection

$$\begin{array}{rccc} \varphi: & \{0, \ldots, 2n\} \smallsetminus \{p, p+1\} & \longrightarrow & \{0, \ldots, 2n-2\} \\ & q \leq p-1 & \longmapsto & q, \\ & q \geq p+2 & \longmapsto & q-2 \end{array}$$

is an isomorphism from $T_{2n+1} - \{p, p+1\}$ onto T_{2n-1}. Hence, φ is an isomorphism from $\sigma(T_{2n+1}) - \{p, p+1\}$ onto $\sigma(T_{2n-1})$. By Fact 2.7, $\sigma(T_{2n-1})$ is prime. Thus, $\sigma(T_{2n+1}) - \{p, p+1\}$ is prime as well. Consequently, (4.28) holds. It follows from (4.28) that

$$E(P_{2n}) \subseteq E(\mathbb{P}(\tau)). \tag{4.29}$$

Lastly, we prove that τ is critical, and $\mathbb{P}(\tau) = P_{2n} \oplus K_{\{2n\}}$. As already observed, $\tau - (2n) = \sigma(L_{2n})$. Hence $\tau - (2n)$ is decomposable. Furthermore, since $\tau - (2n) = \sigma(L_{2n})$, $\tau - \{p, 2n\}$ is decomposable for each $p \in \{0, \ldots, 2n-1\}$. Thus,

$$N_{\mathbb{P}(\sigma)}(2n) = \varnothing. \tag{4.30}$$

Consider $p \in \{1, \dots, 2n-2\}$. It follows from (4.25) and (4.26) that $\{p-1, p+1\}$ is a module of $\tau - p$. Therefore, p is a critical vertex of τ. Moreover, it follows from (4.29) that $\{p-1, p+1\} \subseteq N_{\mathbb{P}(\tau)}(p)$. Since p is a critical vertex of τ, it follows from Lemma 4.4 that $N_{\mathbb{P}(\tau)}(p) = \{p-1, p+1\}$. Hence,

$$\text{for each } p \in \{1, \dots, 2n-2\},\ N_{\mathbb{P}(\tau)}(p) = \{p-1, p+1\}. \tag{4.31}$$

It follows from (4.25) and (4.26) that $\{2, \dots, 2n\}$ is a module of $\tau - 0$. Therefore, 0 is a critical vertex of τ. By (4.31), $1 \in N_{\mathbb{P}(\tau)}(0)$. Since $\{2, \dots, 2n\}$ is a nontrivial module of $\tau - 0$, with $|\{2, \dots, 2n\}| \geq 3$, it follows from Lemma 4.4 that $d_{\mathbb{P}(\tau)}(0) = 1$. Thus,

$$N_{\mathbb{P}(\tau)}(0) = \{1\}. \tag{4.32}$$

Finally, it follows from (4.25) and (4.26) that $\{0, \dots, 2n-3\} \cup \{2n\}$ is a module of $\tau - (2n-1)$. Therefore, $2n-1$ is a critical vertex of τ. By (4.30), $2n \notin N_{\mathbb{P}(\sigma)}(2n-1)$. By (4.31), $2n-2 \in N_{\mathbb{P}(\sigma)}(2n-1)$ and $N_{\mathbb{P}(\sigma)}(2n-1) \cap \{1, \dots, 2n-3\} = \varnothing$. By (4.32), $0 \notin N_{\mathbb{P}(\sigma)}(2n-1)$. Consequently, $N_{\mathbb{P}(\sigma)}(2n-1) = \{2n-2\}$. It follows that τ is critical, and $\mathbb{P}(\tau) = P_{2n} \oplus K_{\{2n\}}$. □

The next characterization is a simple consequence of Proposition 4.23 and its proof.

Theorem 4.24 (Boudabbous and Ille[4.4] [7])**.** *Consider a 2-structure τ defined on $V(\tau) = \{0, \dots, 2n\}$, where $n \geq 2$. The following two statements are equivalent*

- *τ is critical, and $\mathbb{P}(\tau) = P_{2n} \oplus K_{\{2n\}}$;*
- *$\tau = \sigma(T_{2n+1})$ (see Figure 1.2).*

In Remark 4.26, we determine the automorphism group of $\sigma(T_{2n+1})$, where $n \geq 2$. We use the following note.

Note 4.25. Consider a tournament T. We denote by $\mathrm{Iso}(T, T^\star)$ the set of the isomorphism from T onto its dual. We prove that

$$\mathrm{Aut}(\sigma(T)) = \mathrm{Aut}(T) \cup \mathrm{Iso}(T, T^\star). \tag{4.33}$$

Given $x, y, v, w \in V(\sigma(T))$, with $x \neq y$ and $v \neq w$,

$$(x,y) \equiv_{\sigma(T)} (v,w) \text{ if } \begin{cases} (x,y),(v,w) \in A(T) \\ \text{or} \\ (x,y),(v,w) \notin A(T). \end{cases}$$

Therefore,

$$(x,y) \equiv_{\sigma(T)} (v,w) \text{ if } \begin{cases} (x,y),(v,w) \in A(T) \\ \text{or} \\ (x,y),(v,w) \in (A(T))^\star. \end{cases}$$

[4.4] Boudabbous and Ille [7] proved this theorem for digraphs.

It follows that

$$E(\sigma(T)) = \{A(T), (A(T))^\star\}.$$

Given a permutation φ of $V(T)$, we have

$$\varphi \in \mathrm{Aut}(\sigma(T)) \text{if and only if} \quad \begin{cases} \varphi(A(T)) = A(T) \text{ and } \varphi((A(T))^\star) = (A(T))^\star \\ \text{or} \\ \varphi(A(T)) = (A(T))^\star \text{ and } \varphi((A(T))^\star) = A(T). \end{cases}$$

Clearly, if $\varphi(A(T)) = A(T)$, then $\varphi((A(T))^\star) = (A(T))^\star$. Similarly, if $\varphi(A(T)) = (A(T))^\star$, then $\varphi((A(T))^\star) = A(T)$. Therefore,

$$\varphi \in \mathrm{Aut}(\sigma(T)) \text{ if and only if } \begin{cases} \varphi(A(T)) = A(T) \\ \text{or} \\ \varphi(A(T)) = (A(T))^\star. \end{cases}$$

We have $\varphi(A(T)) = A(T)$ if and only if $\varphi \in \mathrm{Aut}(T)$. Analogously, $\varphi(A(T)) = (A(T))^\star$ if and only if $\varphi \in \mathrm{Iso}(T, T^\star)$. It follows that (4.33) holds.

Remark 4.26. Let $n \geq 2$. We verify that T_{2n+1} is rigid. Let $\varphi \in \mathrm{Aut}(T_{2n+1})$. Since $n \geq 2$, $2n$ is the only vertex of T_{2n+1} such that $T_{2n+1} - (2n)$ is a linear order. Consequently, $\varphi(2n) = 2n$. It follows that $\varphi_{\restriction\{0,\dots,2n-1\}} \in \mathrm{Aut}(T_{2n+1} - (2n))$. Since $T_{2n+1} - (2n)$ is a linear order, $T_{2n+1} - (2n)$ is rigid. Therefore, $\varphi_{\restriction\{0,\dots,2n-1\}} = \mathrm{Id}_{\{0,\dots,2n-1\}}$. Since $\varphi(2n) = 2n$, we obtain $\varphi = \mathrm{Id}_{\{0,\dots,2n\}}$.

We denote by $\widehat{\pi_{2n}}$ the extension of π_{2n} to $\{0,\dots,2n\}$ defined by $\widehat{\pi_{2n}}(2n) = 2n$ (see Notation 4.21). Clearly, $\widehat{\pi_{2n}}$ is an isomorphism from T_{2n+1} onto $(T_{2n+1})^\star$. Conversely, consider an isomorphism φ from T_{2n+1} onto $(T_{2n+1})^\star$. Recall that $2n$ is the only vertex of T_{2n+1} such that $T_{2n+1} - (2n)$ is a linear order. Hence, $2n$ is the only vertex of $(T_{2n+1})^\star$ such that $(T_{2n+1})^\star - (2n)$ is a linear order. It follows that $\varphi(2n) = 2n$. Therefore, $\varphi_{\restriction\{0,\dots,2n-1\}}$ is an isomorphism from $T_{2n+1} - (2n)$ onto $(T_{2n+1})^\star - (2n)$. Since $T_{2n+1} - (2n) = L_{2n}$, we obtain $\varphi_{\restriction\{0,\dots,2n-1\}} = \pi_{2n}$. Consequently, we have $\varphi = \widehat{\pi_{2n}}$.

It follows from Note 4.25 that

$$\mathrm{Aut}(\sigma(T_{2n+1})) = \{\mathrm{Id}_{\{0,\dots,2n\}}, \widehat{\pi_{2n}}\}. \tag{4.34}$$

4.2.3. *The type P_{2n+1}.*

Proposition 4.27. *Given $n \geq 2$, consider a 2-structure τ defined on $V(\tau) = \{0,\dots,2n\}$. The following two statements are equivalent*

(1) *τ is critical, and $\mathbb{P}(\tau) = P_{2n+1}$;*
(2) *$(0,1)_\tau \neq (1,0)_\tau$, $[0,1]_\tau \neq [0,2]_\tau$, and for $p, q \in \{0,\dots,2n\}$ such that $p < q$, we have*

$$[p,q]_\tau = \begin{cases} [0,2]_\tau & \text{if } p \text{ and } q \text{ are even,} \\ [0,1]_\tau & \text{otherwise.} \end{cases} \tag{4.35}$$

Proof. To begin, suppose that τ is critical, and $\mathbb{P}(\tau) = P_{2n+1}$. First, we show that (4.35) holds. Consider $p, q \in \{0, \dots, 2n\}$ such that $p < q$. We prove that there exist $p' \in \{0,1\}$ and $q' \in \{2n-1, 2n\}$ such that

$$p' \equiv p \bmod 2,\ q' \equiv q \bmod 2, \text{ and } [p,q]_\tau = [p',q']_\tau. \tag{4.36}$$

For instance, suppose that $p \geq 2$. Since $\mathbb{P}(\tau) = P_{2n+1}$, we have $N_{\mathbb{P}(\tau)}(p-1) = \{p-2, p\}$. By Lemma 4.4, $\{p-2,p\}$ is a module of $\tau - (p-1)$. In particular, we obtain $[p,q]_\tau = [p-2,q]_\tau$. By iteration, we obtain $p' \in \{0,1\}$ such that

$$p' \equiv p \bmod 2 \text{ and } [p,q]_\tau = [p',q]_\tau.$$

Similarly, we obtain $q' \in \{2n-2, 2n-1\}$ such that $q' \equiv q \bmod 2$ and $[p',q]_\tau = [p',q']_\tau$. Therefore, (4.36) holds. It follows from (4.36) that for any $p', q' \in \{0, \dots, 2n\}$ such that $p' < q'$,

$$\text{if } p' \equiv p \bmod 2 \text{ and } q' \equiv q \bmod 2, \text{ then } [p,q]_\tau = [p',q']_\tau. \tag{4.37}$$

We distinguish the following four cases, where $p, q \in \{0, \dots, 2n\}$ such that $p < q$.

CASE 1: p and q are even.

By (4.37), $[p,q]_\tau = [0,2]_\tau$.

CASE 2: p and q are odd.

By (4.37), $[p,q]_\tau = [1,2n-1]_\tau$. Since $\mathbb{P}(\tau) = P_{2n+1}$, we have $N_{\mathbb{P}(\tau)}(2n) = \{2n-1\}$. By Lemma 4.4, $\{0, \dots, 2n-2\}$ is a module of $\tau - (2n)$. In particular, we obtain $[1,2n-1]_\tau = [0,2n-1]_\tau$. By (4.37), $[0,2n-1]_\tau = [0,1]_\tau$. Consequently, we obtain $[p,q]_\tau = [0,1]_\tau$.

CASE 3: p is even and q is odd.

By (4.37), $[p,q]_\tau = [0,1]_\tau$.

CASE 4: p is odd and q is even.

By (4.37), $[p,q]_\tau = [1,2]_\tau$. Since $N_{\mathbb{P}(\tau)}(0) = \{1\}$, we have $\{2, \dots, 2n\}$ is a module of $\tau - 0$. In particular, we obtain $[1,2]_\tau = [1,2n-1]_\tau$. Since $N_{\mathbb{P}(\tau)}(2n) = \{2n-1\}$, we have $\{0, \dots, 2n-2\}$ is a module of $\tau - (2n)$. In particular, we obtain $[1,2n-1]_\tau = [0,2n-1]_\tau$. By (4.37), $[0,2n-1]_\tau = [0,1]_\tau$. Consequently, we obtain $[p,q]_\tau = [0,1]_\tau$.

It follows from the four cases above that (4.35) holds.

Second, we verify that $(0,1)_\tau \neq (1,0)_\tau$ and $[0,1]_\tau \neq [0,2]_\tau$. If $(0,1)_\tau = (1,0)_\tau$, then $\{2i : i \in \{0, \dots, n\}\}$ is a module of τ, which contradicts the fact that τ is critical, and hence prime. Hence $(0,1)_\tau \neq (1,0)_\tau$. Furthermore, if $[0,1]_\tau = [0,2]_\tau$, then $\tau = \sigma(L_{2n+1})$, which contradicts the fact that τ is prime. Thus $[0,1]_\tau \neq [0,2]_\tau$.

Conversely, suppose that $(0,1)_\tau \neq (1,0)_\tau$, $[0,1]_\tau \neq [0,2]_\tau$, and (4.35) holds. To begin, we prove that τ is prime. We show by induction that

$$\text{for each } m \in \{1, \dots, n\} \text{ that } \tau[\{0, \dots, 2m\}] \text{ is prime.} \tag{4.38}$$

Since $[0,1]_\tau \neq [0,2]_\tau$, $\{0,1\}$ and $\{1,2\}$ are not modules of $\tau[\{0,1,2\}]$. Moreover, $\{0,2\}$ is not a module of $\tau[\{0,1,2\}]$ because $(0,1)_\tau \neq (1,0)_\tau$. It follows that $\tau[\{0,1,2\}]$ is prime. Now, consider $m \in \{1,\dots,n-1\}$, and suppose that $\tau[\{0,\dots,2m\}]$ is prime. Set

$$X = \{0,\dots,2m\}.$$

Since (4.35) holds, we obtain $2m+1 \in \langle X\rangle_\tau$ and $2m+2 \in X_\tau(2m)$. Since $(0,1)_\tau \neq (1,0)_\tau$, $X \cup \{2m+2\}$ is not a module of $\tau[\{0,\dots,2m+2\}]$. By statement (P1) of Lemma 3.17, $\tau[\{0,\dots,2m+2\}]$ is prime. Consequently, (4.38) holds for every $m \in \{0,\dots,n\}$. It follows that τ is prime.

To continue, we make the following observation

$$\text{for every } p \in \{0,\dots,2n-1\},\ \tau - \{p,p+1\} \text{ is prime.} \tag{4.39}$$

Indeed, let $p \in \{0,\dots,2n-1\}$. Since (4.35) holds, the bijection

$$\begin{array}{rcc} \{0,\dots,2n\} \setminus \{p,p+1\} & \longrightarrow & \{0,\dots,2n-2\} \\ q \le p-1 & \longmapsto & q, \\ (\text{if } p \le 2n-2)\ q \ge p+2 & \longmapsto & q-2. \end{array}$$

is an isomorphism from $\tau - \{p,p+1\}$ onto $\tau[\{0,\dots,2n-2\}]$. It follows from (4.38) that $\tau[\{0,\dots,2n-2\}]$ is prime, so $\tau - \{p,p+1\}$ is as well.

$$E(P_{2n+1}) \subseteq E(\mathbb{P}(\tau)). \tag{4.40}$$

Lastly, we prove that τ is critical, and $\mathbb{P}(\tau) = P_{2n+1}$. Let $p \in \{1,\dots,2n-1\}$. Since (4.35) holds, $\{p-1,p+1\}$ is a module of $\tau - p$. Thus, p is a critical vertex of τ. By (4.40), $\{p-1,p+1\} \subseteq N_{\mathbb{P}(\tau)}(p)$. Since p is a critical vertex of τ, it follows from Lemma 4.4 that $N_{\mathbb{P}(\tau)}(p) = \{p-1,p+1\}$. Therefore, for each $p \in \{1,\dots,2n-1\}$, we have

$$N_{\mathbb{P}(\tau)}(p) = \{p-1,p+1\}. \tag{4.41}$$

Since (4.35) holds, $\{2,\dots,2n\}$ is a module of $\tau - 0$. Thus 0 is a critical vertex of τ. It follows from (4.41) that $1 \in N_{\mathbb{P}(\tau)}(0)$. Since $\{2,\dots,2n\}$ is a nontrivial module of $\tau - 0$, with $|\{2,\dots,2n\}| \ge 3$, it follows from Lemma 4.4 that $d_{\mathbb{P}(\tau)}(0) = 1$. Therefore,

$$N_{\mathbb{P}(\tau)}(0) = \{1\}. \tag{4.42}$$

Finally, since (4.35) holds, $\{0,\dots,2n-2\}$ is a module of $\tau - (2n)$. Thus $2n$ is a critical vertex of τ. It follows from (4.41) that $2n-1 \in N_{\mathbb{P}(\tau)}(2n)$ and $N_{\mathbb{P}(\tau)}(2n) \cap \{1,\dots,2n-2\} = \varnothing$. By (4.42), $0 \notin N_{\mathbb{P}(\tau)}(2n)$. Therefore, $N_{\mathbb{P}(\tau)}(2n) = \{2n-1\}$. Consequently, τ is critical, and $\mathbb{P}(\tau) = P_{2n+1}$. □

Let τ be a 2-structure defined on $V(\tau) = \{0,\dots,2n\}$, where $n \ge 2$. Suppose that $(0,1)_\tau \neq (1,0)_\tau$ and $[0,1]_\tau \neq [0,2]_\tau$. Suppose also that τ satisfies (4.35). By Proposition 4.27, τ is critical, and $\mathbb{P}(\tau) = P_{2n+1}$. We distinguish the following cases.

Case 1: $(0,2)_\tau = (2,0)_\tau$.

We distinguish the following three subcases.

Subcase a: $(0,2)_\tau = (1,0)_\tau$.

We obtain

$$
\begin{aligned}
E(\tau) = \{&A(L_{2n+1})_{(0,1)} \cup A(L_{2n+1})_{(1,0)} \cup A(L_{2n+1})_{(1,1)}, \\
&(A(L_{2n+1})_{(0,1)} \cup A(L_{2n+1})_{(1,0)} \cup A(L_{2n+1})_{(1,1)})^\star \\
&\cup A(L_{2n+1})_{(0,0)} \cup (A(L_{2n+1})_{(0,0)})^\star\} \quad \text{(see Notation 4.16).}
\end{aligned} \tag{4.43}
$$

In fact, τ is the 2-structure associated with a digraph (see subsection 1.3). Given $m \geq 2$, we consider the digraph D_{2m+1} obtained from the linear order L_{2m+1} by removing all the arcs between the even integers (see Figure 4.4).

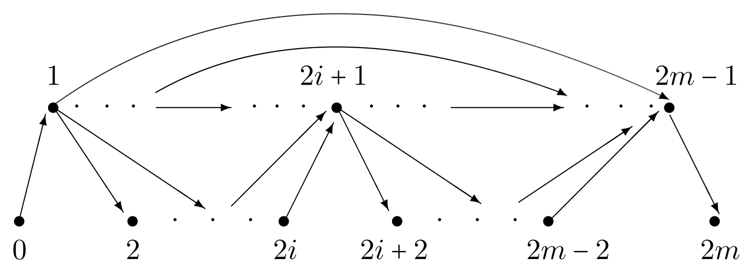

FIGURE 4.4. The digraph D_{2m+1}.

It follows from (4.43) that

$$\tau = \sigma(D_{2n+1}).$$

Subcase b: $(0,2)_\tau = (0,1)_\tau$.

We obtain

$$
\begin{aligned}
E(\tau) = \{&A(L_{2n+1})_{(0,1)} \cup A(L_{2n+1})_{(1,0)} \cup A(L_{2n+1})_{(1,1)} \\
&\cup A(L_{2n+1})_{(0,0)} \cup (A(L_{2n+1})_{(0,0)})^\star, \\
&(A(L_{2n+1})_{(0,1)} \cup A(L_{2n+1})_{(1,0)} \cup A(L_{2n+1})_{(1,1)})^\star\}.
\end{aligned} \tag{4.44}
$$

It follows from (4.44) that

$$\tau = \sigma((D_{2n+1})^\star).$$

Subcase c: $\prec 0,1 \succ_\tau \cap \prec 0,2 \succ_\tau = \varnothing$.

We obtain

$$
\begin{aligned}
E(\tau) = \{&A(L_{2n+1})_{(0,1)} \cup A(L_{2n+1})_{(1,0)} \cup A(L_{2n+1})_{(1,1)}, \\
&(A(L_{2n+1})_{(0,1)} \cup A(L_{2n+1})_{(1,0)} \cup A(L_{2n+1})_{(1,1)})^\star, \\
&A(L_{2n+1})_{(0,0)} \cup (A(L_{2n+1})_{(0,0)})^\star\}.
\end{aligned} \tag{4.45}
$$

It follows from (4.43), (4.44), and (4.45) that

$\tau = \sigma(D_{2n+1}) \wedge \sigma((D_{2n+1})^\star)$ (see Remark 4.11 and Fact 4.14).

Case 2: $(0,2)_\tau \neq (2,0)_\tau$.

Since $[0,1]_\tau \neq [0,2]_\tau$, we have $(0,1)_\tau = (2,0)_\tau$ and $(1,0)_\tau = (0,2)_\tau$ or $<0,1>_\tau \cap <0,2>_\tau = \varnothing$. We distinguish the following two subcases.

Subcase a: $(0,1)_\tau = (2,0)_\tau$ and $(1,0)_\tau = (0,2)_\tau$.

We obtain

$$\begin{aligned} E(\tau) = \{ & A(L_{2n+1})_{(0,1)} \cup A(L_{2n+1})_{(1,0)} \cup A(L_{2n+1})_{(1,1)} \\ & \cup (A(L_{2n+1})_{(0,0)})^\star, \text{ (see Notation 4.16)} \\ & (A(L_{2n+1})_{(0,1)} \cup A(L_{2n+1})_{(1,0)} \cup A(L_{2n+1})_{(1,1)})^\star \\ & \cup A(L_{2n+1})_{(2,0)})\} \text{ (see Notation 1.2).} \end{aligned} \tag{4.46}$$

In fact, τ is the 2-structure associated with a graph (see subsection 1.2). Given $m \geq 2$, we consider the tournament U_{2m+1} obtained from the linear order L_{2m+1} by reversing all the arcs between the even integers (see Figure 4.5).

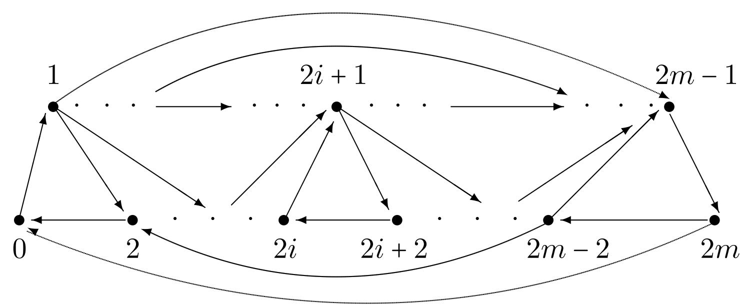

Figure 4.5. The tournament U_{2m+1}.

It follows from (4.46) that

$$\tau = \sigma(U_{2n+1}).$$

Subcase b: $<0,1>_\tau \cap <0,2>_\tau = \varnothing$.

We obtain

$$\begin{aligned} E(\tau) = \{ & A(L_{2n+1})_{(0,1)} \cup A(L_{2n+1})_{(1,0)} \cup A(L_{2n+1})_{(1,1)}, \\ & (A(L_{2n+1})_{(0,1)} \cup A(L_{2n+1})_{(1,0)} \cup (A(L_{2n+1})_{(1,1)})^\star, \\ & A(L_{2n+1})_{(0,0)}, \\ & (A(L_{2n+1})_{(0,0)})^\star\}. \end{aligned} \tag{4.47}$$

It follows from (4.45), (4.46), and (4.47) that

$\tau = \sigma(D_{2n+1}) \wedge \sigma((D_{2n+1})^\star) \wedge \sigma(U_{2n+1})$ (see Remark 4.11 and Fact 4.14).

We summarize the previous examination in the next theorem.

Theorem 4.28 (Boudabbous and Ille[4.5] [7])**.** *Consider a 2-structure τ defined such that $V(\tau) = \{0, \dots, 2n\}$, where $n \geq 2$. The following two statements are equivalent*

- *τ is critical, and $\mathbb{P}(\tau) = P_{2n+1}$;*
- *$\tau = \sigma(D_{2n+1})$, $\sigma((D_{2n+1})^\star)$, $\sigma(D_{2n+1}) \wedge \sigma((D_{2n+1})^\star)$, $\sigma(U_{2n+1})$, or $\sigma(D_{2n+1}) \wedge \sigma((D_{2n+1})^\star) \wedge \sigma(U_{2n+1})$.*

The following result is an immediate consequence of Theorem 4.28.

Corollary 4.29. *Consider a reversible 2-structure τ defined such that $V(\tau) = \{0, \dots, 2n\}$, where $n \geq 2$. The following two statements are equivalent*

- *τ is critical, and $\mathbb{P}(\tau) = P_{2n+1}$;*
- *$\tau = \sigma(U_{2n+1})$, $\sigma(D_{2n+1}) \wedge \sigma((D_{2n+1})^\star)$, or $\sigma(D_{2n+1}) \wedge \sigma((D_{2n+1})^\star) \wedge \sigma(U_{2n+1})$.*

The next remark completes the subsubsection.

Remark 4.30. Given $n \geq 2$, consider a critical 2-structure τ defined on $V(\tau) = \{0, \dots, 2n\}$, and such that $\mathbb{P}(\tau) = P_{2n+1}$. Set

$$\begin{aligned}\mathcal{E}(\tau) = \{&A(L_{2n+1})_{(0,1)} \cup A(L_{2n+1})_{(1,0)} \cup A(L_{2n+1})_{(1,1)},\\ &(A(L_{2n+1})_{(0,1)} \cup A(L_{2n+1})_{(1,0)} \cup A(L_{2n+1})_{(1,1)})^\star,\\ &A(L_{2n+1})_{(0,0)}, A(L_{2n+1})^\star_{(0,0)}\}.\end{aligned}$$

We obtain

$$\pi_{2n+1}(e) = e^\star \text{ for each } e \in \mathcal{E}(\tau) \quad \text{(see Notation 4.21).} \tag{4.48}$$

Consider $e \in E(\tau)$. By Proposition 4.27, there exists $\mathcal{B}_e \subsetneq \mathcal{E}(\tau)$ such that

$$e = \bigcup_{f \in \mathcal{B}_e} f.$$

Thus, we obtain

$$\begin{aligned}\pi_{2n+1}(e) &= \bigcup_{f \in \mathcal{B}_e} \pi_{2n+1}(f)\\ &= \bigcup_{f \in \mathcal{B}_e} f^\star \quad \text{(by (4.48))}\\ &= e^\star.\end{aligned}$$

Consequently, π_{2n+1} is an isomorphism from τ onto $\tau^\star$. If τ is reversible, then $\tau = \tau^\star$, and hence π_{2n+1} is an automorphism of τ.

4.2.4. *The type C_{2n+1}.* Given $m \geq 1$, we consider the tournament W_{2m+1} obtained from the tournament U_{2m+1} by reversing all the arcs between the odd integers (see Figure 4.6). The next remark is useful to establish Proposition 4.36 below.

[4.5]Boudabbous and Ille [7] proved this theorem for digraphs.

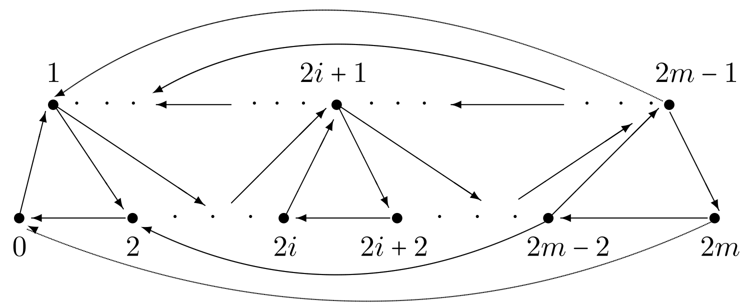

FIGURE 4.6. The tournament W_{2m+1}.

Remark 4.31. Let Γ be a group of odd order with identity element e. Consider $\Omega \subseteq \Gamma \setminus \{e\}$ such that $|\Omega \cap \{x, x^{-1}\}| = 1$ for each $x \in \Gamma \setminus \{e\}$. We associate with Γ and Ω the Cayley tournament $\text{Cay}(\Gamma, \Omega)$ defined on $V(\text{Cay}(\Gamma, \Omega)) = \Gamma$ as follows. Given $x, y \in \Gamma$, $(x, y) \in A(\text{Cay}(\Gamma, \Omega))$ if $yx^{-1} \in \Omega$. For each $a \in \Gamma$, the permutation of Γ, defined by $x \mapsto xa$ for every $x \in \Gamma$, is an automorphism of $\text{Cay}(\Gamma, \Omega)$. Consequently, $\text{Cay}(\Gamma, \Omega)$ is vertex-transitive.

Let $m \geq 1$. We consider the cyclic group $(\mathbb{Z}_{2m+1}, +)$. We consider also the permutation

$$\begin{array}{rccc} \psi_{2m+1}: & \{0, \dots, 2m\} & \longrightarrow & \{0, \dots, 2m\} \\ & p & \longmapsto & (m+1) \times p \mod 2m+1 \end{array}$$

of $\mathbb{Z}_{2m+1}$. We denote by $\psi_{2m+1}(W_{2m+1})$ the unique tournament defined on $\mathbb{Z}_{2m+1}$ such that ψ_{2m+1} is an isomorphism from W_{2m+1} onto $\psi_{2m+1}(W_{2m+1})$.

Fact 4.32. *For $m \geq 1$, we have*

$$(\psi_{2m+1}(W_{2m+1}))^\star = \text{Cay}(\mathbb{Z}_{2m+1}, \{1, \dots, m\}).$$

For convenience, set

$$\text{Cay}_{2m+1} = \text{Cay}(\mathbb{Z}_{2m+1}, \{1, \dots, m\}).$$

Fact 4.33. *Given $m \geq 1$, Cay_{2m+1} is prime.*

Proof. Let M be a module of Cay_{2m+1} such that $|M| \geq 2$. We have to show that $M = \mathbb{Z}_{2m+1}$. As previously noted, the permutation of $\mathbb{Z}_{2m+1}$, defined by $p \mapsto (p+1) \bmod (2m+1)$ for each $p \in \mathbb{Z}_{2m+1}$, is an automorphism of Cay_{2m+1}. Hence, we can assume that $0 \in M$. Moreover, the permutation of $\mathbb{Z}_{2m+1}$, defined by $p \mapsto -p \bmod (2m+1)$ for each $p \in \mathbb{Z}_{2m+1}$, is an isomorphism from Cay_{2m+1} onto $(\text{Cay}_{2m+1})^\star$. Since Cay_{2m+1} and $(\text{Cay}_{2m+1})^\star$ share the same modules, we can assume that there exists $q \in M \cap \{1, \dots, m\}$. Since $\text{Cay}_{2m+1}[\{0, \dots, m\}] = L_{m+1}$, we obtain $\{0, \dots, q\} \subseteq M$. Since $(1, m+1), (m+1, 0) \in A(\text{Cay}_{2m+1})$, we have $m+1 \in M$. Hence

$$\{0, \dots, q\} \cup \{m+1\} \subseteq M. \tag{4.49}$$

Now, we show that

$$\{0,\dots,m+1\} \subseteq M. \tag{4.50}$$

Clearly, (4.50) follows from (4.49) when $q = m$. Thus, suppose that $q \le m-1$. Let $p \in \{q+1,\dots,m\}$. Since $(q,p),(p,m+1) \in A(\mathrm{Cay}_{2m+1})$, we have $p \in M$. It follows that $\{q+1,\dots,m\} \subseteq M$. Since $\{0,\dots,q\} \cup \{m+1\} \subseteq M$ by (4.49), we obtain $\{0,\dots,m+1\} \subseteq M$. Consequently, (4.50) holds. If $m = 1$, then $M = \mathbb{Z}_{2m+1}$ by (4.50). Lastly, suppose that $m \ge 2$. Let $p \in \{m+2,\dots,2m\}$. Since $(m+1,p),(p,0) \in A(\mathrm{Cay}_{2m+1})$, we have $p \in M$. We obtain $\{m+2,\dots,2m\} \subseteq M$. It follows from (4.50) that $M = \mathbb{Z}_{2m+1}$. Consequently, Cay_{2m+1} is prime. □

Fact 4.34. *Given $m \ge 2$, Cay_{2m+1} is critical, and*

$$\mathbb{P}(\mathrm{Cay}_{2m+1}) = \psi_{2m+1}(C_{2m+1}),$$

where $\psi_{2m+1}(C_{2m+1})$ denotes the unique graph defined on $\mathbb{Z}_{2m+1}$ such that ψ_{2m+1} is an isomorphism from C_{2m+1} onto $\psi_{2m+1}(C_{2m+1})$.

Proof. We have $W_{2m+1} \setminus \{2m-1,2m\} = W_{2m-1}$. By Fact 4.33, Cay_{2m-1} is prime. Since $(\psi_{2m-1}(W_{2m-1}))^\star = \mathrm{Cay}_{2m-1}$ by Fact 4.32, W_{2m-1} is prime. Hence $W_{2m+1} \setminus \{2m-1,2m\}$ is prime as well. Since $(\psi_{2m+1}(W_{2m+1}))^\star = \mathrm{Cay}_{2m+1}$ by Fact 4.32, we obtain that $\mathrm{Cay}_{2m+1} \setminus \psi_{2m+1}(\{2m-1,2m\}) = \mathrm{Cay}_{2m+1} \setminus \{m,2m\}$ is prime. The permutation of $\mathbb{Z}_{2m+1}$, defined by $p \mapsto (p-m) \bmod (2m+1)$ for each $p \in \mathbb{Z}_{2m+1}$, is an automorphism of Cay_{2m+1}. Thus $\mathrm{Cay}_{2m+1} \setminus \{0,m\}$ is prime. The permutation of $\mathbb{Z}_{2m+1}$, defined by $p \mapsto -p \bmod (2m+1)$ for each $p \in \mathbb{Z}_{2m+1}$, is an isomorphism from Cay_{2m+1} onto $(\mathrm{Cay}_{2m+1})^\star$. Therefore, $\mathrm{Cay}_{2m+1} \setminus \psi_{2m+1}(\{0,-m\}) = \mathrm{Cay}_{2m+1} \setminus \psi_{2m+1}(\{0,m+1\})$ is prime. It follows that

$$m, m+1 \in N_{\mathbb{P}(\mathrm{Cay}_{2m+1})}(0).$$

Clearly, $\{m,m+1\}$ is a module of $\mathrm{Cay}_{2m+1} - 0$. Thus 0 is a critical vertex of Cay_{2m+1}. By Lemma 4.4, $d_{\mathbb{P}(\mathrm{Cay}_{2m+1})}(0) \le 2$. We obtain

$$N_{\mathbb{P}(\mathrm{Cay}_{2m+1})}(0) = \{m,m+1\}.$$

Let $q \in \mathbb{Z}_{2m+1}$. Since the permutation of $\mathbb{Z}_{2m+1}$, defined by $p \mapsto (p+q) \bmod (2m+1)$ for each $p \in \mathbb{Z}_{2m+1}$, is an automorphism of Cay_{2m+1}, we obtain that q is a critical vertex of Cay_{2m+1}, and $N_{\mathbb{P}(\mathrm{Cay}_{2m+1})}(q) = \{q+m, q+m+1\}$. Consequently, Cay_{2m+1} is critical, and $\mathbb{P}(\mathrm{Cay}_{2m+1}) = \psi_{2m+1}(C_{2m+1})$. □

The next fact is an immediate consequence of Facts 4.32 and 4.34.

Fact 4.35. *Given $m \ge 2$, W_{2m+1} is critical, and $\mathbb{P}(W_{2m+1}) = C_{2m+1}$.*

Proposition 4.36. *Given $n \ge 2$, consider a 2-structure τ defined on $V(\tau) = \{0,\dots,2n\}$. The following two statements are equivalent*

(1) *τ is critical, and $\mathbb{P}(\tau) = C_{2n+1}$;*

(2) $(0,1)_\tau \neq (1,0)_\tau$, *and for* $p,q \in \{0,\dots,2n\}$ *such that* $p<q$, *we have*

$$[p,q]_\tau = \begin{cases} [1,0]_\tau & \text{if } p \text{ and } q \text{ have the same parity,} \\ [0,1]_\tau & \text{otherwise.} \end{cases} \tag{4.51}$$

Proof. To begin, suppose that τ is critical, and $\mathbb{P}(\tau) = C_{2n+1}$. We verify that (4.51) holds in the following manner. Since $E(P_{2n+1}) \subseteq E(C_{2n+1})$, (4.36) holds. It follows that (4.37) holds. We distinguish the following cases, where $p,q \in \{0,\dots,2n\}$ such that $p<q$.

Case 1: p and q are even.

By (4.37), $[p,q]_\tau = [0,2]_\tau$. Since $\mathbb{P}(\tau) = C_{2n+1}$, we have $N_{\mathbb{P}(\tau)}(2n) = \{0,2n-1\}$. By Lemma 4.4, $\{0,2n-1\}$ is a module of $\tau-(2n)$. In particular, we obtain $[0,2]_\tau = [2n-1,2]_\tau$. By (4.37), $[2n-1,2]_\tau = [1,0]_\tau$. Thus

$$[p,q]_\tau = [1,0]_\tau. \tag{4.52}$$

Case 2: p and q are odd.

By (4.37), $[p,q]_\tau = [1,2n-1]_\tau$. Since $\{0,2n-1\}$ is a module of $\tau-(2n)$, we have $[1,2n-1]_\tau = [1,0]_\tau$. Hence

$$[p,q]_\tau = [1,0]_\tau.$$

Case 3: p is even and q is odd.

By (4.37), $[p,q]_\tau = [0,1]_\tau$.

Case 4: p is odd and q is even.

By (4.37), $[p,q]_\tau = [1,2]_\tau$. Since $\mathbb{P}(\tau) = C_{2n+1}$, we have $N_{\mathbb{P}(\tau)}(0) = \{1,2n\}$. By Lemma 4.4, $\{1,2n\}$ is a module of $\tau-0$. In particular, we obtain $[1,2]_\tau = [2n,2]_\tau$. By (4.37), $[2n,2]_\tau = [2,0]_\tau$. By (4.52), $[2,0]_\tau = [0,1]_\tau$. Therefore

$$[p,q]_\tau = [0,1]_\tau.$$

It follows that (4.51) holds. Since τ is prime, τ is not constant. It follows from (4.51) that $(0,1)_\tau \neq (1,0)_\tau$.

Conversely, suppose that (4.51) holds, and $(0,1)_\tau \neq (1,0)_\tau$. We obtain

$$\tau = \sigma(W_{2m+1}).$$

By Fact 4.35, τ is critical, and $\mathbb{P}(\tau) = C_{2m+1}$. □

The next characterization is a simple consequence of Proposition 4.36 and its proof.

Theorem 4.37 (Boudabbous and Ille[4.6] [7]). *Consider a 2-structure* τ *defined on* $V(\tau) = \{0,\dots,2n\}$, *where* $n \geq 2$. *The following two statements are equivalent*

- τ *is critical, and* $\mathbb{P}(\tau) = C_{2n+1}$;
- $\tau = \sigma(W_{2n+1})$ *(see Figure 4.6).*

[4.6] Boudabbous and Ille [7] proved this theorem for digraphs.

In the next remark, we determine the automorphism group of $\sigma(W_{2n+1})$ when $n \geq 2$.

Remark 4.38. Let $n \geq 2$. By Fact 4.32, ψ_{2n+1} is an isomorphism from W_{2n+1} onto $(\mathrm{Cay}_{2n+1})^\star$. To determine $\mathrm{Aut}(\mathrm{Cay}_{2n+1})$, we consider the permutation

$$\begin{array}{rccc} \theta_{2n+1}: & \{0,\dots,2n\} & \longrightarrow & \{0,\dots,2n\} \\ & p & \longmapsto & p+1 \mod 2n+1 \end{array}$$

of $\mathbb{Z}_{2n+1}$. We prove that

$$\mathrm{Aut}(\sigma(W_{2n+1})) = <\theta_{2n+1}, \pi_{2n+1}> . \tag{4.53}$$

To begin, we show that

$$\mathrm{Aut}(\mathrm{Cay}_{2n+1}) = <\theta_{2n+1}> . \tag{4.54}$$

Clearly, $\theta_{2n+1} \in \mathrm{Aut}(\mathrm{Cay}_{2n+1})$, and hence $<\theta_{2n+1}> \subseteq \mathrm{Aut}(\mathrm{Cay}_{2n+1})$. Conversely, consider $\varphi \in \mathrm{Aut}(\mathrm{Cay}_{2n+1})$. Since $<\theta_{2n+1}> \subseteq \mathrm{Aut}(\mathrm{Cay}_{2n+1})$,

$$(\theta_{2n+1})^{-\varphi(0)} \circ \varphi \in \mathrm{Aut}(\mathrm{Cay}_{2n+1}).$$

We have

$$((\theta_{2n+1})^{-\varphi(0)} \circ \varphi)(0) = 0.$$

Since $\mathrm{Cay}_{2n+1}[N^-_{\mathrm{Cay}_{2n+1}}(0)]$ and $\mathrm{Cay}_{2n+1}[N^+_{\mathrm{Cay}_{2n+1}}(0)]$ are linear orders, we obtain $(\theta_{2n+1})^{-\varphi(0)} \circ \varphi = \mathrm{Id}_{\{0,\dots,2n\}}$. Therefore, $\varphi \in <\theta_{2n+1}>$. It follows that (4.54) holds. Moreover, since ψ_{2n+1} is an isomorphism from W_{2n+1} onto $(\mathrm{Cay}_{2n+1})^\star$, we obtain

$$\mathrm{Aut}(W_{2n+1}) = (\psi_{2n+1})^{-1} \circ <\theta_{2n+1}> \circ\, \psi_{2n+1}.$$

We have $(\psi_{2n+1})^{-1} \circ\ \theta_{2n+1} \circ\ \psi_{2n+1} = (\theta_{2n+1})^2$. Furthermore, we have

$$((\theta_{2n+1})^2)^{n+1} = \theta_{2n+1}.$$

It follows that

$$\mathrm{Aut}(W_{2n+1}) = (\psi_{2n+1})^{-1} \circ <\theta_{2n+1}> \circ\, \psi_{2n+1} = <\theta_{2n+1}> . \tag{4.55}$$

Now, we show that

$$\mathrm{Iso}(\mathrm{Cay}_{2n+1}, (\mathrm{Cay}_{2n+1})^\star) = <\theta_{2n+1}> \circ\, \pi_{2n+1}. \tag{4.56}$$

Clearly, $\pi_{2n+1} \in \mathrm{Iso}(\mathrm{Cay}_{2n+1}, (\mathrm{Cay}_{2n+1})^\star)$. It follows from (4.54) that $<\theta_{2n+1}> \circ\, \pi_{2n+1} \subseteq \mathrm{Iso}(\mathrm{Cay}_{2n+1}, (\mathrm{Cay}_{2n+1})^\star)$. Conversely, let φ be an isomorphism from Cay_{2n+1} onto $(\mathrm{Cay}_{2n+1})^\star$. Since $\theta_{2n+1} \in \mathrm{Aut}(\mathrm{Cay}_{2n+1})$ by (4.54), $(\theta_{2n+1})^{n-\varphi(n)} \circ \varphi \in \mathrm{Iso}(\mathrm{Cay}_{2n+1}, (\mathrm{Cay}_{2n+1})^\star)$. Set

$$\varphi' = (\theta_{2n+1})^{n-\varphi(n)} \circ \varphi.$$

We have $\varphi'(n) = n$. Thus, we obtain $\varphi'[N^-_{\mathrm{Cay}_{2n+1}}(n)] = N^+_{\mathrm{Cay}_{2n+1}}(n)$ and $\varphi'[N^+_{\mathrm{Cay}_{2n+1}}(n)] = N^-_{\mathrm{Cay}_{2n+1}}(n)$. Recall that

$$\mathrm{Cay}_{2n+1}[N^-_{\mathrm{Cay}_{2n+1}}(n)] = L_{\{0,\dots,n-1\}} \text{ and } \mathrm{Cay}_{2n+1}[N^+_{\mathrm{Cay}_{2n+1}}(n)] = L_{\{n+1,\dots,2n\}}.$$

It follows that $\varphi' = \pi_{2n+1}$, so $\varphi = (\theta_{2n+1})^{\varphi(n)-n} \circ \pi_{2n+1}$. Consequently, (4.56) holds. Lastly, since ψ_{2n+1} is an isomorphism from W_{2n+1} onto $(\mathrm{Cay}_{2n+1})^\star$, we obtain

$$\mathrm{Iso}(W_{2n+1}, (W_{2n+1})^\star) = (\psi_{2n+1})^{-1} \circ < \theta_{2n+1} > \circ\ \pi_{2n+1} \circ \psi_{2n+1}.$$

Since $\pi_{2n+1} \circ \psi_{2n+1} = (\theta_{2n+1})^n \circ \psi_{2n+1} \circ \pi_{2n+1}$,

$$\begin{aligned}\mathrm{Iso}(W_{2n+1}, (W_{2n+1})^\star) &= (\psi_{2n+1})^{-1} \circ < \theta_{2n+1} > \circ (\theta_{2n+1})^n \circ \psi_{2n+1} \circ \pi_{2n+1} \\ &= (\psi_{2n+1})^{-1} \circ < \theta_{2n+1} > \circ\ \psi_{2n+1} \circ \pi_{2n+1} \\ &= < \theta_{2n+1} > \circ\ \pi_{2n+1} \quad \text{(by (4.55))}. \end{aligned} \tag{4.57}$$

By Note 4.25, $\mathrm{Aut}(\sigma(W_{2n+1})) = \mathrm{Aut}(W_{2n+1}) \cup \mathrm{Iso}(W_{2n+1}, (W_{2n+1})^\star)$. It follows from (4.55) and (4.57) that

$$\mathrm{Aut}(\sigma(W_{2n+1})) = < \theta_{2n+1} > \cup (< \theta_{2n+1} > \circ\ \pi_{2n+1}).$$

Since $\pi_{2n+1} \circ (\theta_{2n+1})^k = (\theta_{2n+1})^{-k} \circ \pi_{2n+1}$, (4.53) holds.

4.3. **Properties of critical 2-structures.**

Lemma 4.39. *Let τ be a critical 2-structure, with $v(\tau) \geq 7$. If u, v, x, y are distinct vertices of τ such that $\{u, x\}, \{x, y\}, \{y, v\} \in E(\mathbb{P}(\tau))$, then $\tau - \{x, y\}$ is critical, τ and $\tau - \{x, y\}$ share the same type, and*

$$E(\mathbb{P}(\tau - \{x, y\})) = (E(\mathbb{P}(\tau)) \setminus \{\{u, x\}, \{x, y\}, \{y, v\}\}) \cup \{\{u, v\}\}.$$

Proof. By Corollary 4.6, there exist $n \geq 3$ and a bijection f defined on $V(\tau)$ such that $f(\mathbb{P}(\tau)) = P_{2n} \oplus K_{\{2n\}}, P_{2n+1}, C_{2n+1}$ or $f(\mathbb{P}(\tau)) = P_{2n}$, with $n \geq 4$.

To begin, suppose that

$$f(\mathbb{P}(\tau)) = C_{2n+1}.$$

We can assume that $f(u) = 2n - 2$, $f(x) = 2n - 1$, $f(y) = 2n$ and $f(v) = 0$. Since $\mathbb{P}(f(\tau)) = f(\mathbb{P}(\tau))$, we have

$$\mathbb{P}(f(\tau)) = C_{2n+1}.$$

By Proposition 4.36, $(0,1)_{f(\tau)} \neq (1,0)_{f(\tau)}$, and $f(\tau)$ satisfies (4.51). Clearly, $(0,1)_{f(\tau)-\{2n-1,2n\}} \neq (1,0)_{f(\tau)-\{2n-1,2n\}}$, and $f(\tau) - \{2n-1, 2n\}$ satisfies (4.51). Since $n \geq 3$, we can apply Proposition 4.36 to $f(\tau) - \{2n-1, 2n\}$. We obtain that $f(\tau) - \{2n-1, 2n\}$ is critical, and $\mathbb{P}(f(\tau) - \{2n-1, 2n\}) = C_{2n-1}$. Since f is an isomorphism from $\mathbb{P}(\tau)$ onto C_{2n+1}, we obtain

$$\begin{aligned} E(\mathbb{P}(\tau)) = &\{\{f^{-1}(p), f^{-1}(p+1)\} : 0 \leq p \leq 2n-1\} \\ &\cup \{\{f^{-1}(2n), f^{-1}(0)\}\}. \end{aligned} \tag{4.58}$$

Clearly, $f_{\restriction V(\tau) \setminus \{x,y\}}$ is an isomorphism from $\tau - \{x, y\}$ onto $f(\tau) - \{2n-1, 2n\}$. Hence, $f_{\restriction V(\tau) \setminus \{x,y\}}$ is an isomorphism from $\mathbb{P}(\tau - \{x, y\})$ onto $\mathbb{P}(f(\tau) - \{2n-1, 2n\})$. Since $\mathbb{P}(f(\tau) - \{2n-1, 2n\}) = C_{2n-1}$, we obtain

$$\begin{aligned} E(\mathbb{P}(\tau - \{x, y\})) = &\{\{f^{-1}(p), f^{-1}(p+1)\} : 0 \leq p \leq 2n-3\} \\ &\cup \{\{f^{-1}(2n-2), f^{-1}(0)\}\}. \end{aligned} \tag{4.59}$$

It follows from (4.58) and (4.59) that

$$\begin{aligned}
&E(\mathbb{P}(\tau - \{x,y\}) = \\
&\quad (E(\mathbb{P}(\tau)) \setminus \{\{f^{-1}(2n-2), f^{-1}(2n-1)\}, \{f^{-1}(2n-1), f^{-1}(2n)\}, \\
&\quad \{f^{-1}(2n), f^{-1}(0)\}\}) \cup \{\{f^{-1}(2n-2), f^{-1}(0)\}\} \\
&\quad = (E(\mathbb{P}(\tau)) \setminus \{\{u,x\}, \{x,y\}, \{y,v\}\}) \cup \{\{u,v\}\}.
\end{aligned}$$

Now, suppose that $f(\mathbb{P}(\tau)) = P_{2n}, P_{2n} \oplus K_{\{2n\}}$, or P_{2n+1}. We proceed in the same way for the three cases. For instance, assume that

$$f(\mathbb{P}(\tau)) = P_{2n+1}.$$

There exists $p \in \{0, \dots, 2n-3\}$ such that $f(u) = p$, $f(x) = p+1$, $f(y) = p+2$ and $f(v) = p+3$. Since $\mathbb{P}(f(\tau)) = f(\mathbb{P}(\tau))$, we have

$$\mathbb{P}(f(\tau)) = P_{2n+1}.$$

By Proposition 4.27, $(0,1)_{f(\tau)} \neq (1,0)_{f(\tau)}$, $[0,1]_{f(\tau)} \neq [0,2]_{f(\tau)}$, and $f(\tau)$ satisfies (4.35). Consider the bijection

$$\begin{array}{rccc}
g: & \{0,\dots,2n\} \setminus \{p+1,p+2\} & \longrightarrow & \{0,\dots,2n-2\} \\
 & q \leq p & \longmapsto & q, \\
 & q \geq p+3 & \longmapsto & q-2.
\end{array}$$

For each $q \in \{0,\dots,2n-2\}$, we have $q \equiv g^{-1}(q) \bmod 2$. Moreover, for any $q,r \in \{0,\dots,2n-2\}$, $q < r$ if and only if $g^{-1}(q) < g^{-1}(r)$. Since $(0,1)_{f(\tau)} \neq (1,0)_{f(\tau)}$, $[0,1]_{f(\tau)} \neq [0,2]_{f(\tau)}$, and $f(\tau)$ satisfies (4.35), we obtain

$$\begin{cases}
(0,1)_{g(f(\tau)-\{p+1,p+2\})} \neq (1,0)_{g(f(\tau)-\{p+1,p+2\})}, \\
[0,1]_{g(f(\tau)-\{p+1,p+2\})} \neq [0,2]_{g(f(\tau)-\{p+1,p+2\})}, \\
\text{and} \\
g(f(\tau)-\{p+1,p+2\}) \text{ satisfies (4.35).}
\end{cases}$$

By Proposition 4.27 applied to $g(f(\tau)-\{p+1,p+2\})$, $g(f(\tau)-\{p+1,p+2\})$ is critical, and

$$\mathbb{P}(g(f(\tau)-\{p+1,p+2\})) = P_{2n-1}.$$

Since f is an isomorphism from $\mathbb{P}(\tau)$ onto P_{2n+1}, we have

$$E(\mathbb{P}(\tau)) = \{\{f^{-1}(q), f^{-1}(q+1)\} : 0 \leq q \leq 2n-1\}. \tag{4.60}$$

Since $\mathbb{P}(g(f(\tau)-\{p+1,p+2\})) = P_{2n-1}$, $g \circ (f_{\restriction V(\tau)\setminus\{x,y\}})$ is an isomorphism from $\mathbb{P}(\tau-\{x,y\})$ onto P_{2n-1}. It follows that

$$\begin{aligned}
&E(\mathbb{P}(\tau-\{x,y\})) = \\
&\quad \{\{(g \circ (f_{\restriction V(\tau)\setminus\{x,y\}}))^{-1}(q), (g \circ (f_{\restriction V(\tau)\setminus\{x,y\}}))^{-1}(q+1)\} : 0 \leq q \leq 2n-3\}.
\end{aligned}$$

We obtain

$$E(\mathbb{P}(\tau-\{x,y\})) = \{\{f^{-1}(p), f^{-1}(p+3)\}\}$$
$$\cup \{\{f^{-1}(q), f^{-1}(q+1)\} : 0 \le q \le p-1\} \quad \text{(when } p \ge 1) \tag{4.61}$$
$$\cup \{\{f^{-1}(q), f^{-1}(q+1)\} : p+3 \le q \le 2n-1\} \quad \text{(when } p \le 2n-4).$$

It follows from (4.60) and (4.61) that

$$\begin{aligned} E(\mathbb{P}(\tau - \{x,y\}) =&(E(\mathbb{P}(\tau)) \smallsetminus \{\{f^{-1}(p), f^{-1}(p+1)\}, \{f^{-1}(p+1), f^{-1}(p+2)\}, \\ &\{f^{-1}(p+2), f^{-1}(p+3)\}\}) \cup \{\{f^{-1}(p), f^{-1}(p+3)\}\} \\ =&(E(\mathbb{P}(\tau)) \smallsetminus \{\{u,x\}, \{x,y\}, \{y,v\}\}) \cup \{\{u,v\}\}. \end{aligned}$$
□

Lemma 4.40. *Let τ be a critical 2-structure, with $v(\tau) \ge 7$. If u, x, y are distinct vertices of τ such that $\{u,x\}, \{x,y\} \in E(\mathbb{P}(\tau))$, and $d_{\mathbb{P}(\tau)}(y) = 1$, then $\tau - \{x,y\}$ is critical, τ and $\tau - \{x,y\}$ share the same type, and*

$$E(\mathbb{P}(\tau - \{x,y\})) = E(\mathbb{P}(\tau)) \smallsetminus \{\{u,x\}, \{x,y\}\}.$$

Proof. By Corollary 4.6, there exist $n \ge 3$ and a bijection f defined on $V(\tau)$ such that $f(\mathbb{P}(\tau)) = P_{2n} \oplus K_{\{2n\}}, P_{2n+1}, C_{2n+1}$, or $f(\mathbb{P}(\tau)) = P_{2n}$, with $n \ge 4$. Since $d_{\mathbb{P}(\tau)}(y) = 1$, $f(\mathbb{P}(\tau)) = P_{2n}, P_{2n} \oplus K_{\{2n\}}$, or P_{2n+1}. As in the proof of Lemma 4.39, we treat only the case $f(\mathbb{P}(\tau)) = P_{2n+1}$. We can assume that $f(u) = 2n-2$, $f(x) = 2n-1$ and $f(y) = 2n$.

Since $\mathbb{P}(f(\tau)) = f(\mathbb{P}(\tau))$, we have

$$\mathbb{P}(f(\tau)) = P_{2n+1}.$$

By Proposition 4.27, $(0,1)_{f(\tau)} \ne (1,0)_{f(\tau)}$, $[0,1]_{f(\tau)} \ne [0,2]_{f(\tau)}$, and $f(\tau)$ satisfies (4.35). Therefore, we have

$$\begin{cases} (0,1)_{f(\tau)-\{2n-1,2n\}} \ne (1,0)_{f(\tau)-\{2n-1,2n\}}, \\ [0,1]_{f(\tau)-\{2n-1,2n\}} \ne [0,2]_{f(\tau)-\{2n-1,2n\}}, \\ \text{and} \\ f(\tau) - \{2n-1,2n\} \text{ satisfies (4.35).} \end{cases}$$

By Proposition 4.27 applied to $f(\tau) - \{2n-1,2n\}$, $f(\tau) - \{2n-1,2n\}$ is critical, and

$$\mathbb{P}(f(\tau) - \{2n-1,2n\}) = P_{2n-1}.$$

Since f is an isomorphism from $\mathbb{P}(\tau)$ onto P_{2n+1}, we have

$$E(\mathbb{P}(\tau)) = \{\{f^{-1}(q), f^{-1}(q+1)\} : 0 \le q \le 2n-1\}. \tag{4.62}$$

Since $\mathbb{P}(f(\tau) - \{2n-1,2n\}) = P_{2n-1}$, $f_{\restriction V(\tau)\smallsetminus\{x,y\}}$ is an isomorphism from $\mathbb{P}(\tau - \{x,y\})$ onto P_{2n-1}. we obtain

$$E(\mathbb{P}(\tau - \{x,y\})) = \{\{f^{-1}(q), f^{-1}(q+1)\} : 0 \le q \le 2n-3\}. \tag{4.63}$$

It follows from (4.62) and (4.63) that

$$\begin{aligned} E(\mathbb{P}(\tau - \{x, y\}) &= E(\mathbb{P}(\tau)) \\ &\qquad \setminus \{\{f^{-1}(2n-2), f^{-1}(2n-1)\}, \{f^{-1}(2n-1), f^{-1}(2n)\}\} \\ &= E(\mathbb{P}(\tau)) \setminus \{\{u, x\}, \{x, y\}\}. \end{aligned}$$

□

The following corollary is an immediate consequence of Lemmas 4.39 and 4.40. It is useful in the next section when disjoint edges of the primality graph of a critical 2-structure are considered.

Corollary 4.41. *Given a critical 2-structure τ, with $v(\tau) \geq 7$, consider distinct vertices x and y of τ such that $\{x, y\} \in E(\mathbb{P}(\tau))$. The following two statements hold.*

(1) *$\tau - \{x, y\}$ is critical;*
(2) *τ and $\tau - \{x, y\}$ share the same type;*
(3) *For every $e \in E(\mathbb{P}(\tau))$, if $e \cap \{x, y\} = \varnothing$, then $e \in E(\mathbb{P}(\tau - \{x, y\}))$;*
(4) *For every $e \in E(\mathbb{P}(\tau - \{x, y\}))$, if $e \setminus (N_{\mathbb{P}(\tau)}(x) \cup N_{\mathbb{P}(\tau)}(y)) \neq \varnothing$, then $e \in E(\mathbb{P}(\tau))$.*

5. Noncritical unordered pair theorems

Given a prime 2-structure, a noncritical unordered pair theorem provides distinct vertices v and w of σ such that $\sigma - \{v,w\}$ is prime as well.

critical support

We refine the notion of a support as follows. Given a 2-structure σ, the *critical support* of σ is the set of the vertices v of σ such that $\sigma - v$ is critical. It is denoted by $\mathscr{S}_c(\sigma)$.

Remark 5.1. Let σ be a prime 2-structure. Suppose that

$$\mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma) \neq \varnothing.$$

Let $v \in \mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma)$. Since $v \in \mathscr{S}(\sigma)$, $\sigma - v$ is prime. Since $v \notin \mathscr{S}_c(\sigma)$, $\sigma - v$ is not critical. Hence, there exists $w \in V(\sigma - v)$ such that $(\sigma - v) - w$ is prime. Therefore, $\{v,w\}$ is a noncritical unordered pair of σ.

Remark 5.2. Let σ be a prime 2-structure, with $v(\sigma) \geq 6$. Suppose that

$$|\mathscr{S}_c(\sigma)| \leq 1.$$

It follows from Theorem 3.11 that there exists $X \subseteq V(\sigma)$ such that $3 \leq |X| \leq 5$ and

$$\mathscr{S}_c(\sigma) \subseteq X. \tag{5.1}$$

By Corollary 3.21, there exist $v, w \in V(\sigma) \setminus X$ such that $\sigma - \{v,w\}$ is prime. Clearly, if $v \neq w$, then $\{v,w\}$ is a noncritical unordered pair of σ. Hence, suppose that $v = w$. We obtain $v \in \mathscr{S}(\sigma)$. Since $\mathscr{S}_c(\sigma) \subseteq X$, we have $v \in \mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma)$, and we conclude as in Remark 5.1.

5.1. The Schmerl–Trotter theorem.

Theorem 5.3 (Schmerl and Trotter [33][5.1]). *Given a prime 2-structure σ such that $v(\sigma) \geq 7$, there exist $v, w \in V(\sigma)$ such that $v \neq w$ and $\sigma - \{v,w\}$ is prime.*

Theorem 5.3 is the second downward hereditary property of primality. We use the properties of critical 2-structures presented in subsection 4.3 to prove it. Our approach is based on Remark 5.1.

In this subsection, we provide a proof of Theorem 5.3 when $v(\sigma) \geq 9$. In section 6, we provide a proof of Theorem 5.3, when $v(\sigma) \geq 7$, by using Theorem 5.23. We begin with the following lemma.

Lemma 5.4. *Let σ be a prime 2-structure, with $v(\sigma) \geq 8$, such that $\mathscr{S}_c(\sigma) \neq \varnothing$. Consider $x \in \mathscr{S}_c(\sigma)$. Let*

$$\mathcal{E} \subseteq E(\mathbb{P}(\sigma - x))$$

such that $e \cap f = \varnothing$ for distinct $e, f \in \mathcal{E}$. If $|\mathcal{E}| \geq 4$, then $\mathcal{E} \cap E(\mathbb{P}(\sigma)) \neq \varnothing$.

[5.1] Schmerl and Trotter [33] proved this theorem for binary relational structures. The cases of partially ordered sets, graphs, and tournaments are specified.

Proof. Suppose that $\mathcal{E} \cap E(\mathbb{P}(\sigma)) = \varnothing$. We have to show that $|\mathcal{E}| \leq 3$. Hence, suppose that $|\mathcal{E}| \geq 3$. We have to show that $|\mathcal{E}| = 3$.

Given $e \in \mathcal{E}$, set

$$X_e = V(\sigma) \smallsetminus (\{x\} \cup e).$$

Since $e \in E(\mathbb{P}(\sigma - x))$, $\sigma[X_e]$ is prime. Since $e \notin E(\mathbb{P}(\sigma))$, $\sigma[X_e \cup \{x\}]$ is decomposable. By Lemma 3.13, $x \in \langle X_e \rangle_\sigma$ or there exists $u_e \in X_e$ such that $x \in (X_e)_\sigma(u_e)$.

Given distinct $e, f \in \mathcal{E}$, set

$$X_{\{e,f\}} = X_e \smallsetminus f.$$

Since $e \cap f = \varnothing$, it follows from Corollary 4.41 applied to $\sigma - x$ that $f \in E(\mathbb{P}(\sigma[X_e]))$, that is,

$$\sigma[X_{\{e,f\}}] \text{ is prime.} \tag{5.2}$$

For a contradiction, suppose that there is $e \in \mathcal{E}$ such that $x \in \langle X_e \rangle_\sigma$. For each $f \in \mathcal{E} \smallsetminus \{e\}$, we have

$$x \in \langle X_{\{e,f\}} \rangle_\sigma. \tag{5.3}$$

If there is $f \in \mathcal{E} \smallsetminus \{e\}$ such that $x \in \langle X_f \rangle_\sigma$, then $V(\sigma) \smallsetminus \{x\}$ is a module of σ, which contradicts the fact that σ is prime. Thus, suppose that for every $f \in \mathcal{E} \smallsetminus \{e\}$, there is $u_f \in X_f$ such that $x \in (X_f)_\sigma(u_f)$. Let $f \in \mathcal{E} \smallsetminus \{e\}$. If $u_f \notin e$, then $x \in (X_{\{e,f\}})_\sigma(u_f)$. By (5.3), $x \in (X_{\{e,f\}})_\sigma(u_f) \cap \langle X_{\{e,f\}} \rangle_\sigma$, which contradicts Lemma 3.13. Therefore, for every $f \in \mathcal{E} \smallsetminus \{e\}$,

$$u_f \in e. \tag{5.4}$$

Since $|\mathcal{E}| \geq 3$, consider distinct $f, g \in \mathcal{E} \smallsetminus \{e\}$. By (5.4), $u_f, u_g \in e$. If $u_f = u_g$, then $\{x, u_f\}$ is a module of σ, which contradicts the fact that σ is prime. Hence $u_f \neq u_g$. Recall that $\sigma[X_{\{f,g\}}]$ is prime by Corollary 4.41. We obtain that $x \in (X_{\{f,g\}})_\sigma(u_f) \cap (X_{\{f,g\}})_\sigma(u_g)$, which contradicts Lemma 3.13.

It follows that for each $e \in \mathcal{E}$, there is $u_e \in X_e$ such that $x \in (X_e)_\sigma(u_e)$. Given distinct $e, f \in \mathcal{E}$, if $u_e = u_f$, then $\{x, u_e\}$ is a module of σ, which contradicts the fact that σ is prime. Hence, for distinct $e, f \in \mathcal{E}$, we have

$$u_e \neq u_f. \tag{5.5}$$

Given distinct $e, f \in \mathcal{E}$, if $u_e \notin f$ and $u_f \notin e$, then $x \in (X_{\{e,f\}})_\sigma(u_e) \cap (X_{\{e,f\}})_\sigma(u_f)$ which contradicts Lemma 3.13 because $u_e \neq u_f$ by (5.5). Thus, for distinct $e, f \in \mathcal{E}$, we have

$$u_e \in f \text{ or } u_f \in e. \tag{5.6}$$

Let $e \in \mathcal{E}$. Recall that the elements of $\mathcal{E}$ are pairwise disjoint. Since $|\mathcal{E}| \geq 3$, there exists $f \in \mathcal{E} \smallsetminus \{e\}$ such that $u_e \notin f$. By (5.6), $u_f \in e$. Let $g \in \mathcal{E} \smallsetminus \{e, f\}$. By (5.6) applied to f and g, $u_g \in f$, because $u_f \notin g$. By (5.6) applied to e and g, $u_e \in g$ because $u_g \notin e$. Therefore, every element of $\mathcal{E} \smallsetminus \{e, f\}$ contains u_e. Consequently, $|\mathcal{E}| = 3$. □

The next result follows from Corollary 4.6 and Lemma 5.4.

Corollary 5.5. *Given a prime 2-structure σ, consider $x \in \mathscr{S}_c(\sigma)$. If $v(\sigma) \geq 9$, then $E(\mathbb{P}(\sigma - x)) \cap E(\mathbb{P}(\sigma)) \neq \varnothing$.*

A first proof of Theorem 5.3 when $v(\sigma) \geq 9$. If σ is critical, then $E(\mathbb{P}(\sigma)) \neq \varnothing$ by Corollary 4.6. Suppose that σ is not critical, so $\mathscr{S}(\sigma) \neq \varnothing$. If $\mathscr{S}_c(\sigma) \neq \varnothing$, then we conclude by using Corollary 5.5. Lastly, if $\mathscr{S}_c(\sigma) = \varnothing$, then we conclude as in Remark 5.1. □

Remark 5.6. By using Corollary 4.6, we can directly verify that Corollary 5.5 holds when $v(\sigma) = 7$ or 8.

The next result improves the Schmerl–Trotter theorem when the critical support is nonempty.

Proposition 5.7. *Let σ be a prime 2-structure such that $v(\sigma) \geq 9$. If $\mathscr{S}_c(\sigma) \neq \varnothing$, then $|E(\mathbb{P}(\sigma))| \geq \lceil \frac{v(\sigma)}{2} \rceil - 4$.*

Proof. Consider $x \in \mathscr{S}_c(\sigma)$. Set

$$n = \lceil \frac{v(\sigma)}{2} \rceil.$$

We have $n \geq 5$. We verify that P_{2n-2} embeds into $\mathbb{P}(\sigma - x)$.

- Suppose that $v(\sigma)$ is even. We obtain $v(\sigma) = 2n$, so $v(\sigma - x) = 2n - 1$. It follows from Corollary 4.6 that $\mathbb{P}(\sigma - x)$ is isomorphic to $P_{2n-2} \oplus K_{\{2n-2\}}$, P_{2n-1}, or C_{2n-1}. Thus, P_{2n-2} embeds into $\mathbb{P}(\sigma - x)$.
- Suppose that $v(\sigma)$ is odd. We obtain $v(\sigma) = 2n-1$, so $v(\sigma - x) = 2n-2$. It follows from Corollary 4.6 that $\mathbb{P}(\sigma - x)$ is isomorphic to P_{2n-2}.

Since P_{2n-2} embeds into $\mathbb{P}(\sigma - x)$, there exists a function $f : \{0, \dots, 2n-3\} \longrightarrow V(\sigma - x)$ such that f is an isomorphism from P_{2n-2} onto $\mathbb{P}(\sigma - x)[\{f(p) : 0 \leq p \leq 2n-3\}]$. Set

$$\mathcal{F} = \{\{f(2m), f(2m+1)\} : 0 \leq m \leq n-2\}.$$

Clearly, $\mathcal{F} \subseteq E(\mathbb{P}(\sigma - x))$. It follows from Lemma 5.4 that $|\mathcal{F} \setminus E(\mathbb{P}(\sigma))| \leq 3$. We obtain

$$\begin{aligned} |E(\mathbb{P}(\sigma))| \geq |\mathcal{F} \cap E(\mathbb{P}(\sigma))| &= |\mathcal{F}| - |\mathcal{F} \setminus E(\mathbb{P}(\sigma))| \\ &= (n-1) - |\mathcal{F} \setminus E(\mathbb{P}(\sigma))| \\ &\geq n - 4. \end{aligned}$$

□

5.2. **Ille's theorem.** Ille [22] succeeded in providing conditions that ensure the existence of a noncritical unordered pair outside a prime substructure of a prime 2-structure.

Theorem 5.8 (Ille [22])**.** *Given a prime 2-structure σ, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. If $|V(\sigma) \setminus X| \geq 6$, then there exist $v, w \in V(\sigma) \setminus X$ such that $v \neq w$ and $\sigma - \{v, w\}$ is prime.*

The first proof of Theorem 5.8 is technical and unclear. A new clearer and shorter proof is provided in subsection 9.6 at the end of section 9. Belkhechine et al. [3] improved Theorem 5.8 in particular cases as follows.

Theorem 5.9 (Belkhechine et al. [3]). *Given a prime 2-structure σ, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that at least one of the following statements holds*

($\mathscr{S}_1$) *there exists $v \in \langle X \rangle_\sigma$ such that $(v, X)_\sigma \neq (X, v)_\sigma$ (see Notation 2.1);*
($\mathscr{S}_2$) *there exist $y \in X$ and $v \in X_\sigma(y)$ such that $(v, y)_\sigma \neq (y, v)_\sigma$.*

Under these assumptions, if $|V(\sigma) \setminus X| \geq 4$, then there exist $v, w \in V(\sigma) \setminus X$ such that $v \neq w$ and $\sigma - \{v, w\}$ is prime.

Sayar [32] proved Theorem 5.9 for tournaments. Obviously, statements ($\mathscr{S}_1$) and ($\mathscr{S}_2$) above are satisfied by tournaments. We provide a proof of Theorem 5.9 in subsection 9.6 as well.

5.3. **The Boudabbous–Ille theorem.** Boubabbous and Ille [7] succeeded in finding a noncritical unordered pair which intersects the support. Note that the proof of the next result uses Theorem 5.3.

Theorem 5.10 (Boubabbous and Ille [7][5.2]). *Consider a prime 2-structure σ such that $v(\sigma) \geq 7$. If $|\mathscr{S}(\sigma)| \geq 2$, then there exists $e \in E(\mathbb{P}(\sigma))$ such that $e \cap \mathscr{S}(\sigma) \neq \varnothing$. (In other words, if $|\mathscr{S}(\sigma)| \geq 2$, then $\mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma) \neq \varnothing$.)*

Proof. By Theorem 5.3, $E(\mathbb{P}(\sigma)) \neq \varnothing$. Hence, $\mathbb{P}(\sigma)$ admits a component C such that $v(C) \geq 2$. Since $|\mathscr{S}(\sigma)| \geq 2$, it follows from Proposition 4.5 that $V(C) \cap \mathscr{S}(\sigma) \neq \varnothing$. Since C is connected, there exist distinct $v, w \in V(C)$ such that $\{v, w\} \in E(\mathbb{P}(\sigma))$ and $v \in \mathscr{S}(\sigma)$. Thus, $v \notin \mathscr{S}_c(\sigma)$. ☐

As shown by the next result, Theorem 5.10 does not hold when $|\mathscr{S}(\sigma)| = 1$. For convenience, we use the following notation.

Notation 5.11. Given $n \geq 3$, set

$$\begin{aligned}\mathcal{R}_{2n} = \{&\sigma(R_{2n}), \sigma((R_{2n})^\star), \sigma(R_{2n}) \wedge \sigma((R_{2n})^\star),\\ &\sigma(Q_{2n}) \wedge \sigma(R_{2n}), \sigma(Q_{2n} \wedge \sigma((R_{2n})^\star), \sigma((Q_{2n})^\star) \wedge \sigma(R_{2n}),\\ &\sigma((Q_{2n})^\star) \wedge \sigma((R_{2n})^\star), \sigma(Q_{2n}) \wedge \sigma((Q_{2n})^\star) \wedge \sigma(R_{2n})\}\end{aligned}$$

(see Figures 4.2 and 4.3).

Remark 5.12. Given $n \geq 3$, it follows from Theorem 4.19 that the elements of $\mathcal{R}_{2n}$ are the critical 2-structures σ defined on $\{0, \ldots, 2n-1\}$ such that $\mathbb{P}(\sigma) = P_{2n}$ and $(0, 2)_\sigma \neq (2, 0)_\sigma$.

Theorem 5.13 (Boubabbous and Ille [7][5.3]). *Consider a prime 2-structure σ such that $v(\sigma) \geq 6$, and $|\mathscr{S}(\sigma)| = 1$. The primality graph $\mathbb{P}(\sigma)$ admits a unique component C such that $v(C) \geq 2$. Moreover, if $V(C) \cap \mathscr{S}(\sigma) = \varnothing$, then $v(\sigma) = 2n + 1$, where $n \geq 3$, and there exists an isomorphism φ from*

[5.2]Boubabbous and Ille [7] proved this theorem for digraphs.
[5.3]Boubabbous and Ille [7] proved this theorem for digraphs.

$\sigma - \mathscr{S}(\sigma)$ onto an element of $\mathcal{R}_{2n}$ satisfying[5.4]

$$(5.7)\quad \begin{cases} [\mathscr{S}(\sigma), \varphi^{-1}(\{2i : i \in \{0,\dots,n-1\}\})]_\sigma = [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma, \\ \textit{and} \\ [\mathscr{S}(\sigma), \varphi^{-1}(\{2i+1 : i \in \{0,\dots,n-1\}\})]_\sigma = [\varphi^{-1}(2), \varphi^{-1}(0)]_\sigma. \end{cases}$$

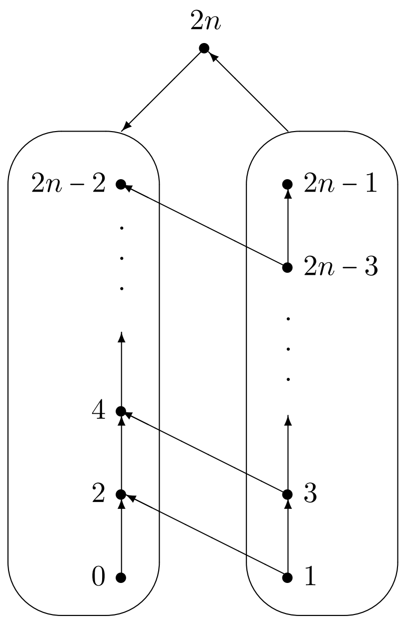

FIGURE 5.1. The digraph R_{2n+1}

Proof. Denote by x the unique element of $\mathscr{S}(\sigma)$. By Theorem 3.11, there exists $X \subseteq V(\sigma)$ such that $x \in X$, $3 \le |X| \le 5$, and $\sigma[X]$ is prime. It follows from Corollary 3.21 that there exist $v, w \in V(\sigma) \smallsetminus X$ such that $\sigma - \{v, w\}$ is prime. Since $\mathscr{S}(\sigma) \subseteq X$, we have $v \neq w$. Denote by C the component of $\mathbb{P}(\sigma)$ containing v and w. For a contradiction, suppose that $\mathbb{P}(\sigma)$ admits a component D such that $v(D) \ge 2$ and $D \neq C$. Since $V(C) \cap V(D) = \varnothing$ and $|\mathscr{S}(\sigma)| = 1$, we have $V(C) \cap \mathscr{S}(\sigma) = \varnothing$ or $V(D) \cap \mathscr{S}(\sigma) = \varnothing$. For instance, assume that $V(C) \cap \mathscr{S}(\sigma) = \varnothing$. Since $V(C) \cap V(D) = \varnothing$ and $v(D) \ge 2$, we obtain $|V(\sigma) \smallsetminus V(C)| \ge 2$, which contradicts Proposition 4.5. Consequently, C is the unique component of $\mathbb{P}(\sigma)$ such that $v(C) \ge 2$.

[5.4] The digraph R_{2n+1} (see Figure 5.1) is the extension of R_{2n} (sse Figure 4.3) to $\{0,\dots,2n\}$ defined by

$$A(R_{2n+1}) = A(R_{2n}) \cup \{(2n, 2i) : 0 \le i \le n-1\} \cup \{(2i+1, 2n) : 0 \le i \le n-1\}.$$

By using the fact that $\sigma(R_{2n+1}) - (2n)$ is prime, it is not difficult to verify that $\sigma(R_{2n+1})$ is prime. We have $\mathscr{S}(\sigma(R_{2n+1})) = \{2n\}$ and $C = P_{2n}$, so $V(C) \cap \mathscr{S}(\sigma) = \varnothing$. Furthermore, $\sigma(R_{2n+1})$ satisfies (5.7) with $\varphi = \mathrm{Id}_{\{0,\dots,2n-1\}}$.

Now, suppose that $V(C) \cap \mathscr{S}(\sigma) = \varnothing$, so $x \notin V(C)$. Since $x \notin V(C)$, we have $d_{\mathbb{P}(\sigma)}(x) = 0$. Therefore, $\sigma - x$ is critical. It follows from Proposition 4.5 that $V(C) = V(\sigma) \smallsetminus \{x\}$, and C is isomorphic to P_{2n}, where $n = (v(\sigma) - 1)/2$. Consider an isomorphism φ from C onto P_{2n}. As in the proof of Proposition 4.15, we verify that

$$<\varphi^{-1}(0), \varphi^{-1}(1)>_\sigma \neq <\varphi^{-1}(0), \varphi^{-1}(2)>_\sigma \text{ (see Notation 1.1)},$$

and for any $p, q \in \{0, \dots, 2n-1\}$ such that $p < q$, we have

$$(5.8) \quad [\varphi^{-1}(p), \varphi^{-1}(q)]_\sigma = \begin{cases} [\varphi^{-1}(0), \varphi^{-1}(1)]_\sigma \text{ if } p \text{ is even and } q \text{ is odd,} \\ [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma \text{ otherwise.} \end{cases}$$

Let $i \in \{0, \dots, n-2\}$. Since φ^{-1} is an isomorphism from P_{2n} onto C, we have $N_{\mathbb{P}(\sigma)}(\varphi^{-1}(2i+1)) = \{\varphi^{-1}(2i), \varphi^{-1}(2i+2)\}$. By Lemma 4.4, $\{\varphi^{-1}(2i), \varphi^{-1}(2i+2)\}$ is a module of $\sigma - \varphi^{-1}(2i+1)$. In particular, we obtain $[x, \varphi^{-1}(2i)]_\sigma = [x, \varphi^{-1}(2i+2)]_\sigma$. It follows that

$$[x, \varphi^{-1}(\{2i : i \in \{0, \dots, n-1\}\}]_\sigma = [x, \varphi^{-1}(2n-2)]_\sigma.$$

Since φ^{-1} is an isomorphism from P_{2n} onto C, we have $N_{\mathbb{P}(\sigma)}(\varphi^{-1}(2n-1)) = \{\varphi^{-1}(2n-2)\}$. By Lemma 4.4, $V(\sigma) \smallsetminus \{\varphi^{-1}(2n-2), \varphi^{-1}(2n-1)\}$ is a module of $\sigma - \varphi^{-1}(2n-1)$. In particular, we obtain $[x, \varphi^{-1}(2n-2)]_\sigma = [\varphi^{-1}(0), \varphi^{-1}(2n-2)]_\sigma$. Moreover, we have $[\varphi^{-1}(0), \varphi^{-1}(2n-2)]_\sigma = [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma$ by (5.8). It follows that

$$[x, \varphi^{-1}(\{2i : i \in \{0, \dots, n-1\}\}]_\sigma = [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma.$$

Similarly, we show that

$$[x, \varphi^{-1}(\{2i+1 : i \in \{0, \dots, n-1\}\}]_\sigma = [\varphi^{-1}(2), \varphi^{-1}(0)]_\sigma.$$

Consequently, (5.7) holds. Since σ is prime, $V(\sigma) \smallsetminus \{x\}$ is not a module of σ. It follows that $[\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma \neq [\varphi^{-1}(2), \varphi^{-1}(0)]_\sigma$. Therefore,

$$[0, 2]_\tau \neq [2, 0]_\tau,$$

where τ is the unique 2-structure defined on $\{0, \dots, 2n-1\}$ such that φ is an isomorphism from $\sigma - x$ onto τ. Hence τ is critical and $\mathbb{P}(\tau) = P_{2n}$. As observed in Remark 5.12, we have $\tau \in \mathcal{R}_{2n}$ because $[0, 2]_\tau \neq [2, 0]_\tau$. □

Since the elements of $\mathcal{R}_{2n}$ are not symmetric, the next result follows from Theorems 5.10 and 5.13.

Corollary 5.14. *Consider a symmetric 2-structure σ such that $v(\sigma) \geq 7$. If σ is prime and noncritical, then there exists $v \in V(\sigma)$ such that $\sigma - v$ is prime and noncritical, as well.*

Proof. Suppose that σ is prime and noncritical. Hence, $\mathscr{S}(\sigma) \neq \varnothing$. If $|\mathscr{S}(\sigma)| \geq 2$, then we conclude by using Theorems 5.10. Therefore, suppose that $\mathscr{S}(\sigma)$ contains a unique element denoted by x. By Theorem 5.13, $\mathbb{P}(\sigma)$ admits a unique component C such that $v(C) \geq 2$. Since σ is symmetric, $\sigma - x$ is not isomorphic to an element of $\mathcal{R}_{2n}$ by Remark 5.12. It follows from Theorem 5.13 that $x \in V(C)$. Since $v(C) \geq 2$, we have $d_{\mathbb{P}(\sigma)}(x) \neq 0$. Thus, $\sigma - x$ is prime and noncritical. □

Theorem 5.13 leads us to introduce the following definition. It is useful to generalize the Chudnovsky–Seymour theorem (see Theorem 5.21).

almost critical

Definition 5.15. Consider a prime 2-structure σ such that $v(\sigma) \geq 5$. Suppose that $\mathscr{S}(\sigma)$ admits a unique element, denoted by x. We say that σ is *almost critical* if $\sigma - x$ is critical (that is, $\mathscr{S}(\sigma) = \mathscr{S}_c(\sigma) = \{x\}$).

Remark 5.16. Consider a prime 2-structure σ such that $v(\sigma) \geq 6$. Suppose that $\mathscr{S}(\sigma)$ admits a unique element denoted by x. By Theorem 5.13, $\mathbb{P}(\sigma)$ admits a unique component C such that $v(C) \geq 2$. Suppose also that σ is almost critical, that is, $\sigma - x$ is critical. Since $\sigma - x$ is critical, we have

$$N_{\mathbb{P}(\sigma)}(x) = \varnothing.$$

As seen in the proof of Theorem 5.13, it follows from Proposition 4.5 that

$$V(C) = V(\sigma) \setminus \{x\},$$

and there exists an isomorphism φ from C onto P_{2n}. Furthermore, φ is an isomorphism from $\sigma - x$ onto a critical 2-structure τ such that $\mathbb{P}(\tau) = P_{2n}$. Thus, φ is an isomorphism from $\mathbb{P}(\sigma - x)$ onto P_{2n}. It follows that

$$\mathbb{P}(\sigma - x) = \mathbb{P}(\sigma) - x = C.$$

The next result is an easy consequence of Theorem 5.13.

Corollary 5.17. *Given a 2-structure σ such that $v(\sigma) \geq 7$, the following two statements are equivalent*

1. *σ is almost critical;*
2. *$v(\sigma) = 2n + 1$, where $n \geq 3$, and there exist $x \in V(\sigma)$ and an isomorphism φ from $\sigma - x$ onto an element of $\mathcal{R}_{2n}$ (see Notation 5.11) such that* (5.7) *holds.*

Proof. To begin, suppose that σ is almost critical. Hence, there exists $x \in V(\sigma)$ such that

$$\mathscr{S}(\sigma) = \mathscr{S}_c(\sigma) = \{x\}.$$

As seen in Remark 5.16, $V(\sigma) \setminus \{x\}$ is the unique component of $\mathbb{P}(\sigma)$ containing at least two elements. Clearly, $(V(\sigma) \setminus \{x\}) \cap \mathscr{S}(\sigma) = \varnothing$, and it suffices to apply Theorem 5.13 to obtain the second statement above.

Conversely, suppose that $v(\sigma) = 2n + 1$, where $n \geq 3$, and suppose that there exist $x \in V(\sigma)$ and an isomorphism φ from $\sigma - x$ onto an element τ of $\mathcal{R}_{2n}$ such that (5.7) holds. As observed in Remark 5.12, τ is critical and $\mathbb{P}(\tau) = P_{2n}$. Hence,

$$\sigma - x \text{ is critical.} \tag{5.9}$$

Set

$$X = V(\sigma) \setminus \{x\}.$$

We prove that σ is prime. As observed in Remark 5.12, we have $(0,2)_\tau \neq (2,0)_\tau$. It follows that $(\varphi^{-1}(0), \varphi^{-1}(2))_\sigma \neq (\varphi^{-1}(2), \varphi^{-1}(0))_\sigma$. Since (5.7) holds, we have $[x, \varphi^{-1}(\{2i : i \in \{0,\dots,n-1\}\})]_\sigma = [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma$ and $[x, \varphi^{-1}(\{2i+1 : i \in \{0,\dots,n-1\}\})]_\sigma = [\varphi^{-1}(2), \varphi^{-1}(0)]_\sigma$. It follows that

$$x \notin \langle X \rangle_\sigma.$$

Now, consider $y \in X$. Set

$$p = \varphi(y).$$

There exists $q \in \{0,\dots,2n-1\}$ such that $\prec p, q \succ_\tau = \prec 0, 1 \succ_\tau$. It follows that

$$\prec \varphi^{-1}(p), \varphi^{-1}(q) \succ_\sigma = \prec \varphi^{-1}(0), \varphi^{-1}(1) \succ_\sigma .$$

Moreover, by Proposition 4.15, $\prec 0, 1 \succ_\tau \neq \prec 0, 2 \succ_\tau$. It follows that

$$\prec \varphi^{-1}(0), \varphi^{-1}(1) \succ_\sigma \neq \prec \varphi^{-1}(0), \varphi^{-1}(2) \succ_\sigma .$$

Since (5.7) holds, we have

$$\prec x, \varphi^{-1}(q) \succ_\sigma = \prec \varphi^{-1}(0), \varphi^{-1}(2) \succ_\sigma .$$

Therefore, we obtain

$$\prec x, \varphi^{-1}(q) \succ_\sigma \neq \prec \varphi^{-1}(p), \varphi^{-1}(q) \succ_\sigma .$$

It follows that

$$x \notin X_\sigma(y).$$

By Lemma 3.13, $x \in \mathrm{Ext}_\sigma(X)$, so σ is prime. Since $\sigma - x$ is critical by (5.9), we obtain

$$x \in \mathscr{S}_c(\sigma). \tag{5.10}$$

Lastly, we show that $(V(\sigma) \smallsetminus \{x\}) \cap \mathscr{S}(\sigma) = \varnothing$. Consider $y \in V(\sigma) \smallsetminus \{x\}$. We have to verify that $\sigma - y$ is decomposable. Set

$$p = \varphi(y).$$

Suppose that $p \in \{1,\dots,2n-2\}$. Since $\mathbb{P}(\tau) = P_{2n}$, we have $N_{\mathbb{P}(\tau)}(p) = \{p-1, p+1\}$. By Lemma 4.4, $\{p-1, p+1\}$ is a module of $\tau - p$. It follows that $\{\varphi^{-1}(p-1), \varphi^{-1}(p+1)\}$ is a module of $\sigma - \{x, \varphi^{-1}(p)\}$. Since (5.7) holds, we have $x \longleftrightarrow_\sigma \{\varphi^{-1}(p-1), \varphi^{-1}(p+1)\}$. It follows that $\{\varphi^{-1}(p-1), \varphi^{-1}(p+1)\}$ is a module of $\sigma - \varphi^{-1}(p)$. Hence, $\sigma - \varphi^{-1}(p)$ is decomposable. Now, suppose that $p = 0$. Since $\mathbb{P}(\tau) = P_{2n}$, we have $N_{\mathbb{P}(\tau)}(0) = \{1\}$. By Lemma 4.4, $\tau - \{0,1\}$ is a module of $\tau - 0$. Precisely, since τ is critical and $\mathbb{P}(\tau) = P_{2n}$, we have $[1, \{2,\dots,2n-1\}]_\tau = [0,2]_\tau$. It follows that $[\varphi^{-1}(1), \varphi^{-1}(\{2,\dots,2n-1\})]_\sigma = [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma$. Since (5.7) holds, we have $[\varphi^{-1}(1), x]_\sigma = [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma$. It follows that $V(\sigma) \smallsetminus \{\varphi^{-1}(0), \varphi^{-1}(1)\}$ is a module of $\sigma - \varphi^{-1}(0)$. Hence, $\sigma - \varphi^{-1}(0)$ is decomposable. Similarly, $\sigma - \varphi^{-1}(2n-1)$ is decomposable. Consequently, we obtain

$$(V(\sigma) \smallsetminus \{x\}) \cap \mathscr{S}(\sigma) = \varnothing.$$

It follows from (5.10) that

$$\mathscr{S}_c(\sigma) = \mathscr{S}(\sigma) = \{x\}.$$

Thus, σ is almost critical. □

We complete the subsection with the following properties of almost critical 2-structures.

Fact 5.18. *Consider an almost critical 2-structure σ such that $v(\sigma) \geq 7$. The following two statements hold, where x denotes the unique element of $\mathscr{S}(\sigma)$,*

1. *given $X \subseteq V(\sigma - x)$, if $\sigma[X]$ is prime, then $\sigma[X \cup \{x\}]$ is prime;*
2. *for $e, f \in E(\mathbb{P}(\sigma - x))$ (or for $e, f \in E(\mathbb{P}(\sigma))$), we have $\sigma - e \simeq \sigma - f$.*

Proof. Consider an isomorphism φ from $\sigma - x$ onto an element ρ of $\mathcal{R}_{2n}$, where $n \geq 3$, satisfying (5.7).

For the first statement, consider $X \subseteq V(\sigma - x)$ such that $\sigma[X]$ is prime. Let $y \in X$. We verify that $x \notin X_\sigma(y)$. Since (5.7) holds, we have

$$\prec x, z \succ_\sigma = \prec \varphi^{-1}(0), \varphi^{-1}(2) \succ_\sigma$$

for every $z \in X \setminus \{y\}$. Since $\rho \in \mathcal{R}_{2n}$, it follows from Proposition 4.15 that there exists $z \in X \setminus \{y\}$ such that

$$\prec y, z \succ_\sigma = \prec \varphi^{-1}(0), \varphi^{-1}(1) \succ_\sigma .$$

Moreover, we have

$$\prec \varphi^{-1}(0), \varphi^{-1}(2) \succ_\sigma \neq \prec \varphi^{-1}(0), \varphi^{-1}(1) \succ_\sigma$$

by Proposition 4.15. It follows that

$$x \notin X_\sigma(y).$$

Now, we verify that $x \notin \langle X \rangle_\sigma$. Consider $i, j \in \{0, \ldots, n-1\}$ such that $i < j$. It follows from Proposition 4.15 that $[\varphi^{-1}(2i), \varphi^{-1}(2j)]_\sigma = [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma$. Therefore, $\sigma[\varphi^{-1}(\{2i : i \in \{0, \ldots, n-1\}\}]$ is constant or linear. Since $\sigma[X]$ is prime, we obtain $X \setminus \varphi^{-1}(\{2i : i \in \{0, \ldots, n-1\}\} \neq \varnothing$. Thus, there exits $p \in \{0, \ldots, n-1\}$ such that $\varphi^{-1}(2p+1) \in X$. Similarly, there exists $q \in \{0, \ldots, n-1\}$ such that $\varphi^{-1}(2q) \in X$. Since (5.7) holds, we have

$$\begin{cases} [x, \varphi^{-1}(2q)]_\sigma = [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma \\ \text{and} \\ [x, \varphi^{-1}(2p+1)]_\sigma = [\varphi^{-1}(2), \varphi^{-1}(0)]_\sigma . \end{cases}$$

Since $\rho \in \mathcal{R}_{2n}$, it follows from Remark 5.12 that

$$[\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma \neq [\varphi^{-1}(2), \varphi^{-1}(0)].$$

Therefore, we have

$$x \notin \langle X \rangle_\sigma .$$

It follows from Lemma 3.13 that

$$x \in \mathrm{Ext}_\sigma(X),$$

that is, $\sigma[X \cup \{x\}]$ is prime.

For the second statement, consider $e, f \in E(\mathbb{P}(\sigma - x))$. To begin, we make the following observation. By Remark 5.16, $\mathbb{P}(\sigma - x) = \mathbb{P}(\sigma) - x$, and x is isolated in $\mathbb{P}(\sigma)$. It follows that

$$E(\mathbb{P}(\sigma - x)) = E(\mathbb{P}(\sigma) - x).$$

Consequently, we can consider $e, f \in E(\mathbb{P}(\sigma))$ as well.

By Remark 5.16, φ is an isomorphism from $\mathbb{P}(\sigma - x)$ onto P_{2n}. By exchanging e and f if necessary, we can suppose that $e = \{\varphi^{-1}(i), \varphi^{-1}(i+1)\}$ and $f = \{\varphi^{-1}(j), \varphi^{-1}(j+1)\}$, where $0 \leq i < j \leq 2n - 2$.

Consider the bijection f from $\{0, \dots, 2n-1\} \smallsetminus \{i, i+1\}$ onto $\{0, \dots, 2n-1\} \smallsetminus \{j, j+1\}$ defined as follows. Given $m \in \{0, \dots, 2n-1\} \smallsetminus \{i, i+1\}$,

$$f(m) = \begin{cases} m \text{ if } i \geq 1 \text{ and } 0 \leq m \leq i-1, \\ m-2 \text{ if } i+2 \leq m \leq j+1, \\ m \text{ if } j \leq 2n-3 \text{ and } j+2 \leq m \leq 2n-1. \end{cases}$$

By Remark 5.12, $\mathbb{P}(\rho) = P_{2n}$. It follows from Proposition 4.15 that for $p, q \in \{0, \dots, 2n-1\}$, with $p < q$, we have

$$[p, q]_\rho = \begin{cases} [0,1]_\rho \text{ if } p \text{ is even and } q \text{ is odd}, \\ [0,2]_\rho \text{ otherwise}. \end{cases}$$

Since f is strictly increasing and preserves the parity, f is an isomorphism from $\rho - \{i, i+1\}$ onto $\rho - \{j, j+1\}$. Now, consider the bijection ψ from $V(\sigma) \smallsetminus e$ onto $V(\sigma) \smallsetminus f$ defined by $\psi(x) = x$, and $\psi(w) = (\varphi^{-1} \circ f \circ \varphi)(w)$ for every $w \in V(\sigma) \smallsetminus (e \cup \{x\})$. Since φ satisfies (5.7) and f preserves the parity, ψ is an isomorphism from $\sigma - e$ onto $\sigma - f$. □

5.4. **The Chudnovsky–Seymour theorem.**

Theorem 5.19 (Chudnovsky and Seymour [10][5.5]). *Let σ be a symmetric 2-structure. If σ is prime and noncritical, then for every prime 2-structure τ such that τ embeds into σ, with $5 \leq v(\tau) < v(\sigma)$, there exists $X \subsetneq V(\sigma)$ such that $\sigma[X] \simeq \tau$ and $\mathrm{Ext}_\sigma(X) \neq \varnothing$ (see Notation 3.12).*

Proof. We consider a prime 2-structure τ, such that $v(\tau) \geq 5$, and we proceed by induction on $v(\sigma) \geq v(\tau) + 1$. The result is obvious when $v(\sigma) = v(\tau) + 1$. Hence, suppose that $v(\sigma) \geq v(\tau) + 2$. We have $v(\sigma) \geq 7$ because $v(\tau) \geq 5$. By Corollary 5.14, we have

$$\mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma) \neq \varnothing.$$

To begin, we prove that there exists $X \subsetneq V(\sigma)$ such that

$$\begin{cases} \sigma[X] \simeq \tau \\ \text{and} \\ (V(\sigma) \smallsetminus X) \cap \mathscr{S}(\sigma) \neq \varnothing. \end{cases} \tag{5.11}$$

[5.5]Chudnovsky and Seymour [10] proved this theorem for graphs.

Consider $Y \subseteq V(\sigma)$ such that $\sigma[Y] \simeq \tau$, and suppose that $\sigma - u$ is decomposable for every $u \in V(\sigma) \setminus Y$. It follows from Corollary 3.21 that there exist distinct $v, w \in V(\sigma) \setminus Y$ such that $\sigma - \{v, w\}$ is prime. Thus, τ embeds into $\sigma - \{v, w\}$. Denote by C the component of $\mathbb{P}(\sigma)$ containing v and w. For a contradiction, suppose that $V(C) \subseteq V(\sigma) \setminus \mathscr{S}(\sigma)$. By Proposition 4.5, $|V(\sigma) \setminus V(C)| \leq 1$, so $|\mathscr{S}(\sigma)| \leq 1$. Since σ is not critical, we have $|\mathscr{S}(\sigma)| = 1$. By Theorem 5.13, C is the unique component of $\mathbb{P}(\sigma)$ such that $v(C) \geq 2$. Since $V(C) \cap \mathscr{S}(\sigma) = \varnothing$, it follows from Theorem 5.13 that σ is almost critical, which contradicts the fact that σ is symmetric (see Remark 5.12). Consequently, we have $V(C) \cap \mathscr{S}(\sigma) \neq \varnothing$. Since $\sigma - u$ is decomposable for every $u \in V(\sigma) \setminus Y$, we have $\{v, w\} \cap \mathscr{S}(\sigma) = \varnothing$. Since C is connected, there exist distinct vertices $c_0, \dots, c_p$ of C satisfying

- $\{c_0, c_1\} = \{v, w\}$;
- $p \geq 2$, $\{c_0, \dots, c_{p-1}\} \subseteq V(\sigma) \setminus \mathscr{S}(\sigma)$, and $c_p \in \mathscr{S}(\sigma)$;
- for $i \in \{0, \dots, p-1\}$, $\{c_i, c_{i+1}\} \in E(\mathbb{P}(\sigma))$.

Let $i \in \{1, \dots, p-1\}$. We have $c_{i-1}, c_{i+1} \in N_{\mathbb{P}(\sigma)}(c_i)$. Since $c_i \notin \mathscr{S}(\sigma)$, it follows from Lemma 4.4 that $N_{\mathbb{P}(\sigma)}(c_i) = \{c_{i-1}, c_{i+1}\}$, and $\{c_{i-1}, c_{i+1}\}$ is a module of $\sigma - c_i$. Thus, $\sigma - \{c_{i-1}, c_i\} \simeq \sigma - \{c_i, c_{i+1}\}$. It follows that $\sigma - \{c_0, c_1\} \simeq \sigma - \{c_{p-1}, c_p\}$, that is, $\sigma - \{v, w\} \simeq \sigma - \{c_{p-1}, c_p\}$. Since τ embeds into $\sigma - \{v, w\}$, τ embeds into $\sigma - \{c_{p-1}, c_p\}$ as well. Since $c_p \in \mathscr{S}(\sigma)$, (5.11) holds.

Now, we consider $X \subseteq V(\sigma)$ such that (5.11) holds. There exists

$$v \in (V(\sigma) \setminus X) \cap \mathscr{S}(\sigma).$$

If there exists $w \in (V(\sigma) \setminus X) \cap (\mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma))$, then it suffices to apply the induction hypothesis to $\sigma - w$. Hence, suppose that

$$(V(\sigma) \setminus X) \cap (\mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma)) = \varnothing.$$

In particular, $\sigma - v$ is critical. Since $\mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma) \neq \varnothing$, there exists

$$x \in X \cap (\mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma)).$$

Since $\sigma - v$ is a critical symmetric 2-structure, it follows from Corollary 4.6 and Propositions 4.15, 4.23, 4.27, and 4.36 that $\mathbb{P}(\sigma - v) = P_{2n}$, where $n \geq 3$. Consequently, there exists $y \in (V(\sigma) - v) \setminus \{x\}$ such that $\{x, y\} \in E(\mathbb{P}(\sigma - v))$. Since $v(\sigma) - v(\tau) \geq 2$, we have $X \subsetneq V(\sigma - v)$. Since $\sigma - v$ is critical, it follows from Corollary 3.21 that there exist distinct $w, w' \in V(\sigma - v) \setminus X$ such that $\{w, w'\} \in E(\mathbb{P}(\sigma - v))$. Thus, τ embeds into $(\sigma - v) - \{w, w'\}$. Since $\{x, y\}, \{w, w'\} \in E(\mathbb{P}(\sigma - v))$, it follows from Corollary 4.8 that $(\sigma - v) - \{x, y\} \simeq (\sigma - v) - \{w, w'\}$. Therefore, τ embeds into $(\sigma - v) - \{x, y\}$ as well. To conclude, it suffices to apply the induction hypothesis to $\sigma - x$. □

Remark 5.20. Theorem 5.19 does not hold for almost critical 2-structures. Indeed, given $n \geq 3$, consider the 2-structure ρ_{2n+1} defined on $\{0, \dots, 2n\}$ by

$$(5.12)\qquad \begin{cases} \rho_{2n+1} - (2n) = \sigma(R_{2n}) \text{ (see Figure 4.3)} \\ [2n, \{2i : i \in \{0, \dots, n-1\}\}]_{\rho_{2n+1}} = [0, 2]_{\rho_{2n+1}}, \\ \text{and} \\ [2n, \{2i+1 : i \in \{0, \dots, n-1\}\}]_{\rho_{2n+1}} = [2, 0]_{\rho_{2n+1}}. \end{cases}$$

By Corollary 5.17, ρ_{2n+1} is almost critical. As observed in Remark 5.16, we have

$$\mathbb{P}(\rho_{2n+1} - (2n)) = \mathbb{P}(\rho_{2n+1}) - (2n) = P_{2n}.$$

Therefore, $\rho_{2n+1} - \{2n-2, 2n-1\}$ is prime. Set

$$\tau = \rho_{2n+1} - \{2n-2, 2n-1\}.$$

Consider $X \subseteq \{0, \dots, 2n\}$ such that τ is isomorphic to $\rho_{2n+1}[X]$. Since τ is prime, $\rho_{2n+1}[X]$ is prime. It follows that

$$V(\rho_{2n+1}) \smallsetminus X \in E(\mathbb{P}(\rho_{2n+1})).$$

As observed in Remark 5.16, we have

$$\mathbb{P}(\rho_{2n+1}) = P_{2n} \oplus K_{\{2n\}}.$$

It follows that there exists $p \in \{0, \dots, 2n-2\}$ such that

$$X = V(\rho_{2n+1}) \smallsetminus \{p, p+1\}.$$

Finally, to establish that Theorem 5.19 does not hold for ρ_{2n+1}, we verify that

$$p, p+1 \notin \mathrm{Ext}_{\rho_{2n+1}}(X).$$

Since $\mathbb{P}(\rho_{2n+1}) = P_{2n} \oplus K_{\{2n\}}$, we have

$$N_{\mathbb{P}(\rho_{2n+1})}(p+1) = \begin{cases} \{p, p+2\} \text{ if } p \leq 2n-3 \\ \text{or} \\ \{p\} \text{ if } p = 2n-2. \end{cases}$$

It follows from Lemma 4.4 that

$$p \in \langle X \rangle_{\rho_{2n+1}} \cup X_{\rho_{2n+1}}(p+2).$$

In the same way, we verify that

$$p+1 \in X_{\rho_{2n+1}}(p-1).$$

We generalize Theorem 5.19 as follows.

Theorem 5.21 (Liu [27][5.6]). *Let σ be a prime 2-structure σ. Suppose that σ is neither critical nor almost critical. For each prime 2-structure τ such that τ embeds into σ, with $5 \leq v(\tau) < v(\sigma)$, there exists $X \subseteq V(\sigma)$ satisfying $X \neq V(\sigma)$, $\sigma[X] \simeq \tau$, and $\mathrm{Ext}_\sigma(X) \neq \varnothing$.*

[5.6] Liu [27] proved this theorem for tournaments.

Theorem 5.21 is proved in appendix A. The next result is obtained by applying Theorem 5.21 several times.

Theorem 5.22. *Let σ be a prime 2-structure. Suppose that σ is neither critical nor almost critical. Consider a prime 2-structure τ such that τ embeds into σ, with $5 \leq v(\tau) < v(\sigma)$. Under these assumptions, there exists $X \subsetneq V(\sigma)$ such that $\sigma[X] \simeq \tau$, and the elements of $V(\sigma) \setminus X$ can be indexed as $z_1, \dots, z_n$ in such a way that $\sigma[X \cup \{z_1, \dots, z_i\}]$ is prime for $i \in \{1, \dots, n\}$.*

5.5. **The critical support.** The purpose of this subsection is to demonstrate the next theorem.

Theorem 5.23 (Sayar[5.7][31])**.** *For every prime 2-structure σ, with $v(\sigma) \geq 7$, we have $|\mathscr{S}_c(\sigma)| \leq 2$.*

Theorem 5.23 is an immediate consequence of Corollary 4.6 and of Propositions 5.26, 5.27, 5.28, and 5.29 below. The proofs of Propositions 5.26, 5.27, 5.28, and 5.29 share the same approach and have similar arguments. Moreover, they are technical and the proofs of the last three ones are long. In order to keep this subsection at a satisfactory length, we provide the proofs of Propositions 5.27, 5.28, and 5.29 in appendix B.

We begin with the following lemma (compare with Corollary 4.10).

Lemma 5.24. *Let σ be a prime 2-structure with $v(\sigma) \geq 6$. Consider $X, Y \subseteq V(\sigma)$ such that $\sigma[X]$ and $\sigma[Y]$ are critical. Suppose that $|X| \leq |Y|$. If there exists $Z \subseteq X \cap Y$ such that $\sigma[Z]$ is prime and $|Z| \geq 5$, then $\sigma[X]$ embeds into $\sigma[Y]$.*

Proof. We can suppose that $|Z| = 5$ or 6. Indeed, suppose that $|Z| \geq 7$. By Theorem 3.10, there exists $Z' \subseteq Z$ such that $\sigma[Z']$ is prime and $|Z'| = 3$ or 4. By Theorem 3.19, there exists $Z'' \subseteq Z$ such that $Z' \subseteq Z''$, $|Z''| = |Z'| + 2$, and $\sigma[Z'']$ is prime. Consequently, suppose that $|Z| = 5$ or 6. Furthermore, $\sigma[Z]$ is critical by Corollary 4.7.

To begin, suppose that $|Z| = 5$. It follows from Corollary 4.7 that there exist $2 \leq m \leq n$ such that $|X| = 2m + 1$ and $|Y| = 2n + 1$. Moreover, it follows from Corollary 4.41 that $\sigma[X]$, $\sigma[Y]$, and $\sigma[Z]$ share the same type. By Corollary 4.6, $\mathbb{P}(\sigma[X])$ is isomorphic to C_{2m+1}, $P_{2m} \oplus K_{\{2m\}}$ or P_{2m+1}. Suppose that $\mathbb{P}(\sigma[X]) \simeq C_{2m+1}$. Hence $\mathbb{P}(\sigma[Y]) \simeq C_{2n+1}$. It follows from Theorem 4.37 that $\sigma[X]$ embeds into $\sigma[Y]$. Similarly, if $\mathbb{P}(\sigma[X]) \simeq P_{2m} \oplus K_{\{2m\}}$, then it follows from Theorem 4.24 that $\sigma[X]$ embeds into $\sigma[Y]$. Therefore, suppose that $\mathbb{P}(\sigma[X]) \simeq P_{2m+1}$. Thus, $\mathbb{P}(\sigma[Y]) \simeq P_{2n+1}$. Consider an isomorphism φ_X from $\mathbb{P}(\sigma[X])$ onto P_{2m+1}. Denote by τ_X the unique 2-structure defined on $\{0, \dots, 2m\}$ such that φ_X is an isomorphism from $\sigma[X]$ onto τ_X. We obtain that τ_X is critical and $\mathbb{P}(\tau_X) = P_{2m+1}$. It follows from Proposition 4.27 that $(0,1)_{\tau_X} \neq (1,0)_{\tau_X}$, $[0,1]_{\tau_X} \neq [0,2]_{\tau_X}$, and

[5.7]Sayar [31] proved this theorem for digraphs.

for any $p, q \in \{0, \dots, 2m\}$ such that $p < q$, we have

$$[p,q]_{\tau_X} = \begin{cases} [0,2]_{\tau_X} \text{ if } p \text{ and } q \text{ are even} \\ [0,1]_{\tau_X} \text{ otherwise.} \end{cases} \tag{5.13}$$

There exist $x_0, \dots, x_4 \in \{0, \dots, 2m\}$ such that $x_0 < \dots < x_4$ and

$$\varphi_X(Z) = \{x_0, \dots, x_4\}.$$

Since $\sigma[Z]$ is isomorphic to $\tau_X[\varphi_X(Z)]$, $\tau_X[\varphi_X(Z)]$ is prime too. By (5.13), if x_0 is odd, then $\varphi_X(Z) \setminus \{x_0\}$ is a module of $\tau_X[\varphi_X(Z)]$. Thus, x_0 is even. Given $i \in \{0, \dots, 3\}$, if $x_i \equiv x_{i+1} \mod 2$, then it follows from (5.13) that $\{x_i, x_{i+1}\}$ is a module of $\tau_X[\varphi_X(Z)]$. Therefore, $x_i \not\equiv x_{i+1} \mod 2$. It follows that

$$\begin{cases} x_0, x_2, x_4 \text{ are even} \\ \text{and} \\ x_1, x_3 \text{ are odd.} \end{cases} \tag{5.14}$$

Let $f_X : \varphi_X(Z) \longleftrightarrow \{0, \dots, 4\}$ defined by $f_X(x_i) = i$ for $i \in \{0, \dots, 4\}$. Clearly, f_X is strictly increasing. By (5.14), f_X preserves the parity. It follows that f_X is an isomorphism from $\tau_X[\varphi_X(Z)]$ onto $\tau_X[\{0, \dots, 4\}]$. Therefore, $\sigma[Z]$ is isomorphic to $\tau_X[\{0, \dots, 4\}]$. Since (5.13) holds, we have

$$\begin{cases} [0,2]_{\tau_X} = [0,4]_{\tau_X} = [2,4]_{\tau_X} \\ \text{and} \\ [0,1]_{\tau_X} = [0,3]_{\tau_X} = [1,2]_{\tau_X} = [1,3]_{\tau_X} = [1,4]_{\tau_X} = [2,3]_{\tau_X} = [3,4]_{\tau_X}. \end{cases}$$

Since $(0,1)_{\tau_X} \neq (1,0)_{\tau_X}$ and $[0,1]_{\tau_X} \neq [0,2]_{\tau_X}$, it follows from Proposition 4.27 applied with $\tau_X[\{0, \dots, 4\}]$ that $\tau_X[\{0, \dots, 4\}]$ is critical and $\mathbb{P}(\tau_X[\{0, \dots, 4\}]) = P_5$. Similarly, let φ_Y be an isomorphism from $\mathbb{P}(\sigma[Y])$ onto P_{2n+1}. Denote by τ_Y the unique 2-structure defined on $\{0, \dots, 2n\}$ such that φ_Y is an isomorphism from $\sigma[Y]$ onto τ_Y. We obtain that τ_Y is critical and $\mathbb{P}(\tau_X) = P_{2n+1}$. It follows from Proposition 4.27 that $(0,1)_{\tau_Y} \neq (1,0)_{\tau_Y}$, $[0,1]_{\tau_Y} \neq [0,2]_{\tau_Y}$, and for any $p, q \in \{0, \dots, 2n\}$ such that $p < q$, we have

$$[p,q]_{\tau_Y} = \begin{cases} [0,2]_{\tau_Y} \text{ if } p \text{ and } q \text{ are even} \\ [0,1]_{\tau_Y} \text{ otherwise.} \end{cases} \tag{5.15}$$

Moreover, $\sigma[Z]$ is isomorphic to $\tau_Y[\{0, \dots, 4\}]$. Consequently, there exists an isomorphism ψ from $\tau_X[\{0, \dots, 4\}]$ onto $\tau_Y[\{0, \dots, 4\}]$. We obtain also that $\tau_Y[\{0, \dots, 4\}]$ is critical and $\mathbb{P}(\tau_Y[\{0, \dots, 4\}]) = P_5$. Since $\mathbb{P}(\tau_X[\{0, \dots, 4\}]) = P_5$ and $\mathbb{P}(\tau_Y[\{0, \dots, 4\}]) = P_5$, ψ is an automorphism of P_5. Therefore, we have

$$\psi = \mathrm{Id}_{\{0, \dots, 4\}} \text{ or } \pi_5 \quad \text{(see Notation 4.21).}$$

To conclude, we distinguish the following two cases.

Case 1: $\psi = \mathrm{Id}_{\{0,\dots,4\}}$.

We obtain $\tau_X[\{0,\dots,4\}] = \tau_Y[\{0,\dots, 4\}]$. It follows from (5.13) and (5.15) that

$$\tau_X = \tau_Y[\{0,\dots,2m\}].$$

Since $\sigma[X] \simeq \tau_X$ and $\sigma[Y] \simeq \tau_Y$, $\sigma[X]$ embeds into $\sigma[Y]$.

Case 2: $\psi = \pi_5$.

By Remark 4.30, ψ is an isomorphism from $\tau_X[\{0,\dots,4\}]$ onto $(\tau_X[\{0,\dots,4\}])^\star$. Since ψ is also an isomorphism from $\tau_X[\{0,\dots,4\}]$ onto $\tau_Y[\{0,\dots,4\}]$, we obtain

$$(\tau_X)^\star[\{0,\dots,4\}] = \tau_Y[\{0,\dots,4\}].$$

Clearly, $(\tau_X)^\star$ is critical and $\mathbb{P}((\tau_X)^\star) = P_{2m+1}$. It follows from Proposition 4.27 that for $p, q \in \{0,\dots,2m\}$ such that $p < q$, we have

$$[p,q]_{(\tau_X)^\star} = \begin{cases} [0,2]_{(\tau_X)^\star} & \text{if } p \text{ and } q \text{ are even} \\ [0,1]_{(\tau_X)^\star} & \text{otherwise.} \end{cases}$$

Since $(\tau_X)^\star[\{0,\dots,4\}] = \tau_Y[\{0,\dots,4\}]$, it follows from (5.15) that

$$(\tau_X)^\star = \tau_Y[\{0,\dots,2m\}].$$

By Remark 4.30, π_{2m+1} is an isomorphism from τ_X onto $(\tau_X)^\star$. Thus, τ_X embeds into τ_Y. Therefore, $\sigma[X]$ embeds into $\sigma[Y]$.

Now, suppose that $|Z| = 6$. It follows from Corollary 4.7 that there exist $3 \le m \le n$ such that $|X| = 2m$ and $|Y| = 2n$. If $m = 3$, then $X = Z$, and hence $\sigma[X]$ embeds into $\sigma[Y]$. Thus, suppose that $m \ge 4$. By Corollary 4.6, $\mathbb{P}(\sigma[X]) \simeq P_{2m}$ and $\mathbb{P}(\sigma[Y]) \simeq P_{2n}$. We proceed as previously, using Proposition 4.15 instead of Proposition 4.27. We obtain a prime 2-structure $\mu = \tau_X[\{0,\dots,5\}]$ such that

$$\begin{cases} <0,1>_\mu \neq <0,2>_\mu \\ [0,1]_\mu = [0,3]_\mu = [0,5]_\mu = [2,3]_\mu = [2,5]_\mu = [4,5]_\mu, \\ [0,2]_\mu = [0,4]_\mu = [1,2]_\mu = [1,3]_\mu = [1,4]_\mu = [1,5]_\mu = [2,4]_\mu \\ \text{and} \\ [0,2]_\mu = [3,4]_\mu = [3,5]_\mu. \end{cases}$$

It follows from Proposition 4.15 that $\mathbb{P}(\mu) = P_6$. We obtain also a prime 2-structure $\nu = \tau_Y[\{0,\dots,5\}]$ such that

$$\begin{cases} <0,1>_\nu \neq <0,2>_\nu \\ [0,1]_\nu = [0,3]_\nu = [0,5]_\nu = [2,3]_\nu = [2,5]_\nu = [4,5]_\nu, \\ [0,2]_\nu = [0,4]_\nu = [1,2]_\nu = [1,3]_\nu = [1,4]_\nu = [1,5]_\nu = [2,4]_\nu \\ \text{and} \\ [0,2]_\nu = [3,4]_\nu = [3,5]_\nu, \end{cases}$$

It follows from Proposition 4.15 that $\mathbb{P}(\nu) = P_6$. Furthermore, there exists an isomorphism ψ from μ onto ν. Thus, ψ is an automorphism of P_6. We obtain

$\psi = \mathrm{Id}_{\{0,\dots,5\}}$ or π_6. As previously, we deduce that $\tau_Y[\{0,\dots,2m-1\}] = \tau_X$ or $(\tau_X)^\star$. Since π_{2m} is an isomorphism from τ_X onto $(\tau_X)^\star$, τ_X embeds into τ_Y. Thus, $\sigma[X]$ embeds into $\sigma[Y]$. □

The next result follows from Lemma 5.24.

Corollary 5.25. *Let σ be a prime 2-structure with $v(\sigma) \geq 7$. Consider distinct $s, t \in \mathscr{S}_c(\sigma)$. We have*

$$N_{\mathbb{P}(\sigma-s)}(t) = N_{\mathbb{P}(\sigma-t)}(s), \text{ and } N_{\mathbb{P}(\sigma-s)}(t) \neq \varnothing.$$

Moreover, if $v(\sigma) \geq 8$, then $\sigma - s \simeq \sigma - t$.

Proof. We have $N_{\mathbb{P}(\sigma-s)}(t) = \{x \in V(\sigma - s) \smallsetminus \{t\} : (\sigma - s) - \{t, x\} \text{ is prime}\}$. Similarly, $N_{\mathbb{P}(\sigma-t)}(s) = \{x \in V(\sigma - t) \smallsetminus \{s\} : (\sigma - t) - \{s, x\} \text{ is prime}\}$. Thus, $N_{\mathbb{P}(\sigma-s)}(t) = N_{\mathbb{P}(\sigma-t)}(s)$.

For a contradiction, suppose that $N_{\mathbb{P}(\sigma-s)}(t) = \varnothing$. It follows from Corollary 4.6 that there exists an isomorphism φ_s from $\mathbb{P}(\sigma - s)$ onto $P_{2n} \oplus K_{\{2n\}}$, where $v(\sigma) = 2n + 2$. Furthermore, since $N_{\mathbb{P}(\sigma-s)}(t) = \varnothing$, $\varphi_s(t) = 2n$. Denote by τ_s the unique 2-structure defined on $\{0,\dots,2n\}$ such that φ_s is an isomorphism from $\sigma - s$ onto τ_s. We obtain that τ_s is critical and $\mathbb{P}(\tau_s) = P_{2n} \oplus K_{\{2n\}}$. By Theorem 4.24, $\tau_s = \sigma(T_{2n+1})$. Similarly, there exists an isomorphism φ_t from $\sigma - t$ onto $\sigma(T_{2n+1})$ such that $\varphi_t(s) = 2n$. Since $\varphi_s(t) = 2n$ and $\varphi_t(s) = 2n$, $(\varphi_s)_{\restriction V(\sigma)\smallsetminus\{s,t\}} \circ ((\varphi_t)_{\restriction V(\sigma)\smallsetminus\{s,t\}})^{-1}$ is an automorphism of $\sigma(T_{2n+1}) - (2n)$. Furthermore, since $T_{2n+1} - (2n) = L_{2n}$, $\sigma(T_{2n+1}) - (2n)$ is linear, and hence $\sigma(T_{2n+1}) - (2n)$ is rigid. Therefore, $(\varphi_s)_{\restriction V(\sigma)\smallsetminus\{s,t\}} = (\varphi_t)_{\restriction V(\sigma)\smallsetminus\{s,t\}}$. It follows that $\{s, t\}$ is a module of σ, which contradicts the fact that σ is prime. Consequently, $N_{\mathbb{P}(\sigma-s)}(t) \neq \varnothing$.

Lastly, suppose that $v(\sigma) \geq 8$. Since $N_{\mathbb{P}(\sigma-s)}(t) \neq \varnothing$ and $N_{\mathbb{P}(\sigma-s)}(t) = N_{\mathbb{P}(\sigma-t)}(s)$, there exists $v \in N_{\mathbb{P}(\sigma-s)}(t) \cap N_{\mathbb{P}(\sigma-t)}(s)$. We have $\sigma - \{s, t, v\}$ is prime. Since $v(\sigma - \{s, t, v\}) \geq 5$, it follows from Lemma 5.24 that $\sigma - s \simeq \sigma - t$. □

Proposition 5.26. *Let σ be a prime 2-structure with $v(\sigma) \geq 7$. If there exists $s \in \mathscr{S}_c(\sigma)$ such that $\mathbb{P}(\sigma - s) \simeq C_{2n+1}$, then $\mathscr{S}_c(\sigma) = \{s\}$.*

Proof. Let $s \in \mathscr{S}_c(\sigma)$ be such that $\mathbb{P}(\sigma - s) \simeq C_{2n+1}$, where $n \geq 3$. Up to isomorphism, we can assume that

$$\begin{cases} V(\sigma) = \{0,\dots,2n+1\}, \\ s = 2n+1, \\ \text{and} \\ \mathbb{P}(\sigma - (2n+1)) = C_{2n+1}. \end{cases}$$

For a contradiction, suppose that

$$|\mathscr{S}_c(\sigma)| \geq 2, \tag{5.16}$$

and consider $t \in \mathscr{S}_c(\sigma) \setminus \{2n+1\}$. Since $\theta_{2n+1}, \pi_{2n+1} \in \mathrm{Aut}(C_{2n+1})$ by Remark 4.38, we can assume that

$$t = 2n.$$

Hence, $N_{\mathbb{P}(\sigma-(2n+1))}(2n) = \{0, 2n-1\}$. By Corollary 5.25, $N_{\mathbb{P}(\sigma-(2n))}(2n+1) = \{0, 2n-1\}$. Moreover, since $v(\sigma) \geq 8$, it follows from Corollary 5.25 that $\sigma - (2n) \simeq \sigma - (2n+1)$. Therefore, $\mathbb{P}(\sigma - (2n)) \simeq C_{2n+1}$. Consider an isomorphism φ from $\mathbb{P}(\sigma-(2n))$ onto C_{2n+1}. Since $\theta_{2n+1}, \pi_{2n+1} \in \mathrm{Aut}(C_{2n+1})$, we can assume that

$$\begin{cases} \varphi(2n+1) = 2n, \\ \varphi(0) = 0, \\ \text{and} \\ \varphi(2n-1) = 2n-1. \end{cases} \tag{5.17}$$

Since $\sigma - (2n+1)$ is critical and $\mathbb{P}(\sigma - (2n+1)) = C_{2n+1}$, it follows from Proposition 4.36 that

$$(0,1)_\sigma \neq (1,0)_\sigma, \tag{5.18}$$

and for $p, q \in \{0, \ldots, 2n\}$ such that $p < q$, we have

$$[p,q]_\sigma = \begin{cases} [0,1]_\sigma \text{ if } p \not\equiv q \bmod 2 \\ [1,0]_\sigma \text{ otherwise.} \end{cases} \tag{5.19}$$

Set

$$\mathbb{E} = \{\{2n-2, 2n-1\}, \{2n-1, 2n\}, \{2n, 0\}\}.$$

Since $\mathbb{E} \subseteq E(\mathbb{P}(\sigma - (2n+1)))$, it follows from Lemma 4.39 that $(\sigma - (2n+1)) - \{2n-1, 2n\}$ is critical, and

$$\begin{aligned} &E(\mathbb{P}((\sigma - (2n+1)) - \{2n-1, 2n\})) \\ &\qquad = (E(\mathbb{P}(\sigma - (2n+1)) \setminus \mathbb{E}) \cup \{\{2n-2, 0\}\}. \end{aligned}$$

We obtain

$$\mathbb{P}(\sigma - \{2n-1, 2n, 2n+1\}) = C_{2n-1}. \tag{5.20}$$

Similarly, we obtain $(\varphi^{-1}(0), \varphi^{-1}(1))_\sigma \neq (\varphi^{-1}(1), \varphi^{-1}(0))_\sigma$, and for $p, q \in \{0, \ldots, 2n\}$ such that $p < q$, we have

$$[\varphi^{-1}(p), \varphi^{-1}(q)]_\sigma = \begin{cases} [\varphi^{-1}(0), \varphi^{-1}(1)]_\sigma \text{ if } p \not\equiv q \bmod 2 \\ [\varphi^{-1}(1), \varphi^{-1}(0)]_\sigma \text{ otherwise.} \end{cases} \tag{5.21}$$

Furthermore, $\varphi_{\restriction(V(\sigma-(2n))\setminus\{\varphi^{-1}(2n-1), \varphi^{-1}(2n)\})}$ is an isomorphism from $\mathbb{P}((\sigma - (2n) - \{\varphi^{-1}(2n-1), \varphi^{-1}(2n)\})$ onto C_{2n-1}. By (5.17), $\varphi_{\restriction\{0,\ldots,2n-2\}}$ is an isomorphism from $\mathbb{P}(\sigma - \{2n-1, 2n, 2n+1\})$ onto C_{2n-1}. It follows from (5.20) that $\varphi_{\restriction\{0,\ldots,2n-2\}} \in \mathrm{Aut}(C_{2n-1})$. Since $\varphi(0) = 0$, we obtain

$$\varphi_{\restriction\{0,\ldots,2n-2\}} = \mathrm{Id}_{\{0,\ldots,2n-2\}} \text{ or } \pi_{2n-1} \text{ (see Notation 4.21).}$$

We distinguish the following two cases. In each of them, we obtain a contradiction.

(1) Suppose that $\varphi_{\restriction\{0,\ldots,2n-2\}} = \mathrm{Id}_{\{0,\ldots,2n-2\}}$. Hence, $\varphi(i) = i$ for $i \in \{0,\ldots,2n-2\}$. Since $\varphi(2n-1) = 2n-1$ by (5.17), we obtain

$$\varphi(i) = i \text{ for } i \in \{0,\ldots,2n-1\}. \tag{5.22}$$

Consider $k \in \{0,\ldots,2n-1\}$. For instance, assume that k is even. We obtain

$$\begin{aligned}[k,2n]_\sigma &= [1,0]_\sigma && \text{by (5.19)}\\ &= [\varphi^{-1}(1),\varphi^{-1}(0)]_\sigma && \text{by (5.22)}\\ &= [\varphi^{-1}(k),\varphi^{-1}(2n)]_\sigma && \text{by (5.21)}\\ &= [k,\varphi^{-1}(2n)]_\sigma && \text{by (5.22)}\\ &= [k,2n+1]_\sigma && \text{by (5.17).}\end{aligned}$$

The same holds when k is odd. It follows that $\{2n,2n+1\}$ is a module of σ, which contradicts the fact that σ is prime.

(2) Suppose that $\varphi_{\restriction\{0,\ldots,2n-2\}} = \pi_{2n-1}$. Therefore, for each $i \in \{0,\ldots,2n-2\}$, we have

$$\varphi^{-1}(i) = 2n-2-i. \tag{5.23}$$

It follows that

$$\begin{aligned}[0,1]_\sigma &= [0,2n-1]_\sigma && \text{by (5.19)}\\ &= [\varphi^{-1}(0),\varphi_t)^{-1}(2n-1)]_\sigma && \text{by (5.17)}\\ &= [\varphi^{-1}(0),\varphi^{-1}(1)]_\sigma && \text{by (5.21)}\\ &= [2n-2,2n-3]_\sigma && \text{by (5.23)}\\ &= [1,0]_\sigma && \text{by (5.19),}\end{aligned}$$

which contradicts (5.18).

Consequently, (5.16) does not hold, and hence $\mathscr{S}_c(\sigma) = \{s\}$. □

Proposition 5.27. *Let τ be a prime 2-structure with $v(\tau) \geq 7$. If there exists $s \in \mathscr{S}_c(\tau)$ such that $\mathbb{P}(\tau - s) \simeq P_{2n} \oplus K_{\{2n\}}$, then $\mathscr{S}_c(\tau) = \{s\}$.*

Proposition 5.28. *Let σ be a prime 2-structure with $v(\sigma) \geq 7$. Suppose that there exists $s \in \mathscr{S}_c(\sigma)$ such that $\mathbb{P}(\sigma - s) \simeq P_{2n+1}$. Also, suppose that there exists $t \in \mathscr{S}_c(\sigma) \setminus \{s\}$. Under these assumptions, the following statements hold*

- *$d_{\mathbb{P}(\sigma-s)}(t) = 2$;*
- *by denoting by x and y the elements of $N_{\mathbb{P}(\sigma-s)}(t)$, the function*

$$\begin{array}{ccc} V(\sigma)\setminus\{s\} & \longrightarrow & V(\sigma)\setminus\{t\}\\ t & \longmapsto & s,\\ x & \longmapsto & y,\\ y & \longmapsto & x,\\ v \in V(\sigma)\setminus\{s,t,x,y\} & \longmapsto & v,\end{array} \tag{5.24}$$

is an isomorphism from $\sigma - s$ onto $\sigma - t$;

- *note that* $(x,y)_\sigma = (y,x)_\sigma$*;*
- $\mathscr{S}_c(\sigma) = \{s,t\}$.

Proposition 5.29. *Let σ be a prime 2-structure with $v(\sigma) \geq 7$. Suppose that there exists $s \in \mathscr{S}_c(\sigma)$ such that $\mathbb{P}(\sigma - s) \simeq P_{2n}$. Also, suppose that there exists $t \in \mathscr{S}_c(\sigma) \smallsetminus \{s\}$. We can suppose that*

$$\begin{cases} V(\sigma) = \{0,\dots,2n\}, \\ s = 2n, \\ t \in \{n,\dots,2n-1\}, \\ \textit{and} \\ \mathbb{P}(\sigma - (2n)) = P_{2n}. \end{cases} \tag{5.25}$$

Under these assumptions, one of the following two cases holds.

(1) *Suppose that $d_{\mathbb{P}(\sigma-(2n))}(t) = 1$. We obtain $t = 2n-1$, $(0,2)_\sigma = (2,0)_\sigma$, and the function*

$$\begin{array}{ccc} \{0,\dots,2n-1\} & \longrightarrow & \{0,\dots,2n-2\} \cup \{2n\} \\ 0 & \longmapsto & 2n, \\ 1 & \longmapsto & 2n-2, \\ 2 \leq k \leq 2n-1 & \longmapsto & k-2, \end{array} \tag{5.26}$$

is an isomorphism from $\sigma - (2n)$ onto $\sigma - t$.

(2) *Suppose that $d_{\mathbb{P}(\sigma-(2n))}(t) = 2$. We obtain $n \leq t \leq 2n-2$ and the function*

$$\begin{array}{ccc} \{0,\dots,2n-1\} & \longrightarrow & \{0,\dots,2n\} \smallsetminus \{t\} \\ t & \longmapsto & 2n, \\ t-1 & \longmapsto & t+1, \\ t+1 & \longmapsto & t-1, \\ v \in V(\sigma) \smallsetminus \{t-1,t,t+1,2n\} & \longmapsto & v, \end{array} \tag{5.27}$$

is an isomorphism from $\sigma - (2n)$ onto $\sigma - t$. In particular, we have $(t-1,t+1)_\sigma = (t+1,t-1)_\sigma$.

In both cases above, we have $\mathscr{S}_c(\sigma) = \{t, 2n\}$.

6. Minimal prime 2-structures

Definition 6.1. Let σ be a prime 2-structure. Consider a vertex subset W of σ. We say that σ is *minimal* for W if for each $W' \subsetneq V(\sigma)$ such that $W \subseteq W'$ and $|W'| \geq 3$, we have $\sigma[W']$ is decomposable. *minimal*

Cournier and Ille [12] characterized the prime digraphs that are minimal for a vertex subset of size 1 or 2. The purpose of this section is to extend their characterization to prime 2-structures. The next question follows naturally.

Question 6.2. Given $k \geq 3$, characterize the prime 2-structures that are minimal for a vertex subset of size k[6.1].

6.1. **Minimal and prime 2-structures for a singleton.** Consider a prime 2-structure σ. Given $v \in V(\sigma)$, suppose that σ is minimal for $\{v\}$. It follows from Theorem 3.11 that

$$v \in \mathscr{R}_3(\sigma) \cup \mathscr{R}_4(\sigma) \cup \mathscr{R}_5(\sigma) \text{ (see Notation 3.1).}$$

Hence, there exists $X \subseteq V(\sigma)$ such that $3 \leq |X| \leq 5$, $v \in X$, and $\sigma[X]$ is prime. Since σ is minimal for $\{v\}$, we obtain $X = V(\sigma)$. Therefore, we have

$$3 \leq v(\sigma) \leq 5.$$

We examine only the minimal and prime 2-structures for one vertex that are defined on five vertices. For instance, it follows from Claims 3.4 and 3.5 that B_5 is prime and minimal for $\{4\}$. We use the following set of 2-structures.

Notation 6.3. We denote by $\mathcal{M}_1$ the set of the 2-structures σ defined on $V(\sigma) = \{0, \ldots, 4\}$ and satisfying the following assertions

(1) $<0,1>_\sigma \neq <0,2>_\sigma$ (see the second statement of Proposition 4.15);
(2) $[0,1]_\sigma = [0,3]_\sigma = [2,3]_\sigma$ and $[0,2]_\sigma = [1,2]_\sigma = [1,3]_\sigma$ (see (4.4) in the second statement of Proposition 4.15 [6.2]);
(3) $[0,4]_\sigma = [4,3]_\sigma = [0,3]_\sigma$ and $[1,4]_\sigma = [4,2]_\sigma = [1,2]_\sigma$.

Remark 6.4. It is easy to verify that the elements of $\mathcal{M}_1$ are prime and minimal for $\{4\}$.

Theorem 6.5 (Cournier and Ille[6.3] [12])**.** *Consider a 2-structure σ such that $v(\sigma) = 5$. Let $v \in V(\sigma)$. The following two assertions are equivalent*

- *σ is prime and minimal for $\{v\}$;*
- *there exists an isomorphism f from σ onto an element of $\mathcal{M}_1$ such that $f(v) = 4$.*

[6.1] Alzohairi and Boudabbous [1] characterized the minimal prime graphs for a vertex subset of size 3 that do not contain $K_{\{0,1,2\}}$ as an induced subgraph.

[6.2] By the first two assertions, $\sigma - 4$ satisfies the second statement of Proposition 4.15. Proposition 4.15 does not hold for 2-structures of size 4 because the primality graph of a prime 2-structure of size 4 is empty. Nevertheless, we can directly verify that a 2-structure of size 4, which satisfies the second statement of Proposition 4.15, is critical. Therefore, we can deduce here that $\sigma - 4$ is critical.

[6.3] Cournier and Ille [12] proved this theorem for digraphs.

The proof of Theorem 6.5 is a long sequence of easy verifications. We omit it, but we provide the following hint.

Hint for a proof of Theorem 6.5. To begin, suppose that there exists an isomorphism from σ onto $\tau \in \mathcal{M}_1$ such that $f(v) = 4$. By Remark 6.4, τ is prime and minimal for $\{4\}$. Thus, σ is prime and minimal for $\{v\}$.

Conversely, suppose that σ is prime and minimal for $\{v\}$. Up to isomorphy, we can assume that $V(\sigma) = \{0, \dots, 4\}$ and $v = 4$. We prove that $\sigma \in \mathcal{M}_1$.

Since σ is minimal for $\{4\}$, we have

$$4 \notin \mathscr{R}_3(\sigma) \cup \mathscr{R}_4(\sigma). \tag{6.1}$$

We show that

$$\mathscr{P}_3(\sigma) \cup \mathscr{P}_4(\sigma) = \{\{0, \dots, 3\}\}. \tag{6.2}$$

By Theorem 3.10, there exists $X \in \mathscr{P}_3(\sigma) \cup \mathscr{P}_4(\sigma)$. By (6.1), $v \notin X$. As in the proof of Theorem 3.11, we obtain that $\sigma[X \cup \{4\}]$ is prime. Since σ is minimal for $\{4\}$, we have $V(\sigma) = X \cup \{4\}$, so $X = \{0, \dots, 3\}$. Consequently, (6.2) holds.

It follows from (6.2) that $\sigma[\{0, \dots, 3\}]$ is critical. Up to isomorphy, we can assume that

- $\{2, 3\}$ is a module of $\sigma[\{0, \dots, 3\}] - 0$;
- $\{0, 2\}$ is a module of $\sigma[\{0, \dots, 3\}] - 1$;
- $\{1, 3\}$ is a module of $\sigma[\{0, \dots, 3\}] - 2$;
- $\{0, 1\}$ is a module of $\sigma[\{0, \dots, 3\}] - 3$.

It follows that $[0, 1]_\sigma = [0, 3]_\sigma = [2, 3]_\sigma$ and $[0, 2]_\sigma = [1, 2]_\sigma = [1, 3]_\sigma$. Therefore, we obtain $<0, 1>_\sigma \neq <0, 2>_\sigma$.

We prove that

$$<i, 4>_\sigma = <0, 1>_\sigma \text{ or } <0, 2>_\sigma \tag{6.3}$$

for $i \in \{0, \dots, 3\}$. By using Proposition 3.8 and (6.2), we show that (6.3) holds for $i = 0$ or 1. Since the permutation $(03)(12)$ is an isomorphism from $\sigma - 4$ onto $(\sigma - 4)^\star$, we obtain that (6.3) holds for $i = 2$ or 3.

Finally, by using Proposition 3.8 and (6.2), we verify that $[0, 4]_\sigma = [4, 3]_\sigma = [0, 3]_\sigma$ and $[1, 4]_\sigma = [4, 2]_\sigma = [1, 2]_\sigma$. Therefore, $\sigma \in \mathcal{M}_1$. □

6.2. **Minimal and prime 2-structures for an unordered pair.** Given $n \geq 4$, it is easy to verify that $\sigma(P_n)$ (see Figure 1.1) is prime and minimal for $\{0, n-1\}$. Furthermore, for $n \geq 3$, M_n is the tournament defined on $V(M_n) = \{0, \dots, n-1\}$ as follows. Given $i, j \in \{0, \dots, n-1\}$, $(i, j) \in A(M_n)$ if $j = i + 1$ or $j < i - 1$ (see Figure 6.1). Given $n \geq 5$, it is easy to verify that $\sigma(M_n)$ is prime and minimal for $\{0, n-1\}$.

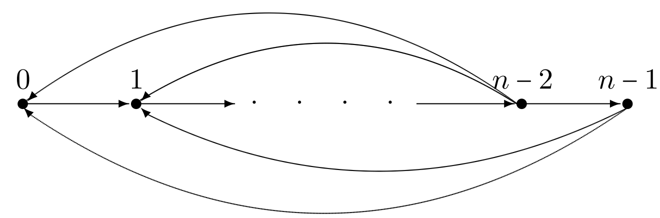

FIGURE 6.1. The tournament M_n.

We generalize $\sigma(P_n)$ and $\sigma(M_n)$ as follows.

Notation 6.6. We denote by $\mathcal{M}_2$ the set of the 2-structures σ defined on $V(\sigma) = \{0, \dots, n-1\}$, where $n \geq 3$, and satisfying the following assertions

$$(6.4) \quad \begin{cases} \text{for } i \in \{0, \dots, n-3\} \text{ and } j \in \{i+2, \dots, n-1\}, \ [i,j]_\sigma = [0, n-1]_\sigma \\ \text{and} \\ \text{for } i \in \{0, \dots, n-2\}, \ [i, i+1]_\sigma \neq [0, n-1]_\sigma. \end{cases}$$

We use the next result to verify that the elements of $\mathcal{M}_2$ are prime.

Lemma 6.7. *Consider a 2-structure $\sigma \in \mathcal{M}_2$. If M is a nontrivial module of σ, then $M = \{0, n-1\}$.*

Proof. Consider $i, j \in M$ such that $i < j$ and $\{m \in M : i \leq m \leq j\} = \{i, j\}$. Suppose that $i \geq 1$. Since $[i-1, i]_\sigma \neq [0, n-1]_\sigma$ and $[i-1, j]_\sigma = [0, n-1]_\sigma$, we have $i - 1 \in M$. By proceeding by induction, we obtain $\{0, \dots, i\} \subseteq M$. Similarly, we have $\{j, \dots, n-1\} \subseteq M$. Therefore, we have $M = \{0, \dots, i\} \cup \{j, \dots, n-1\}$.

Since $M \neq \{0, \dots, n-1\}$, we have $j \geq i+2$. If $i \geq 1$, then $[i-1, i+1]_\sigma = [0, n-1]_\sigma$ and $[i, i+1]_\sigma \neq [0, n-1]_\sigma$, which contradicts the fact that M is a module of σ because $i-1, i \in M$ and $i+1 \notin M$. It follows that $i = 0$. Similarly, we have $j = n-1$. Consequently, we obtain $M = \{0, n-1\}$. □

Lemma 6.8. *Given $\sigma \in \mathcal{M}_2$, if $v(\sigma) \geq 5$, then σ is prime and minimal for $\{0, n-1\}$.*

Proof. Let $\sigma \in \mathcal{M}_2$. We have $V(\sigma) = \{0, \dots, n-1\}$, where $n \geq 5$. First, we verify that σ is prime. For a contradiction, suppose that σ admits a nontrivial module M. By Lemma 6.7, $M = \{0, n-1\}$. We have $[0, 2]_\sigma = [0, n-1]_\sigma$ and $[n-1, 2]_\sigma = [n-1, 0]_\sigma$. It follows that

$$[0, n-1]_\sigma = [n-1, 0]_\sigma.$$

We obtain $[0,1]_\sigma \neq [0,n-1]_\sigma$ and $[n-1,1]_\sigma = [n-1,0]_\sigma = [0,n-1]_\sigma$, which contradicts the fact that $\{0,n-1\}$ is a module of σ. Consequently, σ is prime.

Second, we verify that σ is minimal for $\{0,n-1\}$. Let $W \subsetneq V(\sigma)$ such that $0,n-1 \in W$ and $|W| \geq 3$. Since $W \neq V(\sigma)$, there exists $i \in \{1,\ldots,n-2\}$ such that $i \notin W$. Set

$$W' = W \cap \{0,\ldots,i-1\}.$$

For $j \in W'$ and $k \in W \setminus W'$, we have $k \geq j+2$, and hence $[j,k]_\sigma = [0,n-1]_\sigma$. It follows that W' and $W \setminus W'$ are modules of $\sigma[W]$. Thus, $\sigma[W]$ is decomposable. □

Given $n \geq 6$, the graph Q_n is defined on $V(Q_n) = \{0,\ldots,n-1\}$ in the following way (see Figure 6.2)

(1) $Q_n - \{n-2,n-1\} = P_{n-2}$ (see Figure 1.1);
(2) for $i \in \{0,\ldots,n-4\}$, $\{i,n-2\} \in E(Q_n)$;
(3) $\{n-2,n-1\} \in E(Q_n)$.

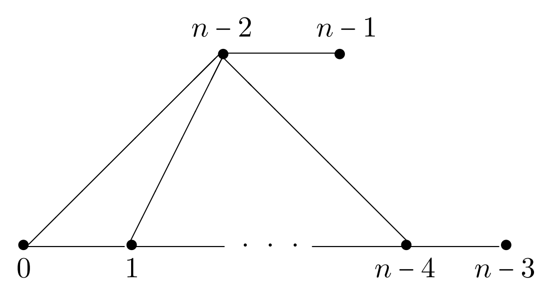

FIGURE 6.2. The graph Q_n

Furthermore, for $n \geq 6$, O_n is the tournament defined on $V(O_n) = \{0,\ldots,n-1\}$ in the following way (see Figure 6.3)

(1) $O_n - \{n-2,n-1\} = M_{n-2}$;
(2) for $i \in \{0,\ldots,n-4\}$, $(i,n-2) \in A(O_n)$;
(3) for $i \in \{0,\ldots,n-3\}$, $(i,n-1) \in A(O_n)$;
(4) $(n-2,n-3), (n-1,n-2) \in A(O_n)$.

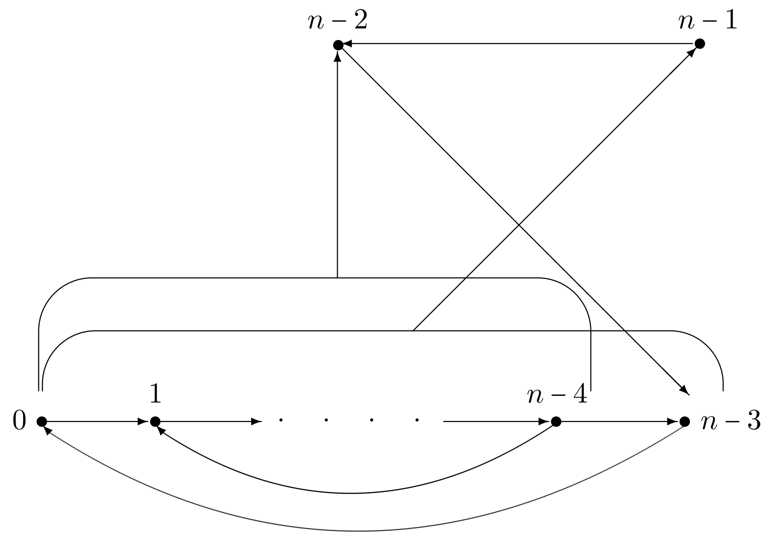

FIGURE 6.3. The tournament O_n

Given $n \geq 6$, it is not difficult to verify that $\sigma(Q_n)$ and $\sigma(O_n)$ are prime and minimal for $\{0, n-1\}$. We generalize $\sigma(Q_n)$ and $\sigma(O_n)$ as follows.

Notation 6.9. We denote by $\mathcal{N}_2$ the set of the 2-structures σ defined on $V(\sigma) = \{0, \ldots, n-1\}$, where $n \geq 5$, and satisfying the following assertions

$$\begin{cases} \sigma - \{n-2, n-1\} \text{ satisfies (6.4)}, \\ \text{for } i \in \{0, \ldots, n-4\},\ [n-2, i]_\sigma = [n-2, n-1]_\sigma, \\ \text{for } i \in \{0, \ldots, n-3\},\ [n-1, i]_\sigma = [0, n-3]_\sigma, \\ \text{and} \\ [n-2, n-1]_\sigma \neq [n-2, n-3]_\sigma \text{ and } [n-2, n-1]_\sigma \neq [n-3, 0]_\sigma. \end{cases} \tag{6.5}$$

Lemma 6.10. *The elements of $\mathcal{N}_2$ are prime and minimal for $\{0, n-1\}$.*

Proof. Let $\sigma \in \mathcal{N}_2$. We have $V(\sigma) = \{0, \ldots, n-1\}$, where $n \geq 5$. First, we verify that σ is prime. We distinguish the following two cases.

CASE 1: $n \in \{5, 6\}$.

Using assertion (M2) of Proposition 2.5 and Lemma 6.7 applied to $\sigma[\{0, \ldots, n-3\}] \in \mathcal{M}_2$, it is not difficult to verify that σ is prime.

CASE 2: $n \geq 7$.

Since $\sigma - \{n-2, n-1\} \in \mathcal{M}_2$, it follows from Lemma 6.8 that $\sigma - \{n-2, n-1\}$ is prime. Set

$$X = \{0, \ldots, n-3\}.$$

It is not difficult to verify that $n-2 \in \mathrm{Ext}_\sigma(X)$ and $n-1 \in \langle X \rangle_\sigma$. Since $[n-2, n-1]_\sigma \neq [n-3, 0]_\sigma$ and $[n-1, n-3]_\sigma = [0, n-3]_\sigma$, we have

$[n-1,n-2]_\sigma \neq [n-1,n-3]_\sigma$. Thus, $X \cup \{n-2\}$ is not a module of $\sigma[X \cup \{n-2,n-1\}]$. It follows from assertion (P2) of Lemma 3.17 that $\sigma[X \cup \{n-2,n-1\}]$, that is σ, is prime.

Second, we verify that σ is minimal for $\{0,n-1\}$. Consider $W \subsetneq V(\sigma)$ such that $0,n-1 \in W$ and $|W| \geq 3$. If $n-2 \notin W$, then $W \setminus \{n-1\}$ is a nontrivial module of $\sigma[W]$. Hence, suppose that $\{0,n-2,n-1\} \subseteq W$. If $n-3 \notin W$, then $W \setminus \{n-2\}$ is a nontrivial module of $\sigma[W]$. Thus, suppose that $\{0,n-3,n-2,n-1\} \subseteq W$. If $1 \notin W$, then $\{0,n-1\}$ is a nontrivial module of $\sigma[W]$. Therefore, suppose that $\{0,1,n-3,n-2,n-1\} \subseteq W$. Since $W \neq \{0,\dots,n-1\}$, there exists $i \in \{2,\dots,n-4\} \setminus W$. We obtain that $W \cap \{0,\dots,i-1\}$ is a nontrivial module of $\sigma[W]$. □

Proposition 6.11. *Consider a prime 2-structure σ such that $v(\sigma) \geq 6$. Let v and w be distinct vertices of σ. Suppose that for every $W \subseteq V(\sigma)$, we have*

(6.6) *if $3 \leq |W| \leq 5$ and $v,w \in W$, then $\sigma[W]$ is decomposable.*

Under these assumptions, there exists an isomorphism φ from an element of $\mathcal{M}_2 \cup \mathcal{N}_2$ onto $\sigma[X]$, where $X \subseteq V(\sigma)$, such that $v,w \in X$ and $|X| \geq 6$, satisfying

$$\varphi(\{0,n-1\}) = \{v,w\}.$$

Proof. Set

$$e = (v,w)_\sigma \text{ and } f = (w,v)_\sigma.$$

Consider

$$Z = \{z \in V(\sigma) \setminus \{v,w\} : z \longleftrightarrow_\sigma \{v,w\}\} \setminus N_\sigma^{(e,f)}(v)$$

(see Notation 2.1 and Notation 3.7).

Denote by $C(v)$ (respectively, $C(w)$) the $\{e,f\}$-component of $\sigma - Z$ (see Definition 2.2) containing v (respectively, w).

To begin, suppose that

$$C(v) = C(w).$$

Let n be the least integer $m \geq 3$ such that there exists a sequence $v_0,\dots,v_{m-1}$ of vertices of $\sigma - Z$ satisfying

- $v_0 = v$ and $v_{m-1} = w$;
- for $0 \leq i \leq m-2$, $[v_i,v_{i+1}]_\sigma \neq (e,f)$.

It follows from the minimality of n that for $i \in \{0,\dots,n-3\}$ and $j \in \{i+2,\dots,n-1\}$, we have

$$[v_i,v_j]_\sigma = (e,f).$$

We consider the bijection $\varphi : \{0,\dots,n-1\} \longrightarrow \{v_0,\dots,v_{n-1}\}$ defined by $\varphi(i) = v_i$ for $i \in \{0,\dots,n-1\}$. Moreover, we denote by τ the unique 2-structure defined on $V(\tau) = \{0,\dots,n-1\}$ such that φ is an isomorphism from τ onto $\sigma[\{v_0,\dots,v_{n-1}\}]$. For $g \in E(\sigma[\{v_0,\dots,v_{n-1}\}])$, set

$$\varphi^{-1}(g) = \{(\varphi^{-1}(x),\varphi^{-1}(y)) : (x,y) \in g\}.$$

We obtain

$$E(\tau) = \{\varphi^{-1}(g) : g \in E(\sigma[\{v_0, \dots, v_{n-1}\}])\}.$$

In particular, we have $[0, n-1]_\tau = (\varphi^{-1}(e), \varphi^{-1}(f))$. Let $i \in \{0, \dots, n-1\}$. Since $[v_i, v_{i+1}]_\sigma \neq (e, f)$, we have $[i, i+1]_\tau \neq [0, n-1]_\tau$. Furthermore, consider $i \in \{0, \dots, n-3\}$ and $j \in \{i+2, \dots, n-1\}$. Since $[v_i, v_j]_\sigma = (e, f)$, we have $[i, j]_\tau = [0, n-1]_\tau$. It follows that τ satisfies (6.4). Hence, $\tau \in \mathcal{M}_2$. Finally, we prove that τ is prime. For a contradiction, suppose that τ admits a nontrivial module M. By Lemma 6.7, $M = \{0, n-1\}$. Hence, $\{v_0, v_{n-1}\}$, that is $\{v, w\}$, is a module of $\sigma[\{v_0, \dots, v_{n-1}\}]$. We obtain $v_1 \longleftrightarrow_\sigma \{v, w\}$ and $[v, v_1]_\sigma \neq (e, f)$. Thus, $v_1 \in Z$, which contradicts the fact that $C(v) \subseteq V(\sigma) \setminus Z$. Consequently, τ is prime. Thus, $\sigma[\{v_0, \dots, v_{n-1}\}]$ is prime too. It follows from (6.6) that $n \geq 6$.

Now, suppose that

$$C(v) \neq C(w).$$

It follows from Lemma 2.4 that $\mathcal{C}_{\{e,f\}}(\sigma - Z)$ (see Definition 2.2) is a modular partition of $\sigma - Z$. In particular, for $c \in C(v)$ and $d \in C(w)$, we have $[c, d]_\sigma = [v, w]_\sigma$.

First, suppose that there exists $z \in Z$ such that $\langle v, z \rangle_\sigma \neq \langle v, w \rangle_\sigma$, we have

$$z \not\longleftrightarrow_\sigma C(v). \tag{6.7}$$

We conclude in the following way. Since $C(v)$ is $\{e, f\}$-connected, there exist $v_0, \dots, v_{k-1} \in C(v)$, where $k \geq 2$, such that

- $\sigma[\{v_0, \dots, v_{k-1}\}]$ satisfies (6.4);
- $v_0 = v$;
- if $k = 2$, then $[v_0, v_{k-1}]_\sigma \neq (f, e)$;
- if $k \geq 3$, then $[v_0, v_{k-1}]_\sigma = (f, e)$;
- $[z, v_0]_\sigma \neq [z, v_{k-1}]_\sigma$;
- for $i \in \{0, \dots, k-2\}$, $[z, v_0]_\sigma = [z, v_i]_\sigma$.

If $k = 2$, then it is not difficult to verify directly that $\sigma[\{v_0, \dots, v_{k-1}\} \cup \{z, w\}]$ is prime, which contradicts (6.6). Therefore, we have $k \geq 3$. Consider the bijection

$$\begin{array}{rccl} \varphi: & \{0, \dots, k+1\} & \longrightarrow & \{v_0, \dots, v_{k-1}\} \cup \{z, w\} \\ & 0 \leq i \leq k-1 & \longmapsto & v_i, \\ & k & \longmapsto & z, \\ & k+1 & \longmapsto & w. \end{array}$$

Denote by τ the unique 2-structure defined on $V(\tau) = \{0, \dots, k+1\}$ such that φ is an isomorphism from τ onto $\sigma[\{v_0, \dots, v_{k-1}\} \cup \{z, w\}]$. We have $\tau \in \mathcal{N}_2$. By Lemma 6.10, τ is prime. Hence, $\sigma[\{v_0, \dots, v_{k-1}\} \cup \{z, w\}]$ is prime too. It follows from (6.6) that $k \geq 4$, so $|\{v_0, \dots, v_{k-1}\} \cup \{z, w\}| \geq 6$.

Second, suppose that for every for $z \in Z$ such that $\langle v, z \rangle_\sigma \neq \langle v, w \rangle_\sigma$, (6.7) does not hold, that is, $z \longleftrightarrow_\sigma C(v)$. Similarly, for $z \in Z$ such that $\langle v, z \rangle_\sigma \neq \langle v, w \rangle_\sigma$, we can suppose that $z \longleftrightarrow_\sigma C(w)$. Since $z \longleftrightarrow_\sigma \{v, w\}$

for every $z \in Z$, we obtain

$$z \longleftrightarrow_{\sigma} C(v) \cup C(w) \text{ for every } z \in Z \text{ such that } < v, z >_{\sigma} \neq < v, w >_{\sigma}. \tag{6.8}$$

For a contradiction, suppose that $e = f$. Since $\mathcal{C}_{\{e\}}(\sigma - Z)$ is a modular partition of $\sigma - Z$, $C(v)$ and $C(w)$ are modules of $\sigma - Z$. By Proposition 2.8, $(\sigma - Z)/\mathcal{C}_{\{e\}}(\sigma - Z)$ is constant. Thus, $C(v) \cup C(w)$ is a module of $\sigma - Z$ as well. It follows from (6.8) that $C(v)$, $C(w)$, and $C(v) \cup C(w)$ are modules of σ, which contradicts the fact that σ is prime. Consequently, we have

$$e \neq f.$$

To continue, suppose that there exists $z \in Z$ such that $z \not\longleftrightarrow_{\sigma} C(w)$. By (6.8), $< v, z >_{\sigma} = < v, w >_{\sigma}$. Since $z \notin N_{\sigma}^{(e,f)}(v)$, we obtain $[z, v]_{\sigma} = (e, f)$. Set

$$C = \{c \in C(w) \setminus \{w\} : z \not\longleftrightarrow_{\sigma} \{c, w\}\}.$$

If there exists $c \in C$ such that $< c, w >_{\sigma} \neq < v, w >_{\sigma}$, then $\sigma[\{c, v, w, z\}]$ is prime, which contradicts (6.6). Furthermore, if there exists $c, d \in C$ such that $[c, w]_{\sigma} = (e, f)$ and $[d, w]_{\sigma} = (f, e)$, then $\sigma[\{c, d, v, w, z\}]$ is prime, which contradicts (6.6). Hence, suppose that $[w, C]_{\sigma} = (e, f)$ or (f, e). We distinguish the following two cases.

Case 1: $[w, C]_{\sigma} = (e, f)$.

Since $C(w)$ is $\{e, f\}$-connected, there exist $w_0, \dots, w_{k-1} \in C(w)$, where $k \geq 3$, such that

- $\sigma[\{w_0, \dots, w_{k-1}\}]$ satisfies (6.4);
- $w_0 = w$ and $[w_0, w_{k-1}]_{\sigma} = (e, f)$;
- $[z, w_0]_{\sigma} \neq [z, w_{k-1}]_{\sigma}$;
- for $i \in \{0, \dots, k-2\}$, $[z, w_i]_{\sigma} = (e, f)$.

Consider the bijection

$$\begin{array}{rccl} \varphi : & \{0, \dots, k+1\} & \longrightarrow & \{w_0, \dots, w_{k-1}\} \cup \{z, v\} \\ & 0 \leq i \leq k-1 & \longmapsto & w_i, \\ & k & \longmapsto & z, \\ & k+1 & \longmapsto & v. \end{array}$$

Denote by τ the unique 2-structure defined on $V(\tau) = \{0, \dots, k+1\}$ such that φ is an isomorphism from τ onto $\sigma[\{w_0, \dots, w_{k-1}\} \cup \{z, v\}]$. We have $\tau \in \mathcal{N}_2$. By Lemma 6.10, τ is prime. Hence, $\sigma[\{w_0, \dots, w_{k-1}\} \cup \{z, v\}]$ is prime too. It follows from (6.6) that $k \geq 4$, so $|\{w_0, \dots, w_{k-1}\} \cup \{z, v\}| \geq 6$.

Case 2: $[w, C]_{\sigma} = (f, e)$.

Since $C(w)$ is $\{e, f\}$-connected, there exist $w_0, \dots, w_{k-1} \in C(w)$, where $k \geq 3$, such that

- $\sigma[\{w_0, \dots, w_{k-1}\}]$ satisfies (6.4);
- $w_0 = w$ and $[w_0, w_{k-1}]_{\sigma} = (f, e)$;
- $[z, w_0]_{\sigma} \neq [z, w_{k-1}]_{\sigma}$;
- for $i \in \{0, \dots, k-2\}$, $[z, w_i]_{\sigma} = (e, f)$.

Consider the bijection

$$\begin{array}{rcccl} \varphi: & \{0,\dots,k+1\} & \longrightarrow & \{w_0,\dots,w_{k-1}\}\cup\{z,v\} \\ & 0\le i\le k-1 & \longmapsto & w_i, \\ & k & \longmapsto & z, \\ & k+1 & \longmapsto & v. \end{array}$$

Denote by τ the unique 2-structure defined on $V(\tau)=\{0,\dots,k+1\}$ such that φ is an isomorphism from τ onto $\sigma[\{w_0,\dots,w_{k-1}\}\cup\{z,v\}]$. We have $\tau\in\mathcal{M}_2$. It follows from Lemma 6.8 that τ and hence $\sigma[\{w_0,\dots,w_{k-1}\}\cup\{z,v\}]$ are prime. By (6.6), $|\{w_0,\dots,w_{k-1}\}\cup\{z,v\}|\ge 6$.

Consequently, we can suppose that $z\longleftrightarrow_\sigma C(w)$ for every $z\in Z$. Since $C(w)$ is a module of $\sigma-Z$, $C(w)$ is a module of σ as well. Since σ is prime, we obtain

$$C(w)=\{w\}.$$

By Proposition 2.8, $(\sigma-Z)/\mathcal{C}_{\{e,f\}}(\sigma-Z)$ is linear. Set

$$I_{\{v,w\}}=\{u\in V(\sigma)\smallsetminus(Z\cup C(v)\cup\{w\}):[v,u]_\sigma=[u,w]_\sigma,[v,u]_\sigma=(e,f)\}.$$

Given $z\in Z$, we verify that

$$\text{(6.9)}\qquad \text{if } <v,z>_\sigma\neq<v,w>_\sigma,\text{ then } z\longleftrightarrow_\sigma(C(v)\cup I_{\{v,w\}}\cup\{w\}).$$

By (6.8), we have $z\longleftrightarrow_\sigma(C(v)\cup\{w\})$. For a contradiction, suppose that there exists $u\in I_{\{v,w\}}$ such that $[z,v]_\sigma\neq[z,u]_\sigma$. It is easy to verify that $\sigma[\{z,u,v,w\}]$ is prime, which contradicts (6.6). It follows that (6.9) holds.

Since σ is prime and $w\longleftrightarrow_\sigma(C(v)\cup I_{\{v,w\}})$, we have $(C(v)\cup I_{\{v,w\}}\cup\{w\})\neq V(\sigma)$. Hence, $C(v)\cup I_{\{v,w\}}\cup\{w\}$ is not a module of σ. Furthermore, since $(\sigma-Z)/\mathcal{C}_{\{e,f\}}(\sigma-Z)$ is linear, $C(v)\cup I_{\{v,w\}}\cup\{w\}$ is a module of $\sigma-Z$. Thus, there exists $z\in Z$ such that $z\not\longleftrightarrow_\sigma(C(v)\cup I_{\{v,w\}}\cup\{w\})$. By (6.9), $<v,z>_\sigma=<v,w>_\sigma$, so $[z,v]_\sigma=(e,f)$. We define by induction a sequence of pairwise disjoint subsets $(Z_p)_{p\ge0}$ of $\{z\in Z:[z,v]_\sigma=(e,f)\}$ as follows. Set

$$Z_0=\{z\in Z:[z,v]_\sigma=(e,f),z\not\longleftrightarrow_\sigma(C(v)\cup I_{\{v,w\}}\cup\{w\})\}.$$

Note that $Z_0\neq\varnothing$. Given $Z_0,\dots,Z_i$, where $i\ge0$, set

$$Z_{i+1}=\{z\in Z\smallsetminus(Z_0\cup\cdots\cup Z_i):[z,v]_\sigma=(e,f),\\ z\not\longleftrightarrow_\sigma(C(v)\cup I_{\{v,w\}}\cup\{w\}\cup(Z_0\cup\cdots\cup Z_i))\}.$$

Denote by p the least integer i such that $Z_i=\varnothing$. As previously noted, $Z_0\neq\varnothing$, so $p\ge1$. We have $[w,C(v)\cup I_{\{v,w\}}\cup(Z_0\cup\cdots\cup Z_{p-1})]_\sigma=(f,e)$. Since σ is prime, we have $C(v)\cup I_{\{v,w\}}\cup\{w\}\cup(Z_0\cup\cdots\cup Z_{p-1})\neq V(\sigma)$. Therefore, there exists $x\in V(\sigma)\smallsetminus(C(v)\cup I_{\{v,w\}}\cup\{w\}\cup(Z_0\cup\cdots\cup Z_{p-1}))$ such that

$$x\not\longleftrightarrow_\sigma(C(v)\cup I_{\{v,w\}}\cup\{w\}\cup(Z_0\cup\cdots\cup Z_{p-1})).$$

Set

$$Z'=\{z'\in Z\smallsetminus(Z_0\cup\cdots\cup Z_{p-1}):[z',v]_\sigma=(e,f)\}.$$

For each $z' \in Z'$, we have $z' \longleftrightarrow_\sigma (C(v) \cup I_{\{v,w\}} \cup \{w\} \cup (Z_0 \cup \cdots \cup Z_{p-1}))$. Thus, $x \notin Z'$. It follows that either $x \in N_\sigma^{(e,f)}(v) \cap N_\sigma^{(e,f)}(w)$ or $x \in Z$ and $<v, x>_\sigma \neq <v, w>_\sigma$. In both cases, we obtain

$$[x, v]_\sigma \neq (e, f).$$

If $x \in Z$ and $<v, x>_\sigma \neq <v, w>_\sigma$, then $x \longleftrightarrow_\sigma (C(v) \cup I_{\{v,w\}} \cup \{w\})$ by (6.9). Furthermore, since $(\sigma - Z)/\mathcal{C}_{\{e,f\}}(\sigma - Z)$ is linear by Proposition 2.8, we have $[C(v) \cup I_{\{v,w\}} \cup \{w\}, N_\sigma^{(e,f)}(v) \cap N_\sigma^{(e,f)}(w)]_\sigma = (e, f)$. Therefore, in both cases, we have

$$x \longleftrightarrow_\sigma (C(v) \cup I_{\{v,w\}} \cup \{w\}).$$

Consequently, there exists $i \in \{0, \ldots, p-1\}$ such that $x \not\longleftrightarrow_\sigma (C(v) \cup I_{\{v,w\}} \cup \{w\}) \cup (Z_0 \cup \cdots \cup Z_i)$. Set

$$j = \min(\{i \in \{0, \ldots, p-1\} : x \not\longleftrightarrow_\sigma (C(v) \cup I_{\{v,w\}} \cup \{w\}) \cup (Z_0 \cup \cdots \cup Z_i)\}).$$

We show that there exists a sequence $(v_0, \ldots, v_{k-1})$ of elements of $C(v) \cup I_{\{v,w\}}$, where $k \geq 3$, such that

$$\begin{cases} \sigma[\{v_0, \ldots, v_{k-1}\}] \text{ satisfies (6.4)}, \\ [v_0, v_{k-1}]_\sigma = (f, e), \\ v_0 = v, \\ x \longleftrightarrow_\sigma \{v_0, \ldots, v_{k-2}\} \\ \text{and} \\ x \not\longleftrightarrow_\sigma \{v_0, v_{k-1}\}. \end{cases} \tag{6.10}$$

By minimality of j, we have $x \longleftrightarrow_\sigma (C(v) \cup I_{\{v,w\}} \cup \{w\}) \cup (Z_0 \cup \cdots \cup Z_{j-1}))$, when $j \geq 1$, and $x \not\longleftrightarrow_\sigma (C(v) \cup I_{\{v,w\}} \cup \{w\}) \cup Z_j)$. There exists a sequence $(z_0, \ldots, z_j)$ satisfying

- for $i \in \{0, \ldots, j\}$, $z_i \in Z_i$;
- $x \not\longleftrightarrow_\sigma (C(v) \cup I_{\{v,w\}} \cup \{w\}) \cup \{z_j\})$;
- if $j \geq 1$, then $x \longleftrightarrow_\sigma (C(v) \cup I_{\{v,w\}} \cup \{w\}) \cup \{z_0, \ldots, z_{j-1}\})$;
- if $j \geq 1$, then $[z_i, z_{i+1}]_\sigma \neq (f, e)$ for $i \in \{0, \ldots, j-1\}$.

Lastly, since $z_0 \in Z_0$, we have $z_0 \in Z$, $[z_0, v]_\sigma = (e, f)$, and $z_0 \not\longleftrightarrow_\sigma (C(v) \cup I_{\{v,w\}} \cup \{w\})$. Therefore, there exists $u \in (C(v) \setminus \{v\}) \cup I_{\{v,w\}}$ such that $[z_0, u]_\sigma \neq (e, f)$. To conclude, we distinguish the following two cases.

Case 1: $u \in I_{\{v,w\}}$.

Set $k = j + 3$ and

$$\begin{cases} v_0 = v, \\ v_1 = u, \\ \text{and} \\ \text{for } l \in \{2, \ldots, k-1\},\ v_l = z_{l-2}. \end{cases}$$

CASE 2: $u \in C(v) \setminus \{v\}$.

Since $C(v)$ is $\{e,f\}$-connected, there exist $u_0, \ldots, u_{m-1} \in C(w)$, where $m \geq 2$, such that

- $u_0 = v$;
- for $l \in \{0, \ldots, m-2\}$, $[u_l, u_{l+1}]_\sigma \neq (f,e)$;
- if $m \geq 3$, then for $l \in \{0, \ldots, m-3\}$ and $l' \in \{l+2, \ldots, m-1\}$, we have $[u_l, u_{l'}]_\sigma = (f,e)$;
- for $l \in \{0, \ldots, m-2\}$, $[u_l, z_0]_\sigma = (f,e)$;
- $[u_{m-1}, z_0]_\sigma \neq (f,e)$.

Set $k = m + j + 1$ and

$$\begin{cases} \text{for } l \in \{0, \ldots, m-1\},\ v_l = u_l, \\ \text{and} \\ \text{for } l \in \{m, \ldots, k-1\},\ v_l = z_{l-m}. \end{cases}$$

In both cases, we obtain $k \geq 3$ and $(v_0, \ldots, v_{k-1})$ satisfies (6.10). Consider the bijection

$$\begin{array}{rccl} \varphi : & \{0, \ldots, k+1\} & \longrightarrow & \{v_0, \ldots, v_{k-1}\} \cup \{x, w\} \\ & 0 \leq i \leq k-1 & \longmapsto & v_i, \\ & k & \longmapsto & x, \\ & k+1 & \longmapsto & w. \end{array}$$

Denote by τ the unique 2-structure defined on $V(\tau) = \{0, \ldots, k+1\}$ such that φ is an isomorphism from τ onto $\sigma[\{v_0, \ldots, v_{k-1}\} \cup \{z, w\}]$. We have $\tau \in \mathcal{N}_2$. It follows from Lemma 6.10 that τ and hence $\sigma[\{v_0, \ldots, v_{k-1}\} \cup \{z, w\}]$ are prime. By (6.6), $|\{v_0, \ldots, v_{k-1}\} \cup \{z, w\}| \geq 6$. □

The next characterization of prime 2-structures that are minimal for an unordered pair follows from Lemma 6.8, Lemma 6.10, and Proposition 6.11.

Theorem 6.12 (Cournier and Ille[6.4] [12]). *Consider a 2-structure σ such that $v(\sigma) \geq 6$. Let v, w be distinct vertices of σ. The following two assertions are equivalent*

- *σ is prime and minimal for $\{v, w\}$;*
- *there exists an isomorphism φ from σ onto an element of $\mathcal{M}_2 \cup \mathcal{N}_2$ defined on $\{0, \ldots, n-1\}$ such that*

$$\varphi(\{v, w\}) = \{0, n-1\}.$$

Remark 6.13. The elements of $\mathcal{M}_2 \cup \mathcal{N}_2$ of size 5 are not the only prime 2-structures that are minimal for an unordered pair. For instance, consider the reversible 2-structure σ defined on $\{0, \ldots, 4\}$ by

$$\begin{aligned} E(\sigma) = \{ & \{(0,1), (2,0), (3,0), (4,0), (4,1), (4,2), (3,4)\}, \\ & \{(1,0), (0,2), (0,3), (0,4), (1,4), (2,4), (4,3)\}, \\ & \{(1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\}\}. \end{aligned}$$

[6.4] Cournier and Ille [12] proved this theorem for digraphs.

It is not difficult to verify that σ is prime and minimal for $\{0,4\}$. Netherthe-less, σ is not isomorphic to an element of $\mathcal{M}_2 \cup \mathcal{N}_2$.

6.3. **Proof of Theorem 5.3.**

A second proof of Theorem 5.3 when $v(\sigma) \geq 7$.

Consider a prime 2-structure σ such that $v(\sigma) \geq 7$. By Theorem 5.23, $|\mathscr{S}_c(\sigma)| \leq 2$. Therefore, there exist distinct $v, w \in V(\sigma)$ such that

$$\mathscr{S}_c(\sigma) \subseteq \{v, w\}.$$

First, suppose that σ is minimal for $\{v, w\}$. By Theorem 6.12, there exists an isomorphism φ from σ onto $\tau \in \mathcal{M}_2 \cup \mathcal{N}_2$ defined on $V(\tau) = \{0, \dots, n-1\}$ such that $\varphi(\{v, w\}) = \{0, n-1\}$. Clearly, we have $\tau - \{0,1\} \in \mathcal{M}_2 \cup \mathcal{N}_2$. It follows from Lemmas 6.8 and 6.10 that $\tau - \{0,1\}$ is prime. Hence, $\sigma - \{\varphi^{-1}(0), \varphi^{-1}(1)\}$ is prime as well.

Second, suppose that σ is not minimal for $\{v, w\}$. There exists $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime and $v, w \in X$. We obtain

$$\mathscr{S}_c(\sigma) \subseteq X \text{ and } X \subsetneq V(\sigma).$$

We conclude as in Remark 5.2 from (5.1) by using Corollary 3.21. $\square$

7. Infinite prime 2-structures

The purpose of this section is to prove the next theorem.

Theorem 7.1 (Ille[7.1][21, 24])**.** *Given an infinite 2-structure σ, the following two assertions are equivalent*

- *σ is prime;*
- *for each finite $F \subseteq V(\sigma)$, there exists $F' \subseteq V(\sigma)$ such that*

$$\begin{cases} F' \text{ is finite}, \\ F \subseteq F', \\ \text{and} \\ \sigma[F'] \text{ is prime.} \end{cases} \tag{7.1}$$

We use the following definition.

Definition 7.2. Let S be a set. A family $\mathcal{F}$ of subsets of S is *up-directed* if for any $X, Y \in \mathcal{F}$, there exists $Z \in \mathcal{F}$ such that $X \cup Y \subseteq Z$. *up-directed*

Lemma 7.3. *Given a 2-structure σ, consider an up-directed family $\mathcal{F}$ of subsets of $V(\sigma)$. If $\sigma[X]$ is prime for each $X \in \mathcal{F}$, then*

$$\sigma[\bigcup_{X \in \mathcal{F}} X] \text{ is prime.}$$

Proof. Let M be a module of $\sigma[\bigcup_{X \in \mathcal{F}} X]$ such that $|M| \geq 2$. We have to show that

$$M = \bigcup_{X \in \mathcal{F}} X.$$

Since $|M| \geq 2$, consider distinct $x, y \in M$. Let $v \in \bigcup_{X \in \mathcal{F}} X$. Since $\mathcal{F}$ is up-directed, there exists $X \in \mathcal{F}$ such that $x, y, v \in X$. By assertion (M2) of Proposition 2.5, $M \cap X$ is a module of $\sigma[X]$. Since $x, y \in M \cap X$, we have $|M \cap X| \geq 2$. Since $\sigma[X]$ is prime, we obtain $M \cap X = X$. Hence $v \in M$. It follows that $M = \bigcup_{X \in \mathcal{F}} X$. □

Lemma 7.3 allows to prove one direction of the equivalence in Theorem 7.1. The use of the next result is decisive in the proof of the other direction. Furthermore, it is also significant in the study of infinite and prime 2-structures.

Theorem 7.4. *Given a prime 2-structure σ, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that $V(\sigma) \setminus X$ is infinite. For each $v \in V(\sigma) \setminus X$, there exists a finite $F \subseteq V(\sigma) \setminus X$ such that $v \in F$ and $\sigma[X \cup F]$ is prime.*

Proof. Consider the set W of $v \in V(\sigma) \setminus X$ such that for every finite $F \subseteq V(\sigma) \setminus X$, we have $\sigma[X \cup F]$ is decomposable whenever $v \in W$. We have to show that $W = \varnothing$.

Recall that

$$p_{(\sigma,X)} = \{\text{Ext}_\sigma(X), \langle X \rangle_\sigma\} \cup \{X_\sigma(y) : y \in X\} \text{ (see Notation 3.12).}$$

[7.1] Ille [21, 24] proved this theorem for digraphs.

By Lemma 3.13, $p_{(\sigma,X)}$ is a partition of $V(\sigma)\setminus X$. Consequently, to prove that $W=\varnothing$, it suffices to show that

$$W\cap \mathrm{Ext}_\sigma(X)=\varnothing, \tag{7.2}$$

$$W\cap \langle X\rangle_\sigma=\varnothing, \tag{7.3}$$

and

$$W\cap X_\sigma(y)=\varnothing \text{ for each } y\in X. \tag{7.4}$$

First, for $v\in \mathrm{Ext}_\sigma(X)$, we have $\sigma[X\cup\{v\}]$ is prime. Thus $v\notin W$. Therefore, (7.2) holds.

Second, we verify that $V(\sigma)\setminus(W\cap\langle X\rangle_\sigma)$ is a module of σ. Consider $w\in W\cap\langle X\rangle_\sigma$. Since $w\in\langle X\rangle_\sigma$, we have $w\longleftrightarrow_\sigma X$ (see Notation 2.1). Consequently, to prove that $w\longleftrightarrow_\sigma (V(\sigma)\setminus(W\cap\langle X\rangle_\sigma))$, it suffices to verify that

$$w\longleftrightarrow_\sigma X\cup\{v\} \text{ for every } v\in(V(\sigma)\setminus X)\setminus(W\cap\langle X\rangle_\sigma). \tag{7.5}$$

Given $v\in(V(\sigma)\setminus X)\setminus(W\cap\langle X\rangle_\sigma)$, we distinguish the following two cases

Case 1: $v\notin\langle X\rangle_\sigma$.

Since $w\in W$, $\sigma[X\cup\{v,w\}]$ is decomposable. It follows from assertions (P1) and (P2) of Lemma 3.17 that $X\cup\{v\}$ is a module of $\sigma[X\cup\{v,w\}]$. Hence, we obtain $w\longleftrightarrow_\sigma X\cup\{v\}$.

Case 2: $v\in\langle X\rangle_\sigma$.

Since $v\in(V(\sigma)\setminus X)\setminus(W\cap\langle X\rangle_\sigma)$, $v\notin W$. Thus, there exists a finite $F\subseteq V(\sigma)\setminus X$ such that $v\in F$ and $\sigma[X\cup F]$ is prime. Set

$$Y=X\cup F.$$

Since $w\in W$, $w\notin Y$. Moreover, since $w\in W$, we have $\sigma[Y\cup\{w\}]$ is decomposable. Thus, $w\notin\mathrm{Ext}_\sigma(Y)$. For a contradiction, suppose that $w\in Y_\sigma(z)$, where $z\in Y$. If $z\in X$, then $w\in X_\sigma(z)$, which contradicts $w\in\langle X\rangle_\sigma$ because $p_{(\sigma,X)}$ is a partition of $V(\sigma)\setminus X$ by Lemma 3.13. Now, suppose that $z\in Y\setminus X$, that is, $z\in F$. Set

$$F'=(F\setminus\{z\})\cup\{w\}.$$

Since $w\in Y_\sigma(z)$, we have $\{z,w\}$ is a module of $\sigma[(X\cup F)\cup\{w\}]$. It follows that $\sigma[X\cup F]$ and $\sigma[X\cup F']$ are isomorphic. Therefore, $\sigma[X\cup F']$ is prime too, which contradicts $w\in W$. Consequently, we obtain $w\in\langle Y\rangle_\sigma$. In particular, we have $w\longleftrightarrow_\sigma X\cup\{v\}$.

It follows from both cases above that (7.5) holds. Consequently, $V(\sigma)\setminus(W\cap\langle X\rangle_\sigma)$ is a module of σ. Since $\sigma[X]$ is prime, we have $|X|\geq 3$. Since $X\subseteq(V(\sigma)\setminus(W\cap\langle X\rangle_\sigma))$, we obtain $V(\sigma)\setminus(W\cap\langle X\rangle_\sigma)=V(\sigma)$, that is, (7.3) holds.

Third, we verify that (7.4) holds. Given $y\in X$, we show that $\{y\}\cup(W\cap X_\sigma(y))$ is a module of σ. Let $w\in W\cap X_\sigma(y)$. We have to verify that

$$v\longleftrightarrow_\sigma\{y,w\} \text{ for every } v\in V(\sigma)\setminus(\{y\}\cup(W\cap X_\sigma(y))). \tag{7.6}$$

Since $w \in X_\sigma(y)$, we have $v \longleftrightarrow_\sigma \{y, w\}$ for $v \in X \setminus \{y\}$. To continue, suppose that

$$v \in (V(\sigma) \setminus X) \setminus (W \cap X_\sigma(y)).$$

We distinguish the following two cases.

CASE 1: $v \notin X_\sigma(y)$.

Since $w \in W$, $\sigma[X \cup \{v, w\}]$ is decomposable. It follows from assertions (P1), (P3), and (P4) of Lemma 3.17 that $\{y, w\}$ is a module of $\sigma[X \cup \{v, w\}]$. In particular, we have $v \longleftrightarrow_\sigma \{y, w\}$.

CASE 2: $v \in X_\sigma(y)$.

Since $v \notin W \cap X_\sigma(y)$, we have $v \notin W$. Thus, there exists a finite $F \subseteq V(\sigma) \setminus X$ such that $v \in F$ and $\sigma[X \cup F]$ is prime. Set

$$Y = X \cup F.$$

Since $w \in W$, $w \notin Y$. Since $w \in W$, we have $\sigma[Y \cup \{w\}]$ is decomposable. Thus, $w \notin \mathrm{Ext}_\sigma(Y)$. Furthermore, if $w \in \langle Y \rangle_\sigma$, then $w \in \langle X \rangle_\sigma$, which contradicts $w \in X_\sigma(y)$ because $p_{(\sigma,X)}$ is a partition of $V(\sigma) \setminus X$ by Lemma 3.13. It follows that $w \in Y_\sigma(z)$, where $z \in Y$. For a contradiction, suppose that $z \in Y \setminus X$, that is, $z \in F$. Set

$$F' = (F \setminus \{z\}) \cup \{w\}.$$

Since $w \in Y_\sigma(z)$, we have $\{z, w\}$ is a module of $\sigma[(X \cup F) \cup \{w\}]$. It follows that $\sigma[X \cup F]$ and $\sigma[X \cup F']$ are isomorphic. Therefore, $\sigma[X \cup F']$ is prime too, which contradicts $w \in W$. Therefore, $z \in X$. We obtain $w \in X_\sigma(z)$ because $w \in Y_\sigma(z)$. It follows from Lemma 3.13 that $z = y$. Hence, $\{y, w\}$ is a module of $\sigma[Y \cup \{w\}]$. In particular, we have $v \longleftrightarrow_\sigma \{y, w\}$.

Consequently, $\{y\} \cup (W \cap X_\sigma(y))$ is a module of σ. Since $(X \setminus \{y\}) \cap (\{y\} \cup (W \cap X_\sigma(y))) = \varnothing$, we have $\{y\} \cup (W \cap X_\sigma(y)) \subsetneq V(\sigma)$. Since σ is prime, we obtain $|\{y\} \cup (W \cap X_\sigma(y))| \leq 1$, that is, $W \cap X_\sigma(y) = \varnothing$. Hence, (7.4) holds. □

Finally, we prove Theorem 7.1 as follows.

Proof of Theorem 7.1. To begin, suppose that σ is prime. Consider a finite $F \subseteq V(\sigma)$. By Theorem 3.10[7.2], there exists $X \subseteq V(\sigma)$ such that $|X| = 3$ or 4, and $\sigma[X]$ is prime. By applying Theorem 7.4 several times from X together with the elements of $F \setminus X$, we obtain $F' \subseteq V(\sigma)$ satisfying (7.1).

Conversely, suppose that

(7.7) for every finite $F \subseteq V(\sigma)$, there exists $F' \subseteq V(\sigma)$ satisfying (7.1).

[7.2] It is not difficult to verify that Proposition 3.8, Corollary 3.9, and hence Theorem 3.10 hold for infinite prime 2-structures as well.

Consider the family $\mathcal{F}$ of finite $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. Since (7.7) holds, we have

$$\bigcup_{X \in \mathcal{F}} X = V(\sigma)$$

and

$$\mathcal{F} \text{ is up-directed.}$$

It follows from Lemma 7.3 that σ is prime. $\square$

8. Critical and nonfinitely critical 2-structures

The path $P_{\mathbb{Z}}$ is defined on $\mathbb{Z}$ as follows. Given $v, w \in \mathbb{Z}$, with $v \neq w$, $\{v, w\} \in E(\mathbb{Z})$ if $|v - w| = 1$. In the sequel, $P_{\mathbb{Z}}[\mathbb{N}]$ is denoted by $P_{\mathbb{N}}$.

For each finite subset F of $\mathbb{Z}$, there exist $m, n \in \mathbb{Z}$ such that $n - m \geq 4$ and $F \subseteq \{m, \dots, n\}$. Since $P_{\mathbb{Z}}[\{m, \dots, n\}] \simeq P_{n-m}$, it follows from Fact 2.6 that $P_{\mathbb{Z}}[\{m, \dots, n\}]$ is prime. Hence, $\sigma(P_{\mathbb{Z}})[\{m, \dots, n\}]$ is prime too. Therefore, it follows from Theorem 7.1 that $\sigma(P_{\mathbb{Z}})$ is prime. Furthermore, for each finite and nonempty subset F of $\mathbb{Z}$, $\mathbb{Z} - F$ is disconnected. Therefore, $\mathbb{Z} - F$ and hence $\sigma(P_{\mathbb{Z}}) - F$ are decomposable. The properties of $\sigma(P_{\mathbb{Z}})$ lead us to introduce the following definition.

Definition 8.1. An infinite prime 2-structure σ is *finitely critical* if for each finite and nonempty subset F of $V(\sigma)$, $\sigma - F$ is decomposable. *finitely critical*

The next result is a direct consequence of Corollary 3.21.

Corollary 8.2. *Given an infinite prime 2-structure σ, σ is critical and nonfinitely critical if and only if the following two assertions hold*

- *for each $v \in V(\sigma)$, $\sigma - v$ is decomposable (i.e. σ is critical);*
- *there exist $x, y \in V(\sigma)$ such that $x \neq y$ and $\sigma - \{x, y\}$ is prime (i.e. $\mathbb{P}(\sigma)$ is nonempty).*

The next result provides a characterization of the nontrivial components of the primality graph of an infinite critical 2-structure. It is an easy consequence of Lemma 4.4 and Proposition 4.5.

Corollary 8.3. *Given an infinite critical 2-structure σ, each nontrivial component of $\mathbb{P}(\sigma)$ is isomorphic to $P_{\mathbb{N}}$ or $P_{\mathbb{Z}}$.*

Proof. Let C be a component of $\mathbb{P}(\sigma)$ such that $v(C) \geq 2$. By Lemma 4.4, C is a cycle or an infinite or finite path. It follows from Proposition 4.5 that C is infinite. Therefore, C is isomorphic to $P_{\mathbb{N}}$ or $P_{\mathbb{Z}}$. □

8.1. The families $\mathscr{F}_{\mathbb{Z}}$ and $\mathscr{F}_{\mathbb{N}}$.

Observation 8.4. Let σ be an infinite critical 2-structure. We denote by Q the partition of $V(\sigma)$ constituted by the vertex sets of the components of $\mathbb{P}(\sigma)$. Using the axiom of choice, it follows from Corollary 8.3 that there exists a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ satisfying

- for each $Y \in Q$ such that $|Y| > 1$, $\varphi_{\restriction Y}$ is an isomorphism from the component $\mathbb{P}(\sigma)[Y]$ of $\mathbb{P}(\sigma)$ onto $P_{\mathbb{N}}$ or $P_{\mathbb{Z}}$.

Denote by ρ the unique 2-structure defined on $\mathbb{N}$ or $\mathbb{Z}$ such that $\varphi_{\restriction Y}$ is an isomorphism from $\sigma[Y]$ onto ρ.

First, consider a nontrivial component C of $\mathbb{P}(\sigma)$ such that $\varphi(V(C)) = \mathbb{Z}$. Let $n \in \mathbb{Z}$. It follows from Lemma 4.4 that $\{n-1, n+1\}$ is a module of $\rho - n$.

Second, consider a nontrivial component C of $\mathbb{P}(\sigma)$ such that $\varphi(V(C)) = \mathbb{N}$. By Lemma 4.4, $\{2, 3, \dots\}$ is a module of $\rho - 0$. Furthermore, by Lemma 4.4, $\{n - 1, n + 1\}$ is a module of $\rho - n$ for every $n \geq 1$.

Observation 8.4 leads us to introduce the following two families of 2-structures.

Notation 8.5. First, we denote by $\mathscr{F}_{\mathbb{Z}}$ the family of the 2-structures τ defined on $V(\tau) = \mathbb{Z}$ and satisfying

- for every $n \in \mathbb{Z}$, $\{n-1, n+1\}$ is a module of $\tau - n$ and not of τ.

For instance, the usual linear order $L_{\mathbb{Z}}$ defined on $\mathbb{Z}$ belongs to $\mathscr{F}_{\mathbb{Z}}$.

Second, we denote by $\mathscr{F}_{\mathbb{N}}$ the family of the 2-structures τ defined on $V(\tau) = \mathbb{N}$ and satisfying

- $\{2, 3, \dots\}$ is a module of $\tau - 0$;
- $\{0\} \cup \{2, 3, \dots\}$ is not a module of τ;
- for every $n \geq 1$, $\{n-1, n+1\}$ is a module of $\tau - n$ and not of τ.

For instance, the usual linear order $L_{\mathbb{N}}$ defined on $\mathbb{N}$ belongs to $\mathscr{F}_{\mathbb{N}}$.

In the next four lemmas, we examine the elements of $\mathscr{F}_{\mathbb{N}} \cup \mathscr{F}_{\mathbb{Z}}$.

Lemma 8.6. *Given a 2-structure τ such that $V(\tau) = \mathbb{Z}$, $\tau \in \mathscr{F}_{\mathbb{Z}}$ if and only if the following two assertions hold*

- $[1, 0]_\tau \neq [1, 2]_\tau$;
- *for $m, n \in \mathbb{Z}$ such that $m < n$, we have $[2m, 2n]_\tau = [0, 2]_\tau$, $[2m, 2n-1]_\tau = [0, 1]_\tau$, $[2m+1, 2n]_\tau = [1, 2]_\tau$, and $[2m+1, 2n+1]_\tau = [1, 3]_\tau$.*

Proof. To begin, suppose that $\tau \in \mathscr{F}_{\mathbb{Z}}$. In particular, $\{0, 2\}$ is a module of $\tau - 1$ and not of τ. It follows that $[1, 0]_\tau \neq [1, 2]_\tau$. For the second assertion, consider $m, n \in \mathbb{Z}$ such that $m < n$. Since $\{n, n+2\}$ is a module of $\tau - (n+1)$, we have

$$[m, n]_\tau = [m, n+2]_\tau.$$

Furthermore, since $\{m, m+2\}$ is a module of $\tau - (m+1)$, we have

$$[m, n+2]_\tau = [m+2, n+2]_\tau.$$

Therefore, we obtain

$$[m, n]_\tau = [m+2, n+2]_\tau.$$

By proceeding by induction, we obtain that the second assertion holds.

Conversely, suppose that both assertions above hold. Since the second assertion holds, we obtain the following. Given $m, n, p, q \in \mathbb{Z}$ such that $m < n$ and $p < q$,

$$\text{if } m \equiv p \bmod 2 \text{ and } n \equiv q \bmod 2, \text{ then } [m, n]_\tau = [p, q]_\tau.$$

It follows that for every $n \in \mathbb{Z}$, $\{n-1, n+1\}$ is a module of $\tau - n$. To conclude, we have to verify that $[n, n-1]_\tau \neq [n, n+1]_\tau$ for every $n \in \mathbb{Z}$. Let $n \in \mathbb{Z}$. For instance, suppose that n is even. We obtain

$$[n, n+1]_\tau = [0, 1]_\tau.$$

Moreover, we have $[n-1, n]_\tau = [1, 2]_\tau$, so

$$[n, n-1]_\tau = [2, 1]_\tau.$$

Since $[1,0]_\tau \neq [1,2]_\tau$, we obtain $[n,n-1]_\tau \neq [n,n+1]_\tau$. The case n odd is handled similarly. □

Lemma 8.7. *Given a 2-structure τ such that $V(\tau) = \mathbb{N}$, $\tau \in \mathscr{F}_{\mathbb{N}}$ if and only if the following three assertions hold*

- $[1,0]_\tau \neq [1,2]_\tau$*;*
- *for $m,n \in \mathbb{N}$ such that $m < n$, we have $[2m,2n]_\tau = [0,2]_\tau$, $[2m,2n-1]_\tau = [0,1]_\tau$, $[2m+1,2n]_\tau = [1,2]_\tau$, and $[2m+1,2n+1]_\tau = [1,3]_\tau$;*
- $[1,2]_\tau = [1,3]_\tau$*.*

Proof. To begin, suppose that $\tau \in \mathscr{F}_{\mathbb{N}}$. As in the proof of Lemma 8.6, we obtain that the first two assertions hold. Since $\tau \in \mathscr{F}_{\mathbb{N}}$, $\{2,3,\dots\}$ is a module of $\tau - 0$. Hence, $[1,2]_\tau = [1,3]_\tau$.

Conversely, suppose that the three assertions above hold. Let $n \geq 1$. As in the proof of Lemma 8.6, we obtain that $\{n-1,n+1\}$ is a module of $\tau - n$ and not of τ. Moreover, it follows from the last two assertions that $\{2,3,\dots\}$ is a module of $\tau - 0$. Lastly, $\{0\} \cup \{2,3,\dots\}$ is not a module of τ because $[1,0]_\tau \neq [1,2]_\tau$. □

Lemma 8.8. *Given $\tau \in \mathscr{F}_{\mathbb{Z}}$, the following four assertions hold*

- *for each $n \in \mathbb{Z}$, $\{n,n+1,\dots\}$ is a module of τ if and only if $[0,1]_\tau = [0,2]_\tau$ and $[1,2]_\tau = [1,3]_\tau$;*
- *for each $n \in \mathbb{Z}$, $\{\dots,n-1,n\}$ is a module of τ if and only if $[1,2]_\tau = [0,2]_\tau$ and $[0,1]_\tau = [1,3]_\tau$;*
- *every module of τ is a module of $L_{\mathbb{Z}}$;*
- *if τ is decomposable and $\tau \neq \sigma(L_{\mathbb{Z}})$, then one of the following two situations holds*
  - ▸ *for each nontrivial module M of τ, there exists $n \geq 1$ such that $M = \{n,n+1,\dots\}$;*
  - ▸ *for each nontrivial module M of τ, there exists $n \in \mathbb{Z}$ such that $M = \{\dots,n-1,n\}$.*

Proof. The first two assertions follow from the second assertion of Lemma 8.6.

For the third assertion, consider a module M of τ. Consider $p,q \in M$ such that $p+2 \leq q$. We verify that

$$\text{if } p \equiv q \mod 2, \text{ then } \{p,\dots,q\} \subseteq M. \tag{8.1}$$

For instance, suppose that p and q are even. To begin, consider r be an odd integer such that $p < r < q$. By the second assertion of Lemma 8.6, $[r,p]_\tau = [1,0]_\tau$ and $[r,q]_\tau = [1,2]_\tau$. Since $[1,0]_\tau \neq [1,2]_\tau$ by the first assertion of Lemma 8.6, we obtain $r \in M$. Now, let r be an even integer such that $p < r < q$. We have $r-1, r+1 \in M$. Moreover, by the second assertion of Lemma 8.6, we have $[r,r-1]_\tau = [2,1]_\tau$ and $[r,r+1]_\tau = [0,1]_\tau$. Since $[0,1]_\tau \neq [2,1]_\tau$ by the first assertion of Lemma 8.6, we obtain $r \in M$. The case p and q both odd follows similarly. Thus, (8.1) holds.

Now, suppose that $p \not\equiv q \bmod 2$. For instance, suppose that p is even and q is odd. For a contradiction, suppose that $q-1 \notin M$ and $q+1 \notin M$. By the second assertion of Lemma 8.6, $[q-1,p]_\tau = [2,0]_\tau$ and $[q-1,q]_\tau = [0,1]_\tau$. Since $q-1 \notin M$, we obtain $[2,0]_\tau = [0,1]_\tau$. Furthermore, by the second assertion of Lemma 8.6, $[q+1,p]_\tau = [2,0]_\tau$ and $[q+1,q]_\tau = [2,1]_\tau$. Since $q+1 \notin M$, we obtain $[2,0]_\tau = [2,1]_\tau$. Therefore, we have $[0,1]_\tau = [2,1]_\tau$, which contradicts the first assertion of Lemma 8.6. Consequently, $q-1 \in M$ or $q+1 \in M$, and we conclude by using (8.1).

For the fourth assertion, suppose that there exist $p, q \in \mathbb{Z}$, with $p < q$, such that $\{p, \ldots, q\}$ is a module of τ. We have to show that $\tau = \sigma(L_\mathbb{Z})$. It follows from the second assertion of Lemma 8.6 that

$$[0,1]_\tau = [0,2]_\tau = [1,2]_\tau = [1,3]_\tau. \tag{8.2}$$

By the first assertion of Lemma 8.6, we have $[1,0]_\tau \neq [1,2]_\tau$. By (8.2), $[0,1]_\tau \neq [1,0]_\tau$. Therefore, $\tau = \sigma(L_\mathbb{Z})$. □

Example 8.9. We consider the tournament $U_\mathbb{Z}$ obtained from the linear order $L_\mathbb{Z}$ by reversing all the arcs between the even integers. By Lemma 8.6, $\sigma(U_\mathbb{Z}) \in \mathscr{F}_\mathbb{Z}$. It follows from Lemma 8.8 that $\sigma(U_\mathbb{Z})$ is prime. We can also see that $\sigma(U_\mathbb{Z})$ is prime by using Theorem 7.1 as follows. Let F be a finite subset of $\mathbb{Z}$. There exists $n \in \mathbb{Z}$ such that $F \subseteq \{-n, \ldots, n\}$. By Theorem 4.28, $\sigma(U_{2n+1})$ is prime (see Figure 4.5). Since $(\sigma(U_\mathbb{Z}))[\{-n, \ldots, n\}]$ and $\sigma(U_{2n+1})$ are isomorphic, $(\sigma(U_\mathbb{Z}))[\{-n, \ldots, n\}]$ is prime too. It follows from Theorem 7.1 that $\sigma(U_\mathbb{Z})$ is prime.

Now, we consider the tournament $W_\mathbb{Z}$ obtained from the linear order $L_\mathbb{Z}$ by reversing all the arcs between the even integers and all the arcs between the odd integers. As previously for $\sigma(U_\mathbb{Z})$, it is not difficult to verify that $\sigma(W_\mathbb{Z})$ is a prime element of $\mathscr{F}_\mathbb{Z}$.

Finally, we consider the bipartite graph $H_\mathbb{Z}$ defined on $\mathbb{Z}$ in the following way. For $p, q \in \mathbb{Z}$, with $p \neq q$, $\{p, q\} \in E(H_\mathbb{Z})$ if there exist $i, j \in \mathbb{Z}$, with $i \leq j$, such that $\{p, q\} = \{2i, 2j+1\}$. Once again, $\sigma(H_\mathbb{Z})$ is a prime element of $\mathscr{F}_\mathbb{Z}$.

Lemma 8.10. *Given $\tau \in \mathscr{F}_\mathbb{N}$, the following four assertions hold*

- *for each $n \geq 1$, $\{n, n+1, \ldots\}$ is a module of τ if and only if $[0,1]_\tau = [0,2]_\tau$;*
- *every module of τ is a module of $L_\mathbb{N}$;*
- *if τ is decomposable and $\tau \neq \sigma(L_\mathbb{N})$, then for each nontrivial module M of τ, there exists $n \geq 1$ such that $M = \{n, n+1, \ldots\}$.*

Proof. The first assertion follows from the last two assertions of Lemma 8.7. We show the second assertion as in the proof of Lemma 8.8.

For the third assertion, suppose that there exist $p \geq 0$ and $q > p$ such that $\{p, \ldots, q\}$ is a module of τ. We have to show that $\tau = \sigma(L_\mathbb{N})$. It follows from the second assertion of Lemma 8.7 that $[1,2]_\tau = [0,2]_\tau$ and $[0,1]_\tau = [1,3]_\tau$. By the third assertion of Lemma 8.7, we have $[1,2]_\tau = [1,3]_\tau$. It follows that (8.2) holds. By the first assertion of Lemma 8.7, we have $[1,0]_\tau \neq [1,2]_\tau$. By (8.2), $[0,1]_\tau \neq [1,0]_\tau$. Therefore, $\tau = \sigma(L_\mathbb{N})$. □

Example 8.11. Set

$$U_{\mathbb{N}} = U_{\mathbb{Z}}[\mathbb{N}],\ W_{\mathbb{N}} = W_{\mathbb{Z}}[\mathbb{N}],\text{ and } H_{\mathbb{N}} = H_{\mathbb{Z}}[\mathbb{N}].$$

As in Example 8.9, we verify easily that $\sigma(U_{\mathbb{N}})$ and $\sigma(H_{\mathbb{N}})$ are prime elements of $\mathscr{F}_{\mathbb{N}}$. Similarly, $\sigma(W_{\mathbb{N}})$ is prime, but $\sigma(W_{\mathbb{N}}) \notin \mathscr{F}_{\mathbb{N}}$ because $[1,2]_{\sigma(W_{\mathbb{N}})} \neq [1,3]_{\sigma(W_{\mathbb{N}})}$.

In fact, $\sigma(W_{\mathbb{N}})$ is also interesting because it shows that the analogue of Theorem 7.1, when the primality is replaced by the criticality, does not hold. Indeed, $\sigma(W_{\mathbb{N}})$ satisfies the second assertion of the analogue. Precisely, for each finite $F \subseteq \mathbb{N}$, there exists $n \geq 2$ such that $F \subseteq \{0,\dots,2n\}$. Clearly, $\sigma(W_{\mathbb{N}})[\{0,\dots,2n\}] = \sigma(W_{2n+1})$ (see Figure 4.6). By Theorem 4.37, $\sigma(W_{2n+1})$ is critical. But, $\sigma(W_{\mathbb{N}})$ does not satisfy the first assertion of the analogue. Clearly, the function $\mathbb{N} \longrightarrow \mathbb{N} \setminus \{0,1\}$, defined by $n \longmapsto n+2$ for each $n \in \mathbb{N}$, is an isomorphism from $\sigma(W_{\mathbb{N}})$ onto $\sigma(W_{\mathbb{N}}) - \{0,1\}$. Thus, $\sigma(W_{\mathbb{N}}) - \{0,1\}$ is prime. Set

$$X = \mathbb{N} \setminus \{0,1\}.$$

Since $(3,1),(1,2) \in A(W_{\mathbb{N}})$, we have

$$1 \notin \langle X \rangle_{\sigma(W_{\mathbb{N}})}.$$

Let $n \geq 1$. Since $W_{\mathbb{N}}[\{1,2n,2n+1\}]$ is a 3-cycle, we have

$$1 \notin (X_{\sigma(W_{\mathbb{N}})}(2n) \cup X_{\sigma(W_{\mathbb{N}})}(2n+1)).$$

By Lemma 3.13,

$$1 \in \mathrm{Ext}_{\sigma(W_{\mathbb{N}})}(X).$$

Hence, $\sigma(W_{\mathbb{N}})[X \cup \{1\}]$, which is $\sigma(W_{\mathbb{N}}) - 0$, is prime. It follows that $\sigma(W_{\mathbb{N}})$ is not critical, so it does not satisfy the first assertion of the analogue. For the opposite direction, we consider $\sigma(P_{\mathbb{Z}})$. As seen at the beginning of this section, $\sigma(P_{\mathbb{Z}})$ is critical. Hence, it satisfies the first assertion of the analogue. Nevertheless, consider $\{0,4\}$ for the finite subset F of $\mathbb{Z}$. Let F' be any subset $\mathbb{Z}$ containing $\{0,4\}$ and such that $\sigma(P_{\mathbb{Z}})[F']$ is prime. Since $\sigma(P_{\mathbb{Z}})[F']$ is prime, $P_{\mathbb{Z}}[F']$ is connected. Thus, there exists $n \geq 4$ such that

$$F' = \{0,\dots,n\}.$$

Clearly, $\sigma(P_{\mathbb{Z}})[F']$ is not critical because $\sigma(P_{\mathbb{Z}})[F'] - n$ is prime. Consequently, $\sigma(P_{\mathbb{Z}})$ does not satisfy the second assertion of the analogue.

Observation 8.4 leads us to introduce the following definition.

Definition 8.12. An infinite 2-structure σ is *locally critical* if there exist a partition Q of $V(\sigma)$ and a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ satisfying the following two assertions *locally critical*

(I1) for every $Y \in Q$ such that $|Y| > 1$, $\varphi_{\restriction Y}$ is an isomorphism from $\sigma[Y]$ onto an element of $\mathscr{F}_{\mathbb{N}} \cup \mathscr{F}_{\mathbb{Z}}$;

(I2) there exists $Y \in Q$ such that $|Y| > 1$.

Note that we do not require a locally critical 2-structure to be prime.

Lemma 8.13. *Given an infinite 2-structure σ, if σ is critical and nonfinitely critical, then σ is locally critical* [8.1]. *Precisely, assertions (I1) and (I2) hold for the partition Q of $V(\sigma)$ and a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ defined as in Observation 8.4.*

Proof. Let Q be the partition of $V(\sigma)$ constituted by the vertex sets of the components of $\mathbb{P}(\sigma)$. Using the axiom of choice, consider also a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ defined as in Observation 8.4.

Consider $Y \in Q$ such that $|Y| > 1$. There exists a nontrivial component C of $\mathbb{P}(\sigma)$ such that $Y = V(C)$. Denote by ρ the unique 2-structure defined on $\mathbb{Z}$ or $\mathbb{N}$ such that $\varphi_{\restriction Y}$ is an isomorphism from $\sigma[Y]$ onto ρ. To verify that assertion (I1) holds, we distinguish the following two cases.

CASE 1: $\varphi_{\restriction Y}$ is an isomorphism from C onto $P_{\mathbb{Z}}$..

We have to verify that $\rho \in \mathscr{F}_{\mathbb{Z}}$. For each $n \in \mathbb{Z}$, we have

$$N_{\mathbb{P}(\sigma)}((\varphi_{\restriction Y})^{-1}(n)) = \{(\varphi_{\restriction Y})^{-1}(n-1), (\varphi_{\restriction Y})^{-1}(n+1)\}.$$

It follows from Lemma 4.4 that $\{(\varphi_{\restriction Y})^{-1}(n-1), (\varphi_{\restriction Y})^{-1}(n+1)\}$ is a module of $\sigma - (\varphi_{\restriction Y})^{-1}(n)$. Thus, $\{n-1, n+1\}$ is a module of $\rho - n$. Since σ is prime, $\{(\varphi_{\restriction Y})^{-1}(n-1), (\varphi_{\restriction Y})^{-1}(n+1)\}$ is not a module of σ. Hence, we have

$$(\varphi_{\restriction Y})^{-1}(n) \not\leftrightarrow_{\sigma} \{(\varphi_{\restriction Y})^{-1}(n-1), (\varphi_{\restriction Y})^{-1}(n+1)\}.$$

It follows that $n \not\leftrightarrow_{\rho} \{n-1, n+1\}$. Therefore, $\{n-1, n+1\}$ is not a module of ρ. Consequently, $\rho \in \mathscr{F}_{\mathbb{Z}}$.

CASE 2: $\varphi_{\restriction Y}$ is an isomorphism from C onto $P_{\mathbb{N}}$.

We have to verify that $\rho \in \mathscr{F}_{\mathbb{N}}$. Let $n \geq 1$. As seen in the first case, $\{n-1, n+1\}$ is a module of $\rho - n$, but not of ρ. Furthermore, we have

$$N_{\mathbb{P}(\sigma)}((\varphi_{\restriction Y})^{-1}(0)) = \{(\varphi_{\restriction Y})^{-1}(1)\}.$$

It follows from Lemma 4.4 that $V(\sigma) \smallsetminus \{(\varphi_{\restriction Y})^{-1}(0), (\varphi_{\restriction Y})^{-1}(1)\}$ is a module of $\sigma - (\varphi_{\restriction Y})^{-1}(0)$. Thus, $\{2, 3, \ldots\}$ is a module of $\rho - 0$. Since σ is prime, $V(\sigma) \smallsetminus \{(\varphi_{\restriction Y})^{-1}(1)\}$ is not a module of σ. Hence, we have

$$(\varphi_{\restriction Y})^{-1}(1) \not\leftrightarrow_{\sigma} \{(\varphi_{\restriction Y})^{-1}(0), (\varphi_{\restriction Y})^{-1}(2)\}.$$

It follows that $1 \not\leftrightarrow_{\rho} \{0, 2\}$. Therefore, $\{0\} \cup \{2, 3, \ldots\}$ is not a module of ρ. Consequently, $\rho \in \mathscr{F}_{\mathbb{N}}$.

It follows that assertion (I1) holds.

By the second assertion of Corollary 8.2, $\mathbb{P}(\sigma)$ is nonempty. Thus, $\mathbb{P}(\sigma)$ admits a nontrivial component C. We obtain that $V(C) \in Q$ and $|V(C)| > 1$. It follows that assertion (I2) holds. □

Notation 8.14. Let σ be a locally critical 2-structure. Consider a partition Q of $V(\sigma)$ and a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ satisfying assertions (I1) and (I2).

Let $Y \in Q$ such that $|Y| > 1$. Since assertion (I1) holds, $\varphi_{\restriction Y}$ is an isomorphism from $\sigma[Y]$ onto an element of $\mathscr{F}_{\mathbb{N}} \cup \mathscr{F}_{\mathbb{Z}}$. We denote $\varphi_{\restriction Y}$ by φ_Y. Also,

[8.1] We use the axiom of choice to prove Lemma 8.13.

we denote by τ_Y the unique 2-structure defined on $\mathbb{Z}$ or $\mathbb{N}$ such that φ_Y is an isomorphism from $\sigma[Y]$ onto τ_Y. Moreover, we denote by C_Y the unique component of $\mathbb{P}(\sigma)$ such that $Y = V(C_Y)$.

Lastly, set

$$V_{\text{even}}(\sigma) = \{v \in V(\sigma) : \varphi(v) \equiv 0 \mod 2\}$$

and

$$V_{\text{odd}}(\sigma) = \{v \in V(\sigma) : \varphi(v) \equiv 1 \mod 2\}.$$

We consider also the partition

$$P = \{Y \in Q : |Y| = 1\} \cup \Big(\bigcup_{\{Y \in Q : |Y| > 1\}} \{Y \cap V_{\text{even}}(\sigma), Y \cap V_{\text{odd}}(\sigma)\}\Big)$$

of $V(\sigma)$.

8.2. **A generalized quotient.**

Observation 8.15. Let σ be an infinite, critical, and nonfinitely critical 2-structure. Consider the partition Q of $V(\sigma)$ and a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ defined as in Observation 8.4.

Let $Y \in Q$ such that $|Y| > 1$. For instance, suppose that $\tau_Y \in \mathscr{F}_{\mathbb{Z}}$. Recall that φ_Y is an isomorphism from C_Y onto $P_{\mathbb{Z}}$. Therefore, for each $n \in \mathbb{Z}$, we have

$$N_{\mathbb{P}(\sigma)}((\varphi_Y)^{-1}(2n+1)) = \{(\varphi_Y)^{-1}(2n), (\varphi_Y)^{-1}(2n+2)\}.$$

By Lemma 4.4, $\{(\varphi_Y)^{-1}(2n), (\varphi_Y)^{-1}(2n+2)\}$ is a module of $\sigma - (\varphi_Y)^{-1}(2n+1)$. In particular, for each $v \in V(\sigma) \smallsetminus Y$, we have $[v, (\varphi_Y)^{-1}(2n)]_\sigma = [v, (\varphi_Y)^{-1}(2n+2)]_\sigma$. It follows that $Y \cap V_{\text{even}}(\sigma)$ is a module of $\sigma[V(\sigma) \smallsetminus (Y \cap V_{\text{odd}}(\sigma))]$. Similarly, $Y \cap V_{\text{odd}}(\sigma)$ is a module of $\sigma[V(\sigma) \smallsetminus (Y \cap V_{\text{even}}(\sigma))]$. The same holds when $\tau_Y \in \mathscr{F}_{\mathbb{N}}$.

Observation 8.15 leads us to introduce the following definition.

Definition 8.16. Let σ be a 2-structure. Consider partitions P and Q of $V(\sigma)$ such that P is finer than Q. Hence, for each $X \in P$, there exists $Y(X) \in Q$ such that $X \subseteq Y(X)$.

We say that P is a *modular partition* of σ *according to* Q [6] if for any $X, X' \in P$ such that $Y(X) \neq Y(X')$, X and X' are modules of $\sigma[X \cup X']$. *modular partition according to*

The generalized quotient is defined in the following manner. Consider partitions P and Q of $V(\sigma)$ such that P is a modular partition of σ according to Q. The *generalized quotient* $\sigma/_Q P$ of σ by P according to Q is defined on $V(\sigma/_Q P) = P$ as follows. Given $X_0, X_1, X_2, X_3 \in V(\sigma/_Q P)$, with $X_0 \neq X_1$ and $X_2 \neq X_3$, *generalized quotient*

$$(X_0, X_1) \equiv_{(\sigma/_Q P)} (X_2, X_3) \text{ if}$$

$$Y(X_0) = Y(X_1) \text{ and } Y(X_2) = Y(X_3) \tag{8.3}$$

or

$$\begin{cases} Y(X_0) \neq Y(X_1),\, Y(X_2) \neq Y(X_3) \\ \text{and} \\ (x_0, x_1) \equiv_\sigma (x_2, x_3), \text{ where } x_i \in X_i \text{ for } i \in \{0,1,2,3\}. \end{cases}$$

A priori, (8.3) might appear arbitrary. In fact, it ensures the following property (see the second assertion of Lemma 8.17). Let R be a module of $\sigma/_Q P$ such that $|\{Y \in Q : Y \cap (\cup R) \neq \varnothing\}| \geq 2$. For each $Y \in Q$ such that $Y \cap (\cup R) \neq \varnothing$, we have $Y \subseteq (\cup R)$.

Two results on the generalized quotient follow.

Lemma 8.17. *Let σ be a 2-structure. Consider two partitions P and Q of $V(\sigma)$ such that P is a modular partition of σ according to Q.*

- *For each $Y \in Q$, Y is a module of σ if and only if $\{X \in P : X \subseteq Y\}$ is a module of $\sigma/_Q P$.*
- *For every $R \subseteq P$ such that $|\{Y \in Q : Y \cap (\cup R) \neq \varnothing\}| \geq 2$, R is a module of $\sigma/_Q P$ if and only if $(\cup\{Y \in Q : Y \cap (\cup R) \neq \varnothing\}) = (\cup R)$ and $\cup R$ is a module of σ (see Notation 2.9).*

Proof. The first assertion follows from the definition of the generalized quotient. For the second assertion, consider $R \subseteq P$ such that $|\{Y \in Q : Y \cap (\cup R) \neq \varnothing\}| \geq 2$. To begin, suppose that $(\cup\{Y \in Q : Y \cap (\cup R) \neq \varnothing\}) = (\cup R)$ and $\cup R$ is a module of σ. It follows from the definition of the generalized quotient that R is a module of $\sigma/_Q P$. Conversely, suppose that R is a module of $\sigma/_Q P$. Clearly, $(\cup R) \subseteq (\cup\{Y \in Q : Y \cap (\cup R) \neq \varnothing\})$. For a contradiction, suppose that

$$(\cup R) \subsetneq (\cup\{Y \in Q : Y \cap (\cup R) \neq \varnothing\}).$$

Let $v \in (\cup\{Y \in Q : Y \cap (\cup R) \neq \varnothing\}) \smallsetminus (\cup R)$. There exist $X_0 \in P \smallsetminus R$ and $Y_0 \in Q$ such that $v \in X_0$, $X_0 \subseteq Y_0$, and $Y_0 \cap (\cup R) \neq \varnothing$. Since $Y_0 \cap (\cup R) \neq \varnothing$, there exists $X_1 \in R$ such that $X_1 \subseteq Y_0$. Since $|\{Y \in Q : Y \cap (\cup R) \neq \varnothing\}| \geq 2$, there exist $X_2 \in R$ and $Y_1 \in Q \smallsetminus \{Y_0\}$ such that $X_2 \subseteq Y_1$. Since $X_0 \cup X_1 \subseteq Y_0$, $X_2 \subseteq Y_1$ and $Y_0 \neq Y_1$, we have

$$(X_0, X_1) \not\equiv_{(\sigma/_Q P)} (X_0, X_2),$$

which contradicts the fact that R is a module of $\sigma/_Q P$. Consequently, we have

$$(\cup R) = (\cup\{Y \in Q : Y \cap (\cup R) \neq \varnothing\}).$$

It follows from the definition of the generalized quotient that $\cup R$ is a module of σ. □

The next result follows easily from Lemma 8.17.

Corollary 8.18. *Let σ be a prime 2-structure. Consider two partitions P and Q of $V(\sigma)$ such that P is a modular partition of σ according to Q. For every nontrivial module R of $\sigma/_Q P$, there exists $Y \in Q$ such that $(\cup R) \subsetneq Y$ and there exists $v \in Y \smallsetminus (\cup R)$ such that $v \not\leftrightarrow_\sigma (\cup R)$.*

8.3. **The main theorem: Theorem 8.26.** In the next lemmas, we continue the study of infinite, critical, and nonfinitely critical 2-structures.

Lemma 8.19. *Let σ be an infinite, critical, and nonfinitely critical 2-structure. Consider the partition Q of $V(\sigma)$ and a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ defined as in Observation 8.4. The following two assertions hold*

(I3) *P (see Notation 8.14) is a modular partition of σ according to Q;*
(I4) *for each $Y \in Q$ such that $V(\tau_Y) = \mathbb{N}$ (see Notation 8.14),*

$$(\varphi_Y)^{-1}(1) \longleftrightarrow_\sigma (\{(\varphi_Y)^{-1}(2)\} \cup (V(\sigma) \smallsetminus Y)).$$

Proof. It follows from Observation 8.15 that assertion (I3) holds. For assertion (I4), consider $Y \in Q$ such that $|Y| > 1$ and $\tau_Y \in \mathscr{F}_\mathbb{N}$. Since φ_Y is an isomorphism from C_Y onto $P_\mathbb{N}$, we have

$$N_{\mathbb{P}(\sigma)}((\varphi_Y)^{-1}(0)) = \{(\varphi_Y)^{-1}(1)\}.$$

By Lemma 4.4, $V(\sigma) \smallsetminus \{(\varphi_Y)^{-1}(0), (\varphi_Y)^{-1}(1)\}$ is a module of $\sigma - (\varphi_Y)^{-1}(0)$. In particular, we have $(\varphi_Y)^{-1}(1) \longleftrightarrow_\sigma (\{(\varphi_Y)^{-1}(2)\} \cup (V(\sigma) \smallsetminus Y))$. Therefore, assertion (I4) holds. □

Lemma 8.20. *Let σ be an infinite, critical, and nonfinitely critical 2-structure. Consider the partition Q of $V(\sigma)$ and a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ defined as in Observation 8.4. The following assertion holds*

(I5) *the generalized quotient $\sigma/_Q P$ is prime.*

Proof. Since σ is prime, assertion (I5) follows easily from Corollary 8.18 because for every $Y \in Q$, $|\{X \in P : X \subseteq Y\}| = 1$ or 2. □

In the next two facts, we consider locally critical 2-structures.

Fact 8.21. *Let σ be a locally critical 2-structure. Consider a partition Q of $V(\sigma)$ and a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ satisfying assertions (I1) and (I2). Suppose also that assertions (I3) and (I4) hold.*

Let $Q' \subseteq Q$ such that

$$\{Y \in Q' : |Y| > 1\} \neq \varnothing.$$

Set

$$P' = \{X \in P : X \subseteq (\cup Q')\}.$$

Suppose that $\sigma[\cup Q']$ admits a nontrivial module M. The following statements hold.

- *If M/Q' possesses a unique element Y, then $\{Y \cap V_{\text{even}}(\sigma), Y \cap V_{\text{odd}}(\sigma)\}$ is a module of $(\sigma/_Q P)[P']$.*
- *Suppose that $|M/Q'| \geq 2$. Given $Y \in (M/Q')$, if $|Y| > 1$, then $(Y \cap M) \cap V_{\text{even}}(\sigma) \neq \varnothing$ and $(Y \cap M) \cap V_{\text{odd}}(\sigma) \neq \varnothing$. It follows that M/P' is a module of $(\sigma/_Q P)[P']$. Moreover, if $M/P' = P'$, then there exists $Y \in (M/Q')$, with $|Y| > 1$, such that $P' \smallsetminus \{Y \cap V_{\text{even}}(\sigma), Y \cap V_{\text{odd}}(\sigma)\}$ is a module of $(\sigma/_Q P)[P']$.*

Proof. To begin, suppose that M/Q' possesses a unique element Y. Hence, $M \subseteq Y$. By assertion (M2) of Proposition 2.5, M is a module of $\sigma[Y]$. Thus, $\varphi_Y(M)$ is a module of τ_Y. By assertion (I1), $\tau_Y \in \mathscr{F}_{\mathbb{N}} \cup \mathscr{F}_{\mathbb{Z}}$. It follows from Lemmas 8.8 and 8.10 that $\varphi_Y(M)$ is a module of $L_{\mathbb{N}}$ or $L_{\mathbb{Z}}$. Therefore, $\varphi_Y(M)$ contains even and odd integers. It follows that $M \cap V_{\text{even}}(\sigma) \neq \varnothing$ and $M \cap V_{\text{odd}}(\sigma) \neq \varnothing$. Since P is a modular partition of σ according to Q by assertion (I3), $\{Y \cap V_{\text{even}}(\sigma), Y \cap V_{\text{odd}}(\sigma)\}$ is a module of $(\sigma/_Q P)[P']$.

Now, suppose that $|M/Q'| \geq 2$. For a contradiction, suppose that there exists $Y \in M/Q'$ such that $|Y| > 1$ and $|Y \cap M| = 1$. We have $\varphi_Y : Y \longrightarrow \mathbb{N}$ or $\varphi_Y : Y \longrightarrow \mathbb{Z}$. Thus, there exists an integer n such that $Y \cap M = \{(\varphi_Y)^{-1}(n)\}$. We distinguish the following two cases. In each of them, we obtain a contradiction.

Case 1: $\varphi_Y : Y \longrightarrow \mathbb{Z}$ or $\varphi_Y : Y \longrightarrow \mathbb{N}$ and $n \geq 1$.

Since $|M| \geq 2$, there exists $x \in M \smallsetminus Y$. Since assertion (I3) holds, P is a modular partition of σ according to Q. Therefore, we have

$$x \longleftrightarrow_\sigma \{(\varphi_Y)^{-1}(n-1), (\varphi_Y)^{-1}(n+1)\}.$$

Since M is a module of $\sigma[\cup Q']$ such that $\{(\varphi_Y)^{-1}(n), x\} \subseteq M$ and $M \cap \{(\varphi_Y)^{-1}(n-1), (\varphi_Y)^{-1}(n+1)\} = \varnothing$, we obtain

$$(\varphi_Y)^{-1}(n) \longleftrightarrow_\sigma \{(\varphi_Y)^{-1}(n-1), (\varphi_Y)^{-1}(n+1)\}.$$

Since φ_Y is an isomorphism from $\sigma[Y]$ onto τ_Y, we obtain

$$n \longleftrightarrow_{\tau_Y} \{n-1, n+1\},$$

which contradicts $\tau_Y \in \mathscr{F}_{\mathbb{Z}} \cup \mathscr{F}_{\mathbb{N}}$.

Case 2: $\varphi_Y : Y \longrightarrow \mathbb{N}$ and $n = 0$.

By considering $x \in M \smallsetminus Y$, we obtain

$$(\varphi_Y)^{-1}(1) \longleftrightarrow_\sigma \{(\varphi_Y)^{-1}(0), x\}.$$

Since assertion (I4) holds, we have

$$(\varphi_Y)^{-1}(1) \longleftrightarrow_\sigma (\{(\varphi_Y)^{-1}(2), x\}.$$

Hence,

$$(\varphi_Y)^{-1}(1) \longleftrightarrow_\sigma (\{(\varphi_Y)^{-1}(0), (\varphi_Y)^{-1}(2)\}.$$

Since φ_Y is an isomorphism from $\sigma[Y]$ onto τ_Y, we obtain

$$1 \longleftrightarrow_{\tau_Y} \{0, 2\},$$

which contradicts $\tau_Y \in \mathscr{F}_{\mathbb{N}}$.

Consequently, for each $Y \in (M/Q')$, we have

$$\text{if } |Y| > 1, \text{ then } |Y \cap M| \geq 2. \tag{8.4}$$

As above, when $|M/Q'| = 1$, we obtain $(Y \cap M) \cap V_{\text{even}}(\sigma) \neq \varnothing$ and $(Y \cap M) \cap V_{\text{odd}}(\sigma) \neq \varnothing$.

Let $Y \in (M/Q')$ such that $|Y| \geq 2$, we have $|Y \cap M| \geq 2$. We obtain $(Y \cap M) \cap V_{\text{even}}(\sigma) \neq \varnothing$ and $(Y \cap M) \cap V_{\text{odd}}(\sigma) \neq \varnothing$. It follows that M/P' is the family of $X \in P'$ such that there exists $Y \in (M/Q')$ satisfying $Y \supseteq X$.

Since P is a modular partition of σ according to Q by assertion (I3), M/P' is a module of $(\sigma/_Q P)[P']$.

Lastly, suppose that $M/P' = P'$. Since M is a nontrivial module of $\sigma[\cup Q']$, there exists $Y \in (M/Q')$ such that $Y \smallsetminus M \neq \varnothing$. Moreover, since $M/P' = P'$, we have $|Y| > 1$. We have $\varphi_Y : Y \longrightarrow \mathbb{N}$ or $\varphi_Y : Y \longrightarrow \mathbb{Z}$. For convenience, set

$$R = P' \smallsetminus \{Y \cap V_{\text{even}}(\sigma), Y \cap V_{\text{odd}}(\sigma)\}.$$

Let $y \in Y \smallsetminus M$. We obtain

$$y \longleftrightarrow_\sigma M.$$

Since P is a modular partition of σ according to Q and $M/P' = P'$, we obtain

$$\{z \in Y : \varphi_Y(z) \equiv \varphi_Y(y) \bmod 2\} \longleftrightarrow_{(\sigma/_Q P)[P']} R. \tag{8.5}$$

Therefore, if $(Y \smallsetminus M) \cap V_{\text{even}}(\sigma) \neq \varnothing$ and $(Y \smallsetminus M) \cap V_{\text{odd}}(\sigma) \neq \varnothing$, then R is a module of $(\sigma/_Q P)[P']$. Thus, suppose that

$$(Y \smallsetminus M) \cap V_{\text{even}}(\sigma) = \varnothing \text{ or } (Y \smallsetminus M) \cap V_{\text{odd}}(\sigma) = \varnothing. \tag{8.6}$$

By assertion (M2) of Proposition 2.5, $M \cap Y$ is a module of $\sigma[Y]$. By (8.4), $|M \cap Y| \geq 2$. Since $Y \smallsetminus M \neq \varnothing$, $M \cap Y$ is a nontrivial module of $\sigma[Y]$. Thus, $\varphi_Y(M \cap Y)$ is a nontrivial module of τ_Y. Since assertion (I1) holds, $\tau_Y \in \mathscr{F}_{\mathbb{N}} \cup \mathscr{F}_{\mathbb{Z}}$. It follows from Lemmas 8.8 and 8.10 that $\varphi_Y(M \cap Y)$ is a nontrivial module of $L_{\mathbb{N}}$ or $L_{\mathbb{Z}}$. It follows from (8.6) that $\tau_Y \in \mathscr{F}_{\mathbb{N}}$ and $M \cap Y = Y \smallsetminus \{(\varphi_Y)^{-1}(0)\}$. Since assertion (I4) holds, we have

$$(\varphi_Y)^{-1}(1) \longleftrightarrow_\sigma (\{(\varphi_Y)^{-1}(2)\} \cup (V(\sigma) \smallsetminus Y)).$$

Since P is a modular partition of σ according Q, we obtain

$$(Y \cap V_{\text{odd}}(\sigma)) \longleftrightarrow_{(\sigma/_Q P)[P']} R.$$

Furthermore, since $(\varphi_Y)^{-1}(0) \in Y \smallsetminus M$, it follows from (8.5) that

$$(Y \cap V_{\text{even}}(\sigma)) \longleftrightarrow_{(\sigma/_Q P)[P']} R.$$

Therefore, R is a module of $(\sigma/_Q P)[P']$. □

The next fact follows easily from Fact 8.21.

Fact 8.22. *Let σ be a locally critical 2-structure. Consider a partition Q of $V(\sigma)$ and a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ satisfying assertions (I1) and (I2). Suppose also that assertions (I3) and (I4) hold.*

Let Q' be a nonempty subset of Q such that

$$\{Y \in Q' : |Y| > 1\} \neq \varnothing.$$

Set

$$P' = \{X \in P : X \subseteq (\cup Q')\}.$$

Suppose that

$$|\{Y \in Q' : |Y| > 1\}| \geq 2 \textit{ or } |Q'| \geq 3. \tag{8.7}$$

If $\sigma[\cup Q']$ *is decomposable, then* $(\sigma/_Q P)[P']$ *is as well.*

We go back to the study of infinite, critical, and nonfinitely critical 2-structures. The next lemma follows easily from Fact 8.22.

Lemma 8.23. *Let* σ *be an infinite, critical, and nonfinitely critical 2-structure. Consider the partition* Q *of* $V(\sigma)$ *and a function* $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ *defined as in Observation 8.4. The following assertion holds*

(I6) *If*

$$|\{Y \in Q : |Y| > 1\}| \geq 2 \text{ or } |Q| \geq 4, \tag{8.8}$$

then

$$\{v \in V(\sigma) : \{v\} \in Q\} \subseteq (P \smallsetminus \mathscr{S}(\sigma/_Q P)).$$

Proof. Consider $v \in V(\sigma)$ such that $\{v\} \in Q$. Furthermore, suppose that (8.8) holds. Set

$$Q' = Q \smallsetminus \{\{v\}\}.$$

Since assertion (I2) holds, $\{Y \in Q : |Y| > 1\} \neq \varnothing$. Furthermore, it follows from (8.8) that (8.7) holds. By Lemma 8.19, assertions (I3) and (I4) hold. To conclude, it suffices to apply Fact 8.22. □

Proposition 8.24. *Let* σ *be an infinite, critical, and nonfinitely critical 2-structure. Consider the partition* Q *of* $V(\sigma)$ *and a function* $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ *defined as in Observation 8.4.*

Suppose that $\mathbb{P}(\sigma)$ *admits a unique nontrivial component* C *and finitely many trivial components. If* $|V(\sigma) \smallsetminus V(C)| \geq 2$*, then*

- $|V(\sigma) \smallsetminus V(C)| = 2$;
- *there exists a unique* $v \in V(\sigma) \smallsetminus V(C)$ *such that* $v \not\leftrightarrow_\sigma V(C)$;
- $\sigma[V(C)]$ *is decomposable.*

Proof. To use Notation 8.14, set $Y = V(C)$. Obviously, $Y \in Q$. To begin, we show that for each $W \subsetneq V(\sigma) \smallsetminus V(C)$, we have

$$\sigma[Y \cup W] \text{ is decomposable.} \tag{8.9}$$

Otherwise, consider $W \subsetneq V(\sigma) \smallsetminus Y$ such that $\sigma[Y \cup W]$ is prime. Since $V(\sigma) \smallsetminus (Y \cup W)$ is finite, it follows from Corollary 3.21 that there exist $v, w \in V(\sigma) \smallsetminus (Y \cup W)$ such that $\sigma - \{v, w\}$ is prime. We cannot have $v = w$ because σ is critical. Moreover, we cannot have $v \neq w$ because v and w are isolated in $\mathbb{P}(\sigma)$. It follows that (8.9) holds.

Set

$$S(Y) = \{v \in V(\sigma) \smallsetminus Y : v \not\leftrightarrow_\sigma Y\}.$$

We prove that either for every $v \in S(Y)$, we have

$$\begin{cases} [v, (\varphi_Y)^{-1}(0)]_\sigma = [(\varphi_Y)^{-1}(0), (\varphi_Y)^{-1}(2)]_\sigma \\ \text{and} \\ [v, (\varphi_Y)^{-1}(1)]_\sigma = [(\varphi_Y)^{-1}(1), (\varphi_Y)^{-1}(3)]_\sigma, \end{cases} \tag{8.10}$$

or for every $v \in S(Y)$, we have

$$(8.11) \qquad \begin{cases} [v,(\varphi_Y)^{-1}(0)]_\sigma = [(\varphi_Y)^{-1}(2),(\varphi_Y)^{-1}(0)]_\sigma \\ \text{and} \\ [v,(\varphi_Y)^{-1}(1)]_\sigma = [(\varphi_Y)^{-1}(3),(\varphi_Y)^{-1}(1)]_\sigma. \end{cases}$$

Indeed, let $v \in S(Y)$. By (8.9), $\sigma[Y \cup \{v\}]$ admits a nontrivial module M. By Lemma 8.19, assertions (I3) and (I4) hold. We use Fact 8.21 in the following manner. For Q' consider $\{Y, \{v\}\}$. We obtain $P' = \{Y \cap V_{\text{even}}(\sigma), Y \cap V_{\text{odd}}(\sigma), \{v\}\}$. For a contradiction, suppose that $|M/Q'| = 1$. We obtain $M/Q' = \{Y\}$. It follows from Fact 8.21 that $\{Y \cap V_{\text{even}}(\sigma), Y \cap V_{\text{odd}}(\sigma)\}$ is a module of $(\sigma/_Q P)[P']$, which contradicts $v \in S(Y)$. Therefore, $|M/Q'| \geq 2$. Hence, we have $v \in M$. Furthermore, it follows from Fact 8.21 that $(Y \cap M) \cap V_{\text{even}}(\sigma) \neq \varnothing$ and $(Y \cap M) \cap V_{\text{odd}}(\sigma) \neq \varnothing$. Thus, there exist $p, q \in \mathbb{Z}$ such that

$$(\varphi_Y)^{-1}(2p), (\varphi_Y)^{-1}(2q+1) \in Y \cap M.$$

In particular, we have $|M \cap Y| \geq 2$. By assertion (M2) of Proposition 2.5, $M \cap Y$ is a module of $\sigma[Y]$. Since $v \in M$ and $M \subsetneq (Y \cup \{v\})$, we have $(M \cap Y) \neq Y$. Moreover, since $|M \cap Y| \geq 2$, $M \cap Y$ is a nontrivial module of $\sigma[Y]$. It follows that $\varphi_Y(Y \cap M)$ is a nontrivial module of τ_Y. We distinguish the following two cases.

Case 1: There exists $n \in \mathbb{Z}$ such that

$$(8.12) \qquad \varphi_Y(Y \cap M) \subseteq \{\dots, n-1, n\}.$$

There exists $m \geq 0$ such that $(\varphi_Y)^{-1}(2m), (\varphi_Y)^{-1}(2m+1) \in (Y \smallsetminus M)$. Since $(\varphi_Y)^{-1}(2p), (\varphi_Y)^{-1}(2q+1) \in Y \cap M$ and $v \in M$, we obtain

$$[v,(\varphi_Y)^{-1}(2m)]_\sigma = [(\varphi_Y)^{-1}(2p),(\varphi_Y)^{-1}(2m)]_\sigma,$$

and hence

$$[v,(\varphi_Y)^{-1}(2m)]_\sigma = [(\varphi_Y)^{-1}(0),(\varphi_Y)^{-1}(2)]_\sigma.$$

Since P is a modular partition of σ according to Q by assertion (I3) (see Lemma 8.19), we have $v \longleftrightarrow_\sigma Y \cap V_{\text{even}}(\sigma)$. It follows that

$$[v,(\varphi_Y)^{-1}(0)]_\sigma = [(\varphi_Y)^{-1}(0),(\varphi_Y)^{-1}(2)]_\sigma.$$

Similarly, we obtain

$$[v,(\varphi_Y)^{-1}(1)]_\sigma = [(\varphi_Y)^{-1}(1),(\varphi_Y)^{-1}(3)]_\sigma.$$

Therefore, v satisfies (8.10). Consequently, if M satisfies (8.12), then v satisfies (8.10).

Case 2: There exists $n \in \mathbb{Z}$ such that

$$(8.13) \qquad \{n, n+1, \dots\} \subseteq \varphi_Y(Y \cap M).$$

By assertion (I1), $\tau_Y \in \mathscr{F}_{\mathbb{N}} \cup \mathscr{F}_{\mathbb{Z}}$. Since $\varphi_Y(Y \cap M)$ is a nontrivial module of τ_Y, it follows from Lemmas 8.8 and 8.10 that there exists $n' \in \mathbb{Z}$ such that

$$\varphi_Y(Y \cap M) = \{n', n'+1, \dots\}.$$

We verify that v satisfies (8.11). We distinguish the following two subcases.

Subcase a: $\tau_Y \in \mathscr{F}_{\mathbb{Z}}$ or $\tau_Y \in \mathscr{F}_{\mathbb{N}}$ and $n' \geq 2$.

There exists $m \geq 0$ such that $(\varphi_Y)^{-1}(2m), (\varphi_Y)^{-1}(2m+1) \in (Y \smallsetminus M)$. Since $(\varphi_Y)^{-1}(2p) \in Y \cap M$ and $v \in M$, we obtain

$$[v, (\varphi_Y)^{-1}(2m)]_\sigma = [(\varphi_Y)^{-1}(2p), (\varphi_Y)^{-1}(2m)]_\sigma,$$

and hence

$$[v, (\varphi_Y)^{-1}(2m)]_\sigma = [(\varphi_Y)^{-1}(2), (\varphi_Y)^{-1}(0)]_\sigma.$$

Since P is a modular partition of σ according to Q by assertion (I3), we obtain $v \longleftrightarrow_\sigma Y \cap V_{\text{even}}(\sigma)$. It follows that

$$[v, (\varphi_Y)^{-1}(0)]_\sigma = [(\varphi_Y)^{-1}(2), (\varphi_Y)^{-1}(0)]_\sigma.$$

Similarly, we obtain

$$[v, (\varphi_Y)^{-1}(1)]_\sigma = [(\varphi_Y)^{-1}(3), (\varphi_Y)^{-1}(1)]_\sigma.$$

Therefore, v satisfies (8.11).

Subcase b: $\tau_Y \in \mathscr{F}_{\mathbb{N}}$ and $n' = 1$..

We have $\varphi_Y(Y \cap M) = \{1, 2, \ldots\}$. Since $v \in M$, we obtain

$$[v, (\varphi_Y)^{-1}(0)]_\sigma = [(\varphi_Y)^{-1}(2), (\varphi_Y)^{-1}(0)]_\sigma.$$

Since $\tau_Y \in \mathscr{F}_{\mathbb{N}}$, φ_Y is an isomorphism from C onto $P_{\mathbb{N}}$. Hence, we have

$$N_{\mathbb{P}(\sigma)}((\varphi_Y)^{-1}(0)) = \{(\varphi_Y)^{-1}(1)\}.$$

By Lemma 4.4, $V(\sigma) \smallsetminus \{(\varphi_Y)^{-1}(0), (\varphi_Y)^{-1}(1)\}$ is a module of $\sigma - (\varphi_Y)^{-1}(0)$. It follows that

$$[v, (\varphi_Y)^{-1}(1)]_\sigma = [(\varphi_Y)^{-1}(3), (\varphi_Y)^{-1}(1)]_\sigma.$$

Thus, v satisfies (8.11).

Consequently, if M satisfies (8.13), then v satisfies (8.11).

It follows that for each $v \in S(Y)$, v satisfies (8.10) or (8.11). Let $v \in S(Y)$. Since $v \not\longleftrightarrow_\sigma Y$, we obtain

$$[v, (\varphi_Y)^{-1}(0)]_\sigma \neq [v, (\varphi_Y)^{-1}(1)]_\sigma.$$

It follows from (8.10) or (8.11) that

$$[(\varphi_Y)^{-1}(0), (\varphi_Y)^{-1}(2)]_\sigma \neq [(\varphi_Y)^{-1}(1), (\varphi_Y)^{-1}(3)]_\sigma.$$

Therefore, $[0,2]_{\tau_Y} \neq [1,3]_{\tau_Y}$, so $\tau_Y \neq \sigma(L(\mathbb{N})$ and $\tau_Y \neq \sigma(L(\mathbb{Z})$. It follows from Lemmas 8.8 and 8.10 that either (8.12) holds or (8.13) holds. Consequently, either (8.10) holds for every $v \in S(Y)$ or (8.11) holds for every $v \in S(Y)$. In particular, we obtain that

(8.14) $$S(Y) \text{ is a module of } \sigma[Y \cup S(Y)].$$

We conclude in the following manner. For a contradiction, suppose that $|V(\sigma) \setminus Y| \geq 3$. We show that

$$Y \cup S(Y) \text{ is a module of } \sigma. \tag{8.15}$$

Let $v \in S(Y)$ and $w \in (V(\sigma) \setminus Y) \setminus S(Y)$. We must verify that $Y \cup \{v\}$ is a module of $\sigma[Y \cup \{v, w\}]$. Since $|V(\sigma) \setminus Y| \geq 3$, it follows from (8.9) that $\sigma[Y \cup \{v, w\}]$ admits a nontrivial module M. We use Fact 8.21 as follows. Consider

$$Q' = \{Y, \{v\}, \{w\}\}.$$

We obtain

$$P' = \{Y \cap V_{\text{even}}(\sigma), Y \cap V_{\text{odd}}(\sigma), \{v\}, \{w\}\}.$$

By Fact 8.21, if $|M/Q'| = 1$, then $\{Y \cap V_{\text{even}}(\sigma), Y \cap V_{\text{odd}}(\sigma)\}$ is a module of $(\sigma/_Q P)[P']$, which contradicts $v \in S(Y)$. Therefore, $|M/Q'| \geq 2$. By Fact 8.21, $(Y \cap M) \cap V_{\text{even}}(\sigma) \neq \varnothing$ and $(Y \cap M) \cap V_{\text{odd}}(\sigma) \neq \varnothing$. It follows that M/P' is a module of $(\sigma/_Q P)[P']$. Furthermore, we have $v \in M$ because $v \in S(Y)$. For a contradiction, suppose that $M/P' = P'$. It follows from Fact 8.21 that $\{\{v\}, \{w\}\}$ is a module of $(\sigma/_Q P)[P']$. Thus, $\{v, w\}$ is a module of $\sigma[Y \cup \{v, w\}]$, which contradicts $v \in S(Y)$ and $w \in (V(\sigma) \setminus Y) \setminus S(Y)$. Consequently, $M/P' \subsetneq P'$. Since $v \in M$, we obtain

$$M/P' = \{Y \cap V_{\text{even}}(\sigma), Y \cap V_{\text{odd}}(\sigma), \{v\}\}.$$

By the second assertion of Lemma 8.17 applied to $\sigma[Y \cup \{v, w\}]$ with Q' and P', $Y \cup \{v\}$ is a module of $\sigma[Y \cup \{v, w\}]$. It follows that (8.15) holds. Finally, it follows from (8.14) and (8.15) that $Y \cup S(Y)$ is a nontrivial module of σ or $S(Y)$ is a nontrivial module of σ, which contradicts the fact that σ is prime. It ensues that

$$|V(\sigma) \setminus Y| = 2.$$

Since Y is not a module of σ, we have $S(Y) \neq \varnothing$. It follows from (8.14) that $|S(Y)| = 1$. Moreover, since both elements of $V(\sigma) \setminus Y$ are isolated in $\mathbb{P}(\sigma)$, we obtain $\sigma[Y]$ is decomposable. $\square$

The next lemma follows from Proposition 8.24.

Lemma 8.25. *Let σ be an infinite, critical, and nonfinitely critical 2-structure. Consider the partition Q of $V(\sigma)$ and a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ defined as in Observation 8.4. The following assertion holds*

(I7) *Suppose that $|Q| \leq 3$ and there exists a unique $Y \in Q$ such that $|Y| > 1$ (i.e. (8.8) does not hold):*
- *if $V(\sigma) = Y$, then $\varphi_{\restriction Y}$ is an isomorphism from $\sigma[Y]$ onto a prime element of $\mathscr{F}_{\mathbb{N}} \cup \mathscr{F}_{\mathbb{Z}}$;*
- *if $|V(\sigma) \setminus Y| = 1$, then $(V(\sigma) \setminus Y) \not\leftrightarrow_\sigma Y$ and $\varphi_{\restriction Y}$ is an isomorphism from $\sigma[Y]$ onto a decomposable element of $\mathscr{F}_{\mathbb{N}} \cup \mathscr{F}_{\mathbb{Z}}$;*
- *if $|V(\sigma) \setminus Y| = 2$, then there exists a unique $v \in V(\sigma) \setminus Y$ such that $v \not\leftrightarrow_\sigma Y$, and $\varphi_{\restriction Y}$ is an isomorphism from $\sigma[Y]$ onto a decomposable element of $\mathscr{F}_{\mathbb{N}} \cup \mathscr{F}_{\mathbb{Z}}$.*

Proof. Suppose that there exists a unique $Y \in Q$ such that $|Y| > 1$. If $V(\sigma) = Y$, then $\varphi_{\restriction Y}$ is an isomorphism from $\sigma[Y]$ onto a prime element of $\mathscr{F}_{\mathbb{N}} \cup \mathscr{F}_{\mathbb{Z}}$ because σ is prime. Suppose that $V(\sigma) \setminus Y$ contains a unique element v. Since σ is prime, Y is not a module of σ, and hence $v \not\leftrightarrow_{\sigma} Y$. Moreover, since σ is critical, $\sigma - v$ is decomposable. Thus, $\varphi_{\restriction Y}$ is an isomorphism from $\sigma[Y]$ onto a decomposable element of $\mathscr{F}_{\mathbb{N}} \cup \mathscr{F}_{\mathbb{Z}}$. Finally, when $|V(\sigma) \setminus Y| \geq 2$, we utilize Proposition 8.24. □

The main theorem follows. It puts together Lemmas 8.13, 8.19, 8.20, 8.23, and 8.25.

Theorem 8.26 (Boubabbous and Ille [6][8.2]). *Consider an infinite, critical, and nonfinitely critical 2-structure σ. Let Q be the partition of $V(\sigma)$ constituted by the vertex sets of the components of $\mathbb{P}(\sigma)$. Using the axiom of choice, consider also a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ defined as in Observation 8.4. Then, assertions (I1),...,(I7) hold.*

8.4. **Locally critical 2-structures.** The purpose of this subsection is to establish the following theorem.

Theorem 8.27 (Boubabbous and Ille [6][8.3]). *Let σ be a locally critical 2-structure. Consider a partition Q of $V(\sigma)$ and a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ satisfying assertions (I1) and (I2). Suppose also that assertions (I3),...,(I6) hold. If* (8.8) *holds, then σ is critical and nonfinitely critical.*

Before proving Theorem 8.27, we establish the following three results.

Lemma 8.28. *Let σ be a locally critical 2-structure. Consider a partition Q of $V(\sigma)$ and a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ satisfying assertions (I1) and (I2). Suppose also that assertions (I3), (I4), and (I5) hold. If*

$$|\{Y \in Q : |Y| > 1\}| \geq 2 \text{ or } |Q| \geq 3, \tag{8.16}$$

then σ is prime.

Proof. We consider the partition P of $V(\sigma)$ defined as in Notation 8.14. Suppose that (8.16) holds. We have to show that σ is prime. We utilize Fact 8.22 as follows. Set $Q' = Q$. We obtain $P' = P$. Since assertion (I2) holds, $\{Y \in Q : |Y| > 1\} \neq \varnothing$. Since (8.16) holds, (8.7) holds too. Since assertion (I5) holds, $\sigma/_Q P$ is prime. It follows from Fact 8.22 that σ is prime. □

Lemma 8.29. *Let σ be a locally critical 2-structure. Consider a partition Q of $V(\sigma)$ and a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ satisfying assertions (I1) and (I2). Suppose also that assertions (I3) and (I4) hold. Consider $Y \in Q$ such that $|Y| > 1$. For every $v \in Y$, $\sigma - v$ is decomposable.*

[8.2] Boubabbous and Ille [6] proved this theorem (see [6, Theorem 12]) for digraphs.

[8.3] Boubabbous and Ille [6] proved this theorem (see [6, Theorem 13]) for digraphs.

Proof. Let $v \in Y$. Set

$$n = \varphi_Y(v).$$

Since assertions (I1) and (I2) hold, we consider the partition P of $V(\sigma)$ defined as in Notation 8.14.

First, suppose that $\tau_Y \in \mathscr{F}_{\mathbb{Z}}$ or $\tau_Y \in \mathscr{F}_{\mathbb{N}}$ and $n \geq 1$. We obtain that $\{n-1, n+1\}$ is a module of $\tau_Y - n$. Since φ_Y is an isomorphism from $\sigma[Y]$ onto τ_Y, $\{(\varphi_Y)^{-1}(n-1), (\varphi_Y)^{-1}(n+1)\}$ is a module of $\sigma[Y] - v$. Since assertion (I3) holds, P is a modular partition of σ according to Q. We obtain

$$\{(\varphi_Y)^{-1}(n-1), (\varphi_Y)^{-1}(n+1)\} \longleftrightarrow_\sigma V(\sigma) \setminus Y.$$

It follows that $\{(\varphi_Y)^{-1}(n-1), (\varphi_Y)^{-1}(n+1)\}$ is a module of $\sigma - v$.

Second, suppose that $\tau_Y \in \mathscr{F}_{\mathbb{N}}$ and $n = 0$. We obtain that $\{2, 3, \dots\}$ is a module of $\tau_Y - 0$. Since φ_Y is an isomorphism from $\sigma[Y]$ onto τ_Y, $Y \setminus \{(\varphi_Y)^{-1}(0), (\varphi_Y)^{-1}(1)\}$ is a module of $\sigma[Y] - (\varphi_Y)^{-1}(0)$. Since assertion (I4) holds, we have

$$(\varphi_Y)^{-1}(1) \longleftrightarrow_\sigma (\{(\varphi_Y)^{-1}(2)\} \cup (V(\sigma) \setminus Y)).$$

It follows that $V(\sigma) \setminus \{(\varphi_Y)^{-1}(0), (\varphi_Y)^{-1}(1)\}$ is a module of $\sigma - v$. □

Proposition 8.30. *Let σ be a locally critical 2-structure. Consider a partition Q of $V(\sigma)$ and a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ satisfying assertions (I1) and (I2). Suppose also that assertions (I3), (I4), and (I5) hold. For every $Y \in Q$ such that $|Y| > 1$, the following two assertions hold*

- (J1) *for each $n \in V(\tau_Y)$ (see Notation 8.14), $\sigma - \{(\varphi_Y)^{-1}(n), (\varphi_Y)^{-1}(n+1)\}$ is isomorphic to σ;*
- (J2) *there exists a nontrivial component C of $\mathbb{P}(\sigma)$ such that $Y = V(C)$, and φ_Y is an isomorphism from C onto $P_{\mathbb{N}}$ or $P_{\mathbb{Z}}$.*

Proof. Let $Y \in Q$ such that $|Y| > 1$. Since assertion (I1) holds, φ_Y is an isomorphism from $\sigma[Y]$ onto $\tau_Y \in \mathscr{F}_{\mathbb{N}} \cup \mathscr{F}_{\mathbb{Z}}$. Furthermore, since assertions (I3), (I4), and (I5) hold, it follows from Lemma 8.28 that σ is prime. We prove that

$$\text{for each } n \in V(\tau_Y),\ \{(\varphi_Y)^{-1}(n), (\varphi_Y)^{-1}(n+1)\} \in E(\mathbb{P}(\sigma)). \tag{8.17}$$

More strongly, we establish that

$$\text{for each } n \in V(\tau_Y),\ \sigma - \{(\varphi_Y)^{-1}(n), (\varphi_Y)^{-1}(n+1)\} \text{ is isomorphic to } \sigma, \tag{8.18}$$

that is, assertion (J1) holds. Let $n \in V(\tau_Y)$. Consider the function

$$\begin{array}{cccc} f: & V(\tau_Y) & \longrightarrow & V(\tau_Y) \setminus \{n, n+1\} \\ & p \leq n-1 & \longmapsto & p, \\ & p \geq n & \longmapsto & p+2. \end{array} \tag{8.19}$$

Clearly, f is strictly increasing and preserves the parity. Since $\tau_Y \in \mathscr{F}_{\mathbb{N}} \cup \mathscr{F}_{\mathbb{Z}}$, it follows from Lemmas 8.6 and 8.7 that f is an isomorphism from τ_Y onto $\tau_Y - \{n, n+1\}$. Since assertion (I1) holds, φ_Y is an isomorphism from $\sigma[Y]$

onto τ_Y. Thus, $((\varphi_Y)^{-1})_{\restriction V(\tau_Y)\smallsetminus\{n,n+1\}} \circ f \circ \varphi_Y$ is an isomorphism from $\sigma[Y]$ onto $\sigma[Y] - \{(\varphi_Y)^{-1}(n), (\varphi_Y)^{-1}(n+1)\}$. For convenience, set

$$\psi = ((\varphi_Y)^{-1})_{\restriction V(\tau_Z)\smallsetminus\{n,n+1\}} \circ f \circ \varphi_Y.$$

Consider the extension $\psi \cup \mathrm{Id}_{(V(\sigma)\smallsetminus Y)}$ of ψ by the identity function on $V(\sigma)\smallsetminus Y$ defined by

$$\begin{array}{rcl} V(\sigma) & \longrightarrow & V(\sigma)\smallsetminus\{(\varphi_Y)^{-1}(n), (\varphi_Y)^{-1}(n+1)\} \\ w \in Y & \longmapsto & \psi(w), \\ w \in (V(\sigma)\smallsetminus Y) & \longmapsto & w. \end{array} \tag{8.20}$$

Since assertion (I3) holds, P is a modular partition of σ according to Q. It follows that $\psi \cup \mathrm{Id}_{(V(\sigma)\smallsetminus Y)}$ is an isomorphism from σ onto $\sigma - \{(\varphi_Y)^{-1}(n), (\varphi_Y)^{-1}(n+1)\}$. Consequently, (8.18) holds, so assertion (J1) holds. Moreover, (8.17) holds because σ is prime.

To prove that assertion (J2) holds, we distinguish the following two cases.

Case 1: $\varphi_Y : Y \longrightarrow \mathbb{Z}$.

Let $n \in Z$. Since (8.17) holds,

$$\{(\varphi_Y)^{-1}(n-1), (\varphi_Y)^{-1}(n+1)\} \subseteq N_{\mathbb{P}(\sigma)}((\varphi_Y)^{-1}(n)).$$

Since assertions (I3) and (I4) hold, it follows from Lemma 8.29 that $\sigma - (\varphi_Y)^{-1}(n)$ is decomposable. By Lemma 4.4,

$$N_{\mathbb{P}(\sigma)}((\varphi_Y)^{-1}(n)) = \{(\varphi_Y)^{-1}(n-1), (\varphi_Y)^{-1}(n+1)\}.$$

It follows that $\mathbb{P}(\sigma)[Y]$ is a component of $\mathbb{P}(\sigma)$, and φ_Y is an isomorphism from $\mathbb{P}(\sigma)[Y]$ onto $P_{\mathbb{Z}}$.

Case 2: $\varphi_Y : Y \longrightarrow \mathbb{N}$.

As previously, we have

$$N_{\mathbb{P}(\sigma)}((\varphi_Y)^{-1}(n)) = \{(\varphi_Y)^{-1}(n-1), (\varphi_Y)^{-1}(n+1)\}$$

for each $n \geq 1$. Furthermore, since (8.17) holds, we have

$$(\varphi_Y)^{-1}(1) \in N_{\mathbb{P}(\sigma)}((\varphi_Y)^{-1}(0)). \tag{8.21}$$

Since assertion (I4) holds, we have

$$(\varphi_Y)^{-1}(1) \longleftrightarrow_\sigma (\{(\varphi_Y)^{-1}(2)\} \cup (V(\sigma)\smallsetminus Y)). \tag{8.22}$$

Moreover, since assertion (I1) holds, $\tau_Y \in \mathscr{F}_{\mathbb{N}}$. It follows from Lemma 8.7 that

$$1 \longleftrightarrow_{\tau_Y} (V(\tau_Y)\smallsetminus\{0,1\}).$$

Since φ_Y is an isomorphism from $\sigma[Y]$ onto τ_Y, we obtain

$$(\varphi_Y)^{-1}(1) \longleftrightarrow_\sigma (Y\smallsetminus\{(\varphi_Y)^{-1}(0), (\varphi_Y)^{-1}(1)\}). \tag{8.23}$$

It follows from (8.22) and (8.23) that

$$(\varphi_Y)^{-1}(1) \longleftrightarrow_\sigma (V(\sigma)\smallsetminus\{(\varphi_Y)^{-1}(0), (\varphi_Y)^{-1}(1)\}).$$

Thus, for every $v \in V(\sigma) \smallsetminus \{(\varphi_Y)^{-1}(0), (\varphi_Y)^{-1}(1)\}$, we have $v \notin N_{\mathbb{P}(\sigma)}((\varphi_Y)^{-1}(0))$. Since $(\varphi_Y)^{-1}(1) \in N_{\mathbb{P}(\sigma)}((\varphi_Y)^{-1}(0))$ by (8.21), we obtain

$$N_{\mathbb{P}(\sigma)}((\varphi_Y)^{-1}(0)) = \{(\varphi_Y)^{-1}(1)\}.$$

Consequently, $\mathbb{P}(\sigma)[Y]$ is a component of $\mathbb{P}(\sigma)$, and φ_Y is an isomorphism from $\mathbb{P}(\sigma)[Y]$ onto $P_{\mathbb{N}}$. □

Proof of Theorem 8.27. Since (8.8) holds, (8.16) holds as well. Since assertions (I3), (I4), and (I5) hold, it follows from Lemma 8.28 that σ is prime.

To continue, we prove that σ is critical. Let $v \in V(\sigma)$. We must verify that $\sigma - v$ is decomposable. Denote by Y the unique element of Q containing v. To begin, suppose that $|Y| > 1$. Since assertions (I3) and (I4) hold, it follows from Lemma 8.29 that $\sigma - v$ is decomposable. Now, suppose that $Y = \{v\}$. Since assertion (I6) and (8.8) hold, $(\sigma/_Q P) - \{v\}$ is decomposable. Let R be a nontrivial module of $(\sigma/_Q P) - \{v\}$. Set

$$Q' = Q \smallsetminus \{\{v\}\} \text{ and } P' = P \smallsetminus \{\{v\}\}.$$

Clearly, P' is a modular partition of $\sigma - v$ according to Q'. Moreover, we have

$$(\sigma/_Q P) - \{v\} = (\sigma - v)/_{(P')}Q'.$$

We apply Lemma 8.17 to $\sigma - v$ together with partitons P' and Q' as follows. We distinguish the following two cases.

CASE 1: $|(\cup R)/Q'| = 1$.

Denote by Z the unique element of $(\cup R)/Q'$. Since $|R| \geq 2$ and $|\{X \in P' : X \subseteq Z\}| \leq 2$, we have $R = \{X \in P' : X \subseteq Z\}$. It follows from the first assertion of Lemma 8.17 that Z is a module of $\sigma - v$.

CASE 2: $|(\cup R)/Q'| \geq 2$.

It follows from the second assertion of Lemma 8.17 that $(\cup\{Y \in Q' : Y \cap (\cup R) \neq \varnothing\}) = (\cup R)$ and $(\cup R)$ is a module of $\sigma - v$. Since $(\cup\{Y \in Q' : Y \cap (\cup R) \neq \varnothing\}) = (\cup R)$ and R is a nontrivial module of $(\sigma - v)/_{(P')}Q'$, $(\cup R)$ is a nontrivial module of $\sigma - v$.

Consequently, σ is critical.

Finally, we verify that σ is not finitely critical. Since assertion (I2) holds, there exists $Y \in Q$ such that $|Y| > 1$. Moreover, since assertions (I3), (I4), and (I5) hold, it follows from assertion (J2) of Proposition 8.30 that there exists a nontrivial component C of $\mathbb{P}(\sigma)$ such that $Y = V(C)$. Hence, there exist distinct $v, w \in Y$ such that $\sigma - \{v, w\}$ is prime. □

Remark 8.31. Consider the tournament T defined on $V(T) = \mathbb{Z} \times \{0, 1\}$ which satisfies

- for $i = 0$ or 1, the function $\psi_i : \mathbb{Z} \longrightarrow \mathbb{Z} \times \{i\}$, defined by $n \longmapsto (n, i)$ for every $n \in \mathbb{Z}$, is an isomorphism from $U_{\mathbb{Z}}$ onto $T[\mathbb{Z} \times \{i\}]$;
- for $p, q \in \mathbb{Z}$, we have $((2p, 0), (2q, 1)) \in A(T)$, $((2p+1, 0), (2q+1, 1)) \in A(T)$, $((2p+1, 1), (2q, 0)) \in A(T)$, and $((2p, 1), (2q+1, 0)) \in A(T)$.

First, consider the partition

$$Q = \{\mathbb{Z} \times \{0\}, \mathbb{Z} \times \{1\}\}$$

of $V(\sigma(T))$ and the function $\varphi : V(\sigma(T)) \longrightarrow \mathbb{Z}$ defined by $\varphi_{\restriction(\mathbb{Z}\times\{0\})} = (\psi_0)^{-1}$ and $\varphi_{\restriction(\mathbb{Z}\times\{1\})} = (\psi_1)^{-1}$. We obtain

$$P = \{(2\mathbb{Z}) \times \{0\}, (2\mathbb{Z}+1) \times \{0\}, (2\mathbb{Z}) \times \{1\}, (2\mathbb{Z}+1) \times \{1\}\}.$$

It is not difficult to verify that $\sigma(T)$ satisfies assertions (I1),...,(I6) with Q and φ. Furthermore, since $|\{Y \in Q : |Y| > 1\}| = 2$, (8.8) holds. By Theorem 8.27, $\sigma(T)$ is critical but not finitely critical. Moreover, it follows from assertion (J2) of Proposition 8.30 that $\mathbb{Z} \times \{0\}$ and $\mathbb{Z} \times \{1\}$ are the vertex sets of the components of $\mathbb{P}(\sigma(T))$.

Second, consider the partition

$$Q = \{\mathbb{Z} \times \{0\}\} \cup \{\{(n,1)\} : n \in \mathbb{Z}\}$$

of $V(\sigma(T))$ and the same function φ as before. We obtain

$$P = \{(2\mathbb{Z}) \times \{0\}, (2\mathbb{Z}+1) \times \{0\}\} \cup \{\{(n,1)\} : n \in \mathbb{Z}\}.$$

Once again, $\sigma(T)$ satisfies assertions (I1),...,(I6) with Q and φ. Furthermore, since Q is infinite, (8.8) holds. Nevertheless, it follows only from Proposition 8.30 that $\mathbb{Z} \times \{0\}$ is the vertex set of a component of $\mathbb{P}(\sigma(T))$.

Consequently, it is not possible to determine the primality graph from assertions (I1),...,(I6) only.

The next result follows from Theorems 8.26 and 8.27.

Corollary 8.32 (Boubabbous and Ille [6][8.4]). *Given an infinite 2-structure σ, if σ is critical and nonfinitely critical, then the following two assertions hold*

- *for any distinct $v, w \in V(\sigma)$, $\sigma - \{v, w\}$ is prime if and only if $\sigma - \{v, w\}$ is isomorphic to σ;*
- *there exist distinct $v, w \in V(\sigma)$ such that $\sigma - \{v, w\}$ is isomorphic to σ.*

Proof. Suppose that σ is an infinite, critical, and nonfinitely critical 2-structure. By the second assertion of Corollary 8.2, there exist distinct $v, w \in V(\sigma)$ such that $\sigma - \{v, w\}$ is prime.

Now, consider any distinct $v, w \in V(\sigma)$ such that $\sigma - \{v, w\}$ is prime. We have to verify that $\sigma - \{v, w\}$ is isomorphic to σ. Consider the partition Q of $V(\sigma)$ and a function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ defined as in Observation 8.4. There exists $Y \in Q$ such that $v, w \in Y$. We have $\{v, w\} \in E(\mathbb{P}(\sigma))$. Since φ_Y is an isomorphism from $\mathbb{P}(\sigma)[Y]$ onto $P_{\mathbb{N}}$ or $P_{\mathbb{Z}}$, there exists $n \in \mathbb{Z}$ such that

$$\{v, w\} = \{(\varphi_Y)^{-1}(n), (\varphi_Y)^{-1}(n+1)\}.$$

Since assertions (I3), (I4), and (I5) hold, it follows from assertion (J1) of Proposition 8.30 that $\sigma - \{(\varphi_Y)^{-1}(n), (\varphi_Y)^{-1}(n+1)\}$ is isomorphic to σ. □

[8.4] Boubabbous and Ille [6] proved this theorem for digraphs.

The next result follows from Corollary 8.32 and Theorem 5.8. It is the analogue of Theorem 5.3 in the infinite case.

Corollary 8.33. *Given an infinite and prime 2-structure σ, if there exists a finite subset F of $V(\sigma)$ such that $|F| \geq 2$ and $\sigma - F$ is prime, then there exist distinct $v, w \in V(\sigma)$ such that $\sigma - \{v, w\}$ is prime.*

Proof. It follows from Corollary 3.20 that there exists $F' \subseteq F$ such that $|F'| = 2$ or 3 and $\sigma - F'$ is prime. The conclusion is obvious when $|F'| = 2$. Hence, suppose that $|F'| = 3$. By Corollary 3.20 again, there exists $x \in F'$ such that $\sigma - x$ is prime. Set

$$\tau = \sigma - x.$$

Clearly, if τ is not critical, then we conclude directly. Thus, suppose that τ is critical. By denoting by y and z the two elements of $F' \setminus \{x\}$, we obtain $\tau - \{y, z\}$ is prime. Therefore, τ is not finitely critical. By applying three times the second assertion of Corollary 8.32 from τ, we obtain $F'' \subseteq V(\tau)$ such that $|F''| = 6$ and $\tau - F''$ is isomorphic to τ. Since $\tau - F'' = \sigma - (\{x\} \cup F'')$, we obtain that $\sigma - (\{x\} \cup F'')$ is prime. It follows from Theorem 5.8 applied to $\sigma - (\{x\} \cup F'')$ that there exist distinct $v, w \in (\{x\} \cup F'')$ such that $\sigma - \{v, w\}$ is prime. □

Remark 8.34. Observe that Corollary 8.32 does not hold if we only suppose that σ is prime but not finitely critical. Similarly, Corollary 8.33 does not hold if we only suppose that the finite subset F of $V(\sigma)$ is nonempty. Indeed, consider the graph G defined on $V(G) = \mathbb{Z} \cup \{\infty\}$ by

$$G[\mathbb{Z}] = P_{\mathbb{Z}} \text{ and } E(G) = E(P_{\mathbb{Z}}) \cup \{\{0, \infty\}\}.$$

As observed at the beginning of this section, $G - \infty = P_{\mathbb{Z}}$ is prime. Hence, $\sigma(G) - \infty$ is prime as well. Set

$$X = V(G) \setminus \{\infty\}.$$

Since $\{0, \infty\} \in E(G)$ and $\{1, \infty\} \notin E(G)$, we have

$$\infty \notin \langle X \rangle_{\sigma(G)}.$$

Furthermore, since $d_G(\infty) = 1$ and $d_{(G-\infty)}(n) = 2$ for every $n \in \mathbb{Z}$, we obtain

$$\infty \notin X_{\sigma(G)}(n) \text{ for each } n \in \mathbb{Z}.$$

It follows from Lemma 3.13 that $\infty \in \mathrm{Ext}_{\sigma(G)}(X)$, so $\sigma(G)$ is prime too. However, for each finite subset F of $\mathbb{Z}$, with $|F| \geq 2$, $G - F$ is disconnected. It follows that $\sigma(G) - F$ is decomposable for each finite subset F of $\mathbb{Z}$ such that $|F| \geq 2$.

We complete this subsection with the following example which is constructed from the graph $H_{\mathbb{Z}}$ (see Example 8.9) and the graph G defined on $V(G) = \mathbb{Z} \cup \{\infty\}$ in Remark 8.34. It shows that Proposition 8.24 does not hold if we do not suppose that the primality graph admits finitely many trivial components.

Example 8.35. Consider the graph H defined on $V(H) = \mathbb{Z} \cup \{\infty_n : n \in \mathbb{Z}\}$ and saisfying

- $H[\mathbb{Z}] = H_{\mathbb{Z}}$;
- the bijection $\mathbb{Z} \longrightarrow \{\infty_n : n \in \mathbb{Z}\}$, defined by $n \longmapsto \infty_n$ for each $n \in \mathbb{Z}$, is an isomorphism from $P_{\mathbb{Z}}$ onto $H[\{\infty_n : n \in \mathbb{Z}\}]$;
- for every $p \in \mathbb{Z}$, $\{2p, \infty_0\} \in E(H)$.

We prove that $\sigma(H)$ is prime, critical, but not finitely critical. Precisely, we show that $\mathbb{P}(\sigma(H))[\mathbb{Z}] = P_{\mathbb{Z}}$, $\mathbb{P}(\sigma(H))[\mathbb{Z}]$ is a component of $\mathbb{P}(\sigma(H))$, and ∞_n is isolated in $\mathbb{P}(\sigma(H))$ for each $n \in \mathbb{Z}$.

Set

$$R = \{\{\infty_n\} : n \in \mathbb{Z}\}.$$

Consider the partition

$$Q = \{\mathbb{Z}\} \cup R$$

of $V(\sigma(H))$ and the function $\varphi : V(\sigma(H)) \longrightarrow \mathbb{Z}$ defined by

$$\varphi_{\restriction \mathbb{Z}} = \mathrm{Id}_{\mathbb{Z}} \text{ and } \varphi(\infty_n) = 0 \text{ for every } n \in \mathbb{Z}.$$

We verify that $\sigma(H)$ satisfies assertions (I1),...,(I6) with Q and φ. As seen at the end of Example 8.9, $\sigma(H_{\mathbb{Z}}) \in \mathscr{F}_{\mathbb{Z}}$. Hence, assertion (I1) holds. Assertion (I2) holds because $\mathbb{Z} \in Q$. For assertion (I3), we obtain

$$P = \{2\mathbb{Z}, 2\mathbb{Z}+1\} \cup R.$$

It follows from the definition of H that for each $n \in \mathbb{Z}$, we have $\infty_n \longleftrightarrow_{\sigma(H)} (2\mathbb{Z})$ and $\infty_n \longleftrightarrow_{\sigma(H)} (2\mathbb{Z}+1)$. Thus, assertion (I3) holds. Obviously, assertion (I4) holds. Clearly, $(\sigma(H)/_Q P)[R] \simeq \sigma(H)[\{\infty_n : n \in \mathbb{Z}\}]$. Since $H[\{\infty_n : n \in \mathbb{Z}\}] \simeq P_{\mathbb{Z}}$, we obtain that $(\sigma(H)/_Q P)[R]$ is prime. Clearly,

$$2\mathbb{Z}+1 \in \langle R \rangle_{(\sigma(H)/_Q P)} \text{ (see Notation 3.12).}$$

Furthermore, the function

$$\begin{array}{ccc} \mathbb{Z} \cup \{\infty\} & \longrightarrow & \{0\} \cup \{\infty_n : n \in \mathbb{Z}\} \\ \infty & \longmapsto & 0, \\ n \in \mathbb{Z} & \longmapsto & \infty_n, \end{array}$$

is an isomorphism from the graph G defined in Remark 8.34 onto $H[\{0\} \cup \{\infty_n : n \in \mathbb{Z}\}]$. Since P is a modular partition of $\sigma(H)$ according to Q, we have $(\sigma(H)/_Q P)[R \cup \{2\mathbb{Z}\}] \simeq \sigma(H)[\{0\} \cup \{\infty_n : n \in \mathbb{Z}\}]$. As seen in Remark 8.34, $\sigma(G)$ is prime. It follows that $(\sigma(H)/_Q P)[R \cup \{2\mathbb{Z}\}]$ is prime, so

$$2\mathbb{Z} \in \mathrm{Ext}_{(\sigma(H)/_Q P)}(R).$$

It follows from the definition of the generalized quotient (see Definition 8.16) that

$$[2\mathbb{Z}+1, \{\infty_0\}]_{(\sigma(H)/_Q P)} \neq [2\mathbb{Z}+1, 2\mathbb{Z}]_{(\sigma(H)/_Q P)}.$$

Hence, $R \cup \{2\mathbb{Z}\}$ is not a module of $\sigma(H)/_Q P$. It follows from assertion (P2) of Lemma 3.17 that $\sigma(H)/_Q P$ is prime. Therefore, assertion (I5) holds. For assertion (I6), consider $n \in \mathbb{Z}$. We must show that $(\sigma(H)/_Q P) - \{\infty_n\}$ is decomposable. Suppose that $n \geq 0$. We obtain that

$$\{\infty_{n+1}, \infty_{n+2}, \ldots\} \text{ is a component of } H - (\infty_n). \tag{8.24}$$

It follows that $\{\{\infty_{n+1}\}, \{\infty_{n+2}\}, \ldots\}$ is a nontrivial module of $(\sigma(H)/_Q P) - \{\infty_n\}$. Suppose that $n \leq 0$. We obtain that

$$\{\ldots, \infty_{n-2}, \infty_{n-1}\} \text{ is a component of } H - (\infty_n). \tag{8.25}$$

It follows that $\{\ldots, \{\infty_{n-2}\}, \{\infty_{n-1}\}\}$ is a nontrivial module of $(\sigma(H)/_Q P) - \{\infty_n\}$. Thus, assertion (I6) holds. Consequently, assertions (I1),…,(I6) hold.

Since Q is infinite, (8.8) holds. It follows from Theorem 8.27 that σ is critical and nonfinitely critical. Precisely, it follows from assertion (J2) of Proposition 8.30 that $\mathbb{P}(\sigma(H))[\mathbb{Z}] = P_{\mathbb{Z}}$ and $\mathbb{P}(\sigma(H))[\mathbb{Z}]$ is a component of $\mathbb{P}(\sigma(H))$. Finally, we verify that for each $n \in \mathbb{Z}$, ∞_n is isolated in $\mathbb{P}(\sigma(H))$. It follows from (8.24) and (8.25) that $H - (\infty_n)$ admits a module M such that M and $(V(\sigma(H)) \setminus \{\infty_n\}) \setminus M$ are infinite. It follows that $(H - (\infty_n)) - v$ is decomposable for every $v \in (V(\sigma(H)) \setminus \{\infty_n\})$. Hence, ∞_n is isolated in $\mathbb{P}(\sigma(H))$.

Consequently, $\mathbb{P}(\sigma(H))$ admits a unique nontrivial component and infinitely many trivial components.

8.5. **Epilogue on assertion (I7).** In the next four facts, we complete the study begun in assertion (I7) of Theorem 8.26, and in Theorem 8.27 when (8.8) does not hold. Precisely, we are interested in the infinite, critical, and nonfinitely critical 2-structures the primality graph of which admits one nontrivial component and one or two trivial ones.

Fact 8.36. *Given a 2-structure σ defined on $V(\sigma) = \mathbb{Z} \cup \{\infty\}$, σ is critical, $\mathbb{P}(\sigma)[\mathbb{Z}] = P_{\mathbb{Z}}$, and ∞ is isolated in $\mathbb{P}(\sigma)$ if and only if the following assertions hold*

- $\sigma - \infty \in \mathscr{F}_{\mathbb{Z}}$;
- $\infty \longleftrightarrow_\sigma (2\mathbb{Z})$, $\infty \longleftrightarrow_\sigma (2\mathbb{Z}+1)$, *and* $[0,\infty]_\sigma \neq [1,\infty]_\sigma$;
- *at least one of the following two cases occurs:*

$$\begin{cases} [0,1]_\sigma = [0,2]_\sigma,\ [1,2]_\sigma = [1,3]_\sigma, \\ \textit{and} \\ [0,1]_\sigma \neq [0,\infty]_\sigma \ \textit{or}\ [1,2]_\sigma \neq [1,\infty]_\sigma, \end{cases} \tag{8.26}$$

or

$$\begin{cases} [1,2]_\sigma = [0,2]_\sigma,\ [0,1]_\sigma = [1,3]_\sigma, \\ \textit{and} \\ [0,2]_\sigma \neq [\infty,2]_\sigma \ \textit{or}\ [0,1]_\sigma \neq [\infty,1]_\sigma. \end{cases} \tag{8.27}$$

Proof. To begin, suppose that σ is critical, $\mathbb{P}(\sigma)[\mathbb{Z}] = P_{\mathbb{Z}}$, and ∞ is isolated in $\mathbb{P}(\sigma)$. First, we verify that $\sigma - \infty \in \mathscr{F}_{\mathbb{Z}}$. Let $n \in \mathbb{Z}$. Since $\mathbb{P}(\sigma)[\mathbb{Z}] = P_{\mathbb{Z}}$

and ∞ is isolated in $\mathbb{P}(\sigma)$, we have

$$N_{\mathbb{P}(\sigma)}(n) = \{n-1, n+1\}.$$

Since σ is critical, it follows from Lemma 4.4 that $\{n-1, n+1\}$ is a module of $\sigma - n$. By assertion (M2) of Proposition 2.5, $\{n-1, n+1\}$ is a module of $(\sigma - \infty) - n$. Since σ is prime, $\{n-1, n+1\}$ is not a module of σ. Since $\{n-1, n+1\}$ is a module of $\sigma - n$, we obtain $n \not\longleftrightarrow_\sigma \{n-1, n+1\}$. It follows that $\{n-1, n+1\}$ is not a module of $\sigma - \infty$. Consequently, $\sigma - \infty \in \mathscr{F}_{\mathbb{Z}}$.

Second, we show that $\infty \longleftrightarrow_\sigma (2\mathbb{Z})$, $\infty \longleftrightarrow_\sigma (2\mathbb{Z}+1)$, and $[0,\infty]_\sigma \neq [1,\infty]_\sigma$. Let $n \in \mathbb{Z}$. As seen above, $\{2n, 2n+2\}$ is a module of $\sigma - (2n+1)$. Hence, $\infty \longleftrightarrow_\sigma \{2n, 2n+2\}$. It follows that $\infty \longleftrightarrow_\sigma (2\mathbb{Z})$. Similarly, we have $\infty \longleftrightarrow_\sigma (2\mathbb{Z}+1)$. Since σ is prime, $\mathbb{Z}$ is not a module of σ. Since $\infty \longleftrightarrow_\sigma (2\mathbb{Z})$ and $\infty \longleftrightarrow_\sigma (2\mathbb{Z}+1)$, we obtain $[0,\infty]_\sigma \neq [1,\infty]_\sigma$.

Third, we prove that (8.26) or (8.27) hold. Since σ is critical, $\sigma - \infty$ is decomposable. Let M be a nontrivial module of $\sigma - \infty$. By the third assertion of Lemma 8.8, M is a nontrivial module of $L_{\mathbb{Z}}$. Thus, M admits a least or a greatest element. In the first instance, there exists $n \in \mathbb{Z}$ such that $n, n+1 \in M$ and $M \subseteq \{n, n+1, \ldots\}$. We obtain $[0,1]_\sigma = [0,2]_\sigma$ and $[1,2]_\sigma = [1,3]_\sigma$ as in the first assertion of Lemma 8.8. Furthermore, since $M \cup \{\infty\}$ is not a module of σ, we obtain $[0,1]_\sigma \neq [0,\infty]_\sigma$ or $[1,2]_\sigma \neq [1,\infty]_\sigma$. Therefore, (8.26) holds. Similarly, when M admits a greatest element, (8.27) holds.

Conversely, suppose that the three assertions above hold. To begin, we verify that assertions (I1),…,(I5) hold. Set

$$Q = \{\mathbb{Z}, \{\infty\}\}$$

and consider the function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ defined by

$$\varphi_{\restriction \mathbb{Z}} = \mathrm{Id}_{\mathbb{Z}} \text{ and } \varphi(\infty) = 0.$$

We obtain

$$P = \{2\mathbb{Z}, 2\mathbb{Z}+1, \{\infty\}\}.$$

Since $\sigma - \infty \in \mathscr{F}_{\mathbb{Z}}$, σ satisfies assertion (I1) with Q and φ. Since $\mathbb{Z} \in Q$, assertion (I2) holds. It follows that σ is locally critical. Furthermore, since $\infty \longleftrightarrow_\sigma (2\mathbb{Z})$ and $\infty \longleftrightarrow_\sigma (2\mathbb{Z}+1)$, P is a modular partition of σ according to Q. Hence, assertion (I3) holds. Since $[0,\infty]_\sigma \neq [1,\infty]_\sigma$, we have

$$[2\mathbb{Z}, \{\infty\}]_{(\sigma/_Q P)} \neq [2\mathbb{Z}+1, \{\infty\}]_{(\sigma/_Q P)}.$$

It follows from the definition of the generalized quotient (see Definition 8.16) that $\sigma/_Q P$ is prime. Thus, assertion (I5) holds. Obviously, assertion (I4) holds.

For a contradiction, suppose that σ admits a nontrivial module M. We utilize Fact 8.21 with $Q' = Q$ as follows. Since $[0,\infty]_\sigma \neq [1,\infty]_\sigma$, $\{2\mathbb{Z}, 2\mathbb{Z}+1\}$ is not a module of $\sigma/_Q P$. It follows from Fact 8.21 that $M \cap (2\mathbb{Z}) \neq \varnothing$, $M \cap (2\mathbb{Z}+1) \neq \varnothing$, and $\infty \in M$. Thus, $M \setminus \{\infty\}$ is a nontrivial module of $\sigma - \infty$. For instance, assume that $M \setminus \{\infty\}$ admits a least element n. Hence, $n, n+1 \in M \setminus \{\infty\}$ and $M \setminus \{\infty\} \subseteq \{n, n+1, \ldots\}$. Since $\sigma - \infty \in \mathscr{F}_{\mathbb{Z}}$, we obtain that $\{n, n+1, \ldots\}$ is a module of $\sigma - \infty$. By the first assertion of Lemma 8.8, we have $[0,1]_\sigma = [0,2]_\sigma$ and $[1,2]_\sigma = [1,3]_\sigma$. Since M is a module of σ, we obtain $[0,1]_\sigma = [0,\infty]_\sigma$ and $[1,2]_\sigma = [1,\infty]_\sigma$. Hence,

$$(8.26) \text{ does not hold.}$$

Since $[0,\infty]_\sigma \neq [1,\infty]_\sigma$, we have $[0,1]_\sigma \neq [1,2]_\sigma$. Since $[0,1]_\sigma = [0,2]_\sigma$, we obtain $[0,2]_\sigma \neq [1,2]_\sigma$. Thus,

$$(8.27) \text{ does not hold.}$$

It follows that σ is prime.

Since assertions (I3) and (I4) hold, it follows from Lemma 8.29 that $\sigma - n$ is decomposable for each $n \in \mathbb{Z}$. Since (8.26) or (8.27) hold, $\{0,1,\ldots\}$ is a module of $\sigma - \infty$ or $\{\ldots,0,1\}$ is a module of $\sigma - \infty$. Hence, $\sigma - \infty$ is decomposable. Consequently, σ is critical.

Finally, since assertions (I3), (I4), and (I5) hold, it follows from assertion (J2) of Proposition 8.30 that $\mathbb{P}(\sigma)[\mathbb{Z}]$ is a component of $\mathbb{P}(\sigma)$ and $\mathbb{P}(\sigma)[\mathbb{Z}] = P_{\mathbb{Z}}$. Since $\mathbb{P}(\sigma)[\mathbb{Z}]$ is a component of $\mathbb{P}(\sigma)$, ∞ is isolated in $\mathbb{P}(\sigma)$. □

Example 8.37. We consider the tournament $T_{\mathbb{Z}}$ defined on $V(T_{\mathbb{Z}}) = \mathbb{Z} \cup \{\infty\}$ and satisfying

- $T_{\mathbb{Z}} - \infty = L_{\mathbb{Z}}$;
- for each $n \in \mathbb{Z}$, $(\infty, 2n) \in A(T_{\mathbb{Z}})$ and $(2n+1, \infty) \in A(T_{\mathbb{Z}})$.

It follows from Fact 8.36 that $\sigma(T_{\mathbb{Z}})$ is critical, $\mathbb{P}(\sigma(T_{\mathbb{Z}}))[\mathbb{Z}] = P_{\mathbb{Z}}$, and ∞ is isolated in $\mathbb{P}(\sigma(T_{\mathbb{Z}}))$. Observe that for $n \in \mathbb{Z}$ and $p \geq 1$, $T_{\mathbb{Z}}[\{2n, \ldots, 2n + 2p - 1\} \cup \{\infty\}]$ is isomorphic to T_{2p+1} (see Figure 1.2).

Fact 8.38. *Given a 2-structure σ defined on $V(\sigma) = \mathbb{N} \cup \{\infty\}$, σ is critical, $\mathbb{P}(\sigma)[\mathbb{N}] = P_{\mathbb{N}}$, and ∞ is isolated in $\mathbb{P}(\sigma)$ if and only if the following assertions hold*

- $\sigma - \infty \in \mathscr{F}_{\mathbb{N}}$;
- $[1,2]_\sigma = [1,\infty]_\sigma$;
- $\infty \longleftrightarrow_\sigma (2\mathbb{N})$, $\infty \longleftrightarrow_\sigma (2\mathbb{N}+1)$, *and* $[0,\infty]_\sigma \neq [1,\infty]_\sigma$;
- *at least one of the following two cases occurs:*

$$[0,1]_\sigma = [0,2]_\sigma \text{ and } [0,1]_\sigma \neq [0,\infty]_\sigma, \tag{8.28}$$

or

$$\begin{cases} [1,2]_\sigma = [0,2]_\sigma, \ [0,1]_\sigma = [1,3]_\sigma, \\ \text{and} \\ [0,2]_\sigma \neq [\infty,2]_\sigma \text{ or } [0,1]_\sigma \neq [\infty,1]_\sigma. \end{cases} \tag{8.29}$$

Although the proof of Fact 8.38 is close to that of Fact 8.36, we provide it because some differences deserve to be pointed out.

Proof of Fact 8.38. To begin, suppose that σ is critical, $\mathbb{P}(\sigma)[\mathbb{N}] = P_{\mathbb{N}}$, and ∞ is isolated in $\mathbb{P}(\sigma)$. First, we verify that $\sigma - \infty \in \mathscr{F}_{\mathbb{N}}$. Let $n \geq 1$. As seen in the proof of Fact 8.36, $\{n-1, n+1\}$ is a module of $(\sigma - \infty) - n$, but not of $\sigma - \infty$. Now, we have to show that $\mathbb{N} \setminus \{0,1\}$ is a module of $(\sigma - \infty) - 0$, but $\mathbb{N} \smallsetminus \{1\}$ is not a module of $\sigma - \infty$. Since $\mathbb{P}(\sigma)[\mathbb{N}] = P_{\mathbb{N}}$ and ∞ is isolated in $\mathbb{P}(\sigma)$, we have

$$N_{\mathbb{P}(\sigma)}(0) = \{1\}.$$

Since σ is critical, it follows from Lemma 4.4 that $V(\sigma) \smallsetminus \{0,1\}$ is a module of $\sigma - 0$. In particular, we have

$$[1,2]_\sigma = [1,\infty]_\sigma.$$

Moreover, by assertion (M2) of Proposition 2.5, $\mathbb{N} \smallsetminus \{0,1\}$ is a module of $(\sigma-\infty)-0$. Since σ is prime, $V(\sigma)\smallsetminus\{1\}$ is not a module of σ. Hence, $[1,2]_\sigma \neq [1,0]_\sigma$. Therefore, $\{0\} \cup \{2,3,\dots\}$ is not a module of $\sigma - \infty$. Consequently, $\sigma - \infty \in \mathscr{F}_{\mathbb{N}}$.

Second, we show that $\infty \longleftrightarrow_\sigma (2\mathbb{N})$, $\infty \longleftrightarrow_\sigma (2\mathbb{N}+1)$, and $[0,\infty]_\sigma \neq [1,\infty]_\sigma$. Let $n \geq 0$. As seen above, $\{2n, 2n+2\}$ is a module of $\sigma - (2n+1)$. Hence, $\infty \longleftrightarrow_\sigma \{2n, 2n+2\}$. It follows that $\infty \longleftrightarrow_\sigma (2\mathbb{N})$. Similarly, we have $\infty \longleftrightarrow_\sigma (2\mathbb{N}+1)$. Since σ is prime, $\mathbb{N}$ is not a module of σ. Since $\infty \longleftrightarrow_\sigma (2\mathbb{N})$ and $\infty \longleftrightarrow_\sigma (2\mathbb{N}+1)$, we obtain $[0,\infty]_\sigma \neq [1,\infty]_\sigma$.

Third, we prove that (8.28) or (8.29) hold. Since σ is critical, $\sigma - \infty$ admits a nontrivial module M. By the second assertion of Lemma 8.10, M is a module of $L_{\mathbb{N}}$. Since $|M| \geq 2$, M contains even and odd integers. We distinguish the following two cases.

CASE 1: $0 \notin M$.

Since M contains even and odd integers, we obtain

$$[0,1]_\sigma = [0,2]_\sigma.$$

Since σ is prime, $V(\sigma) \smallsetminus \{0\}$ is not a module of σ. Thus,

$$[0,1]_\sigma \neq [0,\infty]_\sigma.$$

It follows that (8.28) holds.

CASE 2: $0 \in M$.

Since M is a nontrivial module of $L_{\mathbb{N}}$, there exists $n \geq 1$ such that

$$M = \{0,\dots,n\}.$$

We obtain that (8.29) holds.

Conversely, suppose that the four assertions above hold. To begin, we verify that assertions (I1),...,(I5) hold. Set

$$Q = \{\mathbb{N}, \{\infty\}\}$$

and consider the function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ defined by

$$\varphi_{\restriction \mathbb{N}} = \mathrm{Id}_{\mathbb{N}} \text{ and } \varphi(\infty) = 0.$$

We obtain

$$P = \{2\mathbb{N}, 2\mathbb{N}+1, \{\infty\}\}.$$

Since $\sigma - \infty \in \mathscr{F}_{\mathbb{N}}$, σ satisfies assertion (I1) with Q and φ. Since $\mathbb{N} \in Q$, assertion (I2) holds. It follows that σ is locally critical. Furthermore, since $\infty \longleftrightarrow_\sigma (2\mathbb{N})$ and $\infty \longleftrightarrow_\sigma (2\mathbb{N}+1)$, P is a modular partition of σ according to

Q. Hence, assertion (I3) holds. Moreover, since $[1,2]_\sigma = [1,\infty]_\sigma$, assertion (I4) holds. Lastly, since $[0,\infty]_\sigma \neq [1,\infty]_\sigma$, we have

$$[2\mathbb{N},\{\infty\}]_{(\sigma/_Q P)} \neq [2\mathbb{N}+1,\{\infty\}]_{(\sigma/_Q P)}.$$

It follows from the definition of the generalized quotient (see Definition 8.16) that $\sigma/_Q P$ is prime. Thus, assertion (I5) holds.

For a contradiction, suppose that σ admits a nontrivial module M. We utilize Fact 8.21 with $Q' = Q$ as follows. Since $[0,\infty]_\sigma \neq [1,\infty]_\sigma$, $\{2\mathbb{N},2\mathbb{N}+1\}$ is not a module of $\sigma/_Q P$. It follows from Fact 8.21 that $M \cap (2\mathbb{Z}) \neq \varnothing$, $M \cap (2\mathbb{Z}+1) \neq \varnothing$, and $\infty \in M$. Thus, $M \smallsetminus \{\infty\}$ is a nontrivial module of $\sigma - \infty$. Since $\sigma - \infty \in \mathscr{F}_{\mathbb{N}}$, $M \smallsetminus \{\infty\}$ is a nontrivial module of $L_{\mathbb{N}}$ by the second assertion of Lemma 8.10. In particular, $M \smallsetminus \{\infty\}$ contains even and odd integers. We distinguish the following two cases. In each of them, we obtain a contradiction.

CASE 1: $0 \notin M$.

Since M contains even and odd integers, we obtain

$$[0,1]_\sigma = [0,2]_\sigma.$$

Furthermore, since $\infty \in M$, we obtain

$$[0,1]_\sigma = [0,\infty]_\sigma,$$

which contradicts the fact that (8.28) holds.

CASE 2: $0 \in M$.

Since M is a nontrivial module of $L_{\mathbb{N}}$, there exists $n \geq 1$ such that

$$M = \{0,\dots,n\}.$$

We obtain

$$[0,2]_\sigma = [1,2]_\sigma = [\infty,2]_\sigma$$

and

$$[0,1]_\sigma = [1,3]_\sigma = [\infty,1]_\sigma,$$

which contradicts the fact that that (8.29) holds.

Consequently, σ is prime.

Since assertions (I3) and (I4) hold, it follows from Lemma 8.29 that $\sigma - n$ is decomposable for every $n \in \mathbb{N}$. Moreover, since (8.28) or (8.29) hold, $\{1,2,\dots\}$ is a module of $\sigma - \infty$ or $\{0,1\}$ is a module of $\sigma - \infty$. Hence, $\sigma - \infty$ is decomposable. Consequently, σ is critical.

Finally, since assertions (I3), (I4), and (I5) hold, it follows from assertion (J2) of Proposition 8.30 that $\mathbb{P}(\sigma)[\mathbb{N}]$ is a component of $\mathbb{P}(\sigma)$ and $\mathbb{P}(\sigma)[\mathbb{N}] = P_{\mathbb{N}}$. Since $\mathbb{P}(\sigma)[\mathbb{N}]$ is a component of $\mathbb{P}(\sigma)$, ∞ is isolated in $\mathbb{P}(\sigma)$. □

Example 8.39. We consider the tournament $T_{\mathbb{N}} = T_{\mathbb{Z}}[\mathbb{N}]$. It follows from Fact 8.38 that $\sigma(T_{\mathbb{N}})$ is critical, $\mathbb{P}(\sigma(T_{\mathbb{N}}))[\mathbb{N}] = P_{\mathbb{N}}$, and ∞ is isolated in $\mathbb{P}(\sigma(T_{\mathbb{N}}))$.

Fact 8.40. *Given a 2-structure σ defined on $V(\sigma) = \mathbb{Z} \cup \{\infty,\infty'\}$, σ is critical, $\mathbb{P}(\sigma)[\mathbb{Z}] = P_{\mathbb{Z}}$, and ∞ and ∞' are isolated in $\mathbb{P}(\sigma)$ if and only if, by exchanging ∞ and ∞' if necessary, we have*

- *$\sigma - \{\infty, \infty'\} \in \mathscr{F}_{\mathbb{Z}}$;*
- *$\infty \longleftrightarrow_\sigma (2\mathbb{Z})$, $\infty \longleftrightarrow_\sigma (2\mathbb{Z}+1)$, and $[0,\infty]_\sigma \neq [1,\infty]_\sigma$;*
- *$\infty' \longleftrightarrow_\sigma \mathbb{Z}$ and $[0,\infty']_\sigma \neq [\infty,\infty']_\sigma$;*
- *at least one of the following two cases occurs:*

$$\begin{cases} [0,1]_\sigma = [0,2]_\sigma,\ [1,2]_\sigma = [1,3]_\sigma, \\ \textit{and} \\ [0,1]_\sigma = [0,\infty]_\sigma,\ [1,2]_\sigma = [1,\infty]_\sigma, \end{cases} \tag{8.30}$$

or

$$\begin{cases} [1,2]_\sigma = [0,2]_\sigma,\ [0,1]_\sigma = [1,3]_\sigma, \\ \textit{and} \\ [0,2]_\sigma = [\infty,2]_\sigma,\ [0,1]_\sigma = [\infty,1]_\sigma. \end{cases} \tag{8.31}$$

Proof. To begin, suppose that σ is critical, $\mathbb{P}(\sigma)[\mathbb{Z}] = P_{\mathbb{Z}}$, and ∞ and ∞' are isolated in $\mathbb{P}(\sigma)$. By Proposition 8.24,

$$|\{v \in \{\infty,\infty'\} : v \not\longleftrightarrow_\sigma \mathbb{Z}\}| = 1.$$

By exchanging ∞ and ∞' if necessary, we can assume that $\infty \not\longleftrightarrow_\sigma \mathbb{Z}$ and $\infty' \longleftrightarrow_\sigma \mathbb{Z}$. Since σ is prime, $\mathbb{Z} \cup \{\infty\}$ is not a module of σ. Thus, we have $[0,\infty']_\sigma \neq [\infty,\infty']_\sigma$.

First, we verify that $\sigma - \{\infty,\infty'\} \in \mathscr{F}_{\mathbb{Z}}$. Let $n \in \mathbb{Z}$. Since $\mathbb{P}(\sigma)[\mathbb{Z}] = P_{\mathbb{Z}}$ and ∞ and ∞' are isolated in $\mathbb{P}(\sigma)$, we have

$$N_{\mathbb{P}(\sigma)}(n) = \{n-1, n+1\}.$$

Since σ is critical, it follows from Lemma 4.4 that $\{n-1,n+1\}$ is a module of $\sigma - n$. By assertion (M2) of Proposition 2.5, $\{n-1,n+1\}$ is a module of $(\sigma - \{\infty,\infty'\}) - n$. Since σ is prime, $\{n-1,n+1\}$ is not a module of σ. Hence, $n \not\longleftrightarrow_\sigma \{n-1,n+1\}$. It follows that $\{n-1,n+1\}$ is not a module of $\sigma - \{\infty,\infty'\}$. Consequently, $\sigma - \{\infty,\infty'\} \in \mathscr{F}_{\mathbb{Z}}$.

Second, we verify that $\infty \longleftrightarrow_\sigma (2\mathbb{Z})$ and $\infty \longleftrightarrow_\sigma (2\mathbb{Z}+1)$. Let $n \in \mathbb{Z}$. As seen above, $\{2n, 2n+2\}$ is a module of $\sigma - (2n+1)$. Hence, $\infty \longleftrightarrow_\sigma \{2n, 2n+2\}$. It follows that $\infty \longleftrightarrow_\sigma (2\mathbb{Z})$. Similarly, we have $\infty \longleftrightarrow_\sigma (2\mathbb{Z}+1)$. Since $\infty \not\longleftrightarrow_\sigma \mathbb{Z}$, we obtain $[0,\infty]_\sigma \neq [1,\infty]_\sigma$.

Third, we prove that (8.30) or (8.31) hold. Since σ is critical, $\sigma - \infty'$ is decomposable. Consider a nontrivial module M of $\sigma - \infty'$. Since $[0,\infty]_\sigma \neq [1,\infty]_\sigma$, we have $\infty \in M$. By assertion (M2) of Proposition 2.5, $M \cap \mathbb{Z}$ is a module of $\sigma - \{\infty,\infty'\}$. For a contradiction, suppose that $|M \cap \mathbb{Z}| = 1$. Denote by n the unique element of $M \cap \mathbb{Z}$. Since $\infty \longleftrightarrow_\sigma (2\mathbb{Z})$ and $\infty \longleftrightarrow_\sigma (2\mathbb{Z}+1)$, we obtain $n \longleftrightarrow_\sigma \{n-1, n+1\}$, which contradicts $\sigma - \{\infty,\infty'\} \in \mathscr{F}_{\mathbb{Z}}$. It follows that $|M \cap \mathbb{Z}| \geq 2$. Hence, $M \cap \mathbb{Z}$ is a nontrivial module of $\sigma - \{\infty,\infty'\}$. For instance, suppose that $M \cap \mathbb{Z}$ admits a least element. Since $\infty \in M$, we obtain that (8.30) holds.

Conversely, suppose that the four assertions above hold. To begin, we verify that assertions (I1),…,(I5) hold. Set

$$Q = \{\mathbb{Z}, \{\infty\}, \{\infty'\}\}$$

and consider the function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ defined by

$$\varphi_{\restriction\mathbb{Z}} = \mathrm{Id}_{\mathbb{Z}},\ \varphi(\infty) = 0, \text{ and } \varphi(\infty') = 0.$$

We obtain

$$P = \{2\mathbb{Z}, 2\mathbb{Z}+1, \{\infty\}, \{\infty'\}\}.$$

Since $\sigma - \{\infty, \infty'\} \in \mathscr{F}_{\mathbb{Z}}$, σ satisfies assertion (I1) with Q and φ. Since $\mathbb{Z} \in Q$, assertion (I2) holds. It follows that σ is locally critical. Furthermore, since $\infty' \longleftrightarrow_\sigma \mathbb{Z}$, $\infty \longleftrightarrow_\sigma (2\mathbb{Z})$, and $\infty \longleftrightarrow_\sigma (2\mathbb{Z}+1)$, P is a modular partition of σ according to Q. Hence, assertion (I3) holds. Since $[0,\infty]_\sigma \neq [1,\infty]_\sigma$, $\{2\mathbb{Z}, 2\mathbb{Z}+1\}$ and $\{2\mathbb{Z}, 2\mathbb{Z}+1, \{\infty'\}\}$ are not modules of $\sigma/_Q P$. Moreover, since $[0,\infty']_\sigma \neq [\infty,\infty']_\sigma$, $\{2\mathbb{Z}, 2\mathbb{Z}+1, \{\infty\}\}$ is not a module of $\sigma/_Q P$. Lastly, since $[0,\infty]_\sigma \neq [1,\infty]_\sigma$ and $\infty' \longleftrightarrow_\sigma \mathbb{Z}$, $\{\{\infty\}, \{\infty'\}\}$ is not a module of $\sigma/_Q P$. It follows from the definition of the generalized quotient (see Definition 8.16) that $\sigma/_Q P$ is prime. Thus, assertion (I5) holds. Obviously, assertion (I4) holds.

To verify that σ is prime, we utilize Fact 8.22 with $Q' = Q$ as follows. Clearly, (8.7) holds. Moreover, we have $P' = P$. As previously observed, $\sigma/_Q P$ is prime. Since assertions (I3) and (I4) hold, it follows from Fact 8.22 that σ is prime.

Since assertions (I3) and (I4) hold, it follows from Lemma 8.29 that $\sigma - n$ is decomposable for each $n \in \mathbb{Z}$. Since (8.30) or (8.31) hold, $\{0, 1, \dots\} \cup \{\infty\}$ is a module of $\sigma - \infty$ or $\{\dots, 0, 1\} \cup \{\infty\}$ is a module of $\sigma - \infty'$. Hence, $\sigma - \infty'$ is decomposable. Lastly, since $\infty' \longleftrightarrow_\sigma \mathbb{Z}$, $\sigma - \infty$ is decomposable. Consequently, σ is critical.

Finally, since assertions (I3), (I4), and (I5) hold, it follows from assertion (J2) of Proposition 8.30 that $\mathbb{P}(\sigma)[\mathbb{Z}]$ is a component of $\mathbb{P}(\sigma)$ and $\mathbb{P}(\sigma)[\mathbb{Z}] = P_{\mathbb{Z}}$. Lastly, it follows from Corollary 8.3 that ∞ and ∞' are isolated in $\mathbb{P}(\sigma)$. □

Fact 8.41. *Given a 2-structure σ defined on $V(\sigma) = \mathbb{N} \cup \{\infty, \infty'\}$, σ is critical, $\mathbb{P}(\sigma)[\mathbb{N}] = P_{\mathbb{N}}$, and ∞ and ∞' are isolated in $\mathbb{P}(\sigma)$ if and only if, by exchanging ∞ and ∞' if necessary, we have*

- $\sigma - \{\infty, \infty'\} \in \mathscr{F}_{\mathbb{N}}$;
- $\infty \longleftrightarrow_\sigma (2\mathbb{N})$, $\infty \longleftrightarrow_\sigma (2\mathbb{N}+1)$, *and* $[0,\infty]_\sigma \neq [1,\infty]_\sigma$;
- $\infty' \longleftrightarrow_\sigma \mathbb{N}$ *and* $[0,\infty']_\sigma \neq [\infty,\infty']_\sigma$;
- $[1,2]_\sigma = [1,\infty]_\sigma$ *and* $[1,2]_\sigma = [1,\infty']_\sigma$;
- *at least one of the following two cases occurs:*

$$[0,1]_\sigma = [0,2]_\sigma \text{ and } [0,1]_\sigma = [0,\infty]_\sigma, \tag{8.32}$$

or

$$\begin{cases} [1,2]_\sigma = [0,2]_\sigma,\ [0,1]_\sigma = [1,3]_\sigma, \\ \textit{and} \\ [0,2]_\sigma = [\infty,2]_\sigma,\ [0,1]_\sigma = [\infty,1]_\sigma. \end{cases} \tag{8.33}$$

Proof. To begin, suppose that σ is critical, $\mathbb{P}(\sigma)[\mathbb{N}] = P_{\mathbb{N}}$, and ∞ and ∞' are isolated in $\mathbb{P}(\sigma)$. By Proposition 8.24,

$$|\{v \in \{\infty, \infty'\} : v \not\longleftrightarrow_\sigma \mathbb{N}\}| = 1.$$

By exchanging ∞ and ∞' if necessary, we can assume that $\infty \not\longleftrightarrow_\sigma \mathbb{N}$ and $\infty' \longleftrightarrow_\sigma \mathbb{N}$. Since σ is prime, $\mathbb{N} \cup \{\infty\}$ is not a module of σ. Thus, we have $[0, \infty']_\sigma \neq [\infty, \infty']_\sigma$.

First, we verify that $\sigma - \{\infty, \infty'\} \in \mathscr{F}_{\mathbb{N}}$. Let $n \geq 1$. As seen in the proof of Fact 8.40, $\{n-1, n+1\}$ is a module of $(\sigma - \{\infty, \infty'\}) - n$, but not of $\sigma - \{\infty, \infty'\}$. Since $\mathbb{P}(\sigma)[\mathbb{N}] = P_{\mathbb{N}}$ and ∞ and ∞' are isolated in $\mathbb{P}(\sigma)$, we have

$$N_{\mathbb{P}(\sigma)}(0) = \{1\}.$$

Since σ is critical, it follows from Lemma 4.4 that $V(\sigma) \setminus \{0, 1\}$ is a module of $\sigma - 0$. In particular, we obtain

$$[1, 2]_\sigma = [1, \infty]_\sigma \text{ and } [1, 2]_\sigma = [1, \infty']_\sigma. \tag{8.34}$$

Furthermore, by assertion (M2) of Proposition 2.5, $V(\sigma) \setminus \{0, 1\}$ is a module of $(\sigma - \{\infty, \infty'\}) - 0$. Since σ is prime, $V(\sigma) \setminus \{1\}$ is not a module of σ. Hence, $1 \not\longleftrightarrow_\sigma (V(\sigma) \setminus \{1\})$. It follows that $\mathbb{N} \setminus \{1\}$ is not a module of $\sigma - \{\infty, \infty'\}$. Consequently, $\sigma - \{\infty, \infty'\} \in \mathscr{F}_{\mathbb{N}}$.

Second, we verify that $\infty \longleftrightarrow_\sigma (2\mathbb{N})$ and $\infty \longleftrightarrow_\sigma (2\mathbb{N}+1)$. Let $n \in \mathbb{N}$. As seen above, $\{2n, 2n+2\}$ is a module of $\sigma - (2n+1)$. Hence $\infty \longleftrightarrow_\sigma \{2n, 2n+2\}$. It follows that $\infty \longleftrightarrow_\sigma (2\mathbb{N})$. Similarly, we have $\infty \longleftrightarrow_\sigma (2\mathbb{N}+1)$. Since $\infty \not\longleftrightarrow_\sigma \mathbb{Z}$, we obtain $[0, \infty]_\sigma \neq [1, \infty]_\sigma$.

Third, we prove that (8.32) or (8.33) hold. Since σ is critical, $\sigma - \infty'$ admits a nontrivial module M. Since $\infty \longleftrightarrow_\sigma (2\mathbb{N})$, $\infty \longleftrightarrow_\sigma (2\mathbb{N}+1)$, and $[0, \infty]_\sigma \neq [1, \infty]_\sigma$, we have $\infty \in M$. By assertion (M2) of Proposition 2.5, $M \cap \mathbb{N}$ is a module of $\sigma - \{\infty, \infty'\}$. For a contradiction, suppose that $|M \cap \mathbb{N}| = 1$. Denote by n the unique element of $M \cap \mathbb{N}$. We distinguish the following two cases. In each of them, we obtain a contradiction.

Case 1: $n \geq 1$.

Since $\infty \longleftrightarrow_\sigma (2\mathbb{N})$ and $\infty \longleftrightarrow_\sigma (2\mathbb{N}+1)$, we obtain $n \longleftrightarrow_\sigma \{n-1, n+1\}$, which contradicts $\sigma - \{\infty, \infty'\} \in \mathscr{F}_{\mathbb{N}}$.

Case 2: $n = 0$.

We have

$$[1, 0]_\sigma = [1, \infty]_\sigma.$$

By (8.34), we have $[1, 2]_\sigma = [1, \infty]_\sigma$. It follows that

$$[1, 0]_\sigma = [1, 2]_\sigma,$$

which contradicts $\sigma - \{\infty, \infty'\} \in \mathscr{F}_{\mathbb{N}}$.

It follows that $|M \cap \mathbb{N}| \geq 2$. Hence, $M \cap \mathbb{N}$ is a nontrivial module of $\sigma - \{\infty, \infty'\}$. By the second assertion of Lemma 8.10, $M \cap \mathbb{N}$ is a nontrivial module of $L_{\mathbb{N}}$. Since $|M \cap \mathbb{N}| \geq 2$, M contains even and odd integers. We distinguish the following two cases.

CASE 1: $0 \notin M \cap \mathbb{N}$.

Since $M \cap \mathbb{N}$ contains even and odd integers, we obtain

$$[0,1]_\sigma = [0,2]_\sigma.$$

Moreover, since $\infty \in M$, we obtain

$$[0,1]_\sigma = [0,\infty]_\sigma.$$

It follows that (8.32) holds.

CASE 2: $0 \in M$.

Since $M \cap \mathbb{N}$ is a nontrivial module of $L_{\mathbb{N}}$, there exists $n \geq 1$ such that

$$M \cap \mathbb{N} = \{0,\dots,n\}.$$

Since $\infty \in M$, we obtain

$$M = \{0,\dots,n\} \cup \{\infty\}.$$

We obtain that (8.33) holds.

Conversely, suppose that the five assertions above hold. To begin, we verify that assertions (I1),...,(I5) hold. Set

$$Q = \{\mathbb{N}, \{\infty\}, \{\infty'\}\}$$

and consider the function $\varphi : V(\sigma) \longrightarrow \mathbb{Z}$ defined by

$$\varphi_{\restriction \mathbb{N}} = \mathrm{Id}_{\mathbb{N}},\ \varphi(\infty) = 0 \text{ and } \varphi(\infty') = 0.$$

We obtain

$$P = \{2\mathbb{N}, 2\mathbb{N}+1, \{\infty\}, \{\infty'\}\}.$$

Since $\sigma - \{\infty, \infty'\} \in \mathscr{F}_{\mathbb{Z}}$, σ satisfies assertion (I1) with Q and φ. Since $\mathbb{N} \in Q$, assertion (I2) holds. It follows that σ is locally critical. Furthermore, since $\infty \longleftrightarrow_\sigma (2\mathbb{N})$, $\infty \longleftrightarrow_\sigma (2\mathbb{N}+1)$, and $\infty' \longleftrightarrow_\sigma \mathbb{N}$, P is a modular partition of σ according to Q. Hence, assertion (I3) holds. Since $[0,\infty]_\sigma \neq [1,\infty]_\sigma$, $\{2\mathbb{N}, 2\mathbb{N}+1\}$ and $\{2\mathbb{N}, 2\mathbb{N}+1, \{\infty'\}\}$ are not modules of $\sigma/_Q P$. Moreover, since $[0,\infty']_\sigma \neq [\infty,\infty']_\sigma$, $\{2\mathbb{N}, 2\mathbb{N}+1, \{\infty\}\}$ is not a module of $\sigma/_Q P$. Lastly, since $[0,\infty]_\sigma \neq [1,\infty]_\sigma$ and $\infty' \longleftrightarrow_\sigma \mathbb{N}$, $\{\{\infty\},\{\infty'\}\}$ is not a module of $\sigma/_Q P$. It follows from the definition of the generalized quotient (see Definition 8.16) that $\sigma/_Q P$ is prime. Thus, assertion (I5) holds. To verify that assertion (I4) holds, recall that $[1,2]_\sigma = [1,\infty]_\sigma$ and $[1,2]_\sigma = [1,\infty']_\sigma$. We obtain

$$1 \longleftrightarrow_\sigma \{2,\infty,\infty'\}.$$

It follows that assertions (I4) holds.

To verify that σ is prime, we utilize Fact 8.22 with $Q' = Q$ as follows. Clearly, (8.7) holds. Moreover, we have $P' = P$. As previously observed, $\sigma/_Q P$ is prime. Since assertions (I3) and (I4) hold, it follows from Fact 8.22 that σ is prime.

Since assertions (I3) and (I4) hold, it follows from Lemma 8.29 that $\sigma - n$ is decomposable for each $n \in \mathbb{N}$. Since (8.32) or (8.33) hold, $\{1,2,\dots\} \cup \{\infty\}$ is a module of $\sigma - \infty$ or $\{0,1\} \cup \{\infty\}$ is a module of $\sigma - \infty$. Hence, $\sigma - \infty'$ is decomposable. Lastly, since $\infty' \longleftrightarrow_\sigma \mathbb{N}$, $\sigma - \infty$ is decomposable. Consequently, σ is critical.

Finally, since assertions (I3), (I4), and (I5) hold, it follows from assertion (J2) of Proposition 8.30 that $\mathbb{P}(\sigma)[\mathbb{N}]$ is a component of $\mathbb{P}(\sigma)$ and $\mathbb{P}(\sigma)[\mathbb{N}] = P_{\mathbb{N}}$. Lastly, it follows from Corollary 8.3 that ∞ and ∞' are isolated in $\mathbb{P}(\sigma)$. $\square$

9. Partially critical 2-structures

We consider the following weakening of the notion of a critical structure (see Definition 4.1).

Definition 9.1. Let σ be a prime 2-structure. Given $W \subseteq V(\sigma)$, σ is W-*critical* if all the elements of W are critical vertices of σ.

W-critical

A prime 2-structure σ is *partially critical* if there exists a proper subset X of $V(\sigma)$ such that $\sigma[X]$ is prime and σ is $(V(\sigma) \setminus X)$-critical.

partially critical

Finite partially critical graphs were characterized by Breiner et al. [5]. Furthermore, finite partially critical tournaments were characterized by Sayar [32] who adapted the examination of partial criticality presented in [5] to tournaments. A nice presentation of finite and partially critical tournaments is provided in [2] (see [2, Theorem 2 and Corollary 1]). Lastly, Belkhechine et al. [3] characterized the finite or infinite partially critical 2-structures. In the finite case, they followed the same approach as that of [5].

Theorem 3.19 leads us to introduce the outside graph as follows. The outside graph is the main tool to characterize the partially critical 2-structures. It is frequently used in the study of prime digraphs [22, 25]. We need the next notation.

Notation 9.2. Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. The set of the nonempty subsets Y of $V(\sigma) \setminus X$, such that $\sigma[X \cup Y]$ is prime, is denoted by $\mathscr{P}_{(\sigma,X)}$ (compare with Notation 3.1). Hence, we have

$$\mathrm{Ext}_{\sigma}(X) = \{v \in V(\sigma) \setminus X : \{v\} \in \mathscr{P}_{(\sigma,X)}\} \text{ (see Notation 3.12).}$$

Furthermore, suppose that $|V(\sigma) \setminus X| \geq 2$. By Theorem 3.19, $\mathscr{P}_{(\sigma,X)}$ contains an unordered pair.

Definition 9.3. Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. The *outside graph* $\Gamma_{(\sigma,X)}$ is defined on $V(\Gamma_{(\sigma,X)}) = V(\sigma) \setminus X$ by

outside graph

$$E(\Gamma_{(\sigma,X)}) = \{Y \in \mathscr{P}_{(\sigma,X)} : |Y| = 2\}.$$

By Theorem 3.19, the outside graph $\Gamma_{(\sigma,X)}$ is nonempty when σ is prime and $|V(\sigma) \setminus X| \geq 2$.

Remark 9.4. Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Given $k \in \{1, \ldots, |V(\sigma) \setminus X| - 1\}$, we consider the following statement

$$\text{(S}k\text{)} \qquad \{Y \in \mathscr{P}_{(\sigma,X)} : |Y| = k\} = \varnothing.$$

Clearly, $\mathrm{Ext}_{\sigma}(X) = \varnothing$ means that statement (S1) holds.

First, we make the following observation. Consider $k \in \{1, \ldots, |V(\sigma) \setminus X| - 1\}$ and $m \in \{1, \ldots, k-2\}$ such that $k - m \equiv 0 \bmod 2$. If statement (Sk) holds, then it follows from Corollary 3.20 that statement (Sm) holds.

Second, suppose that σ is $(V(\sigma) \setminus X)$-critical and

$$V(\sigma) \setminus X \text{ is finite.}$$

We verify that statement (Sk) holds for each $k \in \{1, \dots, |V(\sigma) \setminus X| - 1\}$ such that k is odd.

To begin, we verify that

$$|V(\sigma) \setminus X| \text{ is even.} \tag{9.1}$$

Otherwise, it follows from Corollary 3.20 that σ admits a noncritical vertex v such that $v \in V(\sigma) \setminus X$, which contradicts the fact that σ is $(V(\sigma) \setminus X)$-critical.

Now, consider $Y \in \mathscr{P}_{(\sigma,X)}$ such that $Y \neq V(\sigma) \setminus X$. Since σ is $(V(\sigma) \setminus X)$-critical, σ is $(V(\sigma) \setminus (X \cup Y))$-critical as well. It follows from (9.1) that $|V(\sigma) \setminus (X \cup Y)|$ is even. Since $|V(\sigma) \setminus X|$ is even, $|Y|$ is even too. Consequently, statement (Sk) holds for each $k \in \{1, \dots, |V(\sigma) \setminus X| - 1\}$ such that k is odd.

9.1. **Main results.** We begin with a hereditary property of primality through the components of the outside graph.

Theorem 9.5 (Belkhechine et al. [3]). *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S3) holds. The following three assertions are equivalent*

1. *σ is prime;*
2. *for each component C of $\Gamma_{(\sigma,X)}$, $\sigma[X \cup V(C)]$ is prime;*
3. *for each component C of $\Gamma_{(\sigma,X)}$, $v(C) = 2$ or $v(C) \geq 4$ and C is prime.*

Theorem 9.5 allows us to provide a simple and short proof of Theorem 5.8 (see subsection 9.6). Furthermore, Theorem 9.5 is proved for finite graphs in [25] (see [25, Theorem 17] and [25, Corollary 18]). We pursue with a hereditary property of partial criticality through the components of the outside graph. The next theorem also provides a characterization of partially critical 2-structures in terms of criticality of the components of their outside graph.

Theorem 9.6 (Belkhechine et al. [3]). *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S5) holds. The following three assertions are equivalent*

1. *σ is $(V(\sigma) \setminus X)$-critical;*
2. *for each component C of $\Gamma_{(\sigma,X)}$, $\sigma[X \cup V(C)]$ is $V(C)$-critical;*
3. *for each component C of $\Gamma_{(\sigma,X)}$, $v(C) = 2$ or $v(C) \geq 4$ and C is critical.*

Remark 9.7. As seen at the beginning of section 8, $\sigma(P_{\mathbb{Z}})$ is finitely critical. Set

$$X = \{z \in \mathbb{Z} : z \leq 0\}.$$

As for $\sigma(P_{\mathbb{Z}})$, it follows from Theorem 7.1 that $\sigma(P_{\mathbb{Z}})[X]$ is prime. Similarly, $\sigma(P_{\mathbb{Z}})[X \cup \{1, \dots, k\}]$ is prime for every $k \geq 1$. Consequently, for each $k \geq 1$, statement (Sk) does not hold. Moreover, $\{1, 2\}$ is the only edge of $\Gamma_{(\sigma(P_{\mathbb{Z}}),X)}$.

Hence, for every $z \geq 3$, z is an isolated vertex of $\Gamma_{(\sigma(P_{\mathbb{Z}}),X)}$. It follows that Theorem 9.5 does not hold when statement (S3) is not satisfied. Similarly, Theorem 9.6 does not hold when statement (S5) is not satisfied.

We introduce a weakening of the partial criticality in the following way. We obtain the next result by using Theorem 7.4 several times.

Corollary 9.8. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. The following two assertions are equivalent*

(1) *σ is prime;*
(2) *for each finite subset F of $V(\sigma) \setminus X$, there exists $F' \in \mathscr{P}_{(\sigma,X)}$ such that F' is finite and $F \subseteq F'$.*

Corollary 9.8 and the fact that statement (S5) is supposed to be satisfied in Theorem 9.6 lead us to introduce the next definition. The next definition is a weakening of partial criticality (see Theorem 9.10).

Definition 9.9. Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. We say that σ is *finitely $(V(\sigma) \setminus X)$-critical* if for each finite subset F of $V(\sigma) \setminus X$, there exists $F' \in \mathscr{P}_{(\sigma,X)}$ such that F' is finite, $F \subseteq F'$, and $\sigma[X \cup F']$ is (F')-critical.

Theorem 9.10. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. The following two assertions are equivalent*

(1) *Statement (S5) holds and σ is prime;*
(2) *σ is finitely $(V(\sigma) \setminus X)$-critical.*

Theorem 9.10 is discussed in Remark 9.57. Precisely, in Remark 9.57, we provide a prime 2-structure showing that we do not have a compactness theorem with partial criticality. We prove Theorem 9.10 at the end of subsection 9.5. The last main result ends this subsection. It shows that Theorem 5.8 is satisfied in the infinite case when the 2-structure σ is also supposed to be $(V(\sigma) \setminus X)$-critical.

Theorem 9.11. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S5) holds. Suppose also that σ is $(V(\sigma) \setminus X)$-critical. For each $v \in V(\sigma) \setminus X$, there exists $w \in (V(\sigma) \setminus X) \setminus \{v\}$ such that $\sigma - \{v, w\}$ is $((V(\sigma) \setminus \{v, w\}) \setminus X)$-critical. In particular, we obtain*

for each $v \in V(\sigma) \setminus X$, $N_{\mathbb{P}(\sigma)}(v) \cap (V(\sigma) \setminus X) \neq \varnothing$ (see Definition 4.3).

We prove Theorem 9.11 at the end of subsection 9.5.

Remark 9.12. Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S5) holds. Suppose also that σ is $(V(\sigma) \setminus X)$-critical. Lastly, suppose that $V(\sigma) \setminus X$ is infinite. Consider a finite and nonempty subset F of $V(\sigma) \setminus X$. By applying several times Theorem 9.11, we obtain a finite subset F' of $V(\sigma) \setminus X$ such that $F \subseteq F'$ and $\sigma - F'$ is $((V(\sigma) \setminus F') \setminus X)$-critical. Furthermore, it follows from Corollary 3.20 that $|F'|$ is even.

9.2. **Modules of the outside graph.** We begin with two preliminary results on the isolated vertices of an outside graph. We utilize the following remark.

Remark 9.13. Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. It follows from Remark 3.15 that for each $B \in p_{(\sigma,X)} \setminus \{\mathrm{Ext}_\sigma(X)\}$, $\Gamma_{(\sigma,X)}[B]$ is empty. In other words, if $\mathrm{Ext}_\sigma(X) = \varnothing$, then $\Gamma_{(\sigma,X)}$ is multipartite with partition $p_{(\sigma,X)}$ (see Lemma 3.13).

Lemma 9.14 (Breiner et al.[9.1][5]). *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime.*

(1) *If M is a module of σ such that $X \subseteq M$, then the elements of $V(\sigma) \setminus M$ are isolated vertices of $\Gamma_{(\sigma,X)}$.*
(2) *Given $y \in X$, if M is a module of σ such that $M \cap X = \{y\}$, then the elements of $M \setminus \{y\}$ are isolated vertices of $\Gamma_{(\sigma,X)}$.*

Proof. For the first assertion, consider a module M of σ such that $X \subseteq M$. Let $v \in V(\sigma) \setminus M$. Moreover, consider $w \in (V(\sigma) \setminus X) \setminus \{v\}$. We have to verify that

$$\sigma[X \cup \{v,w\}] \text{ is decomposable.}$$

By Remark 3.16, $(V(\sigma) \setminus M) \subseteq \langle X \rangle_\sigma$. It follows from Remark 9.13 that $\sigma[X \cup \{v,w\}]$ is decomposable when $w \notin M$. Now, suppose that $w \in M \setminus X$. By assertion (M2) of Proposition 2.5, $M \cap (X \cup \{v,w\})$, which is $X \cup \{w\}$, is a module of $\sigma[X \cup \{v,w\}]$. Thus, $\sigma[X \cup \{v,w\}]$ is decomposable.

For the second assertion, consider $y \in X$ and a module M of σ such that $M \cap X = \{y\}$. Let $v \in M \setminus \{y\}$. Moreover, consider $w \in (V(\sigma) \setminus X) \setminus \{v\}$. We have to verify that

$$\sigma[X \cup \{v,w\}] \text{ is decomposable.}$$

By Remark 3.16, $M \setminus \{y\} \subseteq X_\sigma(y)$. It follows from Remark 9.13 that $\sigma[X \cup \{v,w\}]$ is decomposable when $w \in M$. Now, suppose that $w \notin M$. By assertion (M2) of Proposition 2.5, $M \cap (X \cup \{v,w\})$, which is $\{y,v\}$, is a module of $\sigma[X \cup \{v,w\}]$. Thus, $\sigma[X \cup \{v,w\}]$ is decomposable. □

The next result is an immediate consequence of Remark 3.16 and Lemma 9.14.

Corollary 9.15. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. If σ admits a nontrivial module M such that $M \cap X \neq \varnothing$, then $\Gamma_{(\sigma,X)}$ possesses isolated vertices.*

Now, we study the modules of the outside graph. We need the following refinement of the outside partition (see Notation 3.12).

Notation 9.16. Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. We consider the following subsets of $V(\sigma) \setminus X$

[9.1] Breiner et al. [5] proved this lemma for (finite) graphs (see [5, Lemma 2.7]).

- for $e, f \in E(\sigma)$,

$$\langle X\rangle_\sigma^{(e,f)} = \langle X\rangle_\sigma \cap N_\sigma^{(e,f)}(y) \text{ (see Notation 3.7)},$$

  where $y \in X$;
- for $e, f \in E(\sigma)$ and $y \in X$,

$$X_\sigma^{(e,f)}(y) = X_\sigma(y) \cap N_\sigma^{(e,f)}(y) \text{ (see Notation 3.7)}.$$

The set $\{\mathrm{Ext}_\sigma(X)\} \cup \{\langle X\rangle_\sigma^{(e,f)} : e, f \in E(\sigma)\} \cup \{X_\sigma^{(e,f)}(y) : e, f \in E(\sigma), y \in X\}$ is denoted by $q_{(\sigma,X)}$.

Lemma 9.17. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S1) holds. Given $M \subseteq (V(\sigma) \smallsetminus X)$, if M is a module of σ, then M is a module of $\Gamma_{(\sigma,X)}$, and there exist $B_p \in p_{(\sigma,X)}$ and $B_q \in q_{(\sigma,X)}$ such that $M \subseteq B_q \subseteq B_p$ and M is a module of $\sigma[B_p]$.*

Proof. Consider a module M of σ such that $M \cap X = \varnothing$. Let $v \in M$. Denote by B_q the unique block of $q_{(\sigma,X)}$ containing v. Consider $w \in M \smallsetminus \{v\}$. Since M is a module of σ such that $M \cap X = \varnothing$, we have $y \longleftrightarrow_\sigma \{v, w\}$ for every $y \in X$. It follows that $w \in B_q$. Consequently, $M \subseteq B_q$. Denote by B_p the unique block of $p_{(\sigma,X)}$ containing B_q. We obtain

$$M \subseteq B_q \subseteq B_p.$$

Since M is a module of σ, M is a module of $\sigma[B_p]$ by assertion (M2) of Proposition 2.5.

Lastly, we prove that M is a module of $\Gamma_{(\sigma,X)}$. Let $w \in (V(\sigma) \smallsetminus X) \smallsetminus M$. Recall that $\mathrm{Ext}_\sigma(X) = \varnothing$ because statement (S1) holds. If $w \in B_p$, then it follows from Remark 9.13 that $\{v, w\} \notin E(\Gamma_{(\sigma,X)})$ for every $v \in M$. Hence, suppose that $w \in (V(\sigma) \smallsetminus X) \smallsetminus B_p$. Since $\mathrm{Ext}_\sigma(X) = \varnothing$, we distinguish the following two cases.

CASE 1: $B_p = \langle X\rangle_\sigma$.

Consider $y \in X$ and $u \in M$.

First, suppose that $u \longleftrightarrow_\sigma \{y, w\}$. Let $v \in M$. Since M is a module of σ, we obtain $v \longleftrightarrow_\sigma \{y, w\}$. Since $v \longleftrightarrow_\sigma X$, we obtain $v \longleftrightarrow_\sigma X \cup \{w\}$. Hence, $X \cup \{w\}$ is a module of $\sigma[X \cup \{v, w\}]$. It follows that $\{v, w\} \notin E(\Gamma_{(\sigma,X)})$.

Second, suppose that $u \not\longleftrightarrow_\sigma \{y, w\}$. Let $v \in M$. Since M is a module of σ, we have $v \not\longleftrightarrow_\sigma \{y, w\}$. Thus, $X \cup \{v\}$ is not a module of $\sigma[X \cup \{y, v\}]$. It follows from assertion (Q1) of Corollary 3.18 that $\{v, w\} \in E(\Gamma_{(\sigma,X)})$.

CASE 2: $B_p = X_\sigma(y)$, where $y \in X$.

Consider $u \in M$.

First, suppose that $w \longleftrightarrow_\sigma \{y, u\}$. Let $v \in M$. Since M is a module of σ, we obtain $w \longleftrightarrow_\sigma \{y, v\}$. Since $\{y, v\}$ is a module of $\sigma[X \cup \{v\}]$, $\{y, v\}$ is a module of $\sigma[X \cup \{v, w\}]$. It follows that $\{v, w\} \notin E(\Gamma_{(\sigma,X)})$ for every $v \in M$.

Second, suppose that $w \not\leftrightarrow_\sigma \{y,u\}$. Let $v \in M$. Since M is a module of σ, we obtain $w \not\leftrightarrow_\sigma \{y,v\}$. Therefore, $\{y,v\}$ is not a module of $\sigma[X \cup \{v,w\}]$. It follows from assertion (Q2) of Corollary 3.18 that $\{v,w\} \in E(\Gamma_{(\sigma,X)})$. □

The opposite direction in Lemma 9.17 is false. Nevertheless, it is true for (finite) graphs (see the second assertion of [5, Lemma 2.6]). Moreover, the opposite direction in Lemma 9.17 is true if we require that statement (S3) holds (see Corollary 9.19 below). We need the following fact.

Fact 9.18 (Breiner et al.[9.2][5])**.** *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S3) holds. Given distinct elements u,v,w of $V(\sigma) \setminus X$, if $\{u,v\},\{u,w\} \in E(\Gamma_{(\sigma,X)})$, then $\{v,w\}$ is a module of $\sigma[X \cup \{u,v,w\}]$, and hence there exists $B_q \in q_{(\sigma,X)}$ such that $v,w \in B_q$.*

Proof. Since $\{u,v\} \in E(\Gamma_{(\sigma,X)})$, $\sigma[X \cup \{u,v\}]$ is prime. Set

$$Y = X \cup \{u,v\}.$$

Since statement (S3) holds,

$$w \notin \mathrm{Ext}_\sigma(Y).$$

For a contradiction, suppose that $w \in \langle Y \rangle_\sigma$. We obtain that $X \cup \{u,v\}$ is a module of $\sigma[X \cup \{u,v,w\}]$. By assertion (M2) of Proposition 2.5, $X \cup \{u\}$ is a module of $\sigma[X \cup \{u,w\}]$, which contradicts $\{u,w\} \in E(\Gamma_{(\sigma,X)})$. Consequently,

$$w \notin \langle Y \rangle_\sigma.$$

It follows from Lemma 3.13 that there exists $z \in Y$ such that

$$w \in Y_\sigma(z).$$

Hence, $\{z,w\}$ is a module of $\sigma[X \cup \{u,v,w\}]$. By assertion (M2) of Proposition 2.5, $(X \cup \{u,w\}) \cap \{z,w\}$ is a module of $\sigma[X \cup \{u,w\}]$. Since $\{u,w\} \in E(\Gamma_{(\sigma,X)})$, $(X \cup \{u,w\}) \cap \{z,w\}$ is a trivial module of $\sigma[X \cup \{u,w\}]$. Since $w \in (X \cup \{u,w\}) \cap \{z,w\}$, we obtain $z \notin X \cup \{u\}$. It follows that $z = v$. Therefore, $w \in Y_\sigma(v)$, that is, $\{v,w\}$ is a module of $\sigma[X \cup \{u,v,w\}]$. By Lemma 9.17, there exists $B_q \in q_{(\sigma,X)}$ such that $v,w \in B_q$. □

The next result follows from Fact 9.18.

Corollary 9.19. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S3) holds. Consider $M \subseteq (V(\sigma) \setminus X)$ such that there exist $B_p \in p_{(\sigma,X)}$ and $B_q \in q_{(\sigma,X)}$ with $M \subseteq B_q \subseteq B_p$. Suppose that M is a module of $\sigma[B_p]$. If M is a module of $\Gamma_{(\sigma,X)}$, then M is a module of σ.*

[9.2] Breiner et al. [5] proved this lemma for (finite) graphs (see [5, Lemma 4.3]).

Proof. Suppose that M is a module of $\Gamma_{(\sigma,X)}$. Consider $u, v \in M$ and $w \in V(\sigma) \setminus M$. It suffices to verify that

$$w \longleftrightarrow_\sigma \{u, v\}. \tag{9.2}$$

Since M is a module of $\sigma[B_p]$, (9.2) holds when $w \in B_p \setminus M$. Furthermore, by Remark 9.4, statement (S1) holds because statement (S3) holds. It follows that

$$\mathrm{Ext}_\sigma(X) = \varnothing.$$

Since u and v belong to the same block of $q_{(\sigma,X)}$, (9.2) holds when $w \in X$. Now, suppose that

$$w \in V(\sigma) \setminus (X \cup B_p).$$

Since M is a module of $\Gamma_{(\sigma,X)}$, we have

$$\begin{array}{c} \{u,w\}, \{v,w\} \in E(\Gamma_{(\sigma,X)}) \\ \text{or} \\ \{u,w\}, \{v,w\} \notin E(\Gamma_{(\sigma,X)}). \end{array} \tag{9.3}$$

Suppose that $\{u,w\}, \{v,w\} \in E(\Gamma_{(\sigma,X)})$. By Fact 9.18, $\{u,v\}$ is a module of $\sigma[X \cup \{u,v,w\}]$, so (9.2) holds.

Lastly, suppose that $\{u,w\}, \{v,w\} \notin E(\Gamma_{(\sigma,X)})$. Since $\mathrm{Ext}_\sigma(X) = \varnothing$, we distinguish the following two cases.

Case 1: $B_p = \langle X \rangle_\sigma$.

Since $\{u,w\}, \{v,w\} \notin E(\Gamma_{(\sigma,X)})$, it follows from assertion (Q1) of Corollary 3.18 that $X \cup \{w\}$ is a module of $\sigma[X \cup \{u,w\}]$ and $\sigma[X \cup \{v,w\}]$. Given $y \in X$, we obtain $u \longleftrightarrow_\sigma \{y,w\}$ and $v \longleftrightarrow_\sigma \{y,w\}$. Since $u, v \in B_q$ and $B_q \subseteq \langle X \rangle_\sigma$, $y \longleftrightarrow_\sigma \{u,v\}$. It follows that (9.2) holds.

Case 2: $B_p = X_\sigma(y)$, where $y \in X$.

Since $\{u,w\}, \{v,w\} \notin E(\Gamma_{(\sigma,X)})$, it follows from assertion (Q2) of Corollary 3.18 that $\{y,u\}$ is a module of $\sigma[X \cup \{u,w\}]$, and $\{y,v\}$ is a module of $\sigma[X \cup \{v,w\}]$. Therefore, we have $w \longleftrightarrow_\sigma \{y,u\}$ and $w \longleftrightarrow_\sigma \{y,w\}$. It follows that (9.2) holds. □

The next fact follows from Lemma 9.17.

Fact 9.20 (Breiner et al.[9.3][5])**.** *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S3) holds. Given $B_p, D_p \in p_{(\sigma,X)}$, consider $u \in B_p$ and $v, w \in D_p$ such that $\{u,v\} \in E(\Gamma_{(\sigma,X)})$ and $\{u,w\} \notin E(\Gamma_{(\sigma,X)})$.*

(1) *If $D_p = \langle X \rangle_\sigma$, then $X \cup \{u,v\}$ is a module of $\sigma[X \cup \{u,v,w\}]$.*
(2) *If $D_p = X_\sigma(y)$, where $y \in X$, then $\{y,w\}$ is a module of $\sigma[X \cup \{u,v,w\}]$.*

Proof. To begin, we establish two preliminary statements (see (9.4) and (9.5)). Since $\{u,v\} \in E(\Gamma_{(\sigma,X)})$, $\sigma[X \cup \{u,v\}]$ is prime. Set

$$Y = X \cup \{u,v\}.$$

[9.3]Breiner et al. [5] proved this lemma for (finite) graphs (see [5, Lemma 4.4]).

Since statement (S3) holds,

$$w \notin \mathrm{Ext}_\sigma(Y).$$

Since $\{u,v\} \in E(\Gamma_{(\sigma,X)})$, we have

$$B_p \neq D_p \tag{9.4}$$

by Remark 9.13. For a contradiction, suppose that $w \in Y_\sigma(u)$. Hence, $\{u,w\}$ is a module of $\sigma[X\cup\{u,v,w\}]$. By Remark 9.4, statement (S1) holds because statement (S3) holds. It follows from Lemma 9.17 applied to $\sigma[X\cup\{u,v,w\}]$ that $B_p = D_p$, which contradicts (9.4). Thus,

$$w \notin Y_\sigma(u).$$

Now, suppose for a contradiction that $w \in Y_\sigma(v)$. Hence, $\{v,w\}$ is a module of $\sigma[X \cup \{u,v,w\}]$. It follows from Lemma 9.17 applied to $\sigma[X \cup \{u,v,w\}]$ that $\{v,w\}$ is a module of $\Gamma_{(\sigma,X)}$, which is impossible because $\{u,v\} \in E(\Gamma_{(\sigma,X)})$ and $\{u,w\} \notin E(\Gamma_{(\sigma,X)})$. Therefore,

$$w \notin Y_\sigma(v).$$

Since $w \notin (Y_\sigma(u) \cup Y_\sigma(v))$, it follows from Lemma 3.13 applied to $\sigma[Y]$ that

$$w \in \langle Y\rangle_\sigma \text{ or } w \in Y_\sigma(y), \text{ where } y \in X. \tag{9.5}$$

First, suppose that $D_p = \langle X\rangle_\sigma$. If $w \in Y_\sigma(y)$, where $y \in X$, then $w \in X_\sigma(y)$, and hence $w \in Y_\sigma(y) \cap \langle X\rangle_\sigma$, which contradicts Lemma 3.13. It follows from (9.5) that $w \in \langle Y\rangle_\sigma$, that is, $X \cup \{u,v\}$ is a module of $\sigma[X \cup \{u,v,w\}]$.

Second suppose that $D_p = X_\sigma(y)$, where $y \in X$. If $w \in \langle Y\rangle_\sigma$, then $w \in \langle X\rangle_\sigma$, and hence $w \in X_\sigma(y)\cap\langle X\rangle_\sigma$, which contradicts Lemma 3.13. It follows from (9.5) that $w \in Y_\sigma(z)$, where $z \in X$. Hence, we have $w \in X_\sigma(z)$. We obtain $w \in X_\sigma(y) \cap X_\sigma(z)$. By Lemma 3.13, we have $y = z$. Consequently, $w \in Y_\sigma(y)$, that is, $\{y,w\}$ is a module of $\sigma[X \cup \{u,v,w\}]$. □

The next two results follow from Fact 9.20.

Corollary 9.21. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S3) holds. Let $B_q \in q_{(\sigma,X)}$. For each $v \in (V(\sigma) \setminus X) \setminus B_q$, $\{u \in B_q : \{u,v\} \in E(\Gamma_{(\sigma,X)})\}$ and $\{u \in B_q : \{u,v\} \notin E(\Gamma_{(\sigma,X)})\}$ are modules of $\sigma[B_q]$. Precisely, if $\{u \in B_q : \{u,v\} \in E(\Gamma_{(\sigma,X)})\} \neq \varnothing$ and $\{u \in B_q : \{u,v\} \notin E(\Gamma_{(\sigma,X)})\} \neq \varnothing$, then the following two assertions hold.*

(1) *If $B_q = \langle X\rangle_\sigma^{(e,f)}$, where $e,f \in E(\sigma)$, then*

$$[\{u \in B_q : \{u,v\} \notin E(\Gamma_{(\sigma,X)})\}, \{u \in B_q : \{u,v\} \in E(\Gamma_{(\sigma,X)})\}]_\sigma = (f,e)$$

(see Notation 2.1).

(2) *If $B_q = X_\sigma^{(e,f)}(\alpha)$, where $\alpha \in X$ and $e,f \in E(\sigma)$, then*

$$[\{u \in B_q : \{u,v\} \notin E(\Gamma_{(\sigma,X)})\}, \{u \in B_q : \{u,v\} \in E(\Gamma_{(\sigma,X)})\}]_\sigma = (e,f).$$

Proof. Let $v \in (V(\sigma) \smallsetminus X) \smallsetminus B_q$. Suppose that $\{u \in B_q : \{u, v\} \in E(\Gamma_{(\sigma,X)})\} \neq \varnothing$ and $\{u \in B_q : \{u, v\} \notin E(\Gamma_{(\sigma,X)})\} \neq \varnothing$. Consider $u^+, u^- \in B_q$ such that $\{u^+, v\} \in E(\Gamma_{(\sigma,X)})$ and $\{u^-, v\} \notin E(\Gamma_{(\sigma,X)})$. We distinguish the following two cases.

CASE 1: $B_q = \langle X \rangle_\sigma^{(e,f)}$, where $e, f \in E(\sigma)$.

By the first assertion of Fact 9.20 applied to u^+, u^-, v, $X \cup \{u^+, v\}$ is a module of $\sigma[X \cup \{u^+, u^-, v\}]$. Since $u^- \in \langle X \rangle_\sigma^{(e,f)}$, we obtain $[u^-, u^+]_\sigma = (f, e)$.

CASE 2: $B_q = X_\sigma^{(e,f)}(y)$, where $y \in X$ and $e, f \in E(\sigma)$.

By the second assertion of Fact 9.20 applied to u^+, u^-, v, $\{y, u^-\}$ is a module of $\sigma[X \cup \{u^+, u^-, v\}]$. Hence, $[u^-, u^+]_\sigma = [y, u^+]_\sigma$. Since $u^+ \in X_\sigma^{(e,f)}(y)$, we obtain $[y, u^+]_\sigma = (e, f)$, so $[u^-, u^+]_\sigma = (e, f)$. □

The proof of the next corollary follows from Corollary 3.18 and Fact 9.20.

Corollary 9.22 (Breiner et al.[9.4][5])**.** *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S3) holds. If σ is prime, then $\Gamma_{(\sigma,X)}$ has no isolated vertices.*

Proof. We denote by $\mathcal{I}$ the set of the isolated vertices of $\Gamma_{(\sigma,X)}$. By Remark 9.4, statement (S1) holds because statement (S3) holds. Therefore, we have $\text{Ext}_\sigma(X) = \varnothing$. By Lemma 3.13, to show that $\mathcal{I} = \varnothing$, it suffices to verify that

$$\mathcal{I} \cap \langle X \rangle_\sigma = \varnothing \tag{9.6}$$

and

$$\mathcal{I} \cap X_\sigma(y) = \varnothing \tag{9.7}$$

for each $y \in X$.

To verify that (9.6) holds, we show that $V(\sigma) \smallsetminus (\mathcal{I} \cap \langle X \rangle_\sigma)$ is a module of σ. Consider $u \in \mathcal{I} \cap \langle X \rangle_\sigma$ and $v \in V(\sigma) \smallsetminus (\mathcal{I} \cap \langle X \rangle_\sigma)$. We verify that $X \cup \{v\}$ is a module of $\sigma[X \cup \{u, v\}]$. This is clear when $v \in X$ because $u \in \langle X \rangle_\sigma$. Hence, suppose that $v \notin X$. We distinguish the following two cases.

CASE 1: $v \notin \langle X \rangle_\sigma$.

Since $u \in \mathcal{I}$, we have $\{u, v\} \notin E(\Gamma_{(\sigma,X)})$. It follows from assertion (Q1) of Corollary 3.18 that $X \cup \{v\}$ is a module of $\sigma[X \cup \{u, v\}]$.

CASE 2: $v \in \langle X \rangle_\sigma$.

Since $v \in V(\sigma) \smallsetminus (\mathcal{I} \cap \langle X \rangle_\sigma)$, we have $v \notin \mathcal{I}$. Since $v \notin \mathcal{I}$, there exists $w \in V(\sigma) \smallsetminus X$ such that $\{v, w\} \in E(\Gamma_{(\sigma,X)})$. Since $u \in \mathcal{I}$, we have $\{u, w\} \notin E(\Gamma_{(\sigma,X)})$. By the first assertion of Fact 9.20, $X \cup \{v, w\}$ is a module of $\sigma[X \cup \{u, v, w\}]$. By assertion (M2) of Proposition 2.5, $X \cup \{v\}$ is a module of $\sigma[X \cup \{u, v\}]$.

In both cases above, $X \cup \{v\}$ is a module of $\sigma[X \cup \{u, v\}]$. It follows that $V(\sigma) \smallsetminus (\mathcal{I} \cap \langle X \rangle_\sigma)$ is a module of σ. Since σ is prime, $V(\sigma) \smallsetminus (\mathcal{I} \cap \langle X \rangle_\sigma)$ is

[9.4]Breiner et al. [5] proved this lemma for (finite) graphs (see [5, Corollary 4.5]).

a trivial module of σ. Thus, we obtain $V(\sigma) \setminus (\mathcal{I} \cap \langle X \rangle_\sigma) = V(\sigma)$. Hence, (9.6) holds.

To verify that (9.7) holds, consider $y \in X$. We show that $\{y\} \cup (\mathcal{I} \cap X_\sigma(y))$ is a module of σ. Consider $u \in \mathcal{I} \cap X_\sigma(y)$ and $v \in V(\sigma) \setminus (\{y\} \cup (\mathcal{I} \cap X_\sigma(y)))$. We verify that $\{y, u\}$ is a module of $\sigma[X \cup \{u, v\}]$. This is clear when $v \in X \setminus \{y\}$ because $u \in X_\sigma(y)$. Hence, suppose that $v \notin X$. We distinguish the following two cases.

Case 1: $v \notin X_\sigma(y)$.

Since $u \in \mathcal{I}$, we have $\{u, v\} \notin E(\Gamma_{(\sigma,X)})$. It follows from assertion (Q2) of Corollary 3.18 that $\{y, u\}$ is a module of $\sigma[X \cup \{u, v\}]$.

Case 2: $v \in X_\sigma(y)$.

Since $v \in V(\sigma) \setminus (\{y\} \cup (\mathcal{I} \cap X_\sigma(y)))$, we have $v \notin \mathcal{I}$. Since $v \notin \mathcal{I}$, there exists $w \in V(\sigma) \setminus X$ such that $\{v, w\} \in E(\Gamma_{(\sigma,X)})$. Since $u \in \mathcal{I}$, we have $\{u, w\} \notin E(\Gamma_{(\sigma,X)})$. By the second assertion of Fact 9.20, $\{y, u\}$ is a module of $\sigma[X \cup \{u, v, w\}]$. By assertion (M2) of Proposition 2.5, $\{y, u\}$ is a module of $\sigma[X \cup \{u, v\}]$.

In both cases above, $\{y, u\}$ is a module of $\sigma[X \cup \{u, v\}]$. It follows that $\{y\} \cup (\mathcal{I} \cap X_\sigma(y))$ is a module of σ. Since σ is prime, $\{y\} \cup (\mathcal{I} \cap X_\sigma(y))$ is a trivial module of σ. Thus, we obtain $\mathcal{I} \cap X_\sigma(y) = \varnothing$. Hence, (9.7) holds. □

9.3. Blocks of the outside partition and of its refinement.

Lemma 9.23. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S3) holds. Consider $e, f \in E(\sigma)$, and $y \in X$. If $\Gamma_{(\sigma,X)}$ does not have isolated vertices, then the following two assertions hold*

1. *if $\langle X \rangle_\sigma^{(e,f)} \neq \varnothing$, then $\langle X \rangle_\sigma^{(e',f')} = \varnothing$ for any $e', f' \in E(\sigma)$ such that $\{e', f'\} \neq \{e, f\}$;*
2. *if $X_\sigma^{(e,f)}(y) \neq \varnothing$, then $X_\sigma^{(e',f')}(y) = \varnothing$ for any $e', f' \in E(\sigma)$ such that $\{e', f'\} \neq \{e, f\}$.*

Proof. Consider $e, f, e', f' \in E(\sigma)$. For the first assertion, suppose that there exist $v \in \langle X \rangle_\sigma^{(e,f)}$ and $v' \in \langle X \rangle_\sigma^{(e',f')}$. We have to prove that

$$\{e, f\} = \{e', f'\}. \tag{9.8}$$

Since $v, v' \in \langle X \rangle_\sigma$, we have $\{v, v'\} \notin E(\Gamma_{(\sigma,X)})$ by Remark 9.13. Furthermore, since $\Gamma_{(\sigma,X)}$ does not have isolated vertices, there exist $w, w' \in (V(\sigma) \setminus X) \setminus \{v, v'\}$ such that $\{v, w\}, \{v', w'\} \in E(\Gamma_{(\sigma,X)})$. Suppose that $w = w'$. We obtain $\{w, v\}, \{w, v'\} \in E(\Gamma_{(\sigma,X)})$. It follows from Fact 9.18 that $(e, f) = (e', f')$, so (9.8) holds. We obtain the same conclusion when $\{v, w'\} \in E(\Gamma_{(\sigma,X)})$ or $\{v', w\} \in E(\Gamma_{(\sigma,X)})$. Thus, suppose that $w \neq w'$, and $\{v, w'\}, \{v', w\} \notin E(\Gamma_{(\sigma,X)})$. It follows from the first assertion of Fact 9.20 applied to v, v', w' that $X \cup \{v', w'\}$ is a module of $\sigma[X \cup \{v, v', w'\}]$. Since $v \in \langle X \rangle_\sigma^{(e,f)}$, we obtain $[v, v']_\sigma = (f, e)$. Similarly, it follows from the first assertion of Fact 9.20 applied to v, v', w that $[v', v]_\sigma = (f', e')$. Therefore, we have $e = f'$ and $e' = f$. Consequently, (9.8) holds.

For the second assertion, suppose that there exist $v \in X_\sigma^{(e,f)}(y)$ and $v' \in X_\sigma^{(e',f')}(y)$, where $y \in X$. We have to prove that (9.8) holds. Since $v, v' \in X_\sigma(y)$, we have $\{v, v'\} \notin E(\Gamma_{(\sigma,X)})$ by Remark 9.13. Furthermore, since $\Gamma_{(\sigma,X)}$ does not have isolated vertices, there exist $w, w' \in (V(\sigma) \smallsetminus X) \smallsetminus \{v, v'\}$ such that $\{v, w\}, \{v', w'\} \in E(\Gamma_{(\sigma,X)})$. Suppose that $w = w'$. We obtain $\{w, v\}, \{w, v'\} \in E(\Gamma_{(\sigma,X)})$. By Fact 9.18, $(e, f) = (e', f')$, so (9.8) holds. We obtain the same conclusion when $\{v, w'\} \in E(\Gamma_{(\sigma,X)})$ or $\{v', w\} \in E(\Gamma_{(\sigma,X)})$. Now, suppose that $w \neq w'$, and $\{v, w'\}, \{v', w\} \notin E(\Gamma_{(\sigma,X)})$. It follows from the second assertion of Fact 9.20 applied to v, v', w' that $\{y, v\}$ is a module of $\sigma[X \cup \{v, v', w'\}]$. We obtain $[v', y]_\sigma = [v', v]_\sigma$. Since $v' \in X_\sigma^{(e',f')}(y)$, we have $[y, v']_\sigma = (e', f')$. Therefore, we obtain $[v', v]_\sigma = (f', e')$. Similarly, it follows from the second assertion of Fact 9.20 applied to v, v', w that $[v, v']_\sigma = (f, e)$. Thus, we have $e = f'$ and $e' = f$. Consequently, (9.8) holds. □

Lemma 9.24. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S3) holds. Consider $e, f \in E(\sigma)$ and $y \in X$. Suppose that*

$$e \neq f. \tag{9.9}$$

If $\Gamma_{(\sigma,X)}$ does not have isolated vertices, then the following two assertions hold

1. *if $\langle X\rangle_\sigma^{(e,f)} \neq \varnothing$ and $\langle X\rangle_\sigma^{(f,e)} \neq \varnothing$, then $[\langle X\rangle_\sigma^{(e,f)}, \langle X\rangle_\sigma^{(f,e)}]_\sigma = (f, e)$;*
2. *if $X_\sigma^{(e,f)}(y) \neq \varnothing$ and $X_\sigma^{(f,e)}(y) \neq \varnothing$, then $[X_\sigma^{(e,f)}(y), X_\sigma^{(f,e)}(y)]_\sigma = (f, e)$.*

Proof. For the first assertion, consider $v \in \langle X\rangle_\sigma^{(e,f)}$ and $v' \in \langle X\rangle_\sigma^{(f,e)}$. Since $v, v' \in \langle X\rangle_\sigma$, we have $\{v, v'\} \notin E(\Gamma_{(\sigma,X)})$ by Remark 9.13. Furthermore, since $\Gamma_{(\sigma,X)}$ does not have isolated vertices, there exists $w' \in (V(\sigma) \smallsetminus X) \smallsetminus \{v, v'\}$ such that $\{v', w'\} \in E(\Gamma_{(\sigma,X)})$. Suppose for a contradiction that $\{v, w'\} \in E(\Gamma_{(\sigma,X)})$. We obtain $\{v, w'\}, \{v', w'\} \in E(\Gamma_{(\sigma,X)})$. It follows from Fact 9.18 that $e = f$, which contradicts (9.9). Therefore, we have $\{v, w'\} \notin E(\Gamma_{(\sigma,X)})$. It follows from the first assertion of Fact 9.20 applied to v, v', w' that $X \cup \{v', w'\}$ is a module of $\sigma[X \cup \{v, v', w'\}]$. Since $v \in \langle X\rangle_\sigma^{(e,f)}$, we obtain $[v', v]_\sigma = (e, f)$.

For the second assertion, consider $v \in X_\sigma^{(e,f)}(y)$ and $v' \in X_\sigma^{(f,e)}(y)$. Since $v, v' \in X_\sigma(y)$, we have $\{v, v'\} \notin E(\Gamma_{(\sigma,X)})$ by Remark 9.13. Furthermore, since $\Gamma_{(\sigma,X)}$ does not have isolated vertices, there exists $w' \in (V(\sigma) \smallsetminus X) \smallsetminus \{v, v'\}$ such that $\{v', w'\} \in E(\Gamma_{(\sigma,X)})$. Suppose for a contradiction that $\{v, w'\} \in E(\Gamma_{(\sigma,X)})$. We obtain $\{v, w'\}, \{v', w'\} \in E(\Gamma_{(\sigma,X)})$. It follows from Fact 9.18 that $e = f$, which contradicts (9.9). Therefore, we have $\{v, w'\} \notin E(\Gamma_{(\sigma,X)})$. It follows from the second assertion of Fact 9.20 applied to v, v', w' that $\{y, v\}$ is a module of $\sigma[X \cup \{v, v', w'\}]$. Thus, we obtain

$[v,v']_\sigma = [y,v']_\sigma$. Since $v' \in X_\sigma^{(f,e)}(y)$, we have $[y,v']_\sigma = (f,e)$, so $[v,v']_\sigma = (f,e)$. □

To state the next result, we use the following notation and definition.

Notation 9.25. Let σ be a 2-structure. For $e \in E(\sigma)$ and $W \subseteq V(\sigma)$, set

$$e[W] = e \cap (W \times W).$$

Given $e \in E(\sigma)$ and $W \subseteq V(\sigma)$, we do not have $e \in E(\sigma[W])$, but we have $e[W] \in E(\sigma[W])$ when $e[W] \neq \varnothing$.

Lemma 9.26. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S3) holds. If σ is prime, then the next two assertions hold.*

(1) *Let $e \in E(\sigma)$. If $|\langle X\rangle_\sigma^{(e,e)}| \geq 2$, then $\sigma[\langle X\rangle_\sigma]$ is constant and*

$$E(\sigma[\langle X\rangle_\sigma]) = \{e[\langle X\rangle_\sigma]\}.$$

Similarly, given $y \in X$, if $|X_\sigma^{(e,e)}(y)| \geq 2$, then $\sigma[X_\sigma(y)]$ is constant and $E(\sigma[X_\sigma(y)]) = \{e[X_\sigma(y)]\}$.

(2) *Consider distinct $e, f \in E(\sigma)$. If $|\langle X\rangle_\sigma^{(e,f)}| \geq 2$, then $\sigma[\langle X\rangle_\sigma]$ is linear and*

$$E(\sigma[\langle X\rangle_\sigma]) = \{e[\langle X\rangle_\sigma], f[\langle X\rangle_\sigma]\}.$$

Similarly, given $y \in X$, if $|X_\sigma^{(e,f)}(y)| \geq 2$, then $\sigma[X_\sigma(y)]$ is linear and $E(\sigma[X_\sigma(y)]) = \{e[X_\sigma(y)], f[X_\sigma(y)]\}$.

Proof. Consider $B_q \in q_{(\sigma,X)}$, with $|B_q| \geq 2$. There exist $e, f \in E(\sigma)$ such that $B_q = \langle X\rangle_\sigma^{(e,f)}$ or $X_\sigma^{(e,f)}(y)$, where $y \in X$. Consider $C \in \mathcal{C}_{\{e,f\}}(\sigma[B_q])$ (see Definition 2.2). We prove that C is a module of σ. We utilize Corollary 9.19 in the following manner. Since $B_q \in q_{(\sigma,X)}$, there exists $B_p \in p_{(\sigma,X)}$ such that $B_q \subseteq B_p$. By Lemma 2.4, C is a module of $\sigma[B_q]$.

Now, we show that C is a module of $\sigma[B_p]$. Suppose that $e = f$. It follows from Lemma 9.23 that $B_q = B_p$. Hence, C is a module of $\sigma[B_p]$. Suppose that $e \neq f$. If $B_q = B_p$, then we proceed as previously. Hence, suppose that $B_q \neq B_p$. It follows from Lemma 9.23 that $B_p \setminus B_q \in q_{(\sigma,X)}$ and

$$B_p \setminus B_q = \begin{cases} \langle X\rangle_\sigma^{(f,e)} \text{ if } B_q = \langle X\rangle_\sigma^{(e,f)} \\ \text{or} \\ X_\sigma^{(f,e)}(y) \text{ if } B_q = X_\sigma^{(e,f)}(y). \end{cases}$$

It follows from Lemma 9.24 that B_q is a module of $\sigma[B_p]$. Since C is a module of $\sigma[B_q]$, it follows from assertion (M3) of Proposition 2.5 that C is a module of $\sigma[B_p]$.

Lastly, we prove that C is a module of $\Gamma_{(\sigma,X)}$. Since $C \subseteq B_p$, we have $\{c,v\} \notin E(\Gamma_{(\sigma,X)})$ for $c \in C$ and $v \in B_p \setminus C$ by Remark 9.13. Therefore, we have to verify that C is a module of $\Gamma_{(\sigma,X)}[C \cup \{v\}]$ for each $v \in (V(\sigma) \setminus X) \setminus B_p$. Let $v \in (V(\sigma) \setminus X) \setminus B_p$. Set

$$C^+ = C \cap N_{\Gamma_{(\sigma,X)}}(v) \text{ and } C^- = C \setminus N_{\Gamma_{(\sigma,X)}}(v).$$

For a contradiction, suppose that $C^- \neq \varnothing$ and $C^+ \neq \varnothing$. It follows from Corollary 9.21 that $[C^-, C^+]_\sigma = (e,f)$ or (f,e), which contradicts $C \in \mathcal{C}_{\{e,f\}}(\sigma[B_q])$. Therefore, $C^- = \varnothing$ or $C^+ = \varnothing$, that is, C is a module of $\Gamma_{(\sigma,X)}[C \cup \{v\}]$ for each $v \in (V(\sigma) \setminus X) \setminus B_p$. Thus, C is a module of $\Gamma_{(\sigma,X)}$.

Consequently, C is a module of $\sigma[B_p]$ and C is a module of $\Gamma_{(\sigma,X)}$. It follows from Corollary 9.19 that C is a module of σ. Since σ is prime, C is trivial. Hence, we obtain $|C| = 1$ because $C \neq \varnothing$ and $C \cap X = \varnothing$. We conclude as follows by distinguishing the following two cases.

CASE 1: $e = f$.

Recall that $B_q = B_p$ by Lemma 9.23. Hence, all the $\{e,f\}$-components of $\sigma[B_p]$ are reduced to singletons. It follows from Proposition 2.8 that $\sigma[B_p]$ is constant. Precisely, it follows from Lemma 2.4 that $(v,w)_\sigma = e$ for distinct $v, w \in B_p$. In other words, $\sigma[B_p]$ is constant and $E(\sigma[B_p]) = \{e[B_p]\}$.

CASE 2: $e \neq f$.

For instance, suppose that $B_q = \langle X \rangle_\sigma^{(e,f)}$. All the $\{e,f\}$-components of $\sigma[\langle X \rangle_\sigma^{(e,f)}]$ are reduced to singletons. It follows from Proposition 2.8 that $\sigma[B_p]$ is linear. Precisely, it follows from Lemma 2.4 that $(v,w)_\sigma = e$ or f for distinct $v, w \in B_p$. In other words, $\sigma[\langle X \rangle_\sigma^{(e,f)}]$ is linear and $E(\sigma[\langle X \rangle_\sigma^{(e,f)}]) = \{e[\langle X \rangle_\sigma^{(e,f)}], f[\langle X \rangle_\sigma^{(e,f)}]\}$.

Lastly, suppose that $B_q \subsetneq B_p$. It follows from Lemma 9.23 that $B_p \setminus B_q = \langle X \rangle_\sigma^{(f,e)}$. Similarly, we have $\sigma[\langle X \rangle_\sigma^{(f,e)}]$ is linear and $E(\sigma[\langle X \rangle_\sigma^{(f,e)}]) = \{e[\langle X \rangle_\sigma^{(f,e)}], f[\langle X \rangle_\sigma^{(f,e)}]\}$. Moreover, we have

$$[\langle X \rangle_\sigma^{(e,f)}, \langle X \rangle_\sigma^{(f,e)}]_\sigma = (f,e)$$

by the first assertion of Lemma 9.24. Consequently, $\sigma[\langle X \rangle_\sigma]$ is linear and $E(\sigma[\langle X \rangle_\sigma]) = \{e[\langle X \rangle_\sigma], f[\langle X \rangle_\sigma]\}$. □

We complete subsection 9.3 with a result on the components of the outside graph, which follows from Fact 9.18 and the following easy consequence of Fact 9.20. We use the following notation.

Notation 9.27. Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. First, the set $\{\langle X \rangle_\sigma^{(e,e)} : e \in E(\sigma)\} \cup \{X_\sigma^{(e,e)}(\alpha) : e \in E(\sigma), \alpha \in X\}$ is denoted by $q^s_{(\sigma,X)}$. Second, the set $q_{(\sigma,X)} \setminus (q^s_{(\sigma,X)} \cup \{\text{Ext}_\sigma(X)\})$ is denoted by $q^a_{(\sigma,X)}$.

Fact 9.28. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S3) holds. Consider distinct $v, v', w, w' \in$*

$V(\sigma) \setminus X$ such that

$$\begin{cases}\{v,w\},\{v',w'\} \in E(\Gamma_{(\sigma,X)}) \\ \text{and} \\ \{v,w'\},\{v',w\} \notin E(\Gamma_{(\sigma,X)}).\end{cases}$$

If there exists $B_q \in q_{(\sigma,X)}$ such that $w, w' \in B_q$, then $B_q \in q^s_{(\sigma,X)}$.

Proof. Since w and w' belong to the same block of $p_{(\sigma,X)}$, we have $\{w,w'\} \notin E(\Gamma_{(\sigma,X)})$ by Remark 9.13. Besides, there exist $e, f \in E(\sigma)$ such that $B_q = \langle X \rangle_\sigma^{(e,f)}$ or $B_q = X_\sigma^{(e,f)}(y)$, where $y \in X$.

First, suppose that $B_q = \langle X \rangle_\sigma^{(e,f)}$. By the first assertion of Fact 9.20 applied to v, w, w', $X \cup \{v,w\}$ is a module of $\sigma[X \cup \{v,w,w'\}]$. Since $w' \in \langle X \rangle_\sigma^{(e,f)}$, we have $[w', X]_\sigma = (f,e)$. It follows that $[w',w]_\sigma = (f,e)$. Similarly, it follows from the first assertion of Fact 9.20 applied to v', w, w' that $[w',w]_\sigma = (e,f)$. Thus, we obtain $e = f$, and hence $B_q \in q^s_{(\sigma,X)}$.

Second, suppose that $B_q = X_\sigma^{(e,f)}(y)$, where $y \in X$. By the second assertion of Fact 9.20 applied to v, w, w', $\{y, w'\}$ is a module of $\sigma[X \cup \{v,w,w'\}]$. Thus, we have $[w,w']_\sigma = [w,y]_\sigma$. Since $w \in X_\sigma^{(e,f)}(y)$, we have $[w,y]_\sigma = (f,e)$. We obtain $[w,w']_\sigma = (f,e)$. Similarly, it follows from the second assertion of Fact 9.20 applied to v', w, w' that $[w',w]_\sigma = (f,e)$. Therefore $e = f$, so $B_q \in q^s_{(\sigma,X)}$. □

Proposition 9.29. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S3) holds. If $\Gamma_{(\sigma,X)}$ does not have isolated vertices, then the following two assertions hold.*

1. *For each component C of $\Gamma_{(\sigma,X)}$, there exist distinct $B_p, D_p \in p_{(\sigma,X)}$ and $B_q, D_q \in q_{(\sigma,X)}$ such that $B_q \subseteq B_p$, $D_q \subseteq D_p$, and C is bipartite with bipartition $\{V(C) \cap B_q, V(C) \cap D_q\}$.*
2. *For a component C of $\Gamma_{(\sigma,X)}$ and for $B_q \in q^a_{(\sigma,X)}$, if $V(C) \cap B_q \neq \varnothing$, then $B_q \subseteq V(C)$.*

Proof. For the first assertion, consider a component C of $\Gamma_{(\sigma,X)}$. Since $\Gamma_{(\sigma,X)}$ does not have isolated vertices, $v(C) \geq 2$. Hence, there exist distinct $c, d \in V(C)$ such that $\{c,d\} \in E(\Gamma_{(\sigma,X)})$. Furthermore, there exist $B_p, D_p \in p_{(\sigma,X)}$ and $B_q, D_q \in q_{(\sigma,X)}$ such that $c \in B_q$, $d \in D_q$, $B_q \subseteq B_p$, and $D_q \subseteq D_p$. Since $\{c,d\} \in E(\Gamma_{(\sigma,X)})$, we have $B_p \neq D_p$ by Remark 9.13. Let $v \in V(C) \setminus \{c,d\}$. Since C is a component of $\Gamma_{(\sigma,X)}$, there exists a sequence $v_0, \ldots, v_n$ of vertices of C satisfying

- $v_0 \in \{c,d\}$;
- $v_n = v$; $\{v_0, \ldots, v_n\} \cap \{c,d\} = \{v_0\}$;
- for $i, j \in \{0, \ldots, n\}$, $\{v_i, v_j\} \in E(\Gamma_{(\sigma,X)})$ if and only if $|i - j| = 1$.

Since $v_0 \in \{c,d\}$ and $v_n \in V(C) \setminus \{c,d\}$, we have $n \geq 1$. We distinguish the following two cases.

Case 1: n is even.

It follows from Fact 9.18 that $v_0, v_2, \ldots, v_n$ belong to the same block of $q_{(\sigma,X)}$. Since $v_0 \in \{c, d\}$ and $v_n = v$, we obtain $v \in B_q \cup D_q$.

CASE 2: n is odd.

Set

$$v_{-1} = \begin{cases} d \text{ if } v_0 = c \\ \text{and} \\ c \text{ if } v_0 = d. \end{cases}$$

We have $v_{-1} \in B_q \cup D_q$. By considering the sequence $v_{-1}v_0 \ldots v_n$, it follows from Fact 9.18 that v and v_{-1} belong to the same block of $q_{(\sigma,X)}$. Hence, $v \in B_q \cup D_q$.

Therefore, we obtain $V(C) \setminus \{c, d\} \subseteq B_q \cup D_q$, and hence $V(C) \subseteq B_q \cup D_q$. By Remark 9.13, C is bipartite with bipartition $\{V(C) \cap B_q, V(C) \cap D_q\}$.

For the second assertion, consider a component C of $\Gamma_{(\sigma,X)}$ and an element B_q of $q^a_{(\sigma,X)}$ such that $V(C) \cap B_q \neq \varnothing$. Consider $v \in V(C) \cap B_q$. For a contradiction, suppose that

$$B_q \setminus V(C) \neq \varnothing,$$

and consider $v' \in B_q \setminus V(C)$. Since $\Gamma_{(\sigma,X)}$ does not have isolated vertices, there exist $u \in (V(\sigma) \setminus X) \setminus \{v\}$ and $u' \in (V(\sigma) \setminus X) \setminus \{v'\}$ such that $\{u, v\}, \{u', v'\} \in E(\Gamma_{(\sigma,X)})$. Furthermore, since C is a component of $\Gamma_{(\sigma,X)}$, with $v \in V(C)$ and $v' \notin V(C)$, we obtain $u \in V(C)$ and $u' \notin V(C)$. Therefore, $u \neq u'$, and $\{u, v'\}, \{u', v\} \notin E(\Gamma_{(\sigma,X)})$. It follows from Fact 9.28 that $B_q \in q^s_{(\sigma,X)}$, which contradicts $B_q \in q^a_{(\sigma,X)}$. Consequently, we have $B_q \subseteq V(C)$. □

9.4. **Proofs of Theorems 9.5 and 9.6.** We use the following notation.

Notation 9.30. Given a graph Γ, $\mathcal{C}(\Gamma)$ denotes the set of the components of Γ.

Proof of Theorem 9.5. To begin, suppose that σ is not prime. We prove that there exists $C \in \mathcal{C}(\Gamma_{(\sigma,X)})$ such that $\sigma[X \cup V(C)]$ is not prime. First, suppose that $\Gamma_{(\sigma,X)}$ admits isolated vertices. Hence, consider $v \in V(\sigma) \setminus X$ such that $\{v\} \in \mathcal{C}(\Gamma_{(\sigma,X)})$. Since statement (S3) holds, $\text{Ext}_\sigma(X) = \varnothing$ by Remark 9.4. Thus $\sigma[X \cup \{v\}]$ is not prime. Second, suppose that $\Gamma_{(\sigma,X)}$ does not have isolated vertices. Since σ is not prime, σ admits a nontrivial module M. It follows from Corollary 9.15 that $M \cap X = \varnothing$. By Lemma 9.17, there exists $B_p \in p_{(\sigma,X)}$ such that $M \subseteq B_p$ and M is a module of $\Gamma_{(\sigma,X)}$. Let $u \in M$. Since $\Gamma_{(\sigma,X)}$ does not have isolated vertices, there exists $v \in (V(\sigma) \setminus X) \setminus \{u\}$ such that $\{u, v\} \in E(\Gamma_{(\sigma,X)})$. Since $M \subseteq B_p$, we have $v \notin M$ by Remark 9.13. Denote by C the component of $\Gamma_{(\sigma,X)}$ containing u. We obtain $v \in V(C)$ because $\{u, v\} \in E(\Gamma_{(\sigma,X)})$. Since M is a module of $\Gamma_{(\sigma,X)}$, we obtain $\{u', v\} \in E(\Gamma_{(\sigma,X)})$ for every $u' \in M$. Therefore, we have $M \subseteq V(C)$. It follows that M is a nontrivial module of $\sigma[X \cup V(C)]$.

Now, we suppose that there exists $C \in \mathcal{C}(\Gamma_{(\sigma,X)})$ such that $\sigma[X \cup V(C)]$ is not prime. Since $\sigma[X \cup V(C)]$ is not prime, we have $v(C) \neq 2$. Assume that $v(C) \geq 4$. We have to prove that C is not prime. Consider a nontrivial module M of $\sigma[X \cup V(C)]$. Clearly, $\sigma[X \cup V(C)]$ satisfies statement (S3). Moreover,

$$\Gamma_{(\sigma[X \cup V(C)], V(C))} = C.$$

Since $v(C) \geq 4$, it follows from Corollary 9.15 applied to $\sigma[X \cup V(C)]$ that $M \subseteq V(C)$. By Lemma 9.17 applied to $\sigma[X \cup V(C)]$, there exists

$$B_p \in p_{(\sigma[X \cup V(C)], V(C))}$$

such that $M \subseteq B_p$ and M is a module of C. We have to verify that $M \neq V(C)$. Let $u \in M$. Since $v(C) \geq 4$, there exists $v \in V(C) \smallsetminus \{u\}$ such that $\{u, v\} \in E(C)$. In particular, we have $v \in V(C)$. Since $u \in B_p$, we have $v \notin B_p$ by Remark 9.13 applied to $\sigma[X \cup V(C)]$. Since $M \subseteq B_p$, we obtain $v \in V(C) \smallsetminus M$.

Lastly, we suppose that there exists $C \in \mathcal{C}(\Gamma_{(\sigma,X)})$ such that $v(C) = 1$ or $v(C) \geq 3$ and C is not prime. We have to prove that σ is not prime. Therefore, by Corollary 9.22, we can suppose that

$$\Gamma_{(\sigma,X)} \text{ does not have isolated vertices.} \tag{9.10}$$

In particular, we obtain $v(C) \geq 3$. Consider a nontrivial module M of C. Clearly, M is a module of $\Gamma_{(\sigma,X)}$ because C is a component of $\Gamma_{(\sigma,X)}$. Since $\Gamma_{(\sigma,X)}$ does not have isolated vertices by (9.10), it follows from the first assertion of Proposition 9.29 that there exist distinct $B_p, D_p \in p_{(\sigma,X)}$ and $B_q, D_q \in q_{(\sigma,X)}$ such that $B_q \subseteq B_p$, $D_q \subseteq D_p$, and C is bipartite with bipartition $\{V(C) \cap B_q, V(C) \cap D_q\}$. Since C is connected, we have $M \subseteq V(C) \cap B_q$ or $M \subseteq V(C) \cap D_q$. For instance, assume that $M \subseteq V(C) \cap B_q$. To conclude, we distinguish the following two cases.

Case 1: $B_q \in q^s_{(\sigma,X)}$.

There exists $e \in E(\sigma)$ such that $B_q = \langle X \rangle_\sigma^{(e,e)}$ or $X_\sigma^{(e,e)}(y)$, where $y \in X$. If $\sigma[B_p]$ is not constant, then it follows from the first assertion of Lemma 9.26 that σ is not prime. Thus, suppose that $\sigma[B_p]$ is constant. It follows that any subset of B_p is a module of $\sigma[B_p]$. In particular, M is a module of $\sigma[B_p]$. Since M is a module of $\Gamma_{(\sigma,X)}$, it follows from Corollary 9.19 that M is a module of σ.

Case 2: $B_q \in q^a_{(\sigma,X)}$.

Since $\Gamma_{(\sigma,X)}$ does not have isolated vertices by (9.10), it follows from the second assertion of Proposition 9.29 that

$$B_q \subseteq V(C).$$

In general, M is not a module of $\sigma[B_q]$, and hence M is not a module of $\sigma[B_p]$. Therefore, we cannot apply Corollary 9.19 to M. Nevertheless, we construct a superset of M, which is a module of $\Gamma_{(\sigma,X)}$ and a module of $\sigma[B_p]$. Consider the set $\mathcal{M}$ of the nontrivial modules M' of C such that $M \subseteq M'$. Set

$$\widetilde{M} = \bigcup \mathcal{M}.$$

Clearly, $M \in \mathcal{M}$. Since $M \neq \varnothing$ and all the elements of $\mathcal{M}$ contain M, it follows from assertion (M5) of Proposition 2.5 that $\widetilde{M}$ is a module of C. Since C is a component of $\Gamma_{(\sigma,X)}$, $\widetilde{M}$ is a module of $\Gamma_{(\sigma,X)}$. As previously seen for M, $\widetilde{M} \subseteq V(C) \cap B_q$ or $\widetilde{M} \subseteq V(C) \cap D_q$. Since $M \subseteq \widetilde{M}$ and $M \subseteq V(C) \cap B_q$, we have $\widetilde{M} \subseteq V(C) \cap B_q$. Therefore, $\widetilde{M} \subseteq B_q$. Set

$$N = \{v \in B_q \smallsetminus \widetilde{M} : v \not\leftrightarrow_\sigma \widetilde{M}\}.$$

We verify that $\widetilde{M} \cup N$ is a module of C. It suffices to show that for any $w \in V(C) \cap D_q$, $u \in \widetilde{M}$ and $v \in N$, we have $\{u,w\},\{v,w\} \in E(\Gamma_{(\sigma,X)})$ or $\{u,w\},\{v,w\} \notin E(\Gamma_{(\sigma,X)})$. Since $v \in N$, there exist $u',u'' \in \widetilde{M}$ such that $v \not\leftrightarrow_\sigma \{u',u''\}$. Furthermore, since $\widetilde{M}$ is a module of C, we have $\{u,w\},\{u',w\},\{u'',w\} \in E(\Gamma_{(\sigma,X)})$ or $\{u,w\},\{u',w\},\{u'',w\} \notin E(\Gamma_{(\sigma,X)})$. For instance, suppose that $\{u,w\},\{u',w\},\{u'',w\} \in E(\Gamma_{(\sigma,X)})$. By Corollary 9.21, $\{z \in B_q : \{z,v\} \in E(\Gamma_{(\sigma,X)})\}$ is a module of $\sigma[B_q]$. Since $u,u',u'' \in \{z \in B_q : \{z,v\} \in E(\Gamma_{(\sigma,X)})\}$ and $v \not\leftrightarrow_\sigma \{x',x''\}$, we obtain $v \in \{z \in B_q : \{z,v\} \in E(\Gamma_{(\sigma,X)})\}$. Hence, we have $\{u,w\},\{u',w\},\{u'',w\},\{v,w\} \in E(\Gamma_{(\sigma,X)})$. Similarly, if $\{u,w\}$, $\{u',w\},\{u'',w\} \notin E(\Gamma_{(\sigma,X)})$, then if follows from Corollary 9.21 that $\{u,w\},\{u',w\},\{u'',w\},\{v,w\} \notin E(\Gamma_{(\sigma,X)})$. Consequently, $\widetilde{M} \cup N$ is a module of C. It follows from the definition of $\widetilde{M}$ that $N \subseteq \widetilde{M}$. Therefore, we have $N = \varnothing$, and hence, $\widetilde{M}$ is a module of $\sigma[B_q]$. Since $\Gamma_{(\sigma,X)}$ does not have isolated vertices by (9.10), it follows from Lemmas 9.23 and 9.24 that $\widetilde{M}$ is a module of $\sigma[B_p]$. Lastly, since $\widetilde{M}$ is a module of $\Gamma_{(\sigma,X)}$, it follows from Corollary 9.19 that $\widetilde{M}$ is a module of σ. □

The next result is an easy consequence of Theorem 9.5.

Corollary 9.31. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. If statement (S5) holds, then $\Gamma_{(\sigma,X)}$ does not embed P_5 (see Figure 1.1).*

Proof. For a contradiction, suppose that there exists $Y \subseteq (V(\sigma) \smallsetminus X)$ such that $\Gamma_{(\sigma,X)}[Y] \simeq P_5$. Since P_5 is connected, there exists a component C of $\Gamma_{(\sigma,X)}$ such that $Y \subseteq V(C)$. We have

$$\Gamma_{(\sigma[X\cup Y],X)} = \Gamma_{(\sigma,X)}[Y].$$

Since $\Gamma_{(\sigma,X)}[Y] = C[Y]$, $\Gamma_{(\sigma[X\cup Y],X)}$ is prime. It follows from Theorem 9.5 applied to $\sigma[X \cup Y]$ that $\sigma[X \cup Y]$ is prime, which contradicts the fact that statement (S5) holds. □

Since the proof of the next observation is obvious, we omit it.

Observation 9.32. Given a connected and bipartite graph Γ, Γ embeds $K_2 \oplus K_2$ if and only if Γ embeds P_5.

Proof of Theorem 9.6. We make a preliminary observation. Since statement (S5) holds, it follows from Remark 9.4 that statement (S3) holds as well.

To begin, suppose that the first assertion holds, that is, σ is $(V(\sigma) \setminus X)$-critical. We have to prove that the second assertion holds. Consider $C \in \mathcal{C}(\Gamma_{(\sigma,X)})$. By Theorem 9.5 applied to σ, $\sigma[X \cup V(C)]$ is prime. We have to show that $\sigma[X \cup V(C)]$ is $V(C)$-critical. Let $c \in V(C)$. Since σ is $(V(\sigma) \setminus X)$-critical, $\sigma - c$ is not prime. We have

$$\Gamma_{(\sigma-c,X)} = \Gamma_{(\sigma,X)} - c.$$

Therefore, we obtain

$$\mathcal{C}(\Gamma_{(\sigma-c,X)}) = (\mathcal{C}(\Gamma_{(\sigma,X)}) \setminus \{C\}) \cup \mathcal{C}(C - c). \tag{9.11}$$

Since $\sigma - c$ is not prime, it follows from Theorem 9.5 applied to $\sigma - c$ that there exists $C' \in \mathcal{C}(\Gamma_{(\sigma-c,X\setminus\{c\})})$ such that $\sigma[X \cup V(C')]$ is not prime. By (9.11), $C' \in (\mathcal{C}(\Gamma_{(\sigma,X)}) \setminus \{C\}) \cup \mathcal{C}(C - c)$. By Theorem 9.5 applied to σ, $\sigma[X \cup V(D)]$ is prime for every $D \in (\mathcal{C}(\Gamma_{(\sigma,X)}) \setminus \{C\})$. Thus, we obtain $C' \in \mathcal{C}(C - c)$. Finally, since

$$\Gamma_{(\sigma[X\cup V(C)]-c,V(C)\setminus\{c\})} = C - c,$$

it follows from Theorem 9.5 applied to $\sigma[X \cup V(C)] - c$ that $\sigma[X \cup V(C)] - c$ is not prime. Consequently, $\sigma[X \cup V(C)]$ is $V(C)$-critical.

To continue, suppose that the second assertion holds. We have to prove that the third assertion holds. Consider $C \in \mathcal{C}(\Gamma_{(\sigma,X)})$. By Theorem 9.5 applied to σ, $v(C) = 2$ or $v(C) \geq 4$ and C is prime. Suppose that $v(C) \geq 4$ and C is prime. We have to show that C is critical. Let $c \in V(C)$. We have to show that $C - c$ is not prime. If $C - c$ is disconnected, then $C - c$ is not prime. Thus, suppose that $C - c$ is connected. It follows that

$$\Gamma_{(\sigma[X\cup V(C)]-c,V(C)\setminus\{c\})} = C - c.$$

Since the second assertion holds, $\sigma[X \cup V(C)] - c$ is not prime. It follows from Theorem 9.5 applied to $\sigma[X \cup V(C)] - c$ that $C - c$ is not prime.

Lastly, suppose that the third assertion holds. Hence, for every $C \in \mathcal{C}(\Gamma_{(\sigma,X)})$, we have

$$v(C) = 2 \text{ or } v(C) \geq 4 \text{ and } C \text{ is critical.} \tag{9.12}$$

We have to prove that σ is $(V(\sigma) \setminus X)$-critical. By Theorem 9.5 applied to σ, σ is prime. Let $v \in V(\sigma) \setminus X$. We have to prove that $\sigma - v$ is not prime. Denote by C the component of $\Gamma_{(\sigma,X)}$ containing v. Since σ is prime, it follows from Corollary 9.22 that $\Gamma_{(\sigma,X)}$ has no isolated vertices. By the first assertion of Proposition 9.29, C is bipartite. Moreover, C does not embed P_5 by Corollary 9.31. It follows from Observation 9.32 that C does not embed $K_2 \oplus K_2$. Therefore,

$$C - v \text{ does not embed } K_2 \oplus K_2. \tag{9.13}$$

As seen in (9.11),

$$\mathcal{C}(C - v) \subseteq \mathcal{C}(\Gamma_{(\sigma-v,X)}). \tag{9.14}$$

Suppose that $C - v$ admits isolated vertices. By (9.14), $\Gamma_{(\sigma-v,X)}$ admits isolated vertices as well. It follows from Corollary 9.22 that $\sigma - v$ is not prime. Finally, suppose that $C - v$ does not admit isolated vertices. Hence, $v(C') \geq 2$ for each $C' \in \mathcal{C}(C - v)$. In particular, we do not have $v(C) = 2$. It follows from (9.12) that

$$v(C) \geq 4 \text{ and } C \text{ is critical.} \tag{9.15}$$

Since $v(C') \geq 2$ for each $C' \in \mathcal{C}(C - v)$, it follows from (9.13) that $C - v$ is connected. By (9.14), $C - v \in \mathcal{C}(\Gamma_{(\sigma-x,X)})$. Furthermore, it follows from (9.15) that $v(C - v) \geq 3$ and $C - v$ is not prime. By Theorem 9.5 applied to $\sigma - v$, $\sigma - v$ is not prime. □

9.5. **Outside graph and half-graph.** Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S5) holds. Suppose also that σ is $(V(\sigma) \setminus X)$-critical. Consider a component C of $\Gamma_{(\sigma,X)}$ such that $v(C) \geq 4$. By Remark 9.4, statement (S3) holds because statement (S5) holds. It follows from Proposition 9.29 that C is bipartite. It follows from Theorem 9.6 that C is critical. Moreover, since statement (S5) holds, C does not embed P_5 by Corollary 9.31. In Theorem 9.38 below, we characterize the bipartite graphs Γ such that Γ does not embed P_5 and Γ is critical. We need the following three definitions (see Definitions 9.33, 9.35, and 9.36).

Definition 9.33. We extend to the infinite case the definition of the half-graph H_{2m} (see Figure 4.1). Given a bipartite graph Γ, with bipartition $\{X, Y\}$, Γ is a *half-graph* [15] if there exist a linear order L defined on X, and a bijection φ from X onto Y such that

$$E(\Gamma) = \{\{x, \varphi(x')\} : x \leq_L x'\}. \tag{9.16}$$

Clearly, a finite half-graph is isomorphic to the graph H_{2m}, where $m \geq 1$ (see Figure 4.1).

Remark 9.34. Given a bipartite graph Γ, with bipartition $\{X, Y\}$. Suppose that Γ is a half-graph. There exist a linear order L defined on X, and a bijection φ from X onto Y such that (9.16) holds. Given $x, y \in X$, we obtain that

$$x \leq_L y \text{ if and only if } N_\Gamma(x) \supseteq N_\Gamma(y).$$

Therefore, the linear order L is unique.

Furthermore, denote by $\varphi(L)$ the unique linear order defined on Y such that φ is an isomorphism from L onto $\varphi(L)$. We obtain

$$E(\Gamma) = \{\{y, \varphi^{-1}(y')\} : y \leq_{(\varphi(L)^\star)} y'\}.$$

Consequently, Γ is also a half-graph by considering the linear order $\varphi(L)^\star$ defined on Y (see subsection 1.3), and the bijection $\varphi^{-1} : Y \longrightarrow X$.

Definition 9.35. A linear order L is *discrete* [30] if the following two conditions are satisfied

(1) for every $v \in V(L)$, if v is not the least element of L, then v admits an immediate predecessor;
(2) for every $v \in V(L)$, if v is not the greatest element of L, then v admits an immediate successor.

Definition 9.36. A half-graph is *discrete* if the linear order L in Definition 9.33 is discrete.

In the next observation, we explain how to decompose suitably a discrete linear order (see Definition 9.35).

Observation 9.37. Given an infinite linear order λ, λ is discrete if and only if λ admits a modular partition P satisfying the following conditions.

(1) If $|P| = 1$, then $\lambda \simeq L_{\mathbb{N}}$ or $(L_{\mathbb{N}})^{\star}$ or $L_{\mathbb{Z}}$.
(2) For each $M \in P$, if M is neither the least nor the greatest element of the quotient λ/P [9.5] , then $\lambda[M] \simeq L_{\mathbb{Z}}$.
(3) If $|P| \geq 2$ and λ/P admits a least element denoted by Min, then $\lambda[\mathrm{Min}] \simeq L_{\mathbb{N}}$ or $L_{\mathbb{Z}}$.
(4) If $|P| \geq 2$ and λ/P admits a greatest element denoted by Max, then $\lambda[\mathrm{Max}] \simeq (L_{\mathbb{N}})^{\star}$ or $L_{\mathbb{Z}}$.

Idea of proof. For a linear order, both notions of an interval and a module coincide. Consider an infinite discrete linear order λ. We define on $V(\lambda)$ the binary relation $\sim$ as follows. Given $v, w \in V(\lambda)$, $v \sim w$ if the smallest module of λ containing v and w is finite. Clearly, $\sim$ is an equivalence relation. Furthermore, the equivalence classes of $\sim$ are modules of λ. Thus, the set P of the equivalence classes of $\sim$ is a modular partition of λ. Since λ is discrete, it is is easy to verify that for each $M \in P$, $\lambda[M]$ is isomorphic to $L_{\mathbb{N}}$, $(L_{\mathbb{N}})^{\star}$, or $L_{\mathbb{Z}}$. □

Theorem 9.38 (Belkhechine et al. [3])**.** *Given a bipartite graph* Γ*, with* $v(\Gamma) \geq 4$*, the following assertions are equivalent*

(1) Γ *is a discrete half-graph;*
(2) Γ *does not embed* P_5 *and* Γ *is critical.*

Now, we examine Theorem 9.38 in the finite case (see Proposition 9.41). We need the following result which follows from the characterization of finite critical 2-structures done in subsection 4.2.

Corollary 9.39. *Given a finite and symmetric 2-structure* τ*, with* $v(\tau) \geq 5$*,* τ *is critical if and only if* τ *is isomorphic to* $\sigma(H_{2n})$*, where* $n \geq 3$*.*

Proof. To begin, suppose that τ is isomorphic to $\sigma(H_{2n})$, where $n \geq 3$. By Corollary 4.20, τ is critical.

Conversely, suppose that τ is critical. By Corollary 4.6, there exists $n \geq 3$ such that $\mathbb{P}(\tau)$ is isomorphic to P_{2n} or there exists $n \geq 3$ such that $\mathbb{P}(\tau)$ is isomorphic to $P_{2n} \oplus K_{\{2n\}}$, P_{2n+1}, or C_{2n+1}. Since τ is symmetric, it

[9.5] It is easy to verify that a quotient of a linear order is a linear order as well.

follows from Propositions 4.23, 4.27, and 4.36 that $\mathbb{P}(\tau)$ is not isomorphic to $P_{2n} \oplus K_{\{2n\}}$, P_{2n+1}, or C_{2n+1}. Consequently, $\mathbb{P}(\tau)$ is isomorphic to P_{2n}. It follows from Corollary 4.20 that τ is isomorphic to $\sigma(H_{2n})$, where $n \geq 3$. □

The next result is an immediate consequence of Corollary 9.39.

Corollary 9.40. *Given a finite and bipartite graph Γ, with $v(\Gamma) \geq 5$, Γ is critical if and only if Γ is a half-graph.*

Proof. To begin, suppose that Γ is critical. We obtain that $\sigma(\Gamma)$ is symmetric and critical. By Corollary 9.39 that $\sigma(\Gamma)$ is isomorphic to $\sigma(H_{2n})$, where $n \geq 3$. We obtain that Γ is isomorphic to H_{2n} or its complement $\overline{H_{2n}}$. Since $n \geq 3$, $\overline{H_{2n}}$ embeds the complete graph K_3. Since Γ is bipartite, we obtain that Γ is isomorphic to H_{2n}. As seen at the end of Definition 9.33, H_{2n} is a half-graph.

Conversely, suppose that Γ is a half-graph. As seen at the end of Definition 9.33, Γ is isomorphic to H_{2n}, where $n \geq 3$. By Corollary 9.39, $\sigma(H_{2n})$ is critical. Hence, H_{2n} is critical too. Therefore, Γ is critical. □

Proposition 9.41. *For a finite and bipartite graph Γ, with $v(\Gamma) \geq 4$, the following assertions are equivalent*

1. Γ *does not embed* P_5 *and* Γ *is prime;*
2. Γ *is critical;*
3. Γ *is a half-graph.*

Proof. First, suppose that $v(\Gamma) = 4$. We have Γ is prime if and only if Γ is isomorphic to P_4, which is isomorphic to the half-graph H_4. Therefore, the three assertions above are equivalent when $v(\Gamma) = 4$.

Second, suppose that $v(\Gamma) = 5$. The first assertion does not hold because

(9.17)
a prime and bipartite graph defined on 5 vertices[9.6] is isomorphic to P_5.

Furthermore, by Corollary 9.40, the last two assertions do not hold because $v(\Gamma)$ is odd. Thus, the three assertions above are equivalent when $v(\Gamma) = 5$.

Now, suppose that $v(\Gamma) \geq 6$. By Corollary 9.40, the last two assertions are equivalent. To begin, suppose that the first assertion holds. By (9.17),

(9.18) Γ does not embed a prime graph of size 5.

It follows from Theorem 5.3 that $v(\Gamma)$ is even. If $v(\Gamma) = 6$, then Γ is critical by (9.18). Hence, suppose that $v(\Gamma) \geq 7$. Since $v(\Gamma)$ is even, it follows from Theorem 5.3 and (9.18) that Γ is critical. Consequently, the first assertion implies the second one.

Lastly, suppose that Γ is both critical and a half-graph. If $v(\Gamma) = 6$, then Γ does not embed P_5. Suppose that $v(\Gamma) \geq 7$. Since Γ is a half-graph, $v(\Gamma)$ is even. Since Γ is critical, it follows from Corollary 3.20 that Γ does not embed P_5. □

The next result is a consequence of Proposition 9.41 and Theorem 7.1.

Corollary 9.42. *A half-graph* Γ*, with* $v(\Gamma) \geq 4$*, is prime.*

Proof. There exists a bipartition $\{X,Y\}$ of $V(\Gamma)$, a linear order L defined on X, and a bijection φ from X onto Y such that $E(\Gamma) = \{\{x,\varphi(x')\} : x \leq_L x'\}$. By Proposition 9.41, we can suppose that Γ is infinite. Consider a finite subset F of $V(\Gamma)$. Let X' be a finite subset of X such that $F \cap X \subseteq X'$, $\varphi^{-1}(F \cap Y) \subseteq X'$, and $|X'| \geq 2$. Set

$$F' = X' \cup \varphi(X').$$

Clearly, we have $F \subseteq F'$. By considering $Y' = \varphi(X')$, the linear order $L' = L[X']$, and the bijection $\varphi_{\restriction X'} : X' \longrightarrow Y'$, we obtain that $\Gamma[F']$ is a half-graph. By Proposition 9.41, $\Gamma[F']$ is prime. To conclude, it suffices to use Theorem 7.1. □

Now, we are ready to demonstrate Theorem 9.38.

Proof of Theorem 9.38. By Proposition 9.41, we can suppose that Γ is infinite.

To begin, suppose that Γ is a discrete half-graph. There exists a bipartition $\{X,Y\}$ of $V(\Gamma)$, a discrete linear order L defined on X, and a bijection φ from X onto Y such that $E(\Gamma) = \{\{x,\varphi(x')\} : x \leq_L x'\}$. By Corollary 9.42, Γ is prime. Hence, Γ is connected. Since Γ is a half-graph, Γ does not embed $K_2 \oplus K_2$. It follows from Observation 9.32 that Γ does not embed P_5. Now, we have to verify that

$$\text{for every } x \in X,\ \Gamma - x \text{ is not prime.} \tag{9.19}$$

First, suppose that x is not the least element of L. Since L is discrete, x admits an immediate predecessor x^-. It is easy to verify that $\{\varphi(x^-),\varphi(x)\}$ is a module of $\Gamma - x$. Second, suppose that x is the least element of L. Clearly, $\varphi(x)$ is an isolated vertex of $\Gamma - x$, so $\Gamma - x$ is not prime. Thus (9.19) holds. Similarly, it follows from Remark 9.34 that $\Gamma - y$ is not prime for each $y \in Y$. Consequently, Γ is critical.

Conversely, suppose that Γ does not embed P_5 and Γ is critical. Since Γ is bipartite, there exists a bipartition $\{X,Y\}$ of $V(\Gamma)$ such that X and Y are stable sets of Γ. To complete the proof, we establish the next claims. To begin, we define a linear order L on X as follows.

Definition 9.43. Since Γ is prime, we have $N_\Gamma(x) \neq N_\Gamma(x')$ for distinct $x,x' \in X$. Moreover, since Γ does not embed P_5, Γ does not embed $K_2 \oplus K_2$ by Observation 9.32. It follows that for distinct $x,x' \in X$, we have $N_\Gamma(x) \subsetneq N_\Gamma(x')$ or $N_\Gamma(x') \subsetneq N_\Gamma(x)$. Therefore, we can define on X a linear order L as follows. Given distinct $x,x' \in X$,

$$x <_L x' \text{ if } N_\Gamma(x) \supsetneq N_\Gamma(x').$$

We show that Γ is the half-graph defined from the linear order L (see Claim 9.51). We have also to define a suitable bijection from X onto Y (see Definition 9.47). We use the fact that Γ is critical.

Claim 9.44. *Given $x \in X$, if $\Gamma - x$ is disconnected, then the following assertions hold*

(1) *$\Gamma - x$ admits a unique isolated vertex i_x and $i_x \in Y$;*
(2) *$N_\Gamma(x) = Y$, so x is the least element of L;*
(3) *i_x is the unique element of $V(\Gamma) \smallsetminus \{x\}$ such that $\Gamma - \{x, i_x\}$ is prime.*

Proof. Since Γ is connected, the set of the isolated vertices of $\Gamma - x$ is a module of Γ. Thus, we have

$$|\{C \in \mathcal{C}(\Gamma - x) : v(C) = 1\}| \le 1.$$

Furthermore, since Γ does not embed $K_2 \oplus K_2$, $\Gamma - x$ admits at most one nontrivial component. Therefore, we have also

$$|\{C \in \mathcal{C}(\Gamma - x) : v(C) \ge 2\}| \le 1.$$

Since $\Gamma - x$ is disconnected, $|\mathcal{C}(\Gamma - x)| \ge 2$. It follows that $\Gamma - x$ admits a unique isolated vertex i_x and $\Gamma - \{x, i_x\}$ is connected. Since i_x is an isolated vertex of $\Gamma - x$, $\{x, i_x\} \in E(\Gamma)$ because Γ is connected. Hence, $i_x \in Y$.

Now, we verify that $N_\Gamma(x) = Y$. Let $y \in Y \smallsetminus \{i_x\}$. Since $\Gamma - \{x, i_x\}$ is connected, there exits $x' \in X \smallsetminus \{x\}$ such that $\{x', y\} \in E(\Gamma)$. Since $\Gamma[\{x, x', y, i_x\}] \not\simeq K_2 \oplus K_2$, we obtain $\{x, y\} \in E(\Gamma)$ or $\{x', i_x\} \in E(\Gamma)$. Since i_x is isolated in $\Gamma - x$, we have $\{x', i_x\} \notin E(\Gamma)$. Therefore, we obtain $\{x, y\} \in E(\Gamma)$. It follows that $N_\Gamma(x) = Y$. Hence, x is the least element of L.

Lastly, we verify that $\Gamma - \{x, i_x\}$ is prime. Otherwise, $\Gamma - \{x, i_x\}$ admits a nontrivial module M. Since $\Gamma - \{x, i_x\}$ is connected and bipartite with bipartition $\{X \smallsetminus \{x\}, Y \smallsetminus \{i_x\}\}$, we have $M \subseteq X \smallsetminus \{x\}$ or $M \subseteq Y \smallsetminus \{i_x\}$. Since $N_\Gamma(x) = Y$ and $N_\Gamma(i_x) = \{x\}$, M is a module of Γ, which contradicts the fact that Γ is prime. Consequently, $\Gamma - \{x, i_x\}$ is prime. Moreover, consider $v \in V(\Gamma) \smallsetminus \{x, i_x\}$. Since i_x is isolated in $\Gamma - x$, it is also isolated in $\Gamma - \{x, v\}$. Therefore $\Gamma - \{x, v\}$ is not prime. It follows that i_x is the unique element of $V(\Gamma) \smallsetminus \{x\}$ such that $\Gamma - \{x, i_x\}$ is prime. □

Claim 9.45. *Let $x \in X$ such that $\Gamma - x$ is connected. For any nontrivial module M of $\Gamma - x$, there exist $x^-, x^+ \in Y$ such that $M = \{x^-, x^+\}$, $\{x, x^-\} \notin E(\Gamma)$, and $\{x, x^+\} \in E(\Gamma)$.*

Proof. Let M be a nontrivial module of $\Gamma - x$. Since $\Gamma - x$ is connected, we have $M \subseteq X \smallsetminus \{x\}$ or $M \subseteq Y$. In the first instance, M is a module of Γ. Therefore, we have $M \subseteq Y$. Set $M^- = \{y \in M : \{x, y\} \notin E(\Gamma)\}$ and $M^+ = \{y \in M : \{x, y\} \in E(\Gamma)\}$. Clearly, M^- and M^+ are modules of Γ. Since Γ is prime and $|M| \ge 2$, we obtain $|M^-| = 1$ and $|M^+| = 1$. Denote by x^- the unique element of M^- and denote by x^+ the unique element of M^+. We obtain $M = \{x^-, x^+\}$. Furthermore, we have $\{x, x^-\} \notin E(\Gamma)$ and $\{x, x^+\} \in E(\Gamma)$. □

Claim 9.46. *Given $x \in X$, if $\Gamma - x$ is connected, then there exist $x^-, x^+ \in Y$ satisfying the following assertions*

(1) $\{x^-, x^+\}$ *is the only nontrivial module of* $\Gamma - x$;
(2) $\{x, x^-\} \notin E(\Gamma)$ *and* $\{x, x^+\} \in E(\Gamma)$;
(3) *for every* $u \in X$, *if* $u <_L x$, *then* $\{u, x^-\} \in E(\Gamma)$;
(4) *for every* $u \in X$, *if* $x <_L u$, *then* $\{u, x^+\} \notin E(\Gamma)$;
(5) $\Gamma - \{x, x^-\}$ *and* $\Gamma - \{x, x^+\}$ *are prime;*
(6) x^+ *is the unique element of* $V(\Gamma) \smallsetminus \{x\}$ *such that* $\{x, x^+\} \in E(\Gamma)$ *and* $\Gamma - \{x, x^+\}$ *is prime.*

Proof. Since Γ is critical, $\Gamma - x$ admits a nontrivial module M. By Claim 9.45, there exist $x^-, x^+ \in Y$ such that $M = \{x^-, x^+\}$, $\{x, x^-\} \notin E(\Gamma)$, and $\{x, x^+\} \in E(\Gamma)$. Hence, $\{x^-, x^+\}$ is a nontrivial module of $\Gamma - x$.

For a contradiction, suppose that M is not the only nontrivial module of $\Gamma - x$. Thus, there exists a nontrivial module N of $\Gamma - x$ such that $N \neq M$. By Claim 9.45, there exist $z^-, z^+ \in Y$ such that $N = \{z^-, z^+\}$, $\{x, z^-\} \notin E(\Gamma)$, and $\{x, z^+\} \in E(\Gamma)$. If $M \cap N \neq \varnothing$, then $M \cup N$ is a nontrivial module of $\Gamma - x$ of size 3, which contradicts Claim 9.45. Hence, we have $M \cap N = \varnothing$. We show that $M \cup N$ is a module of $\Gamma - x$. Let $u \in (X \smallsetminus \{x\})$. It suffices to verify that $M \cup N$ is a module of $\Gamma[M \cup N \cup \{u\}]$. Suppose that there exists $v \in M \cup N$ such that $\{u, v\} \in E(\Gamma)$. For instance, suppose that $v \in M$. Since M is a module of $\Gamma - x$, we have $\{u, x^-\}, \{u, x^+\} \in E(\Gamma)$. We obtain $\{u, x^-\} \in E(\Gamma)$, $\{x, x^-\} \notin E(\Gamma)$, and $\{x, z^+\} \in E(\Gamma)$. Since Γ does not embed $K_2 \oplus K_2$, we obtain $\{u, z^+\} \in E(\Gamma)$. Since $\{z^-, z^+\}$ is a module of $\Gamma - x$, we have $\{u, z^-\} \in E(\Gamma)$. Therefore, $\{u, w\} \in E(\Gamma)$ for every $w \in M \cup N$. It follows that $M \cup N$ is a module of $\Gamma - x$, which contradicts Claim 9.45 because $|M \cup N| = 4$. Consequently, $\{x^-, x^+\}$ is the only nontrivial module of $\Gamma - x$. It follows that $\Gamma - \{x, x^-\}$ and $\Gamma - \{x, x^+\}$ are prime.

Let $u \in X$ such that $u <_L x$. Since $u <_L x$, we have $N_\Gamma(u) \supseteq N_\Gamma(x)$. Hence, we have $\{u, x^+\} \in E(\Gamma)$ because $\{x, x^+\} \in E(\Gamma)$. Since $\{x^-, x^+\}$ is a module of $\Gamma - x$, we obtain $\{u, x^-\} \in E(\Gamma)$.

Let $u \in X$ such that $x <_L u$. Since $x <_L u$, we have $N_\Gamma(x) \supseteq N_\Gamma(u)$. Hence, we have $\{u, x^-\} \notin E(\Gamma)$ because $\{x, x^-\} \notin E(\Gamma)$. Since $\{x^-, x^+\}$ is a module of $\Gamma - x$, we obtain $\{u, x^+\} \notin E(\Gamma)$.

As previously seen, $\Gamma - \{x, x^-\}$ and $\Gamma - \{x, x^+\}$ are prime. Now, consider $v \in V(\Gamma) \smallsetminus \{x, x^-, x^+\}$. Clearly, $\{x^-, x^+\}$ is a nontrivial module of $\Gamma - \{x, v\}$, so $\Gamma - \{x, v\}$ is not prime. Since $\{x, x^-\} \notin E(\Gamma)$, x^+ is the unique element of $V(\Gamma) \smallsetminus \{x\}$ such that $\{x, x^+\} \in E(\Gamma)$ and $\Gamma - \{x, x^+\}$ is prime. □

Definition 9.47. We define a function $\varphi : X \longrightarrow Y$ as follows. Given $x \in X$,

$$\varphi(x) = \begin{cases} i_x \text{ if } \Gamma - x \text{ is disconnected (see Claim 9.44),} \\ \text{or} \\ x^+ \text{ if } \Gamma - x \text{ is connected (see Claim 9.46).} \end{cases}$$

The next claim follows easily from Claims 9.44 and 9.46.

Claim 9.48. *For every* $x \in X$, $\varphi(x)$ *is the unique element of* $V(\Gamma) \smallsetminus \{x\}$ *such that* $\{x, \varphi(x)\} \in E(\Gamma)$ *and* $\Gamma - \{x, \varphi(x)\}$ *is prime.*

In the next two claims, we verify that φ is bijective.

Claim 9.49. *φ is injective.*

Proof. Consider distinct $u, v \in X$. For instance, suppose that $u <_L v$. In particular, v is not the least element of L. It follows from Claim 9.44 that $\Gamma - v$ is connected. By Claim 9.46, there exist $v^-, v^+ \in Y$ such that $\{v, v^-\} \notin E(\Gamma)$, $\{v, v^+\} \in E(\Gamma)$, and $\{v^-, v^+\}$ is the only nontrivial module of $\Gamma - v$. We have $\varphi(v) = v^+$.

First, suppose that $\Gamma - u$ is disconnected. We have $\varphi(u) = i_u$, where i_u is the unique isolated vertex of $\Gamma - u$ by Claim 9.44. We obtain $\{v, \varphi(u)\} \notin E(\Gamma)$. Thus, we have $\varphi(u) \neq \varphi(v)$ because $\{v, \varphi(v)\} \in E(\Gamma)$ (see Claim 9.48).

Second, suppose that $\Gamma - u$ is connected. By Claim 9.46, there exist $u^-, u^+ \in Y$ such that $\{u, u^-\} \notin E(\Gamma)$, $\{u, u^+\} \in E(\Gamma)$, and $\{u^-, u^+\}$ is the only nontrivial module of $\Gamma - u$. We have $\varphi(u) = u^+$. Since $u <_L v$, it follows from the fourth assertion of Claim 9.46 applied to u that $\{v, \varphi(u)\} \notin E(\Gamma)$. Since $\{v, \varphi(v)\} \in E(\Gamma)$ (see Claim 9.48), $\varphi(u) \neq \varphi(v)$. □

Claim 9.50. *φ is surjective.*

Proof. Let $v \in Y$. Since Γ is critical, $\Gamma - v$ is not prime. First, suppose that $\Gamma - v$ is disconnected. As in Claim 9.44, we obtain that $\Gamma - v$ admits an isolated vertex i_v. Thus, we have $N_\Gamma(i_v) = \{v\}$. Since $\{i_v, \varphi(i_v)\} \in E(\Gamma)$, we obtain $\varphi(i_v) = (i_v)^+ = v$.

Second, suppose that $\Gamma - v$ is connected. As in Claim 9.46, there exist $v^-, v^+ \in X$ such that $\{v^-, v^+\}$ is the only nontrivial module of $\Gamma - v$, $\{v, v^-\} \notin E(\Gamma)$, and $\{v, v^+\} \in E(\Gamma)$. Furthermore, $\Gamma - \{v, v^-\}$ and $\Gamma - \{v, v^+\}$ are prime. Thus, we obtain $\Gamma - \{v, v^+\}$ is prime and $\{v, v^+\} \in E(\Gamma)$. It follows from Claim 9.48 that $v = \varphi(v^+)$. □

It follows from Claims 9.49 and 9.50 that φ is bijective.

Claim 9.51. *Γ is the half-graph defined from the linear order L, and the bijection φ.*

Proof. Consider distinct $u, x \in X$. We have to verify that

$$\{u, \varphi(x)\} \in E(\Gamma) \text{ if and only if } u \leq_L x.$$

Suppose that $u \leq_L x$. We obtain $N_\Gamma(x) \subseteq N_\Gamma(u)$. By Claim 9.48, we have $\varphi(x) \in N_\Gamma(x)$. Hence, we obtain $\varphi(x) \in N_\Gamma(u)$. Conversely, suppose that $x <_L u$. In particular, u is not the least element of L. It follows from Claim 9.44 that $\Gamma - u$ is connected. By the fourth assertion of Claim 9.46 applied to x, $\{u, x^+\} \notin E(\Gamma)$, that is, $\{u, \varphi(x)\} \notin E(\Gamma)$. □

Claim 9.52. *Given $x \in X$, if x is not the least element of L, then x admits an immediate predecessor in L.*

Proof. Let $x \in X$. Suppose that x is not the least element of L. It follows from Claim 9.44 that $\Gamma - x$ is connected. By Claim 9.46, there exist $x^-, x^+ \in Y$

such that $\{x^-, x^+\}$ is the only nontrivial module of $\Gamma - x$, $\{x, x^-\} \notin E(\Gamma)$, and $\{x, x^+\} \in E(\Gamma)$. Furthermore, for every $u \in X$, we have

$$\text{if } u <_L x, \text{ then } \{u, x^-\} \in E(\Gamma), \tag{9.20}$$

by the third assertion of Claim 9.46 applied to x. Set

$$t = \varphi^{-1}(x^-).$$

By Claim 9.48, $\{t, \varphi(t)\} \in E(\Gamma)$, that is, $\{t, x^-\} \in E(\Gamma)$. We obtain $x^- \in N_\Gamma(t) \smallsetminus N_\Gamma(x)$. Hence, we have $N_\Gamma(t) \supsetneq N_\Gamma(x)$, so $t <_L x$. We prove that t is the immediate predecessor of x. We must verify that

$$\{u \in X : t <_L u <_L x\} = \varnothing.$$

First, suppose that $\Gamma - t$ is disconnected. By Claim 9.44, there exists $i_t \in Y$ such that i_t is an isolated vertex of $\Gamma - t$. Since $\varphi(t) = i_t$, $i_t = x^-$. We obtain that $\{u, x^-\} \notin E(\Gamma)$ for every $u \in V(\Gamma) \smallsetminus \{t, x^-\}$. It follows from (9.20) that $\{u \in X : t <_L u <_L x\} = \varnothing$.

Second, suppose that $\Gamma - t$ is connected. By Claim 9.46, there exist $t^-, t^+ \in Y$ such that $\{t^-, t^+\}$ is the only nontrivial module of $\Gamma - t$, $\{t, t^-\} \notin E(\Gamma)$, and $\{t, t^+\} \in E(\Gamma)$. Furthermore, for every $u \in X$ such that $t <_L u$, we have $\{u, t^+\} \notin E(\Gamma)$ by the fourth assertion of Claim 9.46 applied to t. Recall that $t^+ = \varphi(t)$. Since $t = \varphi^{-1}(x^-)$, we obtain $t^+ = x^-$. Therefore, for every $u \in X$ such that $t <_L u$, we have $\{u, x^-\} \notin E(\Gamma)$. It follows from (9.20) that $\{u \in X : t <_L u <_L x\} = \varnothing$. □

By Remark 9.34, Γ is also the half-graph defined from the linear order $\varphi(L)^\star$ defined on Y, and the bijection $\varphi^{-1} : Y \longrightarrow X$. The analogue of Claim 9.52 for $\varphi(L)^\star$ follows.

Claim 9.53. *Given $y \in Y$, if y is not the least element of $\varphi(L)^\star$, then y admits an immediate predecessor in $\varphi(L)^\star$.*

The next claim is an immediate consequence of Claims 9.53.

Claim 9.54. *Given $x \in X$, if x is not the greatest element of L, then x admits an immediate successor in L.*

It follows from Claims 9.52 and 9.54 that L is discrete, which completes the proof of Theorem 9.38. □

The next theorem follows from Theorems 9.6 and 9.38.

Theorem 9.55. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that statement (S5) holds. Suppose also that σ is $(V(\sigma) \smallsetminus X)$-critical. For each component C of $\Gamma_{(\sigma,X)}$, with $v(C) \geq 3$, C is a discrete half-graph.*

Proof. Let C be a component of $\Gamma_{(\sigma,X)}$ such that $v(C) \geq 3$. By Theorem 9.6, $v(C) \geq 4$ and C is critical. Furthermore, since statement (S5) holds, C does not embed P_5 by Corollary 9.31. Finally, to use Theorem 9.38, we must verify that C is bipartite. Indeed, since σ is prime, it follows from Corollary 9.22 that $\Gamma_{(\sigma,X)}$ has no isolated vertices. Furthermore, since statement (S5) holds, statement (S3) holds too by Remark 9.4. Therefore, it follows from Proposition 9.29 that there exist distinct $B_p, D_p \in p_{(\sigma,X)}$ and $B_q, D_q \in q_{(\sigma,X)}$ such that $B_q \subseteq B_p$, $D_q \subseteq D_p$, and C is bipartite with bipartition $\{V(C) \cap B_q, V(C) \cap D_q\}$. Therefore, it follows from Theorem 9.38 that C is a discrete half-graph. □

The next result follows from Theorems 9.5 and 9.6, Proposition 9.41, and Corollary 9.31. It is the finite version of Theorem 9.10. Moreover, we use it in the proof of Theorem 9.10.

Corollary 9.56. *Given a 2-structure σ, consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that*

$$V(\sigma) \setminus X \text{ is finite.}$$

The following two assertions are equivalent

(1) *Statement (S5) holds and σ is prime;*
(2) *σ is $(V(\sigma) \setminus X)$-critical.*

Proof. To begin, suppose that σ is $(V(\sigma) \setminus X)$-critical. In particular, σ is prime. Furthermore, by Remark 9.4, statement (S5) holds.

Conversely, suppose that statement (S5) holds and σ is prime. Since statement (S5) holds, we can use Theorem 9.6 to prove that σ is $(V(\sigma) \setminus X)$-critical. Consider a component C of $\Gamma_{(\sigma,X)}$ such that $v(C) \geq 3$. We have to show that C is critical. Since σ is prime, it follows from Theorem 9.5 that $v(C) \geq 4$ and C is prime. Moreover, since statement (S5) holds, it follows from Corollary 9.31 that C does not embed P_5. By Proposition 9.41, Γ is critical. It follows from Theorem 9.6 that σ is $(V(\sigma) \setminus X)$-critical. □

Proof of Theorem 9.10. To begin, suppose that σ is finitely $(V(\sigma) \setminus X)$-critical. Let $v \in V(\sigma) \setminus X$. Since σ is finitely $(V(\sigma) \setminus X)$-critical, there exists a finite subset F of $V(\sigma) \setminus X$ such that $\sigma[X \cup F]$ is F-critical. It follows from Corollary 9.56 that statement (S5) holds and σ is prime.

Conversely, suppose that statement (S5) holds and σ is prime. We prove that σ is finitely $(V(\sigma) \setminus X)$-critical. Let F be a finite subset of $V(\sigma) \setminus X$. We have to find a finite subset F' of $V(\sigma) \setminus X$ such that $F \subseteq F'$ and $\sigma[X \cup F']$ is (F')-critical. We distinguish the following two cases.

CASE 1: $V(\sigma) \setminus X$ is finite.
It follows from Corollary 9.56 that σ is $(V(\sigma) \setminus X)$-critical. Hence, we can consider $V(\sigma) \setminus X$ for F'.

CASE 2: $V(\sigma) \setminus X$ is infinite.
By Corollary 9.8, there exists a finite subset F' of $V(\sigma) \setminus X$ such that $F \subseteq F'$ and $\sigma[X \cup F']$ is prime. Since statement (S5) holds, it follows from Corollary 9.56 applied to $\sigma[X \cup F']$ that $\sigma[X \cup F']$ is (F')-critical. □

As announced in subsection 9.1, we discuss Theorem 9.10 in the next remark by using Theorems 9.6 and 9.38.

Remark 9.57. We denote by $L_{\mathbb{Q}}$ the usual linear order on the set of rational numbers. Obviously, $L_{\mathbb{Q}}$ is not discrete. We consider the graph G defined on $\{0,1,2,3\} \cup (\{0,1\} \times \mathbb{Q})$ by

$$E(G) = \{\{0,1\},\{1,2\},\{2,3\}\} \cup \{\{1,(1,q)\} : q \in \mathbb{Q}\} \cup \Big(\bigcup_{q\in\mathbb{Q}} \{\{(0,q),(1,r)\} : r \geq q\}\Big).$$

Set $X = \{0,1,2,3\}$, $Y = \{0\} \times \mathbb{Q}$ and $Z = \{1\} \times \mathbb{Q}$. We have $G[X]$ is prime because $G[X] = P_4$ (see Fact 2.6). We consider the 2-structure $\sigma(G)$ associated with G. Since $G[X]$ is prime, $\sigma(G)[X]$ is prime too. We have

$$Y = \langle X \rangle_{\sigma(G)},\ Z = X_{\sigma(G)}(0), \text{ and } p_{(\sigma(G),X)} = \{Y, Z\}.$$

Furthermore, it follows from Corollary 3.18 that

$$\Gamma_{(\sigma(G),X)} = G[Y \cup Z]. \tag{9.21}$$

We verify that $\sigma(G)$ is finitely $(V(\sigma) \smallsetminus X)$-critical (see Definition 9.9) without being $(V(\sigma) \smallsetminus X)$-critical.

First, we show that statement (Sk) holds for every odd integer $k \geq 1$. Let W be a finite and nonempty subset of $Y \cup Z$ such that $W \in \mathscr{P}_{(\sigma,X)}$ (see Notation 9.2). We have to show that W is even. If $W \cap Y = \varnothing$, then $\{0\} \cup W$ is a module of $\sigma(G)[X \cup W]$ because $Z = X_{\sigma_G}(0)$. Hence, we have $W \cap Y \neq \varnothing$. We denote the elements of $W \cap Y$ by $(0,q_0),\dots,(0,q_m)$, where $m \geq 0$, in such a way that $q_0 < \cdots < q_m$, when $m \geq 1$. Set

$$Z^- = \{j < q_0 : (1,j) \in W\}.$$

Since $Z = X_{\sigma(G)}(0)$, $\{0\} \cup (\{1\} \times Z^-)$ is a module of $\sigma(G)[X \cup W]$. Hence, we have $Z^- = \varnothing$. Set

$$Z^+ = \{j \geq q_m : (1,j) \in W\}.$$

We obtain that $\{1\} \times Z^+$ is a module of $\sigma(G)[X \cup W]$. Hence, we have $|Z^+| \leq 1$. If $Z^+ = \varnothing$, then $(X \cup W) \smallsetminus \{(0,q_m)\}$ is a module of $\sigma(G)[X \cup W]$ because $(0,q_m) \in \langle X \rangle_{\sigma(G)}$. Thus, we obtain $|Z^+| = 1$. Therefore, $|W| = 2$ if $m = 0$. Now, suppose that $m \geq 1$. Set

$$Z_i = \{q_i \leq j < q_{i+1} : (1,j) \in W\}$$

for $i = 0,\dots,m-1$. Given $i = 0,\dots,m-1$, we have $\{1\} \times Z_i$ is a module of $\sigma(G)[X \cup W]$. Hence, we have $|Z_i| \leq 1$. Moreover, $\{(0,q_i),(0,q_{i+1})\}$ is a module of $\sigma(G)[X \cup W]$ if $Z_i = \varnothing$. Therefore, we obtain $|Z_i| = 1$. Consequently, $Z^- = \varnothing$, $|Z^+| = 1$, and $|Z_i| = 1$ for $i = 0,\dots,m-1$. Thus, $|W \cap Z| = m+1$, and hence $|W| = 2m+2$.

Second, we prove that σ_G is finitely $(V(\sigma) \smallsetminus X)$-critical. Let F be a finite subset of $Y \cup Z$. There exists a finite subset F' of $\mathbb{Q}$ such that $|F'| \geq 2$ and $F \subseteq (\{0,1\} \times F')$. We have $G[\{0,1\} \times F'] \simeq H_{2\times|F'|}$ (see Figure 4.1). It follows from Proposition 9.41 that $G[\{0,1\} \times F']$ is critical. Set $\widetilde{F} = \{0,1\} \times F'$. We obtain that

$$F \subseteq \widetilde{F} \text{ and } G[\widetilde{F}] \text{ is critical.} \tag{9.22}$$

It follows from (9.21) and (9.22) that $\Gamma_{(\sigma(G)[X\cup\widetilde{F}],\widetilde{F})}$ is critical. Since statement (S5) holds, it follows from Theorem 9.6 that $\sigma(G)[X\cup\widetilde{F}]$ is $\widetilde{F}$-critical. Consequently, $\sigma(G)$ is finitely $(V(\sigma)\smallsetminus X)$-critical.

Third, we verify that $\sigma(G)$ is not $(V(\sigma(G))\smallsetminus X)$-critical. To begin, we verify that $G[Y\cup Z]$ is a nondiscrete half-graph. Clearly, $G[Y\cup Z]$ is bipartite with bipartition $\{Y,Z\}$. Consider the bijection $\varphi : Y \longrightarrow Z$, which maps $(0,q)$ to $(1,q)$ for each $q\in\mathbb{Q}$. Moreover, consider the linear order L_Y defined on Y as follows. Given distinct $q,r\in\mathbb{Q}$, $(0,q) <_{L_Y} (0,r)$ if $q <_{L_{\mathbb{Q}}} r$. Clearly, $G[Y\cup Z]$ is the half-graph defined from L_Y and φ. Recall that the linear order L_Y is unique by Remark 9.34. Since $L_Y\simeq L_{\mathbb{Q}}$, $G[Y\cup Z]$ is not discrete.

Since statement (S5) holds, $\Gamma_{(\sigma(G),X)}$ does not embed P_5 by Corollary 9.31. Since $G[Y\cup Z]$ is a nondiscrete half-graph, $\Gamma_{(\sigma(G),X)}$ is a nondiscrete half-graph by (9.21). It follows from Theorem 9.38 that $\Gamma_{(\sigma(G),X)}$ is not critical. Clearly, $G[Y\cup Z]$ is connected. Therefore, $\Gamma_{(\sigma(G),X)}$ is connected by (9.21). Since statement (S5) holds, it follows from Theorem 9.6 that $\sigma(G)$ is not $(V(\sigma)\smallsetminus X)$-critical. Since $\sigma(G)$ is finitely $(V(\sigma(G))\smallsetminus X)$-critical, it follows from Theorem 9.10 that $\sigma(G)$ is prime. Consequently, there exists $v\in V(\sigma(G))\smallsetminus X$ such that $\sigma(G)-v$ is prime. In fact, we have $\sigma(G)-w$ is prime for every $w\in V(\sigma(G))\smallsetminus X$.

Proof of Theorem 9.11. Since statement (S5) holds, we can use Theorem 9.6 as follows. Let $v\in V(\sigma)\smallsetminus X$. Denote by C the component of $\Gamma_{(\sigma,X)}$ such that $v\in V(C)$. By Theorem 9.6 applied to σ, $v(C)=2$ or $v(C)\geq 4$ and C is critical.

First, suppose that $v(C)=2$. We have

$$\Gamma_{(\sigma-V(C),X)}=\Gamma_{(\sigma,X)}-V(C).$$

Therefore, the components of $\Gamma_{(\sigma-V(C),X)}$ are the components of $\Gamma_{(\sigma,X)}$ that are distinct from C. Let D be a component of $\Gamma_{(\sigma,X)}$ such that $D\neq C$. By Theorem 9.6 applied to σ, $v(D)=2$ or $v(D)\geq 4$ and D is critical. It follows from Theorem 9.6 applied to $\sigma-V(C)$ that $\sigma-V(C)$ is $((V(\sigma)\smallsetminus X)\smallsetminus V(C))$-critical. Hence, we consider for w the unique element of $V(C)\smallsetminus\{v\}$.

Second, suppose that $v(C)\geq 4$ and C is critical. By Theorem 9.55, C is a discrete half-graph. As seen in the proof of Theorem 9.55, there exist distinct $B_q, D_q\in q_{(\sigma,X)}$ such that C is bipartite with bipartition $\{V(C)\cap B_q, V(C)\cap D_q\}$. For instance, assume that $v\in V(C)\cap B_q$. Since C is a discrete half-graph, there exists a discrete linear order L defined on $V(C)\cap B_q$ and a bijection $\varphi : V(C)\cap B_q\longrightarrow V(C)\cap D_q$ such that C is defined from L and φ (see Definition 9.33). By Claim 9.48, $C-\{v,\varphi(v)\}$ is prime. Hence, $C-\{v,\varphi(v)\}$ is connected. Consequently, the components of $\Gamma_{(\sigma-\{v,\varphi(v)\},X)}$ are the components of $\Gamma_{(\sigma,X)}$ that are distinct from C and $C-\{v,\varphi(v)\}$. Let D be a component of $\Gamma_{(\sigma,X)}$ such that $D\neq C$. By Theorem 9.6 applied to σ, $v(D)=2$ or $v(D)\geq 4$ and D is critical. If $v(C)=4$, then $v(C-\{v,\varphi(v)\})=2$ and it follows from Theorem 9.6 applied to $\sigma-\{v,\varphi(v)\}$ that $\sigma-\{v,\varphi(v)\}$ is

$((V(\sigma)\smallsetminus X)\smallsetminus\{v,\varphi(v)\})$-critical. Lastly, suppose that $v(C)\geq 5$. To apply Theorem 9.6 to $\sigma-\{v,\varphi(v)\}$, we must verify that $v(C-\{v,\varphi(v)\})\geq 4$ and $C-\{v,\varphi(v)\}$ is critical. Since C is a half-graph with $v(C)\geq 5$, we have $v(C)\geq 6$, and hence $v(C-\{v,\varphi(v)\})\geq 4$. Clearly, $L-v$ is a discrete linear order. Moreover, $C-\{v,\varphi(v)\}$ is the half-graph defined from $L-x$ and the bijection $\varphi_{\restriction(V(C)\cap B_q)\smallsetminus\{v\}}:(V(C)\cap B_q)\smallsetminus\{v\}\longrightarrow(V(C)\cap D_q)\smallsetminus\{\varphi(v)\}$. Therefore, $C-\{v,\varphi(v)\}$ is a discrete half-graph. By Theorem 9.38, $C-\{v,\varphi(v)\}$ is critical. Consequently, it follows from Theorem 9.6 applied to $\sigma-\{v,\varphi(v)\}$ that $\sigma-\{v,\varphi(v)\}$ is $((V(\sigma)\smallsetminus X)\smallsetminus\{v,\varphi(v)\})$-critical. □

9.6. **Proofs of Theorems 5.8 and 5.9.**

Proof of Theorem 5.8. Let σ be a prime 2-structure. Consider $X\subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Suppose that $V(\sigma)\smallsetminus X$ is finite and $|V(\sigma)\smallsetminus X|\geq 6$.

For a contradiction, suppose that for each proper subset Y of $V(\sigma)\smallsetminus X$, we have

$$\text{if } \sigma[X\cup Y] \text{ is prime, then } |V(\sigma)\smallsetminus(X\cup Y)| \text{ is odd.} \tag{9.23}$$

For $Y=\varnothing$ in (9.23), we obtain $|V(\sigma)\smallsetminus X|$ is odd. Hence, we have $|V(\sigma)\smallsetminus X|\geq 7$. For $Y\subsetneq(V(\sigma)\smallsetminus X)$, with $|Y|=5$, it follows from (9.23) that $\sigma[X\cup Y]$ is not prime. Consequently, statement (S5) holds. Since $|V(\sigma)\smallsetminus X|$ is odd, there exists a component C of $\Gamma_{(\sigma,X)}$ such that $v(C)$ is odd. Since statement (S5) holds, statement (S3) holds too by Remark 9.4. Since σ is prime, it follows from Theorem 9.5 that $\sigma[X\cup V(C)]$ is prime. We have

$$V(\sigma)\smallsetminus X=V(C)\cup(V(\sigma)\smallsetminus(X\cup V(C))).$$

Since $|V(\sigma)\smallsetminus X|$ and $v(C)$ are odd, we obtain that $|V(\sigma)\smallsetminus(X\cup V(C))|$ is even. It follows from (9.23) that $V(C)=V(\sigma)\smallsetminus X$. Thus, $\Gamma_{(\sigma,X)}$ is connected. Since σ is prime, it follows from Theorem 9.5 that $\Gamma_{(\sigma,X)}$ is prime. Furthermore, since σ is prime, it follows from Corollary 9.22 that $\Gamma_{(\sigma,X)}$ has no isolated vertices. Since statement (S3) holds, it follows from Proposition 9.29 that C is bipartite. Finally, since statement (S5) holds, $\Gamma_{(\sigma,X)}$ does not embed P_5 by Corollary 9.31. It follows from Proposition 9.41 that $\Gamma_{(\sigma,X)}$ is a half-graph, which is impossible because $v(\Gamma_{(\sigma,X)})=|V(\sigma)\smallsetminus X|$ and $|V(\sigma)\smallsetminus X|$ is odd.

Consequently (9.23) does not hold. Therefore, there exists $Y\subsetneq(V(\sigma)\smallsetminus X)$ such that $\sigma[X\cup Y]$ is prime and $|V(\sigma)\smallsetminus(X\cup Y)|$ is even. Recall that $V(\sigma)\smallsetminus X$ is finite, so $V(\sigma)\smallsetminus(X\cup Y)$ is as well. It follows from Corollary 3.20 applied to $\sigma[X\cup Y]$ that there exist distinct $v,w\in V(\sigma)\smallsetminus(X\cup Y)$ such that $\sigma-\{v,w\}$ is prime. □

Proof of Theorem 5.9. Since statement $\mathscr{S}_1$ or statement $\mathscr{S}_2$ hold, we have

$$q^a_{(\sigma,X)}\neq\varnothing. \tag{9.24}$$

By Theorem 5.8, we can assume that $|V(\sigma)\smallsetminus X|=4$ or 5. If $|V(\sigma)\smallsetminus X|=4$, then it suffices to apply Theorem 3.19. Hence, suppose that $|V(\sigma)\smallsetminus X|=5$.

For a contradiction, suppose that statement (S3) holds. It follows from Theorem 9.5 that for each component C of $\Gamma_{(\sigma,X)}$, we have $v(C) = 2$ or $v(C) \geq 4$ and C is prime. Since $|V(\sigma) \setminus X| = 5$, we obtain that $\Gamma_{(\sigma,X)}$ is connected. Thus, $\Gamma_{(\sigma,X)}$ is prime. Since σ is prime, it follows from Corollary 9.22 that $\Gamma_{(\sigma,X)}$ has no isolated vertices. Since statement (S3) holds, it follows from the first assertion of Proposition 9.29 that $p_{(\sigma,X)} = q_{(\sigma,X)}$, and $q_{(\sigma,X)}$ has two elements, denoted by B_q and D_q. Moreover, $\Gamma_{(\sigma,X)}$ is bipartite, with bipartition $\{B_q, D_q\}$. Since $\Gamma_{(\sigma,X)}$ is prime and bipartite, we have $\Gamma_{(\sigma,X)} \simeq P_5$. Hence, $\Gamma_{(\sigma,X)}$ embeds $K_2 \oplus K_2$. Thus, there exists distinct $v, v' \in B_q$ and distinct $w, w' \in D_q$ such that $\{v, w\}, \{v', w'\} \in E(\Gamma_{(\sigma,X)})$ and $\{v, w'\}, \{v', w\} \notin E(\Gamma_{(\sigma,X)})$. It follows from Fact 9.28 that $B_q, D_q \in q^s_{(\sigma,X)}$, which contradicts (9.24).

Consequently, statement (S3) does not hold. Therefore, there exists $Y \subseteq (V(\sigma) \setminus X)$ such that $|Y| = 3$ and $\sigma[X \cup Y]$ is prime, which completes the proof because $|\overline{X}| = 5$. □

10. The Rigollet-Thomassé theorem

The aim of this section is to demonstrate the following theorem.

Theorem 10.1 (Rigollet and Thomassé[10.1] [29])**.** *Given an infinite prime 2-structure σ, there exists $X \subseteq V(\sigma)$ such that* [10.2]

$$\text{(RT)} \qquad \begin{cases} X \neq V(\sigma), \\ X \textit{ is equipotent to } V(\sigma), \\ \textit{and} \\ \sigma[X] \textit{ is prime.} \end{cases}$$

Observation 10.2. Let σ be an infinite prime 2-structure. Suppose that σ is not finitely critical (see Definition 8.1). Hence, there exists a finite and nonempty subset F of $V(\sigma)$ such that $\sigma - F$ is prime. Clearly, $V(\sigma) \smallsetminus F$ is a proper subset of $V(\sigma)$ and $V(\sigma) \smallsetminus F$ is equipotent to $V(\sigma)$. Therefore, Theorem 10.1 holds for infinite prime 2-structure that are not finitely critical.

Rigollet and Thomassé [29] associated the following digraph with a critical 2-structure.

criticality digraph

Definition 10.3. Consider an infinite prime 2-structure σ. The *criticality digraph* $\mathbb{C}(\sigma)$ of σ is defined on $V(\mathbb{C}(\sigma)) = V(\sigma)$ as follows. Given distinct $v, w \in V(\sigma)$, $(w, v) \in A(\mathbb{C}(\sigma))$ if $\sigma - v$ admits a nontrivial module containing w.

10.1. Modular decomposition in the infinite case.

Notation 10.4. We associate with each 2-structure σ the set $\Upsilon(\sigma)$ of the modules of σ that are maximal under inclusion among the proper modules of σ. (Note that $\Upsilon(\sigma)$ can be empty when σ is infinite.)

Proposition 10.5. *Let σ be a connected 2-structure. If $\Upsilon(\sigma) \neq \varnothing$, then $\Upsilon(\sigma)$ is a modular partition of σ and $\sigma/\Upsilon(\sigma)$ is prime.*

Proof. Since σ is connected, it follows from Proposition 2.12 that σ is uncuttable.

First, we prove that

$$\bigcup \Upsilon(\sigma) = V(\sigma). \tag{10.1}$$

Consider $\mathcal{M} \in \Upsilon(\sigma)$. Let $v \in V(\sigma) \smallsetminus \mathcal{M}$. Consider the family $\mathcal{N}_v$ of the proper modules of σ containing v. Set

$$\mathcal{N} = \bigcup \mathcal{N}_v.$$

[10.1] Rigollet and Thomassé [29] proved this theorem for infinite digraphs.

[10.2] We use the axiom of choice to prove Theorem 10.1. We also use the axiom of choice to prove some of the preliminary results that follow, and we mention its use in their proofs only.

It is easy to verify that $\mathcal{N}$ is a proper module of σ. Indeed, consider $x, y \in \mathcal{N}$ and $w \in V(\sigma) \smallsetminus \mathcal{N}$. We have to verify that

$$w \longleftrightarrow_\sigma \{x, y\}.$$

Since $x, y \in \mathcal{N}$, there exist $N, N' \in \mathcal{N}_v$ such that $x \in N$ and $y \in N'$. Since $N, N' \in \mathcal{N}_v$, we have $v \in N \cap N'$. By assertion (M5) of Proposition 2.5, $N \cup N'$ is a module of σ. Since $w \notin \mathcal{N}$, we have $w \notin N \cup N'$. It follows that $w \longleftrightarrow_\sigma N \cup N'$. In particular, we have $w \longleftrightarrow_\sigma \{x, y\}$. Therefore, $\mathcal{N}$ is a module of σ. For a contradiction, suppose that $\mathcal{N} = V(\sigma)$. Hence, there exists $N \in \mathcal{N}_v$ such that $N \cap \mathcal{M} \neq \varnothing$. By assertion (M5) of Proposition 2.5, $N \cup \mathcal{M}$ is a module of σ. Since $v \in N \smallsetminus \mathcal{M}$, we have $\mathcal{M} \subsetneq N \cup \mathcal{M}$. It follows from the maximality of $\mathcal{M}$ that $N \cup \mathcal{M} = V(\sigma)$. Since $N \smallsetminus \mathcal{M} \neq \varnothing$, it follows from assertion (M6) of Proposition 2.5 that $\mathcal{M} \smallsetminus N$ is a module of σ. Since $N \cup \mathcal{M} = V(\sigma)$, we have $\mathcal{M} \smallsetminus N = V(\sigma) \smallsetminus N$. Consequently, N is a modular cut of σ. Since $v \in N$ and $N \neq V(\sigma)$, N is a nontrivial modular cut of σ, which contradicts the fact that σ is uncuttable. It follows that

$$\mathcal{N} \neq V(\sigma).$$

Hence, (10.1) holds.

Second, we show that $\Upsilon(\sigma)$ is a modular partition of σ. Since (10.1) holds, it suffices to verify that the elements of $\Upsilon(\sigma)$ are pairwise disjoint. Consider $\mathcal{M}, \mathcal{N} \in \Upsilon(\sigma)$ such that $\mathcal{M} \cap \mathcal{N} \neq \varnothing$. By assertion (M5) of Proposition 2.5, $\mathcal{M} \cup \mathcal{N}$ is a module of σ. For a contradiction, suppose that $\mathcal{M} \cup \mathcal{N} = V(\sigma)$. Since $\mathcal{N} \neq V(\sigma)$, we have $\mathcal{M} \smallsetminus \mathcal{N} \neq \varnothing$. By assertion (M6) of Proposition 2.5, $\mathcal{N} \smallsetminus \mathcal{M}$ is a module of σ. Since $\mathcal{M} \cup \mathcal{N} = V(\sigma)$, we have $\mathcal{N} \smallsetminus \mathcal{M} = V(\sigma) \smallsetminus \mathcal{M}$. Thus, $\mathcal{M}$ is a nontrivial modular cut of σ, which contradicts the fact that σ is uncuttable. It follows that

$$\mathcal{M} \cup \mathcal{N} \neq V(\sigma).$$

It follows from the maximality of $\mathcal{M}$ and $\mathcal{N}$ that $\mathcal{M} = \mathcal{M} \cup \mathcal{N}$ and $\mathcal{N} = \mathcal{M} \cup \mathcal{N}$. Consequently, we have $\mathcal{M} = \mathcal{N}$. It follows that $\Upsilon(\sigma)$ is a modular partition of σ.

Third, we prove that $\sigma/\Upsilon(\sigma)$ is prime. Since σ is uncuttable, we have $|\Upsilon(\sigma)| \geq 3$. Let Ψ be a module of $\sigma/\Upsilon(\sigma)$ such that $|\Psi| \geq 2$. We must verify that $\Psi = \Upsilon(\sigma)$. By the second assertion of Lemma 2.10, $\bigcup \Psi$ is a module of σ. Let $\mathcal{M} \in \Psi$. Since $|\Psi| \geq 2$, we have $\mathcal{M} \subsetneq \bigcup \Psi$. It follows from the maximality of $\mathcal{M}$ that $\bigcup \Psi = V(\sigma)$. Hence, we obtain $\Psi = \Upsilon(\sigma)$. □

The following fact is useful to utilize Proposition 10.5.

Fact 10.6. *Let σ be a 2-structure. Consider $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. Let M be a module of σ. If $X \subseteq M$, then*

$$(V(\sigma) \smallsetminus \langle X \rangle_\sigma) \subseteq M \quad \textit{(see Notation 3.12).}$$

Consequently, $\sigma - \langle X \rangle_\sigma$ is connected.

Proof. To begin, consider a module M of σ such that $X \subseteq M$. Let $v \in (V(\sigma) \setminus \langle X\rangle_\sigma)$. By Lemma 3.13, $v \in \mathrm{Ext}_\sigma(X)$ or $v \in X_\sigma(y)$, where $y \in X$. First, suppose that $v \in \mathrm{Ext}_\sigma(X)$. Set

$$Y = X \cup \{v\}.$$

Since $v \in \mathrm{Ext}_\sigma(X)$, $\sigma[Y]$ is prime. By assertion (M2) of Proposition 2.5, $M \cap Y$ is a module of $\sigma[Y]$. Since $X \subseteq (M \cap Y)$, we obtain $M \cap Y = Y$. Thus, $v \in M$. Second, suppose that $v \in X_\sigma(y)$, where $y \in X$. Set

$$Y = (X \setminus \{y\}) \cup \{v\}.$$

Since $v \in X_\sigma(y)$, $\{y, v\}$ is a module of $\sigma[X \cup \{v\}]$. It follows that $\sigma[Y]$ is isomorphic to $\sigma[X]$. Hence, $\sigma[Y]$ is prime. By assertion (M2) of Proposition 2.5, $M \cap Y$ is a module of $\sigma[Y]$. Since $(X \setminus \{y\}) \subseteq (M \cap Y)$, we obtain $M \cap Y = Y$. Therefore, $v \in M$. Consequently, $(V(\sigma) \setminus \langle X\rangle_\sigma) \subseteq M$.

Now, we prove that $\sigma - \langle X\rangle_\sigma$ is uncuttable. Consider a modular cut C of $\sigma - \langle X\rangle_\sigma$. By exchanging C and $(V(\sigma) \setminus \langle X\rangle_\sigma) \setminus C$ if necessary, we can assume that $|C \cap X| \geq 2$. Since $C \cap X$ is a module of $\sigma[X]$ by assertion (M2) of Proposition 2.5, we obtain $X \subseteq C$. It follows from the first assertion above that $(V(\sigma) \setminus \langle X\rangle_\sigma) \subseteq C$. Hence, C is a trivial modular cut of $\sigma - \langle X\rangle_\sigma$. It follows that $\sigma - \langle X\rangle_\sigma$ is uncuttable. By Proposition 2.12, $\sigma - \langle X\rangle_\sigma$ is connected. □

10.2. **Extreme vertices.** Rigollet and Thomassé [29] introduced the following definition.

extreme

Definition 10.7. Consider a critical 2-structure σ. A vertex v of σ is *extreme* if there exists $w \in V(\sigma) \setminus \{v\}$ such that $V(\sigma) \setminus \{v, w\}$ is a module of $\sigma - v$. The set of the extreme vertices of σ is denoted by $\mathscr{E}(\sigma)$.

For instance, as seen in Example 8.11, $\sigma(H_\mathbb{N})$ is a prime element of $\mathscr{F}_\mathbb{N}$. Hence, $\sigma(H_\mathbb{N})$ is critical. Furthermore, $\{2, 3, \ldots\}$ is a module of $\sigma(H_\mathbb{N}) - 0$. Therefore, 0 is an extreme vertex of $\sigma(H_\mathbb{N})$.

We use the next notation to prove Proposition 10.9.

Notation 10.8. Consider an infinite critical 2-structure σ. By using the axiom of choice, we obtain a function

$$F_{\mathscr{E}(\sigma)} : \mathscr{E}(\sigma) \longrightarrow V(\sigma)$$

satisfying for each $v \in \mathscr{E}(\sigma)$, $v \neq F_{\mathscr{E}(\sigma)}(v)$ and

$$F_{\mathscr{E}(\sigma)}(v) \longleftrightarrow_\sigma V(\sigma) \setminus \{v, F_{\mathscr{E}(\sigma)}(v)\} \text{ (see Notation 2.1)},$$

that is, $V(\sigma) \setminus \{v, F_{\mathscr{E}(\sigma)}(v)\}$ is a module of $\sigma - v$.

Lastly, observe that $F_{\mathscr{E}(\sigma)}$ is injective. Indeed, consider distinct $v, v' \in \mathscr{E}(\sigma)$. If $F_{\mathscr{E}(\sigma)}(v) = F_{\mathscr{E}(\sigma)}(v')$, then $V(\sigma) \setminus \{F_{\mathscr{E}(\sigma)}(v)\}$ is a module of σ, which contradicts the fact that σ is prime. Consequently, $F_{\mathscr{E}(\sigma)}$ is injective.

Proposition 10.9. *Given an infinite critical 2-structure σ, $V(\sigma)$ and $V(\sigma) \setminus \mathscr{E}(\sigma)$ are equipotent.*

Proof. Clearly, if $\mathscr{E}(\sigma)$ is finite, then $V(\sigma)$ and $V(\sigma) \smallsetminus \mathscr{E}(\sigma)$ are equipotent because $V(\sigma)$ is infinite. Thus, suppose that $\mathscr{E}(\sigma)$ is infinite.

To begin, we show that $F_{\mathscr{E}(\sigma)}$ does not contain cycles. Otherwise, there exists extreme vertices $v_0, \dots, v_n$ of σ, where $n \geq 1$, such that $F_{\mathscr{E}(\sigma)}(v_0) = v_1, \dots, F_{\mathscr{E}(\sigma)}(v_{n-1}) = v_n$, and $F_{\mathscr{E}(\sigma)}(v_n) = v_0$. We obtain that $V(\sigma) \smallsetminus \{F_{\mathscr{E}(\sigma)}(v_i) : i \in \{0, \dots, n\}\}$ is a module of σ, which contradicts the fact that σ is prime. Consequently, $F_{\mathscr{E}(\sigma)}$ does not contain cycles.

Now, given $v \in \mathscr{E}(\sigma)$, we prove that

(10.2)

$$\text{if } F_{\mathscr{E}(\sigma)}(v) \in \mathscr{E}(\sigma) \text{ and } (F_{\mathscr{E}(\sigma)})^2(v) \in \mathscr{E}(\sigma), \text{ then } (F_{\mathscr{E}(\sigma)})^3(v) \notin \mathscr{E}(\sigma).$$

For a contradiction, suppose that there exists $v \in \mathscr{E}(\sigma)$ such that

$$F_{\mathscr{E}(\sigma)}(v), (F_{\mathscr{E}(\sigma)})^2(v), (F_{\mathscr{E}(\sigma)})^3(v) \in \mathscr{E}(\sigma).$$

Set

$$v = (F_{\mathscr{E}(\sigma)})^0(v) \text{ and } F_{\mathscr{E}(\sigma)}(v) = (F_{\mathscr{E}(\sigma)})^1(v).$$

Since $F_{\mathscr{E}(\sigma)}$ does not contain cycles, $(F_{\mathscr{E}(\sigma)})^0(v)$, $(F_{\mathscr{E}(\sigma)})^1(v)$, $(F_{\mathscr{E}(\sigma)})^2(v)$, $(F_{\mathscr{E}(\sigma)})^3(v)$, and $(F_{\mathscr{E}(\sigma)})^4(v)$ are pairwise distinct. For $i = 0, 1, 2, 3$, there exist $e_{i+1}, f_{i+1} \in E(\sigma)$ such that

$$[(F_{\mathscr{E}(\sigma)})^{i+1}(v), V(\sigma) \smallsetminus \{(F_{\mathscr{E}(\sigma)})^i(v), (F_{\mathscr{E}(\sigma)})^{i+1}(v)\}]_\sigma = (e_{i+1}, f_{i+1}). \tag{10.3}$$

Moreover, for $i = 0, 1, 2, 3$, we have

$$(e_{i+1}, f_{i+1}) \neq [(F_{\mathscr{E}(\sigma)})^{i+1}(v), (F_{\mathscr{E}(\sigma)})^i(v)]_\sigma \tag{10.4}$$

because $V(\sigma) \smallsetminus \{(F_{\mathscr{E}(\sigma)})^{i+1}(v)\}$ is not a module of σ. Using (10.3) and (10.4), we obtain

$$\begin{cases} [(F_{\mathscr{E}(\sigma)})^1(v), (F_{\mathscr{E}(\sigma)})^3(v)]_\sigma = (e_1, f_1), \\ [(F_{\mathscr{E}(\sigma)})^2(v), (F_{\mathscr{E}(\sigma)})^3(v)]_\sigma = (e_2, f_2), \\ [(F_{\mathscr{E}(\sigma)})^3(v), (F_{\mathscr{E}(\sigma)})^1(v)]_\sigma = (e_3, f_3), \\ \text{and} \\ [(F_{\mathscr{E}(\sigma)})^3(v), (F_{\mathscr{E}(\sigma)})^2(v)]_\sigma \neq (e_3, f_3). \end{cases}$$

Therefore, we have

$$(e_1, f_1) \neq (e_2, f_2). \tag{10.5}$$

Using (10.3), we obtain

$$\begin{cases} [(F_{\mathscr{E}(\sigma)})^1(v), (F_{\mathscr{E}(\sigma)})^4(v)]_\sigma = (e_1, f_1), \\ [(F_{\mathscr{E}(\sigma)})^2(v), (F_{\mathscr{E}(\sigma)})^4(v)]_\sigma = (e_2, f_2), \\ [(F_{\mathscr{E}(\sigma)})^4(v), (F_{\mathscr{E}(\sigma)})^1(v)]_\sigma = (e_4, f_4), \\ \text{and} \\ [(F_{\mathscr{E}(\sigma)})^4(v), (F_{\mathscr{E}(\sigma)})^2(v)]_\sigma = (e_4, f_4). \end{cases}$$

It follows that

$$(e_1, f_1) = (e_2, f_2),$$

which contradicts (10.5). Consequently, (10.2) holds.

To conclude, set

$$\begin{cases}\mathscr{E}^0(\sigma) = \{v \in \mathscr{E}(\sigma) : F_{\mathscr{E}(\sigma)}(v) \notin \mathscr{E}(\sigma)\},\\ \mathscr{E}^1(\sigma) = \{v \in \mathscr{E}(\sigma) \setminus \mathscr{E}^0(\sigma) : (F_{\mathscr{E}(\sigma)})^2(v) \notin \mathscr{E}(\sigma)\},\\ \text{and}\\ \mathscr{E}^2(\sigma) = \{v \in \mathscr{E}(\sigma) \setminus (\mathscr{E}^0(\sigma) \cup \mathscr{E}^1(\sigma)) : (F_{\mathscr{E}(\sigma)})^3(v) \notin \mathscr{E}(\sigma)\}.\end{cases}$$

By (10.2), $\{\mathscr{E}^0(\sigma), \mathscr{E}^1(\sigma), \mathscr{E}^2(\sigma)\}$ is a partition of $\mathscr{E}(\sigma)$. Since $\mathscr{E}(\sigma)$ is infinite, we obtain

$$|\mathscr{E}(\sigma)| = \max(|\mathscr{E}^0(\sigma)|, |\mathscr{E}^1(\sigma)|, |\mathscr{E}^2(\sigma)|). \tag{10.6}$$

We obtain

$$\begin{cases}F_{\mathscr{E}(\sigma)}(\mathscr{E}^0(\sigma)) \subseteq V(\sigma) \setminus \mathscr{E}(\sigma),\\ F_{\mathscr{E}(\sigma)}(\mathscr{E}^1(\sigma)) \subseteq \mathscr{E}^0(\sigma),\\ \text{and}\\ F_{\mathscr{E}(\sigma)}(\mathscr{E}^2(\sigma)) \subseteq \mathscr{E}^1(\sigma).\end{cases}$$

Since $F_{\mathscr{E}(\sigma)}$ is injective, we obtain

$$|\mathscr{E}^2(\sigma)| \leq |\mathscr{E}^1(\sigma)| \leq |\mathscr{E}^0(\sigma)| \leq |V(\sigma) \setminus \mathscr{E}(\sigma)|.$$

It follows from (10.6) that $|\mathscr{E}(\sigma)| \leq |V(\sigma) \setminus \mathscr{E}(\sigma)|$. Therefore, we have $|V(\sigma)| = |V(\sigma) \setminus \mathscr{E}(\sigma)|$. □

The next result follows from Proposition 10.5.

Corollary 10.10. *Let σ be an infinite critical 2-structure. Consider distinct $v, w \in V(\sigma)$. If $v \notin \mathscr{E}(\sigma)$ and $(w, v) \notin A(\mathbb{C}(\sigma))$, then $\{w\} \in \Upsilon(\sigma - v)$, $\Upsilon(\sigma - v)$ is a modular partition of $\sigma - v$, and $(\sigma - v)/\Upsilon(\sigma - v)$ is prime.*

Proof. Since $(w, v) \notin A(\mathbb{C}(\sigma))$, we have $\{w\} \in \Upsilon(\sigma - v)$. For a contradiction, suppose that $\sigma - v$ is not connected. There exist $e, f \in E(\sigma - v)$ such that $\sigma - v$ is not $\{e, f\}$-connected. Consider $X \in \mathcal{C}_{\{e,f\}}(\sigma - v)$ (see Definition 2.2) such that $w \in X$. It follows from Lemma 2.4 that X is a module of $\sigma - v$. Since $(w, v) \notin A(\mathbb{C}(\sigma))$, we have

$$X = \{w\}.$$

Using Proposition 2.8, we distinguish the following two cases. In each of them, we obtain a contradiction.

Case 1: $e = f$.

By Proposition 2.8, $(\sigma - v)/\mathcal{C}_{\{e\}}(\sigma - v)$ is constant. Since $\{w\} \in \mathcal{C}_{\{e\}}(\sigma - v)$, we obtain $\mathcal{C}_{\{e\}}(\sigma - v) \setminus \{\{w\}\}$ is a module of $(\sigma - v)/\mathcal{C}_{\{e\}}(\sigma - v)$. By the second assertion of Lemma 2.10, $(V(\sigma) \setminus \{v\}) \setminus \{w\}$ is a module of $\sigma - v$, which contradicts $v \notin \mathscr{E}(\sigma)$.

CASE 2: $e \neq f$.

By Proposition 2.8, $(\sigma - v)/\mathcal{C}_{\{e,f\}}(\sigma - v)$ is linear. Given $\varepsilon \in E((\sigma - v)/\mathcal{C}_{\{e,f\}}(\sigma - v))$, $(\sigma - v)/\mathcal{C}_{\{e,f\}}(\sigma - v)$ is the 2-structure associated to the linear order $(\mathcal{C}_{\{e,f\}}(\sigma - v), \varepsilon)$ (see Remark 1.3). Set

$$\mathcal{X}^- = \{Y \in \mathcal{C}_{\{e,f\}}(\sigma - v) : (Y, X) \in \varepsilon\}.$$

Clearly, $\mathcal{X}^- \cup \{\{w\}\}$ is a module of $(\sigma - v)/\mathcal{C}_{\{e,f\}}(\sigma - v)$. By the second assertion of Lemma 2.10, $\bigcup(\mathcal{X}^- \cup \{\{w\}\})$ is a module of $\sigma - v$. Since $(w, v) \notin A(\mathbb{C}(\sigma))$, we obtain

$$\bigcup(\mathcal{X}^- \cup \{\{w\}\}) = \{w\} \text{ or } \bigcup(\mathcal{X}^- \cup \{\{w\}\}) = V(\sigma) \setminus \{v\}.$$

We obtain that $\{w\}$ is the least vertex or the greatest vertex of the linear order $(\mathcal{C}_{\{e,f\}}(\sigma - v), \varepsilon)$. In both cases, it follows from the second assertion of Lemma 2.10 that $(V(\sigma) \setminus \{v\}) \setminus \{w\}$ is a module of $\sigma - v$, which contradicts $v \notin \mathscr{E}(\sigma)$.

Consequently, $\sigma - v$ is connected. It follows from Proposition 10.5 that $\Upsilon(\sigma - v)$ is a modular partition of $\sigma - v$ and $(\sigma - v)/\Upsilon(\sigma - v)$ is prime. □

Corollary 10.10 leads us to introduce the following notation.

Notation 10.11. Consider an infinite critical 2-structure σ. Set

$$W(\sigma) = V(\sigma) \setminus \mathscr{E}(\sigma).$$

We consider the following subsets of $W(\sigma)$. First, we denote by $W_\varnothing(\sigma)$ the set of $v \in W(\sigma)$ such that $\Upsilon(\sigma - v) = \varnothing$. Second, we denote by $W_\delta(\sigma)$ the set of $v \in W(\sigma)$ such that $\sigma - v$ is not connected. Third, we denote by $W_\pi(\sigma)$ the set of $v \in W(\sigma)$ such that $\Upsilon(\sigma - v)$ is a modular partition of $\sigma - v$ and $(\sigma - v)/\Upsilon(\sigma - v)$ is prime.

Let $v \in W_\delta(\sigma)$. Since $\sigma - v$ is not connected, there exist $e_v, f_v \in E(\sigma - v)$ such that $\sigma - v$ is not $\{e_v, f_v\}$-connected. Hence, there exist $e, f \in E(\sigma)$ such that $e_v = e \cap (V(\sigma - v) \times V(\sigma - v))$ and $f_v = f \cap (V(\sigma - v) \times V(\sigma - v))$. We denote $\{e, f\}$ by $\lambda(v)$.

Observation 10.12. Consider an infinite critical 2-structure σ. It follows from Proposition 10.5 that

$$W(\sigma) = W_\varnothing(\sigma) \cup W_\delta(\sigma) \cup W_\pi(\sigma).$$

Furthermore, we have

$$(W_\varnothing(\sigma) \cup W_\delta(\sigma)) \cap W_\pi(\sigma) = \varnothing.$$

Clearly, we have $W_\varnothing(\sigma) \cap W_\pi(\sigma) = \varnothing$. To verify that $W_\delta(\sigma) \cap W_\pi(\sigma) = \varnothing$, it suffices to show that if $v \in W_\delta(\sigma)$ and $\Upsilon(\sigma - v)$ is a modular partition of $\sigma - v$, then $|\Upsilon(\sigma - v)| = 2$. Indeed, consider $v \in W_\delta(\sigma)$ and $\Upsilon(\sigma - v)$ is a modular partition of $\sigma - v$. Since $v \in W_\delta(\sigma)$, $\sigma - v$ is not connected. It follows from Proposition 2.12 that $\sigma - v$ admits a nontrivial modular cut C. Set

$$P = \{M \in \Upsilon(\sigma - v) : M \cap C \neq \varnothing\}.$$

Since C is a nontrivial modular cut, $C \neq \varnothing$, and hence, $P \neq \varnothing$. By the first assertion of Lemma 2.10, P is a module of $(\sigma - v)/\Upsilon(\sigma - v)$. By the second assertion of Lemma 2.10, $\cup P$ is a module of $(\sigma - v)$. It follows from the maximality of the elements of $\Upsilon(\sigma - v)$ that $|P| = 1$ or $P = \Upsilon(\sigma - v)$. For a contradiction, suppose that $P = \Upsilon(\sigma - v)$. Since C is a nontrivial modular cut of $\sigma - v$, there exists $M \in \Upsilon(\sigma - v)$ such that $M \smallsetminus C \neq \varnothing$. Consider $N \in \Upsilon(\sigma - v) \smallsetminus \{M\}$. Since $P = \Upsilon(\sigma - v)$, $N \cap C \neq \varnothing$. By assertion (M5) of Proposition 2.5, $C \cup N$ is a module of $\sigma - v$. Since $P = \Upsilon(\sigma - v)$, $M \cap C \neq \varnothing$. Thus, we have $N \subsetneq C \cup N$. Furthermore, since $M \smallsetminus C \neq \varnothing$, we obtain

$$N \subsetneq (C \cup N) \subsetneq V(\sigma - v),$$

which contradicts the maximality of N. It follows that $|P| = 1$. Therefore, there exists $M \in \Upsilon(\sigma - v)$ such that $C \subseteq M$. Similarly, there exists $N \in \Upsilon(\sigma - v)$ such that $(V(\sigma - v) \smallsetminus C) \subseteq N$. Since $\Upsilon(\sigma - v)$ is a modular partition of $\sigma - v$, we obtain $\Upsilon(\sigma - v) = \{M, N\}$.

Lastly, note that we can have

$$W_{\varnothing}(\sigma) \cap W_{\delta}(\sigma) \neq \varnothing.$$

Consider the 2-structure $\sigma(T_{\mathbb{Z}})$ introduced in Example 8.37. We have

$$\sigma(T_{\mathbb{Z}}) - \infty = \sigma(L_{\mathbb{Z}}).$$

Consequently, we have $\infty \in W_{\varnothing}(\sigma(T_{\mathbb{Z}})) \cap W_{\delta}(\sigma(T_{\mathbb{Z}}))$.

10.3. **The criticality digraph.**

Fact 10.13. *Let σ be an infinite prime 2-structure. Consider distinct $v, w \in V(\sigma)$. Suppose that $\sigma - v$ admits a nontrivial module M_v and $\sigma - w$ admits a nontrivial module M_w. If $w \notin M_v$ and $v \notin M_w$, then $|M_v \cap M_w| \leq 1$.*

Proof. Suppose that $w \notin M_v$ and $v \notin M_w$. We obtain that $M_v \cap M_w$ is a module of σ. Since σ is prime, we have $|M_v \cap M_w| \leq 1$. ☐

Fact 10.14. *Let σ be an infinite prime 2-structure. Consider distinct $v, w \in V(\sigma)$. Suppose that $\sigma - v$ admits a nontrivial module M_v and $\sigma - w$ admits a nontrivial module M_w. If $w \in M_v$ and $v \notin M_w$, then $M_v \cap M_w \neq \varnothing$.*

Proof. For a contradiction, suppose that $M_v \cap M_w = \varnothing$. We verify that M_w is a module of σ. Since M_w is a module of $\sigma - w$, we have only to verify that $w \longleftrightarrow_{\sigma} M_w$ (see Notation 2.1). Thus, consider $x, y \in M_w$. Since M_v is a nontrivial module of $\sigma - v$, $|M_v| \geq 2$, and hence there exists $w' \in M_v \smallsetminus \{w\}$. Since M_v is a module of $\sigma - v$, we have $[w, x]_{\sigma} = [w', x]_{\sigma}$ and $[w, y]_{\sigma} = [w', y]_{\sigma}$. Furthermore, we have $[w', x]_{\sigma} = [w', y]_{\sigma}$ because M_w is a module of $\sigma - w$. Therefore, we obtain $[w, x]_{\sigma} = [w, y]_{\sigma}$. It follows that M_w is a module of σ, which contradicts the fact that σ is prime. Consequently, $M_v \cap M_w \neq \varnothing$. ☐

Fact 10.15. *Let σ be an infinite prime 2-structure. Consider distinct u,v, $w \in V(\sigma)$. Suppose that $u, v \in W(\sigma)$ and $(v, u) \in A(\mathbb{C}(\sigma))$. If $\sigma - v$ admits a nontrivial module M_v such that $w \in M_v$ and $u \notin M_v$, then $\sigma - u$ admits a nontrivial module containing v and w.*

Proof. Since $(v,u) \in A(\mathbb{C}(\sigma))$, $\sigma - u$ admits a nontrivial module M_u containing v. We can conclude if $w \in M_u$. Hence, suppose that $w \notin M_u$. Thus, we have $w \in M_v \setminus M_u$. By Fact 10.14, $M_u \cap M_v \neq \varnothing$. Since $u \notin (M_u \cup M_v)$, we obtain that $M_u \cup M_v$ is a module of $\sigma - u$. We distinguish the following two cases to verify that $M_u \cup M_v$ is a nontrivial module of $\sigma - u$.

CASE 1: $|M_u \setminus M_v| \geq 2$.

We show that $M_v \setminus M_u$ is a module of σ. Clearly, for every $x \in V(\sigma) \setminus (M_v \cup \{v\})$, we have $x \longleftrightarrow_\sigma M_v \setminus M_u$ (see Notation 2.1). Consider $x \in (M_u \cap M_v) \cup \{v\}$. Since $|M_u \setminus M_v| \geq 2$, there exists $x' \in (M_u \setminus M_v) \setminus v$. Let $y, z \in M_v \setminus M_u$. Since M_u is a module of $\sigma - u$, we have $[x,y]_\sigma = [x',y]_\sigma$ and $[x,z]_\sigma = [x',z]_\sigma$. Furthermore, since M_v is a module of $\sigma - v$, we have $[x',y]_\sigma = [x',z]_\sigma$. It follows that $[x,y]_\sigma = [x,z]_\sigma$. Thus, $x \longleftrightarrow_\sigma M_v \setminus M_u$ for every $x \in (M_u \cap M_v) \cup \{v\}$. Consequently, $M_v \setminus M_u$ is a module of σ. Since σ is prime, $M_v \setminus M_u$ is a trivial module of σ. Hence, we obtain $M_v \setminus M_u = \{w\}$. If $M_u \cup M_v$ is a trivial module of $\sigma - u$, then $M_u \cup M_v = V(\sigma) \setminus \{u\}$, and hence $M_u = V(\sigma) \setminus \{u, w\}$, which contradicts $u \notin \mathscr{E}(\sigma)$. Therefore, $M_u \cup M_v$ is a nontrivial module of $\sigma - u$.

CASE 2: $|M_u \setminus M_v| \leq 1$.

Since $v \in M_u \setminus M_v$, we obtain $M_u \setminus M_v = \{v\}$. If $M_u \cup M_v$ is a trivial module of $\sigma - u$, then $M_u \cup M_v = V(\sigma) \setminus \{u\}$, and hence $M_v = V(\sigma) \setminus \{u, v\}$, which contradicts $v \notin \mathscr{E}(\sigma)$. Therefore, $M_u \cup M_v$ is a nontrivial module of $\sigma - u$.

Consequently, $M_u \cup M_v$ is a nontrivial module of $\sigma - u$ containing v and w. □

The next result follows from Fact 10.15.

Corollary 10.16. *Let σ be an infinite prime 2-structure. Consider distinct $u, v, w \in V(\sigma)$. Suppose that $u, v \in W(\sigma)$. If $(v,u), (w,v) \in A(\mathbb{C}(\sigma))$ and $(u,v) \notin A(\mathbb{C}(\sigma))$, then $\sigma - u$ admits a nontrivial module containing v and w.*

The next result follows from Corollary 10.16.

Corollary 10.17. *Let σ be an infinite prime 2-structure. Consider distinct $u, v, w \in W(\sigma)$ (see Notation 10.11). If $(v,u), (w,v) \in A(\mathbb{C}(\sigma))$ and $(u,v), (v,w) \notin A(\mathbb{C}(\sigma))$, then $(w,u) \in A(\mathbb{C}(\sigma))$ and $(u,w) \notin A(\mathbb{C}(\sigma))$.*

Proof. Suppose that $(v,u), (w,v) \in A(\mathbb{C}(\sigma))$ and

$$(u,v), (v,w) \notin A(\mathbb{C}(\sigma)). \tag{10.7}$$

Since $(v,u), (w,v) \in A(\mathbb{C}(\sigma))$ and $(u,v) \notin A(\mathbb{C}(\sigma))$, we obtain $(w,u) \in A(\mathbb{C}(\sigma))$ by Corollary 10.16.

For a contradiction, suppose that $(u,w) \in A(\mathbb{C}(\sigma))$. Since $(w,v) \in A(\mathbb{C}(\sigma))$ and $(v,w) \notin A(\mathbb{C}(\sigma))$, it follows from Corollary 10.16 that $(u,v) \in A(\mathbb{C}(\sigma))$, which contradicts (10.7). Consequently, we have $(u,w) \notin A(\mathbb{C}(\sigma))$. □

The next result is an immediate consequence of Corollary 10.10 and Notation 10.11.

Corollary 10.18. *Let σ be an infinite critical 2-structure. Consider $v \in W(\sigma)$. If $v \in W_{\varnothing}(\sigma) \cup W_{\delta}(\sigma)$, then $(w, v) \in A(\mathbb{C}(\sigma))$ for every $w \in V(\sigma) \setminus \{v\}$.*

Corollary 10.18 leads us to introduce the following notation.

Notation 10.19. Let σ be an infinite critical 2-structure. Given $v, w \in W(\sigma)$, $v \sim_{\sigma} w$ means $v = w$ or $v \neq w$ and $(v, w), (w, v) \in A(\mathbb{C}(\sigma))$. Clearly, $\sim_{\sigma}$ is a symmetric and reflexive binary relation defined on $V(\sigma) \setminus \mathscr{E}(\sigma)$. It follows from Corollary 10.18 that $v \sim_{\sigma} w$ for any $v, w \in W_{\varnothing}(\sigma) \cup W_{\delta}(\sigma)$.

In the next lemmas, we examine the binary relation $\sim_{\sigma}$.

Lemma 10.20. *Let σ be an infinite critical 2-structure. Consider distinct $v, w \in W(\sigma)$. If $v \sim_{\sigma} w$ and $v \in W_{\pi}(\sigma)$, then $w \in W_{\pi}(\sigma)$.*

Proof. Since $(w, v) \in A(\mathbb{C}(\sigma))$ and $v \in W_{\pi}(\sigma)$, there exists $M_v \in \Upsilon(\sigma - v)$ such that $w \in M_v$ and $|M_v| \geq 2$. Consider the set $\mathcal{X}$ of $X \subseteq (V(\sigma) \setminus \{v, w\})$ such that $|X \cap M| = 1$ for each $M \in \Upsilon(\sigma - v)$. Using the axiom of choice, we obtain

$$\mathcal{X} \neq \varnothing.$$

Since $v \in W_{\pi}(\sigma)$, $(\sigma - v)/\Upsilon(\sigma - v)$ is prime. It follows that $\sigma[X]$ is prime for each $X \in \mathcal{X}$.

Let $X \in \mathcal{X}$. We show that

$$(V(\sigma) \setminus (X \cup \{v\})) \subseteq \bigcup_{z \in X} X_{\sigma}(z) \quad \text{(see Notation 3.12)}. \tag{10.8}$$

Let $u \in (V(\sigma) \setminus (X \cup \{v\}))$. Since $u \neq v$, there exists $N_v \in \Upsilon(\sigma - v)$ such that $u \in N_v$. By denoting by z the unique element of $N_v \cap X$, we obtain $u \in X_{\sigma}(z)$. Hence, (10.8) holds.

Consider a nontrivial module M_w of $\sigma - w$ containing v. For each $X \in \mathcal{X}$, we show that

$$|M_w \cap X| \leq 1. \tag{10.9}$$

As seen in Remark 3.16, we have $|M_w \cap X| \leq 1$ or $X \subseteq M_w$. It follows from Fact 10.6 and (10.8) that $|M_w \cap X| \leq 1$. Hence, (10.9) holds.

Given $X \in \mathcal{X}$, we show that there exists $y \in X$ such that

$$v \in X_{\sigma}(y). \tag{10.10}$$

Since $(v, w) \in A(\mathbb{C}(\sigma))$, $\sigma - w$ admits a nontrivial module M_w containing v. By (10.9), $|M_w \cap X| \leq 1$. Clearly, if $|M_w \cap X| = 1$, then (10.10) holds by denoting the unique element of $M_w \cap X$ by y. Hence, suppose that $M_w \cap X = \varnothing$. Let $u \in M_w \setminus \{v\}$. Since $u \neq v$ and $u \neq w$, it follows from (10.8) that $u \in X_{\sigma}(y)$, where $y \in X$. We verify that $v \in X_{\sigma}(y)$ too. Let $z \in X \setminus \{y\}$. Since $u \in X_{\sigma}(y)$, we have $z \longleftrightarrow_{\sigma} \{y, u\}$ (see Notation 2.1). Furthermore, since $M_w \cap X = \varnothing$, we have $z \notin M_w$. It follows that $z \longleftrightarrow_{\sigma} \{u, v\}$. Therefore, we obtain $z \longleftrightarrow_{\sigma} \{y, v\}$. Consequently, $\{y, v\}$ is a module of $\sigma[X \cup \{v\}]$. Hence, $v \in X_{\sigma}(y)$, so (10.10) holds.

Now, we show that $\Upsilon(\sigma - w) \neq \varnothing$. Consider any nontrivial module M_w of $\sigma - w$ containing v. By (10.9), we have $|M_w \cap X| \leq 1$. By (10.10), there exists $y \in X$ such that $v \in X_\sigma(y)$. It follows from Remark 3.16 that $M_w \subseteq (\{y\} \cup X_\sigma(y))$. Consequently, there exists $\mathcal{M}_w \in \Upsilon(\sigma - w)$ such that $v \in \mathcal{M}_w$.

Lastly, we prove that $w \in W_\pi(\sigma)$. Consider again $M_v \in \Upsilon(\sigma - v)$ such that $w \in M_v$ and $|M_v| \geq 2$. There exists $X \in \mathcal{X}$ such that $w \notin X \cap M_v$. It follows from (10.8) and (10.10) that

$$(V(\sigma) \setminus X) \subseteq \bigcup_{z \in X} X_\sigma(z).$$

Therefore, it follows from Fact 10.6 that $\sigma - w$ is connected. Since $\Upsilon(\sigma - w) \neq \varnothing$, it follows from Proposition 10.5 that $w \in W_\pi(\sigma)$. ☐

The next result follows from Lemma 10.20.

Corollary 10.21. *Let σ be an infinite critical 2-structure. Given distinct $v, w \in W_\pi(\sigma)$, if $v \sim_\sigma w$, then there exists $X \subseteq (V(\sigma) \setminus \{v, w\})$ satisfying*

- *$\sigma[X]$ is prime;*
- *there exist distinct $y, z \in X$ such that $v \in X_\sigma(y)$ and $w \in X_\sigma(z)$;*
- *$\Upsilon(\sigma - v) = \{(\{y\} \cup X_\sigma(y)) \setminus \{v\}, \{z\} \cup X_\sigma(z))\} \cup \{\{u\} : u \in X \setminus \{y, z\}\}$;*
- *$\Upsilon(\sigma - w) = \{\{y\} \cup X_\sigma(y)), (\{z\} \cup X_\sigma(z)) \setminus \{w\}\} \cup \{\{u\} : u \in X \setminus \{y, z\}\}$;*
- *$p_{(\sigma, X)} = \{X_\sigma(y), X_\sigma(z)\}$ and $E(\Gamma_{(\sigma, X)}) = \{\{v, w\}\}$.*

Proof. As in the proof of Lemma 10.20, consider the set $\mathcal{X}$ of $X \subseteq V(\sigma) \setminus \{v, w\}$ such that $|X \cap M_v| = 1$ for each $M_v \in \Upsilon(\sigma - v)$. Using the axiom of choice, we obtain $\mathcal{X} \neq \varnothing$. Since $v \in W_\pi(\sigma)$, $\Upsilon(\sigma - v)$ is a modular partition of $\sigma - v$ and $(\sigma - v)/\Upsilon(\sigma - v)$ is prime. It follows that $\sigma[X]$ is prime for each $X \in \mathcal{X}$.

Let $X \in \mathcal{X}$. It follows from (10.8) and (10.10) (see the proof of Lemma 10.20) that

$$(V(\sigma) \setminus X) \subseteq \bigcup_{y \in X} X_\sigma(y) \quad \text{(see Notation 3.12)}. \tag{10.11}$$

In particular, there exist $y, z \in X$ such that $v \in X_\sigma(y)$ and $w \in X_\sigma(z)$. For each $t \in X \setminus \{y\}$, we have

$$(\{t\} \cup X_\sigma(t)) \in \Upsilon(\sigma - v). \tag{10.12}$$

Indeed, consider $u \in X_\sigma(t)$. Since $t \neq y$, we have $u \neq v$. Thus, there exists $M_v^u \in \Upsilon(\sigma - v)$ such that $u \in M_v^u$. Since $X \in \mathcal{X}$, there exists $t' \in X$ such that $M_v^u \cap X = \{t'\}$. We obtain $u \in X_\sigma(t')$. It follows from Lemma 3.13 that $t' = t$. Therefore, $t \in M_v^u$ for each $u \in X_\sigma(t)$. Since M_v^u is a module of $\sigma - v$ for each $u \in X_\sigma(t)$, it is not difficult to verify that

$$\bigcup_{u \in X_\sigma(t)} M_v^u$$

is a module of $\sigma - v$. Let $u \in X_\sigma(t)$. Since $t \in M_v^u$, it follows from Remark 3.16 that $(M_v^u \setminus \{t\}) \subseteq X_\sigma(t)$. We obtain

$$\bigcup_{u \in X_\sigma(t)} M_v^u = \{t\} \cup X_\sigma(t).$$

It follows from the maximality of the elements of $\Upsilon(\sigma - v)$ that $(\{t\} \cup X_\sigma(t)) \in \Upsilon(\sigma - v)$. Hence, (10.12) holds. Similarly, we have

$$((\{y\} \cup X_\sigma(y)) \setminus \{v\}) \in \Upsilon(\sigma - v). \tag{10.13}$$

By Lemma 10.20, we have $w \in W_\pi(\sigma)$. Hence, $\Upsilon(\sigma - w)$ is a modular partition of $\sigma - w$ and $(\sigma - w)/\Upsilon(\sigma - w)$ is prime. Now, we establish the analogues of (10.12) and (10.13) for $\Upsilon(\sigma - w)$. We verify that for each $M_w \in \Upsilon(\sigma - w)$, we have

$$|M_w \cap X| \leq 1 \quad \text{(see (10.9) in the proof of Lemma 10.20).} \tag{10.14}$$

We can assume that $|M_w| \geq 2$, so M_w is a nontrivial module of $\sigma - w$. As seen in Remark 3.16, we have $|M_w \cap X| \leq 1$ or $X \subseteq M_w$. For a contradiction, suppose that $X \subseteq M_w$. Let $u \in V(\sigma - w) \setminus M_w$. Since $X \subseteq M_w$, we have $u \in \langle X \rangle_\sigma$ (see Notation 3.12). By (10.11), $u \in X_\sigma(t)$, where $t \in X$. We obtain $u \in X_\sigma(t) \cap \langle X \rangle_\sigma$, which contradicts Lemma 3.13. It follows that $|M_w \cap X| \leq 1$. Hence, (10.14) holds. Consequently, there exists $Y \subseteq V(\sigma) \setminus \{w\}$ such that for each $M_w \in \Upsilon(\sigma - w)$, the following three assertions hold

- $|Y \cap M_w| = 1$;
- if $|M_w \cap X| = 1$, then $M_w \cap Y = M_w \cap X$;
- if $v \in M_w$, then $|M_w| \geq 2$ because $(v, w) \in A(\mathbb{C}(\sigma))$, so that we can require that $|Y \cap (M_w \setminus \{v\})| = 1$.

It follows that

$$X \subseteq Y \subseteq (V(\sigma) \setminus \{v, w\}).$$

Since $(\sigma - w)/\Upsilon(\sigma - w)$ is prime, $\sigma[Y]$ is prime. We show that

$$X = Y. \tag{10.15}$$

For a contradiction, suppose that there exists $u \in Y \setminus X$. There exists $M_w^u \in \Upsilon(\sigma - w)$ such that $u \in M_w^u$. Since $M_w^u \cap Y = \{u\}$ and $u \notin X$, we have $M_w^u \cap X = \varnothing$. It follows from (10.11) that $M_w^u \subseteq X_\sigma(t)$, where $t \in X$. Moreover, it follows from (10.12) and (10.13) that $(\{t\} \cup X_\sigma(t)) \setminus \{v\}$ is a module of $\sigma - v$. By assertion (M2) of Proposition 2.5, $((\{t\} \cup X_\sigma(t)) \setminus \{v\}) \cap Y$ is a module of $\sigma[Y]$. Since $X \subseteq Y \subseteq (V(\sigma) \setminus \{v, w\})$ and $M_w^u \subseteq X_\sigma(t)$, we obtain $((\{t\} \cup X_\sigma(t)) \setminus \{v\}) \cap Y = (\{t\} \cup X_\sigma(t)) \cap Y$ and $u, t \in (\{t\} \cup X_\sigma(t)) \cap Y$. Furthermore, given $t' \in (X \setminus \{t\})$, we have $t' \in Y \setminus (\{t\} \cup X_\sigma(t))$. Therefore, $(\{t\} \cup X_\sigma(t)) \cap Y$ is a nontrivial module of $\sigma[Y]$, which contradicts the fact that $\sigma[Y]$ is prime. Consequently, (10.15) holds. The analogues of (10.12) and (10.13) for $\Upsilon(\sigma - w)$ follow. They are proved as previously. For each $t \in X \setminus \{z\}$, we have

$$(\{t\} \cup X_\sigma(t)) \in \Upsilon(\sigma - w). \tag{10.16}$$

Similarly, we have

$$(10.17)\qquad ((\{z\}\cup X_\sigma(z))\smallsetminus\{w\})\in\Upsilon(\sigma-w).$$

It follows that for every $t\in X\smallsetminus\{y,z\}$, we have

$$(10.18)\qquad X_\sigma(t)=\varnothing.$$

Indeed, let $t\in X\smallsetminus\{y,z\}$. It follows from (10.12) that $\{t\}\cup X_\sigma(t)$ is a module of $\sigma-v$. Moreover, $\{t\}\cup X_\sigma(t)$ is a module of $\sigma-w$ by (10.16). Therefore, $\{t\}\cup X_\sigma(t)$ is a module of σ. Since σ is prime, we have $X_\sigma(t)=\varnothing$. Hence, (10.18) holds. It follows from (10.12) and (10.13) that

$$\Upsilon(\sigma-v)=\{(\{y\}\cup X_\sigma(y))\smallsetminus\{v\},\{z\}\cup X_\sigma(z))\}\cup\{\{u\}:u\in X\smallsetminus\{y,z\}\}.$$

Similarly, we have

$$\Upsilon(\sigma-w)=\{\{y\}\cup X_\sigma(y)),(\{z\}\cup X_\sigma(z))\smallsetminus\{w\}\}\cup\{\{u\}:u\in X\smallsetminus\{y,z\}\}.$$

It follows from (10.11) that

$$p_{(\sigma,X)}=\{X_\sigma(y),X_\sigma(z)\}.$$

Suppose for a contradiction that $y=z$. Since $\{y\}\cup X_\sigma(y)$ is a module of $\sigma-w$ by (10.16) and $\{z\}\cup X_\sigma(z)$ is a module of $\sigma-v$ by (10.12), $\{y\}\cup X_\sigma(y)\cup X_\sigma(z)$ is a module of σ, which contradicts the fact that σ is prime. It follows that

$$y\neq z.$$

Since $\{y\}\cup X_\sigma(y)$ is a module of $\sigma-w$, we obtain $\{u,u'\}\notin E(\Gamma_{(\sigma,X)})$ (see Definition 9.3) for any $u\in X_\sigma(y)$ and $u'\in X_\sigma(z)\smallsetminus\{w\}$. Similarly, since $\{z\}\cup X_\sigma(z)$ is a module of $\sigma-v$, we obtain $\{u,u'\}\notin E(\Gamma_{(\sigma,X)})$ for any $u\in X_\sigma(y)\smallsetminus\{v\}$ and $u'\in X_\sigma(z)$. By Theorem 3.19, $\Gamma_{(\sigma,X)}$ is nonempty. Therefore, we have

$$E(\Gamma_{(\sigma,X)})=\{\{v,w\}\}.$$

□

Lemma 10.22. *Let σ be an infinite critical 2-structure. Consider distinct $v,w\in W(\sigma)$. If $v\sim_\sigma w$ and $v\in W_\delta(\sigma)$, then $w\in W_\delta(\sigma)$ and $\lambda(v)=\lambda(w)$.*

Proof. Suppose that $v\sim_\sigma w$ and $v\in W_\delta(\sigma)$. Since $v\in W_\delta(\sigma)$, there exist a nontrivial modular cut C_v of $\sigma-v$ and $e,f\in E(\sigma)$ such that $[C_v,V(\sigma-v)\smallsetminus C_v]_\sigma=(e,f)$, where $\lambda(v)=\{e,f\}$. For instance, assume that $w\in C_v$. Since $v\notin\mathscr{E}(\sigma)$, we have $|C_v|\geq 2$ and $|V(\sigma-v)\smallsetminus C_v|\geq 2$.

First, we prove that $w\in W_\delta(\sigma)$. Since $(W_\varnothing(\sigma)\cup W_\delta(\sigma))\cap W_\pi(\sigma)=\varnothing$ (see Observation 10.12), it follows from Lemma 10.20 that $w\notin W_\pi(\sigma)$. Suppose that $w\in W_\varnothing(\sigma)$. We must show that $w\in W_\delta(\sigma)$. Since $w\in W_\varnothing(\sigma)$, $\sigma-w$ admits a proper module M_w such that $v\in M_w$, $M_w\cap C_v\neq\varnothing$, and $M_w\cap(V(\sigma-v)\smallsetminus C_v)\neq\varnothing$. For a contradiction, suppose that $(V(\sigma-v)\smallsetminus C_v)\smallsetminus M_w\neq\varnothing$. Since $M_w\cap C_v\neq\varnothing$, we obtain $[M_w,(V(\sigma-v)\smallsetminus C_v)\smallsetminus M_w]_\sigma=(e,f)$. Since $[C_v,V(\sigma-v)\smallsetminus C_v]_\sigma=(e,f)$, we obtain $[M_w\cup C_v,(V(\sigma-v)\smallsetminus C_v)\smallsetminus M_w]_\sigma=(e,f)$. Since $v,w\in M_w\cup C_v$, $M_w\cup C_v$ is a nontrivial module of σ, which contradicts the fact that σ is prime. Consequently, we have $(V(\sigma-v)\smallsetminus C_v)\subseteq M_w$. Since M_w is a proper module of $\sigma-w$, we have $(C_v\smallsetminus\{w\})\smallsetminus M_w\neq\varnothing$.

Since $(V(\sigma - v) \setminus C_v) \subseteq M_w$ and $[C_v, V(\sigma - v) \setminus C_v]_\sigma = (e, f)$, we obtain $[(C_v \setminus \{w\}) \setminus M_w, M_w]_\sigma = (e, f)$. It follows that $w \in W_\delta(\sigma)$.

Second, suppose for a contradiction that $\lambda(v) \neq \lambda(w)$. There exist a nontrivial modular cut C_w of $\sigma - w$ and $e', f' \in E(\sigma)$ such that $[C_w, V(\sigma - w) \setminus C_w]_\sigma = (e', f')$, where $\lambda(w) = \{e,' f'\}$. Since $w \notin \mathscr{E}(\sigma)$, we have $|C_w| \geq 2$ and $|V(\sigma - w) \setminus C_w| \geq 2$. For instance, assume that $C_v \cap C_w \neq \varnothing$. Let $u \in C_v \cap C_w$. Since $[u, V(\sigma - w) \setminus C_w]_\sigma = (e', f')$, $[u, V(\sigma - v) \setminus C_v]_\sigma = (e, f)$, and $\{e, f\} \neq \{e,' f'\}$, we obtain $(V(\sigma - w) \setminus C_w) \cap (V(\sigma - v) \setminus C_v) = \varnothing$. Therefore, we have $(V(\sigma) \setminus (C_w \cup \{w\})) \subseteq (C_v \cup \{v\})$. Since $w \in C_v$, we obtain $(V(\sigma) \setminus C_w) \subseteq (C_v \cup \{v\})$. Hence, we have also $(V(\sigma - v) \setminus C_v) \subseteq C_w$. Let $u' \in (V(\sigma - v) \setminus C_v)$. Since $[u', C_v]_\sigma = (f, e)$ and $[u', V(\sigma - w) \setminus C_w]_\sigma = (e', f')$, we obtain $C_v \cap (V(\sigma - w) \setminus C_w) = \varnothing$. We obtain $C_v \subseteq (C_w \cup \{w\})$. It follows that $V(\sigma - v) \subseteq (C_w \cup \{w\})$. Thus, we have $(V(\sigma - w) \setminus C_w) \subseteq \{v\}$, which contradicts the fact that $w \notin \mathscr{E}(\sigma)$. It follows that $\lambda(v) = \lambda(w)$. □

Lemma 10.23. *Given an infinite critical 2-structure σ, we have $|W_\varnothing(\sigma) \setminus W_\delta(\sigma)| \leq 1$. Moreover, if $|W_\varnothing(\sigma) \setminus W_\delta(\sigma)| = 1$, then $|W_\varnothing(\sigma)| = 1$ and $W_\delta(\sigma) = \varnothing$.*

Proof. To begin, suppose that there exist distinct $v, w \in W_\varnothing(\sigma) \setminus W_\delta(\sigma)$. Let $x \in V(\sigma) \setminus \{v, w\}$. Since $\Upsilon(\sigma - v) = \varnothing$, there exists a proper module M_v of $\sigma - v$ such that $x, w \in M_v$. Similarly, there exists a proper module M_w of $\sigma - w$ such that $x, v \in M_w$. It is easy to verify that $M_v \cup M_w$ is a module of σ. Since σ is prime, we obtain

$$M_v \cup M_w = V(\sigma). \tag{10.19}$$

We show that M_v is a nontrivial modular cut of $\sigma - v$. Recall that M_v is a proper module of $\sigma - v$. By (10.19),

$$V(\sigma - v) \setminus M_v = (M_w \setminus \{v\}) \setminus M_v. \tag{10.20}$$

By assertion (M2) of Proposition 2.5, $M_w \setminus \{v\}$ and $M_v \setminus \{w\}$ are modules of $\sigma - \{v, w\}$. Since $w \notin \mathscr{E}(\sigma)$, it follows from (10.19) that $|M_v \setminus M_w| \geq 2$. Hence, we have $(M_v \setminus \{w\}) \setminus M_w$, which is $(M_v \setminus \{w\}) \setminus (M_w \setminus \{v\})$, is nonempty. It follows from assertion (M6) of Proposition 2.5 that $(M_w \setminus \{v\}) \setminus (M_v \setminus \{w\})$, which is $(M_w \setminus \{v\}) \setminus M_v$, is a module of $\sigma - \{v, w\}$. To prove that $(M_w \setminus \{v\}) \setminus M_v$ is a module of $\sigma - v$, it remains to verify that $w \longleftrightarrow_\sigma ((M_w \setminus \{v\}) \setminus M_v)$ (see Notation 2.1). Let $y, z \in ((M_w \setminus \{v\}) \setminus M_v)$. Since M_v is a module of $\sigma - v$ and $y \neq v$, we have $[w, y]_\sigma = [x, y]_\sigma$. Similarly, we have $[w, z]_\sigma = [x, z]_\sigma$. Since $(M_w \setminus \{v\}) \setminus M_v$ is a module of $\sigma - \{v, w\}$, we have $[x, y]_\sigma = [x, z]_\sigma$. It follows that $[w, y]_\sigma = [w, z]_\sigma$. Thus, $(M_w \setminus \{v\}) \setminus M_v$ is a module of $\sigma - v$. It follows from (10.20) that M_v is a modular cut of $\sigma - v$. Since M_v is a proper module of $\sigma - v$ containing x and w, M_v is a nontrivial modular cut of $\sigma - v$. Therefore, $v \in W_\delta(\sigma)$. It follows that

$$|W_\varnothing(\sigma) \setminus W_\delta(\sigma)| \leq 1.$$

Now, suppose that there exists $v \in W_\varnothing(\sigma) \smallsetminus W_\delta(\sigma)$. Suppose for a contradiction that $W_\delta(\sigma) \neq \varnothing$, and consider $w \in W_\delta(\sigma)$. It follows from Corollary 10.18 that $v \sim_\sigma w$, which contradicts Lemma 10.22. Consequently, we have

$$W_\delta(\sigma) = \varnothing,$$

and hence

$$W_\varnothing(\sigma) = \{v\}.$$

□

10.4. **Proof of Theorem 10.1.** The next result follows from Corollary 10.10 and Fact 10.13.

Proposition 10.24. *Given an infinite critical 2-structure σ, consider distinct $v, w \in W(\sigma)$. If $(w,v),(v,w) \notin A(\mathbb{C}(\sigma))$, then* (RT) *holds (see Theorem 10.1).*

Proof. Suppose that $(w,v),(v,w) \notin A(\mathbb{C}(\sigma))$. It follows from Corollary 10.10 that $\{w\} \in \Upsilon(\sigma - v)$ and $(\sigma - v)/\Upsilon(\sigma - v)$ is prime. Using the axiom of choice, consider $X \subseteq V(\sigma) \smallsetminus \{v\}$ such that $|X \cap \mathcal{M}| = 1$ for each $\mathcal{M} \in \Upsilon(\sigma - v)$. We have $\sigma[X] \simeq (\sigma - v)/\Upsilon(\sigma - v)$. Hence, $\sigma[X]$ is prime. Consequently, (RT) holds when X is equipotent to $V(\sigma)$.

Now, suppose that X is strictly subpotent to $V(\sigma)$. We have

$$|V(\sigma)| = \sup\{|\mathcal{M}| : \mathcal{M} \in \Upsilon(\sigma - v)\}. \tag{10.21}$$

We show that

$$|\Upsilon(\sigma - w)| \geq |\mathcal{M}| \tag{10.22}$$

for every $\mathcal{M} \in \Upsilon(\sigma - v)\}$. This is obvious when $|\mathcal{M}| = 1$. Hence, consider $\mathcal{M} \in \Upsilon(\sigma - v)\}$ such that $|\mathcal{M}| \geq 2$. Since $(v,w) \notin A(\mathbb{C}(\sigma))$, it follows from Corollary 10.10 that $\{v\} \in \Upsilon(\sigma - w)$ and $(\sigma - w)/\Upsilon(\sigma - w)$ is prime. Let $\mathcal{N} \in \Upsilon(\sigma - w)$ such that $\mathcal{M} \cap \mathcal{N} \neq \varnothing$. Since $\{v\} \in \Upsilon(\sigma - w)$ and $v \notin \mathcal{M}$, we have $v \notin \mathcal{N}$. It follows from Fact 10.13 that $|\mathcal{M} \cap \mathcal{N}| = 1$. Therefore, we obtain

$$|\Upsilon(\sigma - w)| \geq |\mathcal{M}|,$$

so (10.22) holds. It follows from (10.21) that $|\Upsilon(\sigma - w)| \geq |V(\sigma)|$. Using the axiom of choice, consider $Y \subseteq V(\sigma) \smallsetminus \{w\}$ such that $|Y \cap \mathcal{O}| = 1$ for each $\mathcal{O} \in \Upsilon(\sigma - w)$. We obtain that $|Y| = |\Upsilon(\sigma - w)|$ and $\sigma[Y]$ is prime. Since $|\Upsilon(\sigma - w)| \geq |V(\sigma)|$, we have $|Y| = |V(\sigma)|$. Consequently, (RT) holds. □

The next result follows from Corollary 10.16. We use the following notation.

Notation 10.25. Recall that $L_\mathbb{N}$ denotes the usual linear order on $\mathbb{N}$. We denote by $\widehat{L_\mathbb{N}}$ the linear order defined on $\mathbb{N} \cup \{\infty\}$ such that $\widehat{L_\mathbb{N}}[\mathbb{N}] = L_\mathbb{N}$ and $(n, \infty) \in A(\widehat{L_\mathbb{N}})$ for every $n \in \mathbb{N}$.

Lemma 10.26. *Given an infinite critical 2-structure σ, $\mathbb{C}(\sigma) - \mathscr{E}(\sigma)$ embeds neither $\widehat{L_\mathbb{N}}$ nor its dual $(\widehat{L_\mathbb{N}})^\star$.*

Proof. First, suppose for a contradiction that there exist a sequence $(v_n)_{n\geq 0}$ of elements of $V(\sigma) \setminus \mathscr{E}(\sigma)$ and $v_\infty \in V(\sigma) \setminus \mathscr{E}(\sigma)$ such that the bijection

$$\begin{array}{ccl} \mathbb{N} \cup \{\infty\} & \longrightarrow & \{v_n : n \geq 0\} \cup \{v_\infty\} \\ n \geq 0 & \longmapsto & v_n, \\ \infty & \longmapsto & v_\infty \end{array}$$

is an isomorphism from $\widehat{L_{\mathbb{N}}}$ onto $\mathbb{C}(\sigma)[\{v_n : n \geq 0\} \cup \{v_\infty\}]$. Let $n \geq 2$. We have $(v_n, v_\infty), (v_0, v_n) \in A(\mathbb{C}(\sigma))$, and $(v_\infty, v_n) \notin A(\mathbb{C}(\sigma))$. By Corollary 10.16, $\sigma - v_n$ admits a nontrivial module M_n containing v_0 and v_{n-1}. Note that $v_\infty \notin M_n$ because $(v_\infty, v_n) \notin A(\mathbb{C}(\sigma))$. Set

$$M = \bigcup_{n\geq 2} M_n.$$

Since

$$v_0 \in \bigcap_{n\geq 2} M_n$$

and $\{v_n : n \geq 2\} \subseteq M$, it is not difficult to verify that M is a module of σ. Since $v_\infty \notin M_n$ for every $n \geq 2$, we have $M \neq V(\sigma)$. Moreover, since $M_2 \subseteq M$, we have $|M| \geq 2$. It follows that M is a nontrivial module of σ, which contradicts the fact that σ is prime. Consequently, $\mathbb{C}(\sigma) - \mathscr{E}(\sigma)$ does not embed $\widehat{L_{\mathbb{N}}}$.

Second, suppose for a contradiction that there exist a sequence $(v_n)_{n\geq 0}$ of elements of $V(\sigma) \setminus \mathscr{E}(\sigma)$ and $v_\infty \in V(\sigma) \setminus \mathscr{E}(\sigma)$ such that the bijection

$$\begin{array}{ccl} \mathbb{N} \cup \{\infty\} & \longrightarrow & \{v_n : n \geq 0\} \cup \{v_\infty\} \\ n \geq 0 & \longmapsto & v_n, \\ \infty & \longmapsto & v_\infty \end{array}$$

is an isomorphism from $(\widehat{L_{\mathbb{N}}})^\star$ onto $\mathbb{C}(\sigma)[\{v_n : n \geq 0\} \cup \{v_\infty\}]$. Since σ is critical, $\sigma - v_\infty$ admits a nontrivial module M_∞. Let $w \in M_\infty$. We have $(w, v_\infty) \in A(\mathbb{C}(\sigma))$. Moreover, for each $n \geq 0$, we have $(v_\infty, v_n) \in A(\mathbb{C}(\sigma))$ and $(v_n, v_\infty) \notin A(\mathbb{C}(\sigma))$. By Corollary 10.16, $\sigma - v_n$ admits a nontrivial module M_n containing v_∞ and w. Set

$$M = \bigcap_{n\geq 2} M_n.$$

It is not difficult to verify that M is a module of σ. Since $v_0 \notin M_0$, we have $v_0 \notin M$, so $M \neq V(\sigma)$. Moreover, since $w, v_\infty \in M$, we have $|M| \geq 2$. It follows that M is a nontrivial module of σ, which contradicts the fact that σ is prime. Consequently, $\mathbb{C}(\sigma) - \mathscr{E}(\sigma)$ does not embed $(\widehat{L_{\mathbb{N}}})^\star$. □

Remark 10.27. Let L be an infinite linear order. Suppose that L embeds neither $\widehat{L_{\mathbb{N}}}$ nor its dual $(\widehat{L_{\mathbb{N}}})^\star$. We show that L is isomorphic to $L_{\mathbb{N}}$, $(L_{\mathbb{N}})^\star$, or $L_{\mathbb{Z}}$. Indeed, using the axiom of countable choice, we obtain a countable subset W of $V(L)$. It follows from the infinite Ramsey's theorem that $L[W]$ embeds $L_{\mathbb{N}}$ or $(L_{\mathbb{N}})^\star$. By exchanging L and $L^\star$ if necessary, we can assume that L embeds $L_{\mathbb{N}}$. Hence, there exists a sequence $(v_n)_{n\geq 0}$ of vertices of L

$$\begin{array}{rcl} \mathbb{N} & \longrightarrow & \{v_n : n \geq 0\} \\ n \geq 0 & \longmapsto & v_n \end{array}$$

is an isomorphism from $L_{\mathbb{N}}$ onto $L[\{v_n : n \geq 0\}]$. Set

$$V^- = \{v \in V(\sigma) \setminus \{v_n : n \geq 0\} : v < v_0 \bmod L\}$$

and for each $n \geq 0$, set

$$V_n = \{v \in V(\sigma) \setminus \{v_n : n \geq 0\} : v_n < v < v_{n+1} \bmod L\}.$$

Let $v \in V(\sigma) \setminus \{v_n : n \geq 0\}$. Since L does not embed $\widehat{L_{\mathbb{N}}}$, there exists $n \geq 0$ such that $v < v_n \bmod L$. Therefore, we have

$$(V(\sigma) \setminus \{v_n : n \geq 0\}) = V^- \cup (\bigcup_{n\geq 0} V_n).$$

For a contradiction, suppose that there exists $n \geq 0$ such that V_n is infinite. As previously, $L[V_n]$ embeds $L_{\mathbb{N}}$ or $(L_{\mathbb{N}})^\star$, which contradicts the fact that L embeds neither $\widehat{L_{\mathbb{N}}}$ nor its dual $(\widehat{L_{\mathbb{N}}})^\star$. Therefore, V_n is finite for every $n \geq 0$. Set

$$V^+ = \{v_n : n \geq 0\} \cup (\bigcup_{n\geq 0} V_n).$$

It follows that

$$L[V^+] \simeq L_{\mathbb{N}}. \tag{10.23}$$

Moreover, we have

$$V(L) = V^- \cup V^+. \tag{10.24}$$

If V^- is finite, then $L \simeq L_{\mathbb{N}}$ too. Hence, suppose that V^- is infinite. As previously, $L[V^-]$ embeds $L_{\mathbb{N}}$ or $(L_{\mathbb{N}})^\star$. Since L does not embed $\widehat{L_{\mathbb{N}}}$, $L[V^-]$ embeds $(L_{\mathbb{N}})^\star$. Therefore, there exists a sequence $(w_n)_{n\geq 0}$ of element of V^-

$$\begin{array}{rcl} \mathbb{N} & \longrightarrow & \{v_n : n \geq 0\} \\ n \geq 0 & \longmapsto & w_n \end{array}$$

is an isomorphism from $(L_{\mathbb{N}})^\star$ onto $L[\{w_n : n \geq 0\}]$. Set

$$W_0 = \{v \in V^- : w_0 < v \bmod L\}$$

and for each $n \geq 1$, set

$$W_n = \{v \in V^- : w_n < v < w_{n-1} \bmod L\}.$$

Since L does not embed neither $\widehat{L_{\mathbb{N}}}$ nor its dual $(\widehat{L_{\mathbb{N}}})^\star$, we have

$$V^- = \bigcup_{n\geq 0} W_n$$

and W_n is finite for each $n \geq 0$. Consequently, $L[V^-] \simeq (L_{\mathbb{N}})^\star$. It follows from (10.23) and (10.24) that $L \simeq L_{\mathbb{Z}}$.

The next result follows from Corollary 10.21.

Proposition 10.28. *Let σ be an infinite critical 2-structure. Given distinct $v, w \in W_\pi(\sigma)$, if $v \sim_\sigma w$, then* (RT) *holds (see Theorem 10.1).*

Proof. It follows from Corollary 10.21 that there exists $X \subseteq V(\sigma) \setminus \{v, w\}$ satisfying

- $\sigma[X]$ is prime;
- there exist distinct $y, z \in X$ such that $v \in X_\sigma(y)$ and $w \in X_\sigma(z)$;
- $\Upsilon(\sigma - v) = \{(\{y\} \cup X_\sigma(y)) \setminus \{v\}, \{z\} \cup X_\sigma(z))\} \cup \{\{u\} : u \in X \setminus \{y, z\}\}$;
- $\Upsilon(\sigma - w) = \{\{y\} \cup X_\sigma(y)), (\{z\} \cup X_\sigma(z)) \setminus \{w\}\} \cup \{\{u\} : u \in X \setminus \{y, z\}\}$;
- $p_{(\sigma,X)} = \{X_\sigma(y), X_\sigma(z)\}$ and $E(\Gamma_{(\sigma,X)}) = \{\{v, w\}\}$.

We verify that

$$\begin{cases} (\{y\} \cup X_\sigma(y)) \longleftrightarrow_\sigma ((\{z\} \cup X_\sigma(z)) \setminus \{w\}) \text{ (see Notation 2.1)} \\ \text{and} \\ ((\{y\} \cup X_\sigma(y)) \setminus \{v\}) \longleftrightarrow_\sigma (\{z\} \cup X_\sigma(z)). \end{cases} \tag{10.25}$$

Indeed, consider $u \in X_\sigma(y)$ and $u' \in X_\sigma(z)$. Suppose that $u \neq v$ or $u' \neq w$. Since $E(\Gamma_{(\sigma,X)}) = \{\{v, w\}\}$, we have $\{u, u'\} \notin E(\Gamma_{(\sigma,X)})$. It follows from assertion (P3) of Lemma 3.17 that $\{y, u\}$ and $\{z, u'\}$ are modules of $\sigma[X \cup \{u, u'\}]$. Therefore, we obtain $[u, u']_\sigma = [y, z]_\sigma$. Moreover, since $u \in X_\sigma(y)$, we have $[u, z]_\sigma = [y, z]_\sigma$. Similarly, we have $[y, u']_\sigma = [y, z]_\sigma$ because $u' \in X_\sigma(z)$. It follows that (10.25) holds. Moreover, consider $W \subseteq (\{y\} \cup X_\sigma(y))$ and $W' \subseteq (\{z\} \cup X_\sigma(z))$ such that $v \in W$ and $w \in W'$. If $|W| \geq 2$ or $|W'| \geq 2$, then

$$W \not\longleftrightarrow_\sigma W'. \tag{10.26}$$

Indeed, it follows from (10.25) that $(W \setminus \{v\}) \longleftrightarrow_\sigma W'$ and $W \longleftrightarrow_\sigma (W' \setminus \{w\})$. Precisely, since $W \subseteq (\{y\} \cup X_\sigma(y))$ and $W' \subseteq (\{z\} \cup X_\sigma(z))$, we have $[W \setminus \{v\}), W']_\sigma = [y, z]_\sigma$ and $[W, W' \setminus \{w\}]_\sigma = [y, z]_\sigma$. Since $\{v, w\} \in E(\Gamma_{(\sigma,X)})$, it follows from assertion (P3) of Lemma 3.17 that $\{y, v\}$ is not a module of $\sigma[X \cup \{v, w\}]$ or $\{z, w\}$ is not a module of $\sigma[X \cup \{v, w\}]$. Furthermore, $\{y, v\}$ is a module of $\sigma[X \cup \{v, w\}]$ if and only if $\{z, w\}$ is a module of $\sigma[X \cup \{v, w\}]$. For instance, assume that $\{y, v\}$ is not a module of $\sigma[X \cup \{v, w\}]$. We obtain $[y, w]_\sigma \neq [v, w]_\sigma$. Since $w \in X_\sigma(z)$, we have $[y, w]_\sigma = [y, z]_\sigma$. Therefore, we obtain $[v, w]_\sigma \neq [y, z]_\sigma$. It follows that (10.26) holds.

Clearly, if X is equipotent to $V(\sigma)$, then (RT) holds. Hence, suppose that X is strictly subpotent to $V(\sigma)$. Consequently, $\{y\} \cup X_\sigma(y)$ or $\{z\} \cup X_\sigma(z)$ are equipotent to $V(\sigma)$. For instance, assume that $\{y\} \cup X_\sigma(y)$ is equipotent to $V(\sigma)$.

We prove that

$$|X_\sigma(z)| \geq 2. \tag{10.27}$$

Otherwise, suppose that $X_\sigma(z) = \{w\}$. We verify that $z \notin \mathscr{E}(\sigma)$. Set

$$Y = (X \setminus \{z\}) \cup \{w\}.$$

Since $w \in X_\sigma(z)$, $\sigma[Y]$ is isomorphic to $\sigma[X]$, so $\sigma[Y]$ is prime. Let $u' \in (X_\sigma(y) \setminus \{v\})$. Since $E(\Gamma_{(\sigma,X)}) = \{\{v, w\}\}$, we have $\{w, u'\} \notin E(\Gamma_{(\sigma,X)})$. It follows from assertion (P3) of Lemma 3.17 that $\{y, u'\}$ is a module of

$\sigma[X \cup \{u', w\}]$. By assertion (M2) of Proposition 2.5, $\{y, u'\}$ is a module of $\sigma[Y \cup \{u'\}]$. Therefore, $u' \in Y_\sigma(y)$. It follows that

$$(X_\sigma(y) \smallsetminus \{v\}) \subseteq Y_\sigma(y).$$

Since $\{v, w\} \in E(\Gamma_{(\sigma,X)})$, we obtain $[v, w]_\sigma \neq [y, w]_\sigma$. Thus, $v \notin Y_\sigma(y)$. It follows that

$$\begin{cases} Y_\sigma(y) = X_\sigma(y) \smallsetminus \{v\} \\ \text{and} \\ p_{(\sigma - z, Y)} = \{X_\sigma(y) \smallsetminus \{v\}, \{v\}\}. \end{cases} \tag{10.28}$$

Given a nontrivial module M_z of $\sigma - z$, we verify that

$$\begin{array}{l} M_z \subseteq ((\{y\} \cup X_\sigma(y)) \smallsetminus \{v\}) \\ \text{or} \\ \text{there exists } t \in (Y \smallsetminus \{y, w\}) \text{ such that } M_z = \{t, v\}. \end{array} \tag{10.29}$$

Let M_z be a nontrivial module of $\sigma - z$. As seen in Remark 3.16, we have $|M_z \cap Y| \leq 1$ or $Y \subseteq M_z$. Suppose that $|M_z \cap Y| \leq 1$. It follows from (10.28) that $M_z \subseteq ((\{y\} \cup X_\sigma(y)) \smallsetminus \{v\})$ or there exists $t \in (Y \smallsetminus \{y\})$ such that $M_z = \{t, v\}$. If $t = w$, then $\{y, w\}$ is a module of $\sigma[Y]$, which contradicts the fact that $\sigma[Y]$ is prime. Thus, $t \in (Y \smallsetminus \{y, w\})$. Consequently, (10.29) holds when $|M_z \cap Y| \leq 1$. In the other case, we have a nontrivial module M_z of $\sigma - z$ such that $Y \subseteq M_z$. By Fact 10.6, we have $Y_\sigma(y) \subseteq M_z$, and hence

$$v \longleftrightarrow_\sigma (V(\sigma) \smallsetminus \{z, w\}).$$

In particular, we have

$$v \longleftrightarrow_\sigma ((\{y\} \cup X_\sigma(y)) \smallsetminus \{v\}).$$

Clearly, we have

$$t \longleftrightarrow_\sigma ((\{y\} \cup X_\sigma(y)) \smallsetminus \{v\})$$

for every $t \in X \smallsetminus \{y, z\}$. Moreover, by (10.25), we have

$$w \longleftrightarrow_\sigma ((\{y\} \cup X_\sigma(y)) \smallsetminus \{v\}).$$

It follows that $(\{y\} \cup X_\sigma(y)) \smallsetminus \{v\}$ is a module of σ, which contradicts the fact that σ is prime. Therefore, (10.29) holds. It follows that $\sigma - z$ is uncuttable. In particular, we have $z \notin \mathscr{E}(\sigma)$.

To obtain a contradiction when $X_\sigma(z) = \{w\}$, we distinguish the following two cases.

Case 1: $(v, z) \notin A(\mathbb{C}(\sigma))$.

Consider a nontrivial module M_z of $\sigma - z$. Since $(v, z) \notin A(\mathbb{C}(\sigma))$, it follows from (10.29) that $M_z \subseteq (\{y\} \cup (X_\sigma(y) \smallsetminus \{v\}))$. Since $M_z \subseteq (\{y\} \cup X_\sigma(y))$, we have $z \longleftrightarrow_\sigma M_z$. It follows that M_z is a nontrivial module of σ, which contradicts the fact that σ is prime.

CASE 2: $(v,z) \in A(\mathbb{C}(\sigma))$.

Since $(z,v) \in A(\mathbb{C}(\sigma))$, we have $v \sim_\sigma z$. By Lemma 10.20, $z \in W_\pi(\sigma)$. Furthermore, by Corollary 10.21, there exists $Z \subseteq (V(\sigma) \setminus \{v,z\})$ satisfying

- $\sigma[Z]$ is prime;
- there exist distinct $y', z' \in Z$ such that $v \in Z_\sigma(y')$ and $z \in Z_\sigma(z')$;
- $\Upsilon(\sigma - v) = \{(\{y'\} \cup Z_\sigma(y')) \setminus \{v\}, \{z'\} \cup Z_\sigma(z')\} \cup \{\{u\} : u \in Z \setminus \{y', z'\}\}$;
- $\Upsilon(\sigma - z) = \{\{y'\} \cup Z_\sigma(y')), (\{z'\} \cup Z_\sigma(z')) \setminus \{z\}\} \cup \{\{u\} : u \in Z \setminus \{y', z'\}\}$;
- $p_{(\sigma,Z)} = \{Z_\sigma(y'), Z_\sigma(z')\}$ and $E(\Gamma_{(\sigma,Z)}) = \{\{v,z\}\}$.

Recall that

$$\Upsilon(\sigma - v) = \{(\{y\} \cup X_\sigma(y)) \setminus \{v\}, \{z\} \cup X_\sigma(z))\} \cup \{\{u\} : u \in X \setminus \{y,z\}\}.$$

Since $X_\sigma(z) = \{w\}$, we have $\{z,w\} \in \Upsilon(\sigma - v)$. Since $\{y\} \cup X_\sigma(y)$ is equipotent to $V(\sigma)$, $(\{y\} \cup X_\sigma(y)) \setminus \{v\}$ is an infinite element of $\Upsilon(\sigma - v)$. It follows that

$$\{z'\} \cup Z_\sigma(z') = \{z,w\}.$$

Moreover, since $(\{y\} \cup X_\sigma(y)) \setminus \{v\}$ is an infinite element of $\Upsilon(\sigma - v) \setminus \{\{z'\} \cup Z_\sigma(z') = \{z,w\}\}$, we have

$$(\{y\} \cup X_\sigma(y)) \setminus \{v\} = (\{y'\} \cup Z_\sigma(y')) \setminus \{v\}.$$

Therefore, we obtain

$$\begin{cases} y' \in ((\{y\} \cup X_\sigma(y)) \setminus \{v\}), \\ v \in Z_\sigma(y'), \\ \text{and} \\ w \in Z, \end{cases}$$

which contradicts (10.26) with $W = \{y', v\}$ and $W' = \{w\}$.

Consequently, (10.27) holds. Consider $u' \in X_\sigma(z) \setminus \{w\}$. We show that $u' \in W_\pi(\sigma)$ and

$$\{\{u\} : u \in \{y\} \cup X_\sigma(y)\} \subseteq \Upsilon(\sigma - u'), \tag{10.30}$$

which allows us to conclude because $\{y\} \cup X_\sigma(y)$ is equipotent to $V(\sigma)$. Let $M_{u'}$ be a nontrivial module of $\sigma - u'$. For a contradiction, suppose that $X \subseteq M_{u'}$. Since $M_{u'}$ is a nontrivial module of $\sigma - u'$, there exists $u \in (V(\sigma) \setminus \{u'\}) \setminus M_{u'}$. Since $p_{(\sigma,X)} = \{X_\sigma(y), X_\sigma(z)\}$, we get $u \in X_\sigma(t)$, where $t = y$ or z. Set

$$Y = (X \setminus \{t\}) \cup \{u\}.$$

Since $\{t,u\}$ is a module of $\sigma[X \cup \{u\}]$, $\sigma[Y]$ is isomorphic to $\sigma[X]$. It follows that $\sigma[Y]$ is prime. By assertion (M2) of Proposition (2.5), $M_{u'} \cap Y$ is a module of $\sigma[Y]$. We have $(X \setminus \{t\}) \subseteq (M_{u'} \cap Y)$, so $|M_{u'} \cap Y| \geq 2$. Furthermore, we have $u \in (Y \setminus (M_{u'} \cap Y))$. Therefore, $M_{u'} \cap Y$ is a nontrivial module of $\sigma[Y]$, which contradicts the fact that $\sigma[Y]$ is prime. It follows that $X \smallsetminus M_{u'} \neq \varnothing$. It follows from Remark 3.16 that we obtain $|M_{u'} \cap X| \leq 1$. Since $p_{(\sigma,X)} = \{X_\sigma(y), X_\sigma(z)\}$, we obtain

$$M_{u'} \subseteq (\{y\} \cup X_\sigma(y)) \text{ or } M_{u'} \subseteq (\{z\} \cup X_\sigma(z)).$$

It follows that $u' \notin \mathscr{E}(\sigma)$ and $\sigma - u'$ is uncuttable. By Proposition 2.12, $\sigma - u'$ is connected. Moreover, for $t \in X \smallsetminus \{y, z\}$, we obtain $\{t\} \in \Upsilon(\sigma - u')$. It follows from Proposition 10.5 that $u' \in W_\pi(\sigma)$. Finally, we establish (10.30). For a contradiction, suppose that there exists $M_{u'} \in \Upsilon(\sigma - u')$ such that $M_{u'} \subseteq (\{y\} \cup X_\sigma(y))$ and $|M_{u'}| \geq 2$. Since $w \notin M_{u'}$, we have $w \longleftrightarrow_\sigma M_{u'}$. Since $|M_{u'}| \geq 2$, it follows from (10.26) that $v \notin M_{u'}$. Therefore, we obtain $u' \longleftrightarrow_\sigma M_{u'}$ by (10.25). Hence, $M_{u'}$ is a module of σ, which contradicts the fact that σ is prime. It follows that (10.30) holds. Since $\{y\} \cup X_\sigma(y)$ is equipotent to $V(\sigma)$, $\Upsilon(\sigma - u')$ is equipotent to $V(\sigma)$ as well. Since $u' \in W_\pi(\sigma)$, $(\sigma - u')/\Upsilon(\sigma - u')$ is prime. Consequently, (RT) holds. ☐

The next result follows from Corollary 10.16, Remark 10.27, and Proposition 10.28.

Proposition 10.29. *Let σ be an infinite critical 2-structure. If $W_\pi(\sigma) \neq \varnothing$, then* (RT) *holds.*

Proof. Suppose that $W_\pi(\sigma) \neq \varnothing$. To begin, we show that (RT) holds when $W_\pi(\sigma)$ is finite. Indeed, suppose that $W_\pi(\sigma)$ is finite. By Proposition 10.9, $W(\sigma)$ is equipotent to $V(\sigma)$. Thus, $W(\sigma) \smallsetminus W_\pi(\sigma)$ is equipotent to $V(\sigma)$. Let $v \in W_\pi(\sigma)$. Since $(\sigma - v)/\Upsilon(\sigma - v)$ is prime, it suffices to verify that $\Upsilon(\sigma - v)$ is equipotent to $W(\sigma) \smallsetminus W_\pi(\sigma)$. Consider $w \in W(\sigma) \smallsetminus W_\pi(\sigma)$. By Corollary 10.18, we have $(v, w) \in A(\mathbb{C}(\sigma))$. It follows from Lemma 10.20 that $(w, v) \notin A(\mathbb{C}(\sigma))$. Hence, $\{w\} \in \Upsilon(\sigma - v)$. Therefore, we have

$$\{\{w\} : w \in W(\sigma) \smallsetminus W_\pi(\sigma)\} \subseteq \Upsilon(\sigma - v).$$

It follows that $\Upsilon(\sigma - v)$ is equipotent to $W(\sigma) \smallsetminus W_\pi(\sigma)$, so $\Upsilon(\sigma - v)$ is equipotent to $V(\sigma)$. Consequently, (RT) holds.

In the sequel, we suppose that $W_\pi(\sigma)$ is infinite. By Corollary 10.17 and Propositions 10.24 and 10.28, we can assume that $\mathbb{C}(\sigma)[W_\pi(\sigma)]$ is a linear order. Furthermore, by Lemma 10.26, $\mathbb{C}(\sigma)[W_\pi(\sigma)]$ embeds neither $\widehat{L_\mathbb{N}}$ nor its dual $(\widehat{L_\mathbb{N}})^\star$. It follows from Remark 10.27 that $\mathbb{C}(\sigma)[W_\pi(\sigma)]$ is isomorphic to $L_\mathbb{N}$, $(L_\mathbb{N})^\star$, or $L_\mathbb{Z}$.

First, suppose that $\mathbb{C}(\sigma)[W_\pi(\sigma)]$ is isomorphic to $L_\mathbb{N}$ or $L_\mathbb{Z}$. In particular, observe that $W_\pi(\sigma)$ is countable. There exists a sequence $(v_n)_{n\geq 0}$ of elements of $W_\pi(\sigma)$ such that the function

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & \{v_n : n \geq 0\} \\ n \geq 0 & \longmapsto & v_n \end{array}$$

is an isomorphism from $L_\mathbb{N}$ onto $\mathbb{C}(\sigma)[\{v_n : n \geq 0\}]$. For a contradiction, suppose that $W_\varnothing(\sigma) \cup W_\delta(\sigma) \neq \varnothing$. Consider $w \in W_\varnothing(\sigma) \cup W_\delta(\sigma)$. Let $n \geq 0$. By Corollary 10.18, we have $(v_n, w) \in A(\mathbb{C}(\sigma))$. It follows from Lemma 10.20

that $(w,v_n)\notin A(\mathbb{C}(\sigma))$. Consequently, we obtain $\mathbb{C}(\sigma)[\{v_n : n\geq 0\}\cup\{w\}]\simeq \widehat{L_{\mathbb{N}}}$, which contradicts Lemma 10.26. It follows that

$$W_{\varnothing}(\sigma)\cup W_{\delta}(\sigma)=\varnothing,$$

so

$$W(\sigma)=W_{\pi}(\sigma).$$

It follows from Proposition 10.9 that $V(\sigma)$ is countable. Since $v_0\in W_{\pi}(\sigma)$, $\Upsilon(\sigma-v_0)$ is a modular partition of $\sigma-v_0$ and $(\sigma-v_0)/\Upsilon(\sigma-v_0)$ is prime. Let $n\geq 1$. Since $(v_n,v_0)\notin A(\mathbb{C}(\sigma))$, we have $\{v_n\}\in\Upsilon(\sigma-v_0)$. Consider $X\subseteq V(\sigma-v_0)$ such that $|X\cap M|=1$ for every $M\in\Upsilon(\sigma-v_0)$. We obtain that X is a countable proper subset of $V(\sigma)$ such that $\sigma[X]$ is prime. Thus, (RT) holds.

Second, suppose that $\mathbb{C}(\sigma)[W_{\pi}(\sigma)]$ is isomorphic to $(L_{\mathbb{N}})^{\star}$. There exists a sequence $(v_n)_{n\geq 0}$ of elements of $W_{\pi}(\sigma)$ such that

$$W_{\pi}(\sigma)=\{v_n : n\geq 0\}$$

and the function

$$\begin{array}{ccc}\mathbb{N} & \longrightarrow & \{v_n : n\geq 0\}\\ n\geq 0 & \longmapsto & v_n\end{array}$$

is an isomorphism from $(L_{\mathbb{N}})^{\star}$ onto $\mathbb{C}(\sigma)[W_{\pi}(\sigma)]$.

Let $w\in W_{\varnothing}(\sigma)\cup W_{\delta}(\sigma)$. As seen previously, we obtain that $\{w\}\in\Upsilon(\sigma-v_0)$. Thus, $W_{\varnothing}(\sigma)\cup W_{\delta}(\sigma)$ is subpotent to $\Upsilon(\sigma-v_0)$. Suppose that $W_{\varnothing}(\sigma)\cup W_{\delta}(\sigma)$ is infinite. Since $W_{\pi}(\sigma)$ is countable, we obtain that $W(\sigma)$ is equipotent to $W_{\varnothing}(\sigma)\cup W_{\delta}(\sigma)$. It follows from Proposition 10.9 that $V(\sigma)$ is equipotent to $W_{\varnothing}(\sigma)\cup W_{\delta}(\sigma)$. Since $W_{\varnothing}(\sigma)\cup W_{\delta}(\sigma)$ is subpotent to $\Upsilon(\sigma-v_0)$, it follows from Bernstein–Schröder theorem that $\Upsilon(\sigma-v_0)$ is equipotent to $V(\sigma)$. Consider $X\subseteq V(\sigma-v_0)$ such that $|X\cap M|=1$ for every $M\in\Upsilon(\sigma-v_0)$. We obtain that X is proper subset of $V(\sigma)$ such that X is equipotent to $V(\sigma)$ and $\sigma[X]$ is prime. Therefore, (RT) holds. In the sequel of the proof, we suppose that $W_{\varnothing}(\sigma)\cup W_{\delta}(\sigma)$ is finite. Since $W_{\pi}(\sigma)$ is countable, $W(\sigma)$ is countable as well. It follows from Proposition 10.9 that $V(\sigma)$ is countable.

To continue, we show that for each $p\geq 0$,

$$\{v_m : m>p\} \text{ is a module of } \sigma[W_{\pi}(\sigma)]-v_p. \tag{10.31}$$

Indeed, consider $p\geq 0$. Let $q\geq p+2$. We have $(v_{p+1},v_p),(v_q,v_p)\in A(\mathbb{C}(\sigma))$ and $(v_p,v_{p+1})\notin A(\mathbb{C}(\sigma))$. By Corollary 10.16, $\sigma-v_p$ admits a nontrivial module M_p^q containing v_{p+1} and v_q. Since $v_p\in W_{\pi}(\sigma)$, $\Upsilon(\sigma-v_p)$ is a modular partition of $\sigma-v_p$ and $(\sigma-v_p)/\Upsilon(\sigma-v_p)$ is prime. Denote by M_p the unique element of $\Upsilon(\sigma-v_p)$ containing v_{p+1}. It follows from the maximality of M_p that $M_p^q\subseteq M_p$. By assertion (M2) of Proposition 2.5, $M_p\cap(\{v_n : n\geq 0\}\smallsetminus\{v_p\})$ is a module of $\sigma[\{v_n : n\geq 0\}]-v_p$. We obtain

$$\{v_m : m>p\}\subseteq(M_p\cap(\{v_n : n\geq 0\}\smallsetminus\{v_p\})).$$

Let $m < p$. Since $(v_m, v_p) \notin A(\mathbb{C}(\sigma))$, we have $\{v_m\} \in \Upsilon(\sigma - v_p)$, so $v_m \notin M_p$. Hence, when $p \geq 1$, we have

$$\{v_0, \dots, v_{p-1}\} \cap M_p = \varnothing. \tag{10.32}$$

It follows that

$$(M_p \cap (W_\pi(\sigma) \smallsetminus \{v_p\})) = \{v_m : m > p\}. \tag{10.33}$$

Hence, (10.31) holds. Consider $X^p \subseteq V(\sigma - v_p)$ such that $|X^p \cap N_p| = 1$ for every $N_p \in \Upsilon(\sigma - v_p)$. We have $\sigma[X^p]$ is prime. Since $M_p \in \Upsilon(\sigma - v_p)$, denote by y_p the unique element of $X^p \cap M_p$. We obtain

$$(M_p \smallsetminus \{y_p\}) \subseteq (X^p)_\sigma(y_p) \text{ (see Notation 3.12).} \tag{10.34}$$

Since $\{v_m\} \in \Upsilon(\sigma - v_p)$ for each $m < p$, we obtain

$$\{v_m : m < p\} \subseteq X^p. \tag{10.35}$$

As previously seen, it follows from Lemma 10.20 that

$$(W_\varnothing(\sigma) \cup W_\delta(\sigma)) \subseteq X^p. \tag{10.36}$$

Let $w \in V(\sigma - v_p) \smallsetminus X^p$. There exists $N_p \in \Upsilon(\sigma - v_p)$ such that $w \in N_p$. We obtain $w \in (X^p)_\sigma(z)$, where z denotes the unique element of $X^p \cap N_p$. It follows from Lemma 3.13 that $V(\sigma - v_p) \cap \langle X^p \rangle_\sigma = \varnothing$. For a contradiction, suppose that there exists $N_p \in (\Upsilon(\sigma - v_p) \smallsetminus \{M_p\})$ such that $|N_p| \geq 2$. Let $w \in N_p \smallsetminus X^p$. It follows from (10.33), (10.35), and (10.36) that $w \notin W(\sigma)$, that is, $w \in \mathscr{E}(\sigma)$. Hence, $V(\sigma) \smallsetminus \{w, F_{\mathscr{E}(\sigma)}(w)\}$ is a module of $\sigma - w$ (see Notation 10.8). We have $|X^p| \geq 3$ because $\sigma[X^p]$ is prime. Thus, $|(V(\sigma) \smallsetminus \{w, F_{\mathscr{E}(\sigma)}(w)\}) \cap X^p| \geq 2$. Since $\sigma[X^p]$ is prime, we obtain $X^p \subseteq (V(\sigma) \smallsetminus \{w, F_{\mathscr{E}(\sigma)}(w)\})$. It follows from Fact 10.6 that $(V(\sigma) \smallsetminus \{v_p, w\}) \subseteq (V(\sigma) \smallsetminus \{w, F_{\mathscr{E}(\sigma)}(w)\})$. We obtain $v_p = F_{\mathscr{E}(\sigma)}(w)$. In particular, we obtain $v_p \longleftrightarrow_\sigma M_p$ (see Notation 2.1). Hence, M_p is a nontrivial module of σ, which contradicts the fact that σ is prime. Consequently, we have

$$(V(\sigma - v_p) \smallsetminus M_p) \subseteq X^p. \tag{10.37}$$

Finally, we show that for each $p \geq 1$,

$$v_p \not\longleftrightarrow_\sigma \{v_{p+1}, v_{p+2}\} \text{ (see Notation 2.1).} \tag{10.38}$$

Otherwise, there exists $p \geq 1$ such that $v_p \longleftrightarrow_\sigma \{v_{p+1}, v_{p+2}\}$. Recall that M_p denotes the unique element of $\Upsilon(\sigma - v_p)$ containing v_{p+1}. By (10.31), $\{v_m : m > p+1\}$ is a module of $\sigma[W_\pi(\sigma)] - v_{p+1}$. We obtain $v_p \longleftrightarrow_\sigma \{v_m : m > p+1\}$. Since $v_p \longleftrightarrow_\sigma \{v_{p+1}, v_{p+2}\}$, we have $v_p \longleftrightarrow_\sigma \{v_m : m > p\}$. It follows from (10.33) that $v_p \longleftrightarrow_\sigma (M_p \cap (W_\pi(\sigma) \smallsetminus \{v_p\}))$. Since σ is prime, M_p is not a module of σ. Therefore, $v_p \not\longleftrightarrow_\sigma M_p$. Thus, there exists $w \in M_p$ such that $v_p \not\longleftrightarrow_\sigma \{v_{p+1}, w\}$. Since $v_p \longleftrightarrow_\sigma \{v_m : m > p\}$, we have $w \notin \{v_m : m > p\}$. It follows from (10.35) that $w \notin \{v_m : m < p\}$. We obtain $w \notin W_\pi(\sigma)$. Furthermore, it follows from (10.36) that $w \notin W_\varnothing(\sigma) \cup W_\delta(\sigma)$. We obtain $w \in \mathscr{E}(\sigma)$. Hence, $V(\sigma) \smallsetminus \{w, F_{\mathscr{E}(\sigma)}(w)\}$ is a module of $\sigma - w$ (see Notation 10.8). As previously, it follows from Fact 10.6 that

$(V(\sigma) \smallsetminus \{v_p, w\}) \subseteq (V(\sigma) \smallsetminus \{w, F_{\mathscr{E}(\sigma)}(w)\})$ and we obtain $v_p = F_{\mathscr{E}(\sigma)}(w)$. By (10.33), we have $v_p, v_{p+1} \in M_0$. Recall that X^0 is a subset of $V(\sigma - v_0)$ such that $|X^0 \cap N_0| = 1$ for every $N_0 \in \Upsilon(\sigma - v_0)$. Furthermore, y_0 denotes the unique element of $X^0 \cap M_0$. Hence, we can assume that $y_0 = v_{p+1}$ so that $y_0 \neq v_p$ and $y_0 \neq w$. It follows from (10.34) that $v_p \in (X^0)_\sigma(v_{p+1})$. Since $(v_p, v_0), (w, v_p) \in A(\mathbb{C}(\sigma))$ and $(v_0, v_p) \notin A(\mathbb{C}(\sigma))$, it follows from Corollary 10.16 that $\sigma - v_0$ admits a nontrivial module N_0 containing v_p and w. It follows from the maximality of M_0 that $N_0 \subseteq M_0$. In particular, we have $w \in M_0$. Since $V(\sigma) \smallsetminus \{w, v_p\}$ is a module of $\sigma - w$, we obtain $v_p \in \langle X^0 \rangle_\sigma$. Therefore, we have $v_p \in (X^0)_\sigma(v_{p+1}) \cap \langle X^0 \rangle_\sigma$, which contradicts Lemma 3.13. Consequently, (10.38) holds.

To conclude, we verify that $\sigma[W_\pi(\sigma) \smallsetminus \{v_0\}]$ is prime. Let M be a module of $\sigma[W_\pi(\sigma) \smallsetminus \{v_0\}]$ such that $|M| \geq 2$. We have to show that $M = \{v_n : n \geq 1\}$. Given $m < n$, we prove that

(10.39) if $v_m, v_n \in M$ and $v_{n+1} \notin M$, then $v_{n+2} \in M$.

Suppose that $v_m, v_n \in M$ and $v_{n+1} \notin M$. It follows from (10.31) and (10.38) that $v_m \longleftrightarrow_\sigma \{v_{n+1}, v_{n+2}\}$ and $v_n \not\longleftrightarrow_\sigma \{v_{n+1}, v_{n+2}\}$. Since $v_m, v_n \in M$ and $v_{n+1} \notin M$, we obtain $v_{n+2} \in M$. Hence, (10.39) holds. Given $m < n$, we verify that

(10.40) if $v_m, v_n \in M$, then $v_{n+1} \in M$.

Suppose that $v_m, v_n \in M$. For a contradiction, suppose that $v_{n+1} \notin M$. By (10.39), $v_{n+2} \in M$. Let $r \geq n+3$. By (10.38), $v_{n+1} \not\longleftrightarrow_\sigma \{v_{n+2}, v_{n+3}\}$. By (10.31), $\{v_m : m > n+2\}$ is a module of $\sigma[W_\pi(\sigma)] - v_{n+2}$. Thus, we have $v_{n+1} \longleftrightarrow_\sigma \{v_{n+3}, v_r\}$. It follows that $v_{n+1} \not\longleftrightarrow_\sigma \{v_{n+2}, v_r\}$. Therefore, $v_r \notin M$ for every $r \geq n+3$, which contradicts (10.39). It follows that $v_{n+1} \in M$. Hence, (10.40) holds.

Consider
$$p = \min(\{n \geq 1 : v_n \in M\}),$$
and
$$q = \min(\{n \geq 1 : v_n \in (M \smallsetminus v_p\}\}).$$
For a contradiction, suppose that $p \geq 2$. It follows from (10.31) and (10.38) that $v_{p-1} \not\longleftrightarrow_\sigma \{v_p, v_q\}$. Thus, we have $p = 1$. Proceeding by induction, it follows from (10.40) that $v_r \in M$ for every $r \geq q+1$. By (10.38), $v_{q-1} \not\longleftrightarrow_\sigma \{v_q, v_{q+1}\}$. Since $v_q, v_{q+1} \in M$, we have $v_{q-1} \in M$. It follows from the minimality of q that $q - 1 = 1$. Therefore, we obtain $M = \{v_n : n \geq 1\}$. It follows that $\sigma[W_\pi(\sigma) \smallsetminus \{v_0\}]$ is prime. Since $V(\sigma)$ is countable, (RT) holds. □

We use the following definition and remark in the proof of the next proposition.

directed path

Definition 10.30. A 2-structure σ is a *directed path* on $\mathbb{Z}$ if $V(\sigma) = \mathbb{Z}$ and there exist distinct $e, f \in E(\sigma)$ satisfying

$$\begin{cases} \text{for any } n \in \mathbb{Z} \text{ and } p \geq 2,\ [n, n+p]_\sigma = (e, f) \\ \text{and} \\ \text{for every } n \in \mathbb{Z},\ [n, n+1]_\sigma \neq (e, f). \end{cases} \tag{10.41}$$

Remark 10.31. Let σ be a directed path on $\mathbb{Z}$. There exist distinct $e, f \in E(\sigma)$ satisfying (10.41). Let M be a module of $L_{\mathbb{Z}}$ such that $|M| \geq 5$. We verify that $\sigma[M]$ is prime. Indeed, let N be a module of $\sigma[M]$ such that $|N| \geq 2$. We must show that $N = M$. Consider $m, n \in N$ such that $m < n$ and $N \cap \{p \in \mathbb{Z} : m \leq p \leq n\} = \{m, n\}$. Suppose that $m - 1 \in M$. Since $[m-1, m]_\sigma \neq (e, f)$ and $[m-1, n]_\sigma = (e, f)$, we obtain $m - 1 \in N$. By proceeding by induction, we obtain

$$(M \cap \{\ldots, m-1, m\}) \subseteq N.$$

Similarly, we obtain

$$(M \cap \{n, n+1, \ldots\}) \subseteq N.$$

Since $N \cap \{p \in \mathbb{Z} : m \leq p \leq n\} = \{m, n\}$, we have

$$N = M \cap (\{\ldots, m-1, m\} \cup \{n, n+1, \ldots\}).$$

For a contradiction, suppose that $n > m + 1$. Since $[m, m+1]_\sigma \neq (e, f)$ and $[m-1, m+1]_\sigma = (e, f)$, we obtain $m - 1 \notin M$. Similarly, we have $n + 1 \notin M$. It follows that $M = \{p \in \mathbb{Z} : m \leq p \leq n\}$ and $N = \{m, n\}$. Since $|M| \geq 5$, we have $n \geq m + 4$. Since $[m, m+2]_\sigma = (e, f)$, $[n, m+2]_\sigma = (f, e)$, and $e \neq f$, we obtain $[m, m+2]_\sigma \neq [n, m+2]_\sigma$, which contradicts the fact that N is a module of $\sigma[M]$. Consequently, we have $n = m + 1$. Since $N = M \cap (\{\ldots, m-1, m\} \cup \{n, n+1, \ldots\})$, we have $N = M$.

Proposition 10.32. *Let σ be an infinite critical 2-structure. If there exists $v \in W_\delta(\sigma)$ such that $|\lambda(v)| = 2$ (see Notation 10.11), then* (RT) *holds.*

Proof. Suppose that there exists $u \in W_\delta(\sigma)$ such that $|\lambda(u)| = 2$. Hence, there exist distinct $e, f \in E(\sigma)$ such that $\lambda(u) = \{e, f\}$. By Corollary 10.18 and Lemma 10.22, we have

$$\begin{cases} W(\sigma) = W_\delta(\sigma) \\ \text{and} \\ \text{for every } v \in W_\delta(\sigma),\ \lambda(v) = \{e, f\}. \end{cases} \tag{10.42}$$

Let $v \in W_\delta(\sigma)$. There exist $e_v, f_v \in E(\sigma - v)$ such that $\sigma - v$ is not $\{e_v, f_v\}$-connected. Moreover, we have

$$\begin{cases} e_v = e \cap (V(\sigma - v) \times V(\sigma - v)) \\ \text{and} \\ f_v = f \cap (V(\sigma - v) \times V(\sigma - v)). \end{cases}$$

We show that

$$|\mathcal{C}_{\{e_v, f_v\}}(\sigma - v)| = 2. \tag{10.43}$$

Otherwise, suppose that $|\mathcal{C}_{\{e_v,f_v\}}(\sigma - v)| \geq 3$ (see Definition 2.2). It follows from Proposition 2.8 there exist a modular partition $\{X_v, Y_v, Z_v\}$ of $\sigma - v$ such that $[X_v, Y_v \cup Z_v]_\sigma = (e,f)$ and $[Y_v, Z_v]_\sigma = (e,f)$. Since σ is prime, σ is $\{e,f\}$-connected. Thus, there exists $x_v \in X_v$ and $z_v \in Z_v$ such that $[x_v, v]_\sigma \neq (e,f)$ and $[v, z_v]_\sigma \neq (e,f)$. Therefore, for every $w \in Y_v$, $\sigma - w$ is $\{e,f\}$-connected. It follows from (10.42) that $Y_v \subseteq \mathscr{E}(\sigma)$. Consider $w \in Y_v$. Since $w \in \mathscr{E}(\sigma)$, there exists $F_{\mathscr{E}(\sigma)}(w) \in V(\sigma) \smallsetminus \{w\}$ (see Notation 10.8) such that

$$F_{\mathscr{E}(\sigma)}(w) \longleftrightarrow_\sigma V(\sigma) \smallsetminus \{w, F_{\mathscr{E}(\sigma)}(w)\} \text{ (see Notation 2.1).}$$

Suppose that $|Y_v| \geq 2$. Let $w \in Y_v$. Since $[X_v, Y_v \smallsetminus \{w\}]_\sigma = (e,f)$ and $[Y_v \smallsetminus \{w\}, Z_v]_\sigma = (e,f)$, we have $F_{\mathscr{E}(\sigma)}(w) \notin Y_v$. For a contradiction, suppose that $F_{\mathscr{E}(\sigma)}(w) \in Z_v$. Since $[X_v, Z_v]_\sigma = (e,f)$ and $[w, Z_v]_\sigma = (e,f)$, we obtain that $V(\sigma) \smallsetminus \{F_{\mathscr{E}(\sigma)}(w)\}$ is a module of σ, which contradicts the fact that σ is prime. It follows that $F_{\mathscr{E}(\sigma)}(w) \notin Z_v$. Similarly, we have $F_{\mathscr{E}(\sigma)}(w) \notin X_v$. Therefore, we obtain $F_{\mathscr{E}(\sigma)}(w) = v$. Since $F_{\mathscr{E}(\sigma)}$ is injective (see Notation 10.8), we have $|Y_v| = 1$. Denote by w the unique element of Y_v. We have $F_{\mathscr{E}(\sigma)}(w) = v$. Since $[v, x_v]_\sigma \neq (f,e)$ and $[v, z_v]_\sigma \neq (e,f)$, there exist $e', f' \in E(\sigma)$ such that $[v, V(\sigma) \smallsetminus \{v,w\}]_\sigma = (e', f')$ and $\{e', f'\} \neq \{e,f\}$. It follows that $\sigma - u$ is $\{e,f\}$-connected for every $u \in X_v \cup Z_v$. By (10.42), $u \notin W(\sigma)$ for every $u \in X_v \cup Z_v$. Since $Y_v \subseteq \mathscr{E}(\sigma)$, we obtain $W(\sigma) = \{v\}$, which contradicts Proposition 10.9. Consequently, (10.43) holds for each $v \in W_\delta(\sigma)$.

Let $v \in W_\delta(\sigma)$. By (10.43), there exists a unique nontrivial modular cut C_v of $\sigma - v$ such that

$$[C_v, V(\sigma - v) \smallsetminus C_v]_\sigma = (e,f).$$

We define a digraph $L_\delta(\sigma)$ on $V(L_\delta(\sigma)) = W_\delta(\sigma)$ as follows. Given distinct $v, w \in W_\delta(\sigma)$,

$$(v,w) \in A(L_\delta(\sigma)) \text{ if } (C_v \cup \{v\}) \subseteq C_w.$$

We verify that $L_\delta(\sigma)$ is a linear order. Clearly, $L_\delta(\sigma)$ is a partial order. Consider distinct $v, w \in W_\delta(\sigma)$. We prove that

$$\text{(10.44)} \qquad \begin{cases} w \in C_v \text{ if and only if } (w,v) \in A(L_\delta(\sigma)) \\ \text{and} \\ v \in (V(\sigma - w) \smallsetminus C_w) \text{ if and only if } (w,v) \in A(L_\delta(\sigma)). \end{cases}$$

Clearly, if $(w,v) \in A(L_\delta(\sigma))$, then $(C_w \cup \{w\}) \subseteq C_v$, so $w \in C_v$. Conversely, suppose that $w \in C_v$. Since σ is prime, σ is $\{e,f\}$-connected. Thus, there exists $x_v \in C_v$ and $y_v \in V(\sigma - v) \smallsetminus C_v$ such that $[x_v, v]_\sigma \neq (e,f)$ and $[v, y_v]_\sigma \neq (e,f)$. We distinguish the following two cases.

CASE 1: There exists $t_v \in C_v \smallsetminus \{w\}$ such that $[t_v, v]_\sigma \neq (e,f)$.

$[t_v, v]_\sigma \neq (e,f)$. We obtain that $(V(\sigma - v) \smallsetminus C_v) \cup \{v, t_v\}$ is $\{e,f\}$-connected. Therefore, we have $(V(\sigma - v) \smallsetminus C_v)) \cup \{v, t_v\} \subseteq D_w$, where $D_w = C_w$ or $V(\sigma - w) \smallsetminus C_w$. Let $u \in (V(\sigma - w) \smallsetminus D_w)$. Since $[u, V(\sigma - v) \smallsetminus$

$C_v]_\sigma = (e, f)$, we obtain $D_w = V(\sigma - w) \smallsetminus C_w$, and hence $(V(\sigma) \smallsetminus C_v) \subseteq (V(\sigma - w) \smallsetminus C_w)$. It follows that $(w, v) \in A(L_\delta(\sigma))$.

CASE 2: $w = x_v$ and $[C_v \smallsetminus \{w\}, v]_\sigma = (e, f)$.

Since $w \notin \mathscr{E}(\sigma)$, there exists $z_v \in (V(\sigma) \smallsetminus C_v)$ such that $[z_v, v]_\sigma \neq (e, f)$. Since $[v, y_v]_\sigma \neq (e, f)$, we obtain that $(V(\sigma - v) \smallsetminus C_v) \cup \{v\}$, which is $V(\sigma) \smallsetminus C_v$, is $\{e, f\}$-connected. Therefore, we have $(V(\sigma) \smallsetminus C_v) \subseteq D_w$, where $D_w = C_w$ or $V(\sigma - w) \smallsetminus C_w$. Let $u \in V(\sigma - w) \smallsetminus D_w$. Since $[u, V(\sigma - v) \smallsetminus C_v]_\sigma = (e, f)$, we obtain $D_w = V(\sigma - w) \smallsetminus C_w$. Thus, we have $(C_w \cup \{w\}) \subseteq C_v$, so $(w, v) \in A(L_\delta(\sigma))$.

It follows that $w \in C_v$ if and only if $(w, v) \in A(L_\delta(\sigma))$. We show similarly that $v \in (V(\sigma - w) \smallsetminus C_w)$ if and only if $(w, v) \in A(L_\delta(\sigma))$. Hence, (10.44) holds. It follows that $(w, v) \in A(L_\delta(\sigma))$ or $(v, w) \in A(L_\delta(\sigma))$. Consequently, $L_\delta(\sigma)$ is a linear order.

Now, we prove that $L_\delta(\sigma)$ does not embed $\widehat{L_{\mathbb{N}}}$. Otherwise, there exist a sequence $(v_n)_{n\geq 0}$ of elements of $W(\sigma)$ and $v_\infty \in W(\sigma)$ such that the bijection

$$\begin{array}{ccl} \mathbb{N} \cup \{\infty\} & \longrightarrow & \{v_n : n \geq 0\} \cup \{v_\infty\} \\ n \geq 0 & \longmapsto & v_n, \\ \infty & \longmapsto & v_\infty \end{array}$$

is an isomorphism from $\widehat{L_{\mathbb{N}}}$ onto $L_\delta(\sigma)[\{v_n : n \geq 0\} \cup \{v_\infty\}]$. For each $n \geq 0$, we have

$$\begin{cases} (C_{v_n} \cup \{v_n\}) \subseteq C_{v_{n+1}} \\ \text{and} \\ (C_{v_{n+1}} \cup \{v_{n+1}\}) \subseteq C_{v_\infty}. \end{cases} \tag{10.45}$$

Set

$$C = \bigcup_{n\geq 0} C_{v_n}.$$

It follows from (10.45) that C is a nontrivial module of σ, which contradicts the fact that σ is prime. Consequently, $L_\delta(\sigma)$ does not embed $\widehat{L_{\mathbb{N}}}$.

To pursue, we prove that $L_\delta(\sigma)$ does not embed $(\widehat{L_{\mathbb{N}}})^\star$. Otherwise, there exist a sequence $(v_n)_{n\geq 0}$ of elements of $W(\sigma)$ and $v_\infty \in W(\sigma)$ such that the bijection

$$\begin{array}{ccl} \mathbb{N} \cup \{\infty\} & \longrightarrow & \{v_n : n \geq 0\} \cup \{v_\infty\} \\ n \geq 0 & \longmapsto & v_n, \\ \infty & \longmapsto & v_\infty \end{array}$$

is an isomorphism from $(\widehat{L_{\mathbb{N}}})^\star$ onto $L_\delta(\sigma)[\{v_n : n \geq 0\} \cup \{v_\infty\}]$. For each $n \geq 0$, we have

$$\begin{cases} (C_{v_{n+1}} \cup \{v_{n+1}\}) \subseteq C_{v_n} \\ \text{and} \\ (C_{v_\infty} \cup \{v_\infty\}) \subseteq C_{v_n}. \end{cases} \tag{10.46}$$

Set

$$C = \bigcap_{n\geq 0} C_{v_n}.$$

It follows from (10.46) that C is a nontrivial module of σ, which contradicts the fact that σ is prime. Consequently, $L_\delta(\sigma)$ does not embed $(\widehat{L_\mathbb{N}})^\star$.

It follows that $L_\delta(\sigma)$ embeds neither $\widehat{L_\mathbb{N}}$ nor its dual $(\widehat{L_\mathbb{N}})^\star$. By (10.42), we have $W(\sigma) = W_\delta(\sigma)$. It follows from Proposition 10.9 that $L_\delta(\sigma)$ is an infinite linear order. By Remark 10.27, $L_\delta(\sigma)$ is isomorphic to $L_\mathbb{N}$, $(L_\mathbb{N})^\star$, or $L_\mathbb{Z}$. Given $v, w \in W_\delta(\sigma)$ such that $v <_{L_\delta(\sigma)} w$, we prove that

$$\begin{aligned} {}_\sigma = (e,f) \text{ if and only if there exists } u \in W_\delta(\sigma) \text{ such that} \\ v <_{L_\delta(\sigma)} u <_{L_\delta(\sigma)} w \end{aligned} \tag{10.47}$$

Consider $v, w \in W_\delta(\sigma)$ such that $v <_{L_\delta(\sigma)} w$. To begin, suppose that there exists $u \in W_\delta(\sigma)$ such that $v <_{L_\delta(\sigma)} u <_{L_\delta(\sigma)} w$. By (10.44), we have $v \in C_u$ and $w \in (V(\sigma - u) \smallsetminus C_u)$. Hence, we have $[v, w]_\sigma = (e, f)$. Conversely, suppose that $[v, w]_\sigma = (e, f)$. By (10.44), we have $w \in (V(\sigma - v) \smallsetminus C_v)$. Hence, we have $[w, C_v]_\sigma = (f, e)$. Since $[w, v]_\sigma = (f, e)$, we obtain $[w, C_v \cup \{v\}]_\sigma = (f, e)$. Since $(v, w) \in A(L_\delta(\sigma))$, we have $C_v \cup \{v\} \subseteq C_w$. Since $w \notin \mathscr{E}(\sigma)$, C_w is a nontrivial module of $\sigma - w$. Since C_w is not a nontrivial module of σ, there exists $u \in (C_w \smallsetminus (C_v \cup \{v\}))$ such that $[w, u]_\sigma \neq (f, e)$. Since $u \in (C_w \smallsetminus (C_v \cup \{v\}))$, it follows from (10.44) that $v <_{L_\delta(\sigma)} u <_{L_\delta(\sigma)} w$.

For a contradiction, suppose that $\mathscr{E}(\sigma) \neq \varnothing$. As seen at the end of the proof of Proposition 10.9, $\{\mathscr{E}^0(\sigma), \mathscr{E}^1(\sigma), \mathscr{E}^2(\sigma)\}$ is a partition of $\mathscr{E}(\sigma)$ and $|\mathscr{E}^2(\sigma)| \leq |\mathscr{E}^1(\sigma)| \leq |\mathscr{E}^0(\sigma)|$. It follows that $\mathscr{E}^0(\sigma) \neq \varnothing$. Hence, there exists $u \in \mathscr{E}(\sigma)$ such that $F_{\mathscr{E}(\sigma)}(u) \notin \mathscr{E}(\sigma)$, so $F_{\mathscr{E}(\sigma)}(u) \in W(\sigma)$. Since $L_\delta(\sigma)$ is isomorphic to $L_\mathbb{N}$, $(L_\mathbb{N})^\star$, or $L_\mathbb{Z}$, there exist $v, w \in W(\sigma)$ such that

$$\begin{cases} w <_{L_\delta(\sigma)} v <_{L_\delta(\sigma)} F_{\mathscr{E}(\sigma)}(u) \\ \text{and} \\ \{t \in W(\sigma) : v <_{L_\delta(\sigma)} t <_{L_\delta(\sigma)} F_{\mathscr{E}(\sigma)}(u)\} = \varnothing. \end{cases}$$

or

$$\begin{cases} F_{\mathscr{E}(\sigma)}(u) <_{L_\delta(\sigma)} v <_{L_\delta(\sigma)} w \\ \text{and} \\ \{t \in W(\sigma) : F_{\mathscr{E}(\sigma)}(u) <_{L_\delta(\sigma)} t <_{L_\delta(\sigma)} v\} = \varnothing. \end{cases}$$

In the first instance, it follows from (10.47) that $[w, F_{\mathscr{E}(\sigma)}(u)]_\sigma = (e, f)$ and $[v, F_{\mathscr{E}(\sigma)}(u)]_\sigma \neq (e, f)$, which contradicts $F_{\mathscr{E}(\sigma)}(u) \longleftrightarrow_\sigma V(\sigma) \smallsetminus \{u, F_{\mathscr{E}(\sigma)}(u)\}$ (see Notation 10.8). Similarly, the second instance leads to a contradiction. It follows that

$$\mathscr{E}(\sigma) = \varnothing,$$

so we have

$$V(\sigma) = W_\delta(\sigma).$$

Lastly, suppose for a contradiction that $L_\delta(\sigma)$ is isomorphic to $L_\mathbb{N}$. There exists a sequence $(v_n)_{n \geq 0}$ of elements of $V(\sigma)$ such that

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & \{v_n : n \geq 0\} \\ n & \longmapsto & v_n \end{array}$$

is an isomorphism from $L_{\mathbb{N}}$ onto $L_\delta(\sigma)$. Let $n \geq 0$. It follows from (10.47) that $[v_n, v_{n+1}]_\sigma \neq (e,f)$ and $[v_n, v_{n+p}]_\sigma = (e,f)$ for every $p \geq 2$. Consequently, there exists a directed path τ on $\mathbb{Z}$ (see Definition 10.30) such that σ is isomorphic to $\tau[\mathbb{N}]$. By Remark 10.31, $\sigma - v_0$ is prime, which contradicts the fact that σ is critical. Similarly, if $L_\delta(\sigma)$ is isomorphic to $(L_{\mathbb{N}})^\star$, then σ is not critical. It follows that $L_\delta(\sigma)$ is isomorphic to $L_{\mathbb{Z}}$. Hence, there exists a sequence $(v_n)_{n\in\mathbb{Z}}$ of elements of $V(\sigma)$ such that

$$\begin{array}{rccc} \varphi: & \mathbb{Z} & \longrightarrow & \{v_n : n \geq 0\} \\ & n & \longmapsto & v_n \end{array}$$

is an isomorphism from $L_{\mathbb{Z}}$ onto $L_\delta(\sigma)$. It follows from (10.47) that φ is an isomorphism from σ onto a directed path on $\mathbb{Z}$. By Remark 10.31, $\sigma[\{v_n : n \geq 0\}]$ is prime, so (RT) holds. □

Proposition 10.33. *Let σ be an infinite critical 2-structure. If there exists $v \in W_\delta(\sigma)$ such that $|\lambda(v)| = 1$ (see Notation 10.11), then* (RT) *holds.*

Proof. Suppose that there exists $u \in W_\delta(\sigma)$ such that $|\lambda(u)| = 1$. Hence, there exists $e \in E(\sigma)$ such that $\lambda(u) = \{e\}$. By Lemma 10.22, we have

$$(10.48)\qquad \begin{cases} W(\sigma) = W_\delta(\sigma) \\ \text{and} \\ \text{for every } v \in W_\delta(\sigma),\ \lambda(v) = \{e\}. \end{cases}$$

Let $v \in W_\delta(\sigma)$. There exists $e_v \in E(\sigma - v)$ such that $\sigma - v$ is not $\{e_v\}$-connected. Moreover, we have $e_v = e \cap (V(\sigma - v) \times V(\sigma - v))$. For each $w \in V(\sigma) \setminus \{v\}$, we denote by C_v^w the unique element of $\mathcal{C}_{\{e_v\}}(\sigma - v)$ containing w.

We prove that $\mathscr{E}(\sigma) = \varnothing$. Otherwise, as observed in the proof of Proposition 10.32, there exists $u \in \mathscr{E}(\sigma)$ such that $F_{\mathscr{E}(\sigma)}(u) \in W(\sigma)$. For convenience, set

$$v = F_{\mathscr{E}(\sigma)}(u).$$

Let $D_v \in (\mathcal{C}_{\{e_v\}}(\sigma - v) \setminus \{C_v^u\})$. Since σ is $\{e\}$-connected, there exists $w \in D_v$ such that $\prec v, w \succ_\sigma \neq \{e\}$. Since $v \longleftrightarrow_\sigma V(\sigma) \setminus \{u, v\}$ (see Notation 10.8), we obtain $\prec v, t \succ_\sigma \neq \{e\}$ for each $t \in V(\sigma) \setminus \{u, v\}$. Hence, for each $t \in V(\sigma - v) \setminus C_v^u$, $\sigma - t$ is $\{e_t\}$-connected, where $e_t = e \cap (V(\sigma - t) \times V(\sigma - t))$. It follows from (10.48) that $t \in \mathscr{E}(\sigma)$. Thus, we have $W(\sigma) \subseteq (C_v^u \cup \{v\})$. It follows from Proposition 10.9 that $W(\sigma) \setminus \{v\}$ is infinite. Let $w \in (W(\sigma) \setminus \{v\})$. Recall that $\prec v, t \succ_\sigma \neq \{e\}$ for each $t \in V(\sigma) \setminus \{u, v\}$. Hence, $\sigma - \{u, w\}$ is $\{e_{\{u,w\}}\}$-connected, where $e_{\{u,w\}} = e \cap (V(\sigma - \{u,w\}) \times V(\sigma - \{u,w\}))$. Since $\sigma - w$ is not $\{e_w\}$-connected, we obtain $[u, V(\sigma - \{u, w\})]_\sigma = (e, e)$, which contradicts the fact that $w \notin \mathscr{E}(\sigma)$. Consequently, we have $\mathscr{E}(\sigma) = \varnothing$, so

$$(10.49)\qquad V(\sigma) = W(\sigma).$$

Consider distinct $v, w \in V(\sigma)$. We show that

$$(10.50)\qquad C_v^w \cup C_w^v = V(\sigma).$$

Let $D_v \in (\mathcal{C}_{\{e_v\}}(\sigma - v) \setminus \{C_v^w\})$. Since σ is $\{e\}$-connected, there exists $u \in D_v$ such that $\prec u, v \succ_\sigma \neq \{e\}$. It follows that

$$V(\sigma) \setminus C_v^w \text{ is } \{e\}\text{-connected.} \tag{10.51}$$

Since $v \in (V(\sigma) \setminus C_v^w)$, we obtain $(V(\sigma) \setminus C_v^w) \subseteq C_w^v$. It follows that (10.50) holds.

We conclude as follows. Consider distinct $v, w \in V(\sigma)$. By (10.50), we have $C_v^w \cup C_w^v = V(\sigma)$. By exchanging v and w if necessary, we can assume that C_v^w is equipotent to $V(\sigma)$. We verify that $\sigma[C_v^w]$ is prime. For a contradiction, suppose that M is a nontrivial module of $\sigma[C_v^w]$. Clearly, M is a module of $\sigma - v$. Since σ is prime, there exists $t \in M$ such that $\prec t, v \succ_\sigma \neq \{e\}$. Let $s \in M \setminus \{t\}$. Since C_v^w is $\{e\}$-connected and M is a module of $\sigma[C_v^w]$, $\sigma[C_v^w] - s$ is $\{e\}$-connected as well. Since $\prec t, v \succ_\sigma \neq \{e\}$ and $t \in (C_v^w \setminus \{s\})$, we obtain that $(C_v^w \setminus \{s\}) \cup \{v\}$ is $\{e\}$-connected. Moreover, by (10.51), $V(\sigma) \setminus C_v^w$ is $\{e\}$-connected. Since $v \in (V(\sigma) \setminus C_v^w) \cap ((C_v^w \setminus \{s\}) \cup \{v\})$,

$$(V(\sigma) \setminus C_v^w) \cup ((C_v^w \setminus \{s\}) \cup \{v\}),$$

which is $V(\sigma) \setminus \{s\}$, is $\{e\}$-connected, which contradicts (10.48) and (10.49). Consequently, $\sigma[C_v^w]$ is prime. It follows that (RT) holds. □

Theorem 10.1 follows easily:

Proof of Theorem 10.1. Let σ be an infinite prime 2-structures. Clearly, if σ is not critical, then (RT) holds. Hence, suppose that σ is critical. By Proposition 10.9, $V(\sigma)$ and $W(\sigma)$ are equipotent.

By Proposition 10.29, if $W_\pi(\sigma) \neq \varnothing$, then (RT) holds. Thus, suppose that

$$W_\pi(\sigma) = \varnothing.$$

By Observation 10.12, we have

$$W(\sigma) = W_\varnothing(\sigma) \cup W_\delta(\sigma).$$

Since $W_\varnothing(\sigma) \cup W_\delta(\sigma)$ is infinite, it follows from Lemma 10.23 that $W_\varnothing(\sigma) \subseteq W_\delta(\sigma)$, so

$$W(\sigma) = W_\delta(\sigma).$$

Finally, it follows from Propositions 10.32 and 10.33 that (RT) holds. □

A. Proof of Theorem 5.21

We need the next two results to prove Theorem 5.21. The next lemma has to be compared with Corollary 5.14.

Lemma A.1. *Let σ be a 2-structure such that $v(\sigma) \geq 7$. If σ is prime, and neither critical nor almost critical, then there exists $v \in V(\sigma)$ such that $\sigma - v$ is prime and noncritical (that is, $v \in \mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma)$).*

Proof. Since σ is not critical, we have $\mathscr{S}(\sigma) \neq \varnothing$. If $|\mathscr{S}(\sigma)| \geq 2$, then it suffices to apply Theorem 5.10. Now, suppose that $\mathscr{S}(\sigma)$ admits a unique element denoted by x. Since σ is not almost critical, we have $\mathscr{S}_c(\sigma) = \varnothing$. It follows that $\sigma - x$ is prime and noncritical. □

Proposition A.2. *Let σ be a 2-structure such that $v(\sigma) \geq 7$. If σ is prime, and neither critical nor almost critical, then there exists $v \in V(\sigma)$ such that $\sigma - v$ is prime, and neither critical nor almost critical, as well.*

The proof of Proposition A.2 is long and technical. We decompose it into several claims.

The beginning of the proof of Proposition A.2. By Lemma A.1,

$$\mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma) \neq \varnothing.$$

If there exists $v \in \mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma)$ such that $|\mathscr{S}(\sigma - v)| \geq 2$, then it follows from Theorem 5.10 applied to $\sigma - v$ that $\sigma - v$ is prime, and neither critical nor almost critical.

To continue, suppose that

$$|\mathscr{S}(\sigma - v)| \leq 1 \text{ for every } v \in \mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma). \tag{A.1}$$

Given $v \in \mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma)$, we have $\mathscr{S}(\sigma - v) \neq \varnothing$ because $v \notin \mathscr{S}_c(\sigma)$. Thus, for every $v \in \mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma)$, $\mathscr{S}(\sigma - v)$ admits a unique element. Consider the function $f : \mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma) \longrightarrow V(\sigma)$, which maps each $v \in \mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma)$ to the unique element of $\mathscr{S}(\sigma - v)$.

Given $v \in \mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma)$, if $f(v) \notin \mathscr{S}_c(\sigma - v)$, then $\sigma - v$ is prime, and neither critical nor almost critical. Lastly, suppose that

$$\mathscr{S}(\sigma - v) = \mathscr{S}_c(\sigma - v) = \{f(v)\} \text{ for every } v \in \mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma). \tag{A.2}$$

Let $v \in \mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma)$. It follows from Theorem 5.13 applied to $\sigma - v$ that

$$v(\sigma) = 2n + 2, \text{ where } n \geq 3, \tag{A.3}$$

and there exists an isomorphism φ_v from $(\sigma - v) - f(v)$ onto an element τ_v of $\mathcal{R}_{2n}$ satisfying (5.7).

Observation A.3. Recall that if (A.1) or (A.2) does not hold, then we can conclude as above. In the sequel, we suppose that (A.1) and (A.2) hold. We establish the new claims below in order to finally obtain a contradiction.

Claim A.4. *We have $f : \mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma) \longrightarrow V(\sigma) \smallsetminus \mathscr{S}(\sigma)$.*

Proof. Otherwise, there exists $v \in \mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma)$ such that $f(v) \in \mathscr{S}(\sigma)$. Since $\sigma - \{v, f(v)\}$ is prime, we have $f(v) \in \mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma)$. It follows that $(f \circ f)(v) = v$.

As seen in the proof of Theorem 5.13, φ_v and $\varphi_{f(v)}$ are isomorphisms from $\mathbb{P}(\sigma - \{v, f(v)\})$ onto P_{2n}. We obtain that $\varphi_{f(v)} \circ (\varphi_v)^{-1}$ is an automorphism of P_{2n}. We have

$$\text{(A.4)} \qquad \text{Aut}(P_{2n}) = \{\text{Id}_{\{0,\dots,2n-1\}}, \pi_{2n}\} \quad \text{(see Notation 4.21).}$$

It follows that

$$\varphi_{f(v)} = \varphi_v \text{ or } \pi_{2n} \circ \varphi_v.$$

Recall that (5.7) holds for φ_v and $\varphi_{f(v)}$. Therefore, if $\varphi_{f(v)} = \varphi_v$, then $\{v, f(v)\}$ is a module of σ, which contradicts the fact that σ is prime. Suppose that $\varphi_{f(v)} = \pi_{2n} \circ \varphi_v$. Since (5.7) holds for $\varphi_{f(v)}$, we have

$$[v, (\varphi_{f(v)})^{-1}(\{2i : i \in \{0, \dots, n-1\}\})]_\sigma = [(\varphi_{f(v)})^{-1}(0), (\varphi_{f(v)})^{-1}(2)]_\sigma.$$

Since $\varphi_{f(v)} = \pi_{2n} \circ \varphi_v$, we obtain

$$[v, (\varphi_v)^{-1}(\{2i+1 : i \in \{0, \dots, n-1\}\})]_\sigma = [(\varphi_v)^{-1}(2n-1), (\varphi_v)^{-1}(2n-3)]_\sigma.$$

Since τ_v is critical, with $\mathbb{P}(\tau_v) = P_{2n}$, it follows from Proposition 4.15 that

$$[(\varphi_v)^{-1}(2n-1), (\varphi_v)^{-1}(2n-3)]_\sigma = [(\varphi_v)^{-1}(2), (\varphi_v)^{-1}(0)]_\sigma.$$

Therefore, we obtain

$$[v, (\varphi_v)^{-1}(\{2i+1 : i \in \{0, \dots, n-1\}\})]_\sigma = [(\varphi_v)^{-1}(2), (\varphi_v)^{-1}(0)]_\sigma.$$

Since (5.7) holds for φ_v, we have

$$[f(v), (\varphi_v)^{-1}(\{2i+1 : i \in \{0, \dots, n-1\}\})]_\sigma = [(\varphi_v)^{-1}(2), (\varphi_v)^{-1}(0)]_\sigma.$$

It follows that

$$[f(v), (\varphi_v)^{-1}(\{2i+1 : i \in \{0, \dots, n-1\}\})]_\sigma = [v, (\varphi_v)^{-1}(\{2i+1 : i \in \{0, \dots, n-1\}\})]_\sigma.$$

Similarly, we have

$$[f(v), (\varphi_v)^{-1}(\{2i : i \in \{0, \dots, n-1\}\})]_\sigma = [v, (\varphi_v)^{-1}(\{2i : i \in \{0, \dots, n-1\}\})]_\sigma.$$

Consequently, $\{v, f(v)\}$ is a module of σ, which contradicts the fact that σ is prime. □

Claim A.5. *The function f is injective.*

Proof. Otherwise, there exist distinct $v, w \in \mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma)$ such that $f(v) = f(w)$. By Claim A.4, $f(v) \in V(\sigma) \setminus \mathscr{S}(\sigma)$. Since $\sigma - \{v, f(v)\}$ and $\sigma -$

$\{w, f(v)\}$ are prime, it follows from Lemma 4.4 that $\{v, w\}$ is a module of $\sigma - f(v)$. Therefore, the bijection

$$\begin{array}{rccl} \psi : & V(\sigma) \setminus \{w, f(v)\} & \longrightarrow & V(\sigma) \setminus \{v, f(v)\} \\ & v & \longmapsto & w \\ & x \in V(\sigma) \setminus \{v, w, f(v)\} & \longmapsto & x, \end{array}$$

is an isomorphism from $(\sigma - w) - f(v)$ onto $(\sigma - v) - f(v)$. Thus, we can choose $\varphi_v \circ \psi$ for φ_w. As shown in the proof of Theorem 5.13, φ_v and $\varphi_v \circ \psi$ satisfy (5.7). Since $w \in \mathscr{S}(\sigma)$ and $f(v) \in V(\sigma) \setminus \mathscr{S}(\sigma)$, there exists $p \in \{0, \dots, 2n-1\}$ such that $\varphi_v(w) = p$. Observe that $\pi_{2n} \circ \varphi_v$ (see (A.4)) is also an isomorphism from $(\sigma - v) - f(v)$ onto $(\tau_v)^\star$, with $(\tau_v)^\star \in \mathcal{R}_{2n}$, satisfying (5.7). Therefore, we can assume that

$$w = (\varphi_v)^{-1}(2t),$$

where $t \in \{0, \dots, n-1\}$. Since φ_v satisfies (5.7), we obtain

$$[f(v), w]_\sigma = [(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(2)]_\sigma.$$

Since $n \geq 3$ by (A.3), there exist $k, l \in \{0, \dots, n-1\}$ such that $k < l$ and $p \in \{2i : i \in \{0, \dots, n-1\}\} \setminus \{2k, 2l\}$. Since φ_v satisfies (5.8), we obtain

$$[f(v), w]_\sigma = [(\varphi_v)^{-1}(2k), (\varphi_v)^{-1}(2l)]_\sigma.$$

Since φ_w, that is, $\varphi_v \circ \psi$ satisfies (5.7), we obtain $[f(v), (\varphi_v \circ \psi)^{-1}(\{2i : i \in \{0, \dots, n-1\}\})]_\sigma = [(\varphi_v \circ \psi)^{-1}(0), (\varphi_v \circ \psi)^{-1}(2)]_\sigma$. Furthermore, since $\varphi_v \circ \psi$ satisfies (5.8), we obtain $[(\varphi_v \circ \psi)^{-1}(0), (\varphi_v \circ \psi)^{-1}(2)]_\sigma = [(\varphi_v \circ \psi)^{-1}(2k), (\varphi_v \circ \psi)^{-1}(2l)]_\sigma$. Since $p \in \{2i : i \in \{0, \dots, n-1\}\} \setminus \{2k, 2l\}$, we obtain

$$(\varphi_v)^{-1}(2k), (\varphi_v)^{-1}(2l) \in V(\sigma) \setminus \{v, f(v), w\}.$$

Thus, $\psi^{-1}((\varphi_v)^{-1}(2k)) = (\varphi_v)^{-1}(2k)$ and $\psi^{-1}((\varphi_v)^{-1}(2l)) = (\varphi_v)^{-1}(2l)$. It follows that

$$[f(v), (\varphi_v \circ \psi)^{-1}(\{2i : i \in \{0, \dots, n-1\}\})]_\sigma = [(\varphi_v)^{-1}(2k), (\varphi_v)^{-1}(2l)]_\sigma.$$

Since $(\varphi_v)^{-1}(2p) = w$ and $\psi^{-1}(w) = v$, we have $v \in (\varphi_v \circ \psi)^{-1}(\{2i : i \in \{0, \dots, n-1\}\})$. Consequently, we obtain

$$[f(v), v]_\sigma = [(\varphi_v)^{-1}(2k), (\varphi_v)^{-1}(2l)]_\sigma.$$

Hence $[f(v), v]_\sigma = [f(v), w]_\sigma$. Since $\{v, w\}$ is a module of $\sigma - f(v)$, $\{v, w\}$ is a module of σ, which contradicts the fact σ is prime. It follows that f is injective. □

Claim A.6. *For every $v \in \mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma)$, $N_{\mathbb{P}(\sigma)}(f(v)) = \{v\}$.*

Proof. Otherwise, consider $v \in \mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma)$ such that $N_{\mathbb{P}(\sigma)}(f(v)) \neq \{v\}$. By Claim A.4, $f(v) \in V(\sigma) \setminus \mathscr{S}(\sigma)$. It follows from Lemma 4.4 that $d_{\mathbb{P}(\sigma)}(f(v)) \leq 2$. Since $\sigma - \{v, f(v)\}$ is prime, we have $v \in N_{\mathbb{P}(\sigma)}(f(v))$, and hence

$$d_{\mathbb{P}(\sigma)}(f(v)) = 2.$$

Since $v \in N_{\mathbb{P}(\sigma)}(f(v))$, there exists $w \in V(\sigma) \setminus \{v, f(v)\}$ such that

$$N_{\mathbb{P}(\sigma)}(f(v)) = \{v, w\}.$$

For a contradiction, suppose that $w \in \mathscr{S}(\sigma)$. Since $\sigma - \{f(v), w\}$ is prime, we obtain $w \in \mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma)$, and $f(w) = f(v)$, which contradicts Claim A.5. It follows that

$$w \notin \mathscr{S}(\sigma).$$

Observe that $\pi_{2n} \circ \varphi_v$ (see (A.4)) is also an isomorphism from $(\sigma - v) - f(v)$ onto $(\tau_v)^\star$, with $(\tau_v)^\star \in \mathcal{R}_{2n}$, satisfying (5.7). Therefore, we can assume that

$$w = (\varphi_v)^{-1}(2p),$$

where $p \in \{0, \dots, n-1\}$. Since $w \notin \mathscr{S}(\sigma)$, we have $d_{\mathbb{P}(\sigma)}(w) \leq 2$ by Lemma 4.4. Since $\sigma - \{f(v), w\}$ is prime, we obtain

$$d_{\mathbb{P}(\sigma)}(w) = 1 \text{ or } 2.$$

We distinguish the following two cases. Each of them leads to a contradiction.

CASE 1: $d_{\mathbb{P}(\sigma)}(w) = 1$.

Since $\sigma - \{f(v), w\}$ is prime, we have

$$N_{\mathbb{P}(\sigma)}(w) = \{f(v)\}.$$

It follows from Lemma 4.4 that

$$f(v) \longleftrightarrow_\sigma \{(\varphi_v)^{-1}(i) : i \in \{0, \dots, 2n-1\}\} \setminus \{(\varphi_v)^{-1}(2p)\}, \tag{A.5}$$

which contradicts the fact that φ_v is an isomorphism satisfying (5.7). Indeed, since $n \geq 3$ by (A.3), there exists $q \in \{0, \dots, n-1\} \setminus \{p\}$. Since (5.7) is satisfied by φ_v, we have

$$[f(v), (\varphi_v)^{-1}(2q)]_\sigma = [(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(2)]_\sigma,$$

and

$$[f(v), (\varphi_v)^{-1}(2q+1)]_\sigma = [(\varphi_v)^{-1}(2), (\varphi_v)^{-1}(0)]_\sigma.$$

Since $\tau_v \in \mathcal{R}_{2n}$, we have $(0,2)_{\tau_v} \neq (2,0)_{\tau_v}$ by Remark 5.12. Hence, we have $[0,2]_{\tau_v} \neq [2,0]_{\tau_v}$. Since φ_v is an isomorphism from $(\sigma - v) - f(v)$ onto τ_v, we obtain

$$[(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(2)]_\sigma \neq [(\varphi_v)^{-1}(2), (\varphi_v)^{-1}(0)]_\sigma. \tag{A.6}$$

It follows that

$$[f(v), (\varphi_v)^{-1}(2q)]_\sigma \neq [f(v), (\varphi_v)^{-1}(2q+1)]_\sigma,$$

which contradicts (A.5).

CASE 2: $d_{\mathbb{P}(\sigma)}(w) = 2$.

Since $f(v) \in N_{\mathbb{P}(\sigma)}(w)$, there exists $u \in V(\sigma) \setminus \{f(v), w\}$ such that

$$N_{\mathbb{P}(\sigma)}(w) = \{u, f(v)\}.$$

Since $\mathscr{S}(\sigma - v) = \{f(v)\}$ and $\sigma - \{u, w\}$ is prime, we obtain $u \neq v$. Therefore, we have

$$u = (\varphi_v)^{-1}(i),$$

where $i \in \{0, \dots, 2n-1\} \smallsetminus \{2p\}$. By Lemma 4.4, $\{f(v), (\varphi_v)^{-1}(i)\}$ is a module of $\sigma - w$, that is,

$$\{f(v), (\varphi_v)^{-1}(i)\} \text{ is a module of } \sigma - (\varphi_v)^{-1}(2p). \tag{A.7}$$

Since (5.7) is satisfied by φ_v, we have

$$\langle f(v), (\varphi_v)^{-1}(j) \rangle_\sigma = \langle (\varphi_v)^{-1}(0), (\varphi_v)^{-1}(2) \rangle_\sigma$$

for every $j \in \{0, \dots, 2n-1\}$. It follows that

$$\langle (\varphi_v)^{-1}(i), (\varphi_v)^{-1}(j) \rangle_\sigma = \langle (\varphi_v)^{-1}(0), (\varphi_v)^{-1}(2) \rangle_\sigma$$

for every $j \in \{0, \dots, 2n-1\} \smallsetminus \{2p, i\}$. Since φ_v is an isomorphism from $(\sigma - v) - f(v)$ onto τ_v, we have $\langle i, j \rangle_{\tau_v} = \langle 0, 2 \rangle_{\tau_v}$ for every $j \in \{0, \dots, 2n-1\} \smallsetminus \{2p, i\}$. Since $\tau_v \in \mathcal{R}_{2n}$, it follows from (4.4) that $\langle i, 2p \rangle_{\tau_v} = \langle 0, 1 \rangle_{\tau_v}$. We obtain $p = 0$ and $i = 1$. It follows from (A.7) that

$$\{f(v), (\varphi_v)^{-1}(1)\} \text{ is a module of } \sigma - (\varphi_v)^{-1}(0). \tag{A.8}$$

Since $\tau_v \in \mathcal{R}_{2n}$, τ_v is critical and $\mathbb{P}(\tau_v) = P_{2n}$. It follows from Proposition 4.15 that $[1, 3]_{\tau_v} = [0, 2]_{\tau_v}$. Since φ_v is an isomorphism from $(\sigma - v) - f(v)$ onto τ_v, we obtain

$$[(\varphi_v)^{-1}(1), (\varphi_v)^{-1}(3)]_\sigma = [(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(2)]_\sigma,$$

Since (5.7) is satisfied by φ_v, we have

$$[f(v), (\varphi_v)^{-1}(3)]_\sigma = [(\varphi_v)^{-1}(2), (\varphi_v)^{-1}(0)]_\sigma.$$

By (A.6), $[(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(2)]_\sigma \neq [(\varphi_v)^{-1}(2), (\varphi_v)^{-1}(0)]_\sigma$. It follows that

$$[(\varphi_v)^{-1}(1), (\varphi_v)^{-1}(3)]_\sigma \neq [f(v), (\varphi_v)^{-1}(3)]_\sigma,$$

which contradicts (A.8). □

Claim A.7. *We have* $|\mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma)| = 1$.

Proof. Otherwise, consider distinct $v, w \in \mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma)$. Since f is injective, we have $f(v) \neq f(w)$. Furthermore, it follows from Claim A.6 that $N_{\mathbb{P}(\sigma)}(f(v)) = \{v\}$ and $N_{\mathbb{P}(\sigma)}(f(w)) = \{w\}$. As previously noted, by considering $\varphi_v \circ \pi_{2n}$ (see (A.4)) instead of φ_v, we can assume that

$$w = (\varphi_v)^{-1}(2p),$$

where $p \in \{0, \dots, n-1\}$. Since $N_{\mathbb{P}(\sigma)}(f(w)) = \{w\}$, it follows from Lemma 4.4 that

$$w \longleftrightarrow_\sigma ((\varphi_v)^{-1}(\{0, \dots, 2n-1\}) \smallsetminus \{w, f(w)\}) \cup \{v, f(v)\}.$$

Since $w = (\varphi_v)^{-1}(2p)$, we obtain $p = n-1$ and $\varphi_v(f(w)) = 2n-1$. Therefore, we have

$$[w, ((\varphi_v)^{-1}(\{0,\dots,2n-1\}) \smallsetminus \{w, f(w)\}) \cup \{v, f(v)\}]_\sigma = [(\varphi_v)^{-1}(2), (\varphi_v)^{-1}(0)]_\sigma.$$

As observed in Remark 5.16, $N_{\mathbb{P}(\sigma-v)}(f(v)) = \varnothing$, and $\mathbb{P}(\sigma - v) - f(v) = \mathbb{P}(\sigma - \{v, f(v)\})$. Since φ_v is an isomorphism from $\mathbb{P}(\sigma - \{v, f(v)\})$ onto P_{2n}, we obtain

$$N_{\mathbb{P}(\sigma-v)}((\varphi_v)^{-1}(2n-2)) = \{(\varphi_v)^{-1}(2n-3), (\varphi_v)^{-1}(2n-1)\}.$$

It follows from Lemma 4.4 that $\{(\varphi_v)^{-1}(2n-3), (\varphi_v)^{-1}(2n-1)\}$ is a module of $(\sigma - v) - ((\varphi_v)^{-1}(2n-2))$, that is, $(\sigma - v) - w$. Since $N_{\mathbb{P}(\sigma)}(f(v)) = \{v\}$, it follows from Lemma 4.4 that

$$v \longleftrightarrow_\sigma V(\sigma) \smallsetminus \{v, f(v)\}.$$

Consequently, $\{(\varphi_v)^{-1}(2n-3), (\varphi_v)^{-1}(2n-1)\}$ is a module of $\sigma - w$, which contradicts the fact that $w \in \mathscr{S}(\sigma)$. □

The end of the proof of Proposition A.2. We conclude as follows. By Claim A.7, $\mathscr{S}(\sigma) \smallsetminus \mathscr{S}_c(\sigma)$ admits a unique element denoted by v. By Claim A.6, $N_{\mathbb{P}(\sigma)}(f(v)) = \{v\}$. Thus, $\sigma - \{v, f(v)\}$ is prime. Set

$$X = V(\sigma) \smallsetminus \{v, f(v)\}.$$

Moreover, we have $f(v) \notin \mathscr{S}(\sigma)$ by Claim A.4. Since $N_{\mathbb{P}(\sigma)}(f(v)) = \{v\}$ by Claim A.6, it follows from Lemma 4.4 that

$$v \longleftrightarrow_\sigma V(\sigma) \smallsetminus \{v, f(v)\}, \tag{A.9}$$

that is, $v \in \langle X \rangle_\sigma$.

We prove that

$$[(\varphi_v)^{-1}(1), V(\sigma) \smallsetminus \{(\varphi_v)^{-1}(0),(\varphi_v)^{-1}(1)\}]_\sigma = [(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(2)]_\sigma. \tag{A.10}$$

Since (5.7) is satisfied by φ_v, we have

$$\begin{cases} [f(v), (\varphi_v)^{-1}(\{2i : i \in \{0,\dots,n-1\}\})]_\sigma = [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma, \\ \text{and} \\ [f(v), (\varphi_v)^{-1}(\{2i+1 : i \in \{0,\dots,n-1\}\})]_\sigma = [\varphi^{-1}(2), \varphi^{-1}(0)]_\sigma. \end{cases}$$

Since $[(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(2)]_\sigma \neq [(\varphi_v)^{-1}(2), (\varphi_v)^{-1}(0)]_\sigma$ by (A.6), we obtain $f(v) \notin \langle X \rangle_\sigma$. Since σ is prime, it follows from (A.9) that

$$[v, V(\sigma) \smallsetminus \{v, f(v)\}]_\sigma \neq [v, f(v)]_\sigma. \tag{A.11}$$

Since φ_v is an isomorphism from $\mathbb{P}(\sigma - \{v, f(v)\})$ onto P_{2n}, we obtain that $\sigma[X] - \{(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(1)\}$ is prime. Set

$$Y = X \smallsetminus \{(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(1)\}.$$

Since $v \in \langle X \rangle_\sigma$, we have $v \in \langle Y \rangle_\sigma$. As previously, since (5.7) is satisfied by φ_v, it follows from (A.6) that $f(v) \notin \langle Y \rangle_\sigma$. Since (A.11) holds, it follows from statements (P1) and (P2) of Lemma 3.17 that $\sigma[Y \cup \{v, f(v)\}] = \sigma - \{(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(1)\}$ is prime. Hence,

$$(\varphi_v)^{-1}(1) \in N_{\mathbb{P}(\sigma)}((\varphi_v)^{-1}(0)).$$

For a contradiction, suppose that $(\varphi_v)^{-1}(0) \in \mathscr{S}(\sigma)$. Since $\sigma - \{(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(1)\}$ is prime, we obtain $(\varphi_v)^{-1}(0) \in \mathscr{S}(\sigma) \setminus \mathscr{S}_c(\sigma)$, which contradicts Claim A.7. It follows that

$$(\varphi_v)^{-1}(0) \notin \mathscr{S}(\sigma).$$

Since $(\varphi_v)^{-1}(1) \in N_{\mathbb{P}(\sigma)}((\varphi_v)^{-1}(0))$, it follows from Lemma 4.4 that

$$d_{\mathbb{P}(\sigma)}((\varphi_v)^{-1}(0)) = 1 \text{ or } 2.$$

For a contradiction, suppose that $d_{\mathbb{P}(\sigma)}((\varphi_v)^{-1}(0)) = 2$. There exists $w \in V(\sigma) \setminus \{(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(1)\}$ such that $N_{\mathbb{P}(\sigma)}((\varphi_v)^{-1}(0)) = \{(\varphi_v)^{-1}(1), w\}$. Since $N_{\mathbb{P}(\sigma)}(f(v)) = \{v\}$ by Claim A.6, we have $(\varphi_v)^{-1}(0) \notin N_{\mathbb{P}(\sigma)}(f(v))$. Thus $w \neq f(v)$. Furthermore, since $\mathscr{S}(\sigma - v) = \{f(v)\}$, we have $(\varphi_v)^{-1}(0) \notin \mathscr{S}(\sigma - v)$, and hence $(\varphi_v)^{-1}(0) \notin N_{\mathbb{P}(\sigma)}(v)$. Therefore, $w \neq v$. It follows that $w \in Y$. Since $\{(\varphi_v)^{-1}(1), w\}$ is a module of $\sigma - (\varphi_v)^{-1}(0)$ by Lemma 4.4, we obtain $(\varphi_v)^{-1}(1) \in Y_\sigma(w)$. But, since φ_v is an isomorphism from $\mathbb{P}(\sigma - \{v, f(v)\})$ onto P_{2n}, we have $N_{\mathbb{P}(\sigma-\{v,f(v)\})}((\varphi_v)^{-1}(0)) = \{(\varphi_v)^{-1}(1)\}$. Since $\sigma - \{v, f(v)\}$ is critical, it follows from Lemma 4.4 applied to $\sigma - \{v, f(v)\}$ that $(V(\sigma) \setminus \{v, f(v)\}) \setminus \{(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(1)\}$ is a module of $(\sigma - \{v, f(v)\}) - (\varphi_v)^{-1}(0)$. We obtain $\varphi^{-1}(1) \in \langle Y \rangle_\sigma$. Consequently, $\varphi^{-1}(1) \in Y_\sigma(w) \cap \langle Y \rangle_\sigma$, which contradicts Lemma 3.13. It follows that

$$N_{\mathbb{P}(\sigma)}((\varphi_v)^{-1}(0)) = \{(\varphi_v)^{-1}(1)\}.$$

Since $(\varphi_v)^{-1}(0) \notin \mathscr{S}(\sigma)$, it follows from Lemma 4.4 that

$$V(\sigma) \setminus \{(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(1)\}$$

is a module of $\sigma - (\varphi_v)^{-1}(0)$. Since φ_v is an isomorphism from $\sigma - \{v, f(v)\}$ onto an element of $\mathcal{R}_{2n}$, we have

$$[(\varphi_v)^{-1}(1), (\varphi_v)^{-1}(3)]_\sigma = [(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(2)]_\sigma.$$

It follows that (A.10) holds. In particular, we have

$$[(\varphi_v)^{-1}(1), v]_\sigma = [(\varphi_v)^{-1}(0), (\varphi_v)^{-1}(2)]_\sigma. \tag{A.12}$$

Similarly, we have

$$[(\varphi_v)^{-1}(2n-2), V(\sigma) \setminus \{(\varphi_v)^{-1}(2n-1), (\varphi_v)^{-1}(2n-2)\}]_\sigma = [(\varphi_v)^{-1}(2), (\varphi_v)^{-1}(0)]_\sigma.$$

In particular, we obtain

$$[(\varphi_v)^{-1}(2n-2), v]_\sigma = [(\varphi_v)^{-1}(2), (\varphi_v)^{-1}(0)]_\sigma.$$

By (A.6), we have $[(\varphi_v)^{-1}(0),(\varphi_v)^{-1}(2)]_\sigma \neq [(\varphi_v)^{-1}(2),(\varphi_v)^{-1}(0)]_\sigma$. Therefore, it follows from (A.12) that

$$v \not\leftrightarrow_\sigma \{(\varphi_v)^{-1}(1),(\varphi_v)^{-1}(2n-2)\},$$

which contradicts (A.9). □

Proof of Theorem 5.21. We proceed by induction on $v(\sigma)-v(\tau)\geq 1$. The result is obvious when $v(\sigma)-v(\tau)=1$. Hence, suppose that $v(\sigma)-v(\tau)\geq 2$. Since $v(\tau)\geq 5$, we have $v(\sigma)\geq 7$.

For convenience, we denote by $\mathscr{N}(\sigma)$ the set of $v\in V(\sigma)$ such that $\sigma - v$ is prime, and neither critical nor almost critical. By Proposition A.2,

$$\mathscr{N}(\sigma)\neq\varnothing.$$

To begin[A.1], we prove that there exists $X \subsetneq V(\sigma)$ such that

$$\text{(A.13)}\qquad \begin{cases}\sigma[X]\simeq\tau\\ \text{and}\\ (V(\sigma)\smallsetminus X)\cap\mathscr{S}(\sigma)\neq\varnothing.\end{cases}$$

Consider $Y\subseteq V(\sigma)$ such that $\sigma[Y]\simeq\tau$, and suppose that $\sigma-u$ is decomposable for every $u\in V(\sigma)\smallsetminus Y$. It follows from Corollary 3.21 that there exist distinct $v,w\in V(\sigma)\smallsetminus Y$ such that $\sigma-\{v,w\}$ is prime. Thus, τ embeds into $\sigma-\{v,w\}$. Denote by C the component of $\mathbb{P}(\sigma)$ containing v and w. For a contradiction, suppose that $V(C)\subseteq V(\sigma)\smallsetminus\mathscr{S}(\sigma)$. By Proposition 4.5, $|V(\sigma)\smallsetminus V(C)|\leq 1$, so $|\mathscr{S}(\sigma)|\leq 1$. Since σ is not critical, we have $|\mathscr{S}(\sigma)|=1$. By Theorem 5.13, C is the unique component of $\mathbb{P}(\sigma)$ such that $v(C)\geq 2$. If $V(C)\cap\mathscr{S}(\sigma)=\varnothing$, then it follows from Theorem 5.13 that σ is almost critical. Consequently, we have $V(C)\cap\mathscr{S}(\sigma)\neq\varnothing$. Therefore, there exist distinct vertices $c_0,\dots,c_p$ of C satisfying

- $\{c_0,c_1\}=\{v,w\}$;
- $p\geq 2$, $\{c_0,\dots,c_{p-1}\}\subseteq V(\sigma)\smallsetminus\mathscr{S}(\sigma)$, and $c_p\in\mathscr{S}(\sigma)$;
- for $i\in\{0,\dots,p-1\}$, $\{c_i,c_{i+1}\}\in E(\mathbb{P}(\sigma))$.

Let $i\in\{1,\dots,p-1\}$. We have $c_{i-1},c_{i+1}\in N_{\mathbb{P}(\sigma)}(c_i)$. Since $c_i\notin\mathscr{S}(\sigma)$, it follows from Lemma 4.4 that $N_{\mathbb{P}(\sigma)}(c_i)=\{c_{i-1},c_{i+1}\}$, and $\{c_{i-1},c_{i+1}\}$ is a module of $\sigma-c_i$. Thus, $\sigma-\{c_{i-1},c_i\}\simeq\sigma-\{c_i,c_{i+1}\}$. It follows that $\sigma-\{c_0,c_1\}\simeq\sigma-\{c_{p-1},c_p\}$, that is, $\sigma-\{v,w\}\simeq\sigma-\{c_{p-1},c_p\}$. Since τ embeds into $\sigma-\{v,w\}$, τ embeds into $\sigma-\{c_{p-1},c_p\}$ as well. Since $c_p\in\mathscr{S}(\sigma)$, (A.13) holds.

Now, we consider $X\subseteq V(\sigma)$ such that (A.13) holds. There exists

$$v\in(V(\sigma)\smallsetminus X)\cap\mathscr{S}(\sigma).$$

If there exists $w\in(V(\sigma)\smallsetminus X)\cap\mathscr{N}(\sigma)\neq\varnothing$, then it suffices to apply the induction hypothesis to $\sigma-w$. Hence, suppose that

$$(V(\sigma)\smallsetminus X)\cap\mathscr{N}(\sigma)=\varnothing.$$

[A.1]From here until ($\star$) (see page 197), the proof is similar to that of Theorem 5.19.

Thus, $v \notin \mathscr{N}(\sigma)$. Since $\mathscr{N}(\sigma) \neq \varnothing$, consider

$$(\star) \qquad x \in X \cap \mathscr{N}(\sigma).$$

Since $\sigma - v$ is prime, $\sigma - v$ is critical or almost critical. We distinguish the following two cases.

CASE 1: $\sigma - v$ is critical.

Since $v(\sigma) - v(\tau) \geq 2$, we have $X \subsetneq V(\sigma - v)$. Since $\sigma - v$ is critical, it follows from Corollary 3.21 that there exist distinct $w, w' \in V(\sigma - v) \setminus X$ such that $\{w, w'\} \in E(\mathbb{P}(\sigma - v))$. Thus, τ embeds into $(\sigma - v) - \{w, w'\}$.

First, suppose that there exists $y \in (V(\sigma) - v) \setminus \{x\}$ such that $\{x, y\} \in E(\mathbb{P}(\sigma - v))$. Since $\{x, y\}, \{w, w'\} \in E(\mathbb{P}(\sigma - v))$, it follows from Corollary 4.8 that $(\sigma - v) - \{x, y\} \simeq (\sigma - v) - \{w, w'\}$. Therefore, τ embeds into $(\sigma - v) - \{x, y\}$ as well. To conclude, it suffices to apply the induction hypothesis to $\sigma - x$.

Second, suppose that x is an isolated vertex of $\mathbb{P}(\sigma - v)$. It follows from Corollary 4.6 that there exists $n \geq 3$ such that $\mathbb{P}(\sigma - v) \simeq P_{2n} \oplus K_{\{2n\}}$. In particular, we obtain that $v(\sigma)$ is even. In another vein, it follows from Corollary 5.5 (and Remark 5.6) that there exists $e \in E(\mathbb{P}(\sigma - v)) \cap E(\mathbb{P}(\sigma))$. Since $e, \{w, w'\} \in E(\mathbb{P}(\sigma - v))$, it follows from Corollary 4.8 that $(\sigma - v) - e \simeq (\sigma - v) - \{w, w'\}$. Therefore, τ embeds into $(\sigma - v) - e$, and hence τ embeds into $\sigma - e$. Since $e \in E(\mathbb{P}(\sigma))$, $\sigma - e$ is prime. Furthermore, since $e \in E(\mathbb{P}(\sigma - v))$, $(\sigma - e) - v$ is prime. Thus, $\sigma - e$ is not critical. Lastly, since $v(\sigma - e)$ is even, it follows from Theorem 5.13 that $\sigma - e$ is not almost critical. To conclude, it suffices to apply the induction hypothesis to $\sigma - e$.

CASE 2: $\sigma - v$ is almost critical.

There exists $w \in V(\sigma - v)$ such that

$$\mathscr{S}(\sigma - v) = \mathscr{S}_c(\sigma - v) = \{w\}.$$

It follows from Theorem 5.13 that $v(\sigma) = 2n + 2$, where $n \geq 3$, and there exists an isomorphism φ from $(\sigma - v) - w$ onto an element ρ of $\mathcal{R}_{2n}$ satisfying (5.7).

We can conclude when $w \notin X$. Indeed, suppose that $X \subseteq V(\sigma) \setminus \{v, w\}$. It follows from the first statement of Fact 5.18 that $w \in \mathrm{Ext}_\sigma(X)$. In what follows, we suppose that

$$w \in X.$$

First, suppose that

$$x \neq w.$$

Since $\mathbb{P}(\rho) = P_{2n}$, φ is an isomorphism from $\mathbb{P}((\sigma - v) - w)$ onto P_{2n}. Since $x \neq w$, there exists $y \in V(\sigma - v) \setminus \{x, w\}$ such that $\{x, y\} \in E(\mathbb{P}((\sigma - v) - w))$. As observed in Remark 5.16, we have $\mathbb{P}((\sigma - v) - w) = \mathbb{P}(\sigma - v) - w$. Thus

$$\{x, y\} \in E(\mathbb{P}(\sigma - v)),$$

so $(\sigma - v) - \{x, y\}$ is prime. Furthermore, since $X \subsetneq V(\sigma - v)$, it follows from Corollary 3.21 that there exist $u, u' \in V(\sigma - v) \setminus X$ such that $(\sigma - v) - \{u, u'\}$ is prime. Since $\mathscr{S}(\sigma - v) = \{w\}$ and $w \in X$, we obtain $u \neq u'$. Hence

$$\{u, u'\} \in E(\mathbb{P}(\sigma - v)).$$

It follows from the second statement of Fact 5.18 that $(\sigma - v) - \{x, y\} \simeq (\sigma - v) - \{u, u'\}$. It follows that τ embeds into $(\sigma - v) - \{x, y\}$. Therefore, τ embeds into $\sigma - x$. To conclude, it suffices to apply the induction hypothesis to $\sigma - x$.

Second, suppose that

$$x = w.$$

Set

$$Y = V(\sigma) \setminus \{v, w\}.$$

Since $\mathscr{S}_c(\sigma - v) = \{w\}$, $\sigma[Y]$ is critical. Furthermore, $\sigma[Y \cup \{v\}]$ is prime because $\sigma[Y \cup \{v\}] = \sigma - x$. It follows that $v \in \mathscr{S}_c(\sigma[Y \cup \{v\}])$. By Corollary 5.5 (and Remark 5.6), there exists $e \in E(\mathbb{P}(\sigma[Y \cup \{v\}])) \cap E(\mathbb{P}(\sigma[Y]))$. Set

$$Z = Y \setminus e.$$

We have $\sigma[Z]$ is prime. Since $e \in E(\mathbb{P}(\sigma[Y \cup \{v\}])$, we obtain $\sigma[Y \cup \{v\}]) - e = (\sigma - e) - x$ is prime. Since $(\sigma - e) - x = \sigma[Z \cup \{v\}]$, we have $v \in \mathrm{Ext}_\sigma(Z)$. Furthermore, it follows from the first statement of Fact 5.18 that $x \in \mathrm{Ext}_\sigma(Z)$. Since $(\sigma - e) - x = \sigma[Z \cup \{v\}]$ and $(\sigma - e) - v = \sigma[Z \cup \{x\}]$, we obtain that

$$(\sigma - e) - v \text{ and } (\sigma - e) - x \text{ are prime.} \tag{A.14}$$

Since $(\sigma - v) - e$ is prime, we have $e \in E(\mathbb{P}(\sigma - v))$. In another vein, it follows from Corollary 3.21 applied to $\sigma - v$ that there exist $u, u' \in V(\sigma - v) \setminus X$ such that $(\sigma - v) - \{u, u'\}$ is prime. Since $\mathscr{S}(\sigma - v) = \{w\}$, we obtain $u \neq u'$. Hence, $\{u, u'\} \in E(\mathbb{P}(\sigma - v))$. Therefore, we obtain $e, \{u, u'\} \in E(\mathbb{P}(\sigma - v))$. By the second statement of Fact 5.18, $(\sigma - v) - e \simeq (\sigma - v) - \{u, u'\}$. Since τ embeds into $(\sigma - v) - \{u, u'\}$,

$$\tau \text{ embeds into } (\sigma - v) - e. \tag{A.15}$$

Finally, we distinguish the following two subcases.

Subcase 2.1: $\sigma - e$ is decomposable.

Since $\sigma - e = \sigma[Z \cup \{v, x\}]$, it follows from statement (P5) of Lemma 3.17 that $\{v, x\}$ is a module of $\sigma - e$. It follows that $(\sigma - e) - x \simeq (\sigma - e) - v$. By (A.15), τ embeds into $(\sigma - e) - v$. Consequently, τ embeds into $(\sigma - e) - x$. To conclude, it suffices to apply the induction hypothesis to $\sigma - x$ because $x \in \mathscr{N}(\sigma)$.

Subcase 2.2: $\sigma - e$ is prime.

It follows from (A.14) that

$$v, x \in \mathscr{S}(\sigma - e).$$

Consequently, $\sigma - e$ is neither critical nor almost critical. Moreover, it follows from (A.15) that τ embeds into $\sigma - e$. To conclude, it suffices to apply the induction hypothesis to $\sigma - e$. □

B. Proofs of Propositions 5.27, 5.28, and 5.29

Proof of Proposition 5.27. Consider $s \in \mathscr{S}_c(\tau)$ such that $\mathbb{P}(\tau - s) \simeq P_{2n} \oplus K_{\{2n\}}$. Therefore, $v(\sigma) = 2n + 2$. Since $v(\sigma) \geq 7$, we obtain

$$n \geq 3.$$

Up to isomorphism, we can assume that

$$\begin{cases} V(\tau) = \{0, \dots, 2n+1\}, \\ s = 2n+1, \\ \text{and} \\ \mathbb{P}(\tau - (2n+1)) = P_{2n} \oplus K_{\{2n\}}. \end{cases}$$

For a contradiction, suppose that

$$|\mathscr{S}_c(\tau)| \geq 2, \tag{B.1}$$

and consider $t \in \mathscr{S}_c(\tau) \setminus \{2n+1\}$. By Corollary 5.25, $N_{\mathbb{P}(\tau-(2n+1))}(t) = N_{\mathbb{P}(\tau-t)}(2n+1)$, and $N_{\mathbb{P}(\tau-(2n+1))}(t) \neq \varnothing$. Since $N_{\mathbb{P}(\tau-(2n+1))}(t) \neq \varnothing$, $t \neq 2n$. Moreover, since $v(\tau) \geq 8$, it follows from Corollary 5.25 that $\tau - t \simeq \tau - (2n+1)$. Therefore, $\mathbb{P}(\tau - t) \simeq P_{2n} \oplus K_{\{2n\}}$. Consider an isomorphism φ from $\mathbb{P}(\tau - t)$ onto $P_{2n} \oplus K_{\{2n\}}$. Since $N_{\mathbb{P}(\tau-t)}(2n+1) \neq \varnothing$, $\varphi_t(2n+1) \neq 2n$.

Since $\tau - (2n+1)$ is critical and $\mathbb{P}(\tau - (2n+1)) = P_{2n} \oplus K_{\{2n\}}$, it follows from Theorem 4.24 that

$$\tau - (2n+1) = \sigma(T_{2n+1}),$$

where $n \geq 3$. Furthermore, since $\widehat{\pi_{2n}} \in \mathrm{Aut}(\sigma(T_{2n+1}))$ by Remark 4.26, we can assume that $n \leq t \leq 2n-1$. Similarly, there exists an isomorphism φ from $\tau - t$ onto $\sigma(T_{2n+1})$ such that $n \leq \varphi(2n+1) \leq 2n-1$. Since $N_{\mathbb{P}(\tau-(2n+1))}(t) \neq \varnothing$, it follows from Lemma 4.4 that $d_{\mathbb{P}(\tau-(2n+1))}(t) = 1$ or 2.

The following observation is useful in what follows. Let $x, y \in \{0, \dots, 2n\}$ such that $x < y < 2n$.

$$\text{If } \{\varphi^{-1}(x), \varphi^{-1}(y)\} \cap \{2n, 2n+1\} = \varnothing, \text{ then } \varphi^{-1}(x) < \varphi^{-1}(y). \tag{B.2}$$

Indeed, we have

$$\begin{aligned} (x, y)_{\sigma(T_{2n+1})} &= (\varphi^{-1}(x), \varphi^{-1}(y))_{\tau - t} \\ &\qquad \text{because } \varphi \text{ is an isomorphism from } \tau - t \text{ onto } \sigma(T_{2n+1}) \\ &= (\varphi^{-1}(x), \varphi^{-1}(y))_{\tau-(2n+1)} \quad \text{because } 2n+1 \notin \{\varphi^{-1}(x), \varphi^{-1}(y)\} \\ &= (\varphi^{-1}(x), \varphi^{-1}(y))_{\sigma(T_{2n+1})} \quad \text{because } \tau - (2n+1) = \sigma(T_{2n+1}) \\ &= (\varphi^{-1}(x), \varphi^{-1}(y))_{\sigma(T_{2n+1})-(2n)} \text{ because } 2n \notin \{\varphi^{-1}(x), \varphi^{-1}(y)\}. \end{aligned}$$

Since $x < y < 2n$, we obtain

$$(x, y)_{\sigma(T_{2n+1})-(2n)} = (\varphi^{-1}(x), \varphi^{-1}(y))_{\sigma(T_{2n+1})-(2n)}.$$

Since $T_{2n+1} - (2n) = L_{2n}$ and $x < y$, we have $\varphi^{-1}(x) < \varphi^{-1}(y)$.

First, suppose that $d_{\mathbb{P}(\tau-(2n+1))}(t) = 1$, and denote by u the unique element of $N_{\mathbb{P}(\tau-(2n+1))}(t)$. Since $n \le t \le 2n-1$, we obtain $t = 2n-1$, and hence $u = 2n-2$. Since $N_{\mathbb{P}(\tau-(2n+1))}(t) = N_{\mathbb{P}(\tau-t)}(2n+1)$, we have $N_{\mathbb{P}(\tau-t)}(2n+1) = \{u\}$, that is, $N_{\mathbb{P}(\tau-t)}(2n+1) = \{2n-2\}$. Since $n \le \varphi(2n+1) \le 2n-1$, we obtain

$$\varphi(2n+1) = 2n-1 \ \text{ and } \ \varphi(2n-2) = 2n-2.$$

Furthermore, since $\{t,u\} \in E(\mathbb{P}(\tau-(2n+1)))$, it follows from Lemma 4.40 that $(\tau-(2n+1)) - \{t,u\}$ is critical, and

$$E(\mathbb{P}((\tau-(2n+1)) - \{t,u\})) = E(\mathbb{P}(\tau-(2n+1))) \smallsetminus \{\{2n-3,u\},\{t,u\}\}.$$

Similarly, $(\tau-t) - \{2n+1,u\}$ is critical, and

$$E(\mathbb{P}((\tau-t) - \{2n+1,u\})) = E(\mathbb{P}(\tau-t)) \smallsetminus \{\{\varphi^{-1}(2n-3),u\},\{2n+1,u\}\}.$$

Thus, $2n$ is the unique isolated vertex of $\mathbb{P}((\tau-(2n+1)) - \{t,u\})$, that is, $\mathbb{P}(\tau-\{t,u,2n+1\})$. Analogously, $\varphi^{-1}(2n)$ is the unique isolated vertex of $\mathbb{P}(\tau-\{t,u,2n+1\})$. Therefore,

$$\varphi(2n) = 2n.$$

Recall that $\tau-(2n+1) = \sigma(T_{2n+1})$, and φ is an isomorphism from $\tau-t$ onto $\sigma(T_{2n+1})$. Consequently, $\varphi_{\restriction\{0,\dots,2n-3\}}$ is an automorphism of $\sigma(T_{2n+1}) - \{2n-2,2n-1,2n\}$, that is, $\sigma(T_{2n+1}-\{2n-2,2n-1,2n\})$. Since $T_{2n+1}-\{2n-2,2n-1,2n\}$ is linear, $\sigma(T_{2n+1}-\{2n-2,2n-1,2n\})$ is rigid. Hence,

$$\varphi_{\restriction\{0,\dots,2n-3\}} = \mathrm{Id}_{\{0,\dots,2n-3\}}.$$

Since $\varphi(2n-2) = 2n-2$ and $\varphi(2n) = 2n$, we obtain that $\{2n-1,2n+1\}$, that is, $\{s,t\}$ is a module of τ, which contradicts the fact that τ is prime. It follows that $d_{\mathbb{P}(\tau-s)}(t) \ne 1$.

Second, suppose that $d_{\mathbb{P}(\tau-s)}(t) = 2$. Since $d_{\mathbb{P}(\tau-s)}(t) \ne 1$, $t \ne 2n-1$. Hence, $n \le t \le 2n-2$. Similarly, by setting $j = \varphi(2n+1)$, we have $n \le j \le 2n-2$. Recall that $N_{\mathbb{P}(\tau-(2n+1))}(t) = N_{\mathbb{P}(\tau-t)}(2n+1)$ by Corollary 5.25. It follows that

$$\{\varphi^{-1}(j-1),\varphi^{-1}(j+1)\} = \{t-1,t+1\}.$$

It follows from (B.2) that

$$\varphi^{-1}(j-1) = t-1 \text{ and } \varphi^{-1}(j+1) = t+1.$$

Since $\{t-2,t-1\},\{t-1,t\},\{t,t+1\} \in E(\mathbb{P}(\tau-(2n+1)))$, it follows from Lemma 4.39 that $(\tau-(2n+1)) - \{t,t+1\}$ is critical, and

$$\begin{aligned} &E(\mathbb{P}((\tau-(2n+1)) - \{t-1,t\})) \\ &\quad = (E(\mathbb{P}(\tau-(2n+1))) \smallsetminus \{\{k,k+1\} : k \in \{t-2,t-1,t\}\}) \\ &\qquad \cup \{\{t-2,t+1\}\} \\ &\quad = \{\{k,k+1\} : k \in \{0,\dots,t-3\} \cup \{t+1,\dots,2n-2\}\} \\ &\qquad \cup \{\{t-2,t+1\}\}. \end{aligned} \tag{B.3}$$

Similarly, we obtain $(\tau - t) - \{2n+1, \varphi^{-1}(j-1)\}$ is critical, and

$$\begin{aligned} &E(\mathbb{P}((\tau - t) - \{2n+1, \varphi^{-1}(j-1)\})) \\ &= \{\{\varphi^{-1}(k), \varphi^{-1}(k+1)\} : k \in \{0, \dots, j-3\} \cup \{j+1, \dots, 2n-2\}\} \\ &\quad \cup \{\{\varphi^{-1}(j-2), \varphi^{-1}(j+1)\}\}. \end{aligned} \tag{B.4}$$

Since $\varphi^{-1}(j-1) = t-1$, we have

$$(\tau - (2n+1)) - \{t-1, t\} = (\tau - t) - \{2n+1, \varphi^{-1}(j-1)\}.$$

Set

$$\mu = \tau - \{t, t+1, 2n+1\}.$$

It follows from (B.3) that

$$N_{\mathbb{P}(\mu)}(t+1) = \begin{cases} \{t-2\} \text{ if } t = 2n-2 \\ \{t-2, t+2\} \text{ if } t < 2n-2. \end{cases} \tag{B.5}$$

Similarly, it follows from (B.4) that

$$N_{\mathbb{P}(\mu)}(\varphi^{-1}(j+1)) = \begin{cases} \{\varphi^{-1}(j-2)\} \text{ if } j = 2n-2 \\ \{\varphi^{-1}(j-2), \varphi^{-1}(j+2)\} \text{ if } j < 2n-2. \end{cases} \tag{B.6}$$

Since $\varphi^{-1}(j+1) = t+1$, we obtain $t = 2n-2$ if and only if $j = 2n-2$.

To begin, suppose that $t = 2n-2$ and $j = 2n-2$. By (B.5) and (B.6), we have $\varphi^{-1}(2n-4) = 2n-4$. By proceeding by induction, it follows from (B.3) and (B.4) that

$$\varphi^{-1}(l) = l$$

for every $l \in \{0, \dots, 2n-4\}$. Since $\varphi^{-1}(j-1) = t-1$ and $\varphi^{-1}(j+1) = t+1$, we obtain

$$\varphi^{-1}(l) = l$$

for every $l \in \{0, \dots, 2n-3\} \cup \{2n-1\}$. Recall that $\tau - (2n+1) = \sigma(T_{2n+1})$, and φ is an isomorphism from $\tau - t$ onto $\sigma(T_{2n+1})$. Since $\varphi(2n+1) = 2n-2$, we obtain $\varphi^{-1}(2n) = 2n$. It follows that $\{2n-2, 2n+1\}$ is a module of τ, which contradicts the fact that τ is prime.

Now, suppose that $t < 2n-2$ and $j < 2n-2$. By (B.5) and (B.6), we have $\{\varphi^{-1}(j-2), \varphi^{-1}(j+2)\} = \{t-2, t+2\}$. It follows from (B.2) that

$$\varphi^{-1}(j-2) = t-2 \text{ and } \varphi^{-1}(j+2) = t+2.$$

It follows from (B.3) that

$$N_{\mathbb{P}(\mu)}(t-2) = \{t-3, t+1\}.$$

Similarly, it follows from (B.4) that

$$N_{\mathbb{P}(\mu)}(\varphi^{-1}(j-2)) = \{\varphi^{-1}(j-3), \varphi^{-1}(j+1)\}.$$

Since $\varphi^{-1}(j+1) = t+1$ and $\varphi^{-1}(j-2) = t-2$, we obtain $\varphi^{-1}(j-3) = t-3$. By proceeding by induction, we obtain

$$\varphi^{-1}(j-k) = t-k$$

for every $k \in \{2, \ldots, \min(t, j)\}$. For instance, suppose that $t \leq j$. We obtain $\varphi^{-1}(j - t) = 0$. Since $d_{\mathbb{P}(\mu)}(0) = 1$, we have $d_{\mathbb{P}(\mu)}(\varphi^{-1}(j - t)) = 1$. Hence, $j - t = 0$ or $2n - 1$. Since $j < 2n - 2$, we obtain $j = t$. Thus, we have

$$\varphi^{-1}(l) = l$$

for every $l \in \{0, \ldots, t - 2\}$. Similarly, we obtain

$$\varphi^{-1}(l) = l$$

for every $l \in \{t + 1, \ldots, 2n - 1\}$. Since $\varphi^{-1}(j - 1) = t - 1$ and $j = t$, we obtain

$$\varphi^{-1}(l) = l$$

for every $l \in \{0, \ldots, 2n - 1\} \setminus \{t\}$. Recall that $\tau - (2n + 1) = \sigma(T_{2n+1})$, and φ is an isomorphism from $\tau - t$ onto $\sigma(T_{2n+1})$. We obtain $\varphi^{-1}(2n) = 2n$. It follows that $\{t, 2n+1\}$ is a module of τ, which contradicts the fact that τ is prime. □

Proof of Proposition 5.28. Consider $s \in \mathscr{S}_c(\sigma)$ such that $\mathbb{P}(\sigma - s) \simeq P_{2n+1}$. Therefore, $v(\sigma) = 2n + 2$. Since $v(\sigma) \geq 7$, we obtain

$$n \geq 3.$$

Up to isomorphism, we can assume that

$$\begin{cases} V(\sigma) = \{0, \ldots, 2n + 1\}, \\ s = 2n + 1, \\ \text{and} \\ \mathbb{P}(\sigma - (2n + 1)) = P_{2n+1}. \end{cases}$$

Since $\sigma - (2n + 1)$ is critical and $\mathbb{P}(\sigma - (2n + 1)) = P_{2n+1}$, it follows from Proposition 4.27 that

$$\text{(B.7)} \qquad \begin{cases} (0, 1)_\sigma \neq (1, 0)_\sigma, \\ \text{and,} \\ [0, 1]_\sigma \neq [0, 2]_\sigma. \end{cases}$$

Furthermore, for any $p, q \in \{0, \ldots, 2n\}$ such that $p < q$, we have

$$\text{(B.8)} \qquad [p, q]_\sigma = \begin{cases} [0, 2]_\sigma \text{ if } p \text{ and } q \text{ are even,} \\ [0, 1]_\sigma \text{ otherwise.} \end{cases}$$

Suppose that $|\mathscr{S}_c(\sigma)| \geq 2$. Consider any element t of $\mathscr{S}_c(\sigma) \setminus \{s\}$. Since $\pi_{2n+1} \in \mathrm{Aut}(P_{2n+1})$ (see Notation 4.21), we can assume that

$$n \leq t \leq 2n.$$

It follows from Corollary 5.25 that

$$\text{(B.9)} \qquad \begin{cases} N_{\mathbb{P}(\sigma-(2n+1))}(t) = N_{\mathbb{P}(\sigma-t)}(2n + 1) \\ \text{and} \\ \mathbb{P}(\sigma - t) \simeq P_{2n+1}. \end{cases}$$

As above, since $\mathbb{P}(\sigma - t) \simeq P_{2n+1}$, there exists an isomorphism φ from $\sigma - t$ onto τ, where τ is a critical 2-structure such that $\mathbb{P}(\tau) = P_{2n+1}$. By Proposition 4.27,

$$\text{(B.10)} \qquad \begin{cases} (\varphi{-}1(0), \varphi^{-1}(1))_\sigma \neq (\varphi^{-1}(1), \varphi^{-1}(0))_\sigma \\ \text{and} \\ [\varphi^{-1}(0), \varphi^{-1}(1)]_\sigma \neq [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma. \end{cases}$$

Furthermore, for any $p, q \in \{0, \dots, 2n\}$ such that $p < q$, we have

$$\text{(B.11)} \qquad [\varphi^{-1}(p), \varphi^{-1}(q)]_\sigma = \begin{cases} [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma \text{ if } p \text{ and } q \text{ are even,} \\ [\varphi^{-1}(0), \varphi^{-1}(1)]_\sigma \text{ otherwise.} \end{cases}$$

Similarly, we can assume that $n \leq \varphi(2n+1) \leq 2n$.

For a contradiction, suppose that

$$\text{(B.12)} \qquad d_{\mathbb{P}(\sigma-(2n+1))}(t) = 1.$$

Since $n \leq t \leq 2n$, we have $t = 2n$. Hence

$$N_{\mathbb{P}(\sigma-(2n+1))}(2n) = \{2n-1\}.$$

It follows from (B.9) that $\varphi(2n+1) = 2n$, $N_{\mathbb{P}(\sigma-(2n))}(2n+1) = \{\varphi^{-1}(2n-1)\}$, and

$$\varphi(2n-1) = 2n-1.$$

It follows from Lemma 4.40 that $(\sigma - (2n+1)) - \{2n-1, 2n\}$ is critical and

$$\begin{aligned} &E(\mathbb{P}((\sigma - (2n+1)) - \{2n-1, 2n\})) \\ &\quad = E(\mathbb{P}(\sigma - (2n+1))) \setminus \{\{k, k+1\} : k \in \{2n-2, 2n-1\}\} \\ &\quad = \{\{k, k+1\} : k \in \{0, \dots, 2n-3\}\} \\ &\quad = E(P_{2n-1}). \end{aligned}$$

Thus, we obtain

$$\mathbb{P}(\sigma - \{2n-1, 2n, 2n+1\}) = P_{2n-1}.$$

Similarly, $\tau - \{2n-1, 2n\}$ is critical and

$$\mathbb{P}(\tau - \{2n-1, 2n\})) = P_{2n-1}.$$

Observe that $(\varphi^{-1})_{\restriction\{0,\dots,2n-2\}}$ is an isomorphism from $\mathbb{P}(\tau - \{2n-1, 2n\}))$, which is P_{2n-1}, onto $\mathbb{P}(\sigma - \{2n, 2n+1, \varphi^{-1}(2n-1)\})$. Since $\varphi(2n-1) = 2n-1$, we have

$$\sigma - \{2n-1, 2n, 2n+1\} = \sigma - \{2n, 2n+1, \varphi^{-1}(2n-1)\}.$$

It follows that

$$\varphi_{\restriction\{0,\dots,2n-2\}} \in \mathrm{Aut}(P_{2n-1}).$$

Therefore, we obtain

$$\varphi_{\restriction\{0,\dots,2n-2\}} = \mathrm{Id}_{\{0,\dots,2n-2\}} \text{ or } \pi_{2n-1}.$$

We distinguish the following two cases. In each of them, we obtain a contradiction.

CASE 1: $\varphi_{\restriction\{0,\ldots,2n-2\}} = \mathrm{Id}_{\{0,\ldots,2n-2\}}$.

Since $\varphi(2n-1) = 2n-1$, we obtain

$$\varphi(k) = k \tag{B.13}$$

for each $k \in \{0,\ldots,2n-1\}$. We verify that $\{2n, 2n+1\}$ is a module of σ. Let $p \in \{0,\ldots,2n-1\}$. For instance, assume that p is even. We obtain

$$\begin{aligned}
[p, 2n+1]_\sigma &= [\varphi^{-1}(p), \varphi^{-1}(2n)]_\sigma && \text{by (B.13)}\\
&= [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma && \text{by (B.11)}\\
&= [0, 2]_\sigma && \text{by (B.13)}\\
&= [p, 2n]_\sigma && \text{by (B.8).}
\end{aligned}$$

Similarly, we have $[p, 2n+1]_\sigma = [p, 2n]_\sigma$ when p is odd. It follows that $\{2n, 2n+1\}$ is a module of σ, which contradicts the fact that σ is prime.

CASE 2: $\varphi_{\restriction\{0,\ldots,2n-2\}} = \pi_{2n-1}$.

We obtain

$$\varphi(k) = 2n-2-k \tag{B.14}$$

for each $k \in \{0,\ldots,2n-2\}$. Therefore, we have

$$\begin{aligned}
(0,1)_\sigma &= (\varphi^{-1}(2n-2), \varphi^{-1}(2n-3))_\sigma && \text{by (B.14)}\\
&= (\varphi^{-1}(1), \varphi^{-1}(0))_\sigma && \text{by (B.11)}\\
&= (\varphi^{-1}(2n-1), \varphi^{-1}(2n-2))_\sigma && \text{by (B.11)}\\
&= (2n-1, \varphi^{-1}(2n-2))_\sigma \\
&\qquad \text{because } \varphi^{-1}(2n-1) = 2n-1 \\
&= (2n-1, 0)_\sigma && \text{by (B.14)}\\
&= (1, 0)_\sigma && \text{by (B.8).}
\end{aligned}$$

It follows that $(0,1)_\sigma = (1,0)_\sigma$, which contradicts (B.7).

Consequently, (B.12) does not hold. Therefore, $d_{\mathbb{P}(\sigma-(2n+1))}(t) = 2$. Since $n \le t \le 2n$, we have $n \le t \le 2n-1$. Set

$$j = \varphi(2n+1).$$

By (B.9), $N_{\mathbb{P}(\sigma-(2n+1))}(t) = N_{\mathbb{P}(\sigma-t)}(2n+1)$. Hence, $n \le j \le 2n-1$ and

$$\{\varphi^{-1}(j-1), \varphi^{-1}(j+1)\} = \{t-1, t+1\}.$$

It follows that

$$\varphi^{-1}(j-1) = t-1 \text{ and } \varphi^{-1}(j+1) = t+1 \tag{B.15}$$

or

$$\varphi^{-1}(j-1) = t+1 \text{ and } \varphi^{-1}(j+1) = t-1. \tag{B.16}$$

Suppose that (B.15) holds. We prove that

$$\begin{cases} t = j, \\ t = n, \\ n \text{ is odd,} \\ \text{and} \\ \text{for each } l \in \{0, \dots, n-2\} \cup \{n+2, \dots, 2n\},\ \varphi(l) = 2n - l. \end{cases} \tag{B.17}$$

Recall that $\mathbb{P}(\sigma - (2n+1)) = P_{2n+1}$. By Lemma 4.39,

$$\begin{aligned} &E(\mathbb{P}((\sigma - (2n+1)) - \{t-1, t\})) \\ &\quad = (E(\mathbb{P}(\sigma - (2n+1))) \\ &\qquad \setminus \{\{k, k+1\} : k \in \{t-2, t-1, t\}\}) \cup \{\{t-2, t+1\}\}. \end{aligned}$$

It follows that

$$\begin{aligned} &E(\mathbb{P}(\sigma - \{t-1, t, 2n+1\})) \\ &\quad = \{\{k, k+1\} : k \in \{0, \dots, t-3\} \cup \{t+1, \dots, 2n-1\}\} \\ &\qquad \cup \{\{t-2, t+1\}\}. \end{aligned} \tag{B.18}$$

(Note that if $t = 2n-1$, then

$$\begin{aligned} &E(\mathbb{P}(\sigma - \{t-1, t, 2n+1\})) \\ &\quad = \{\{k, k+1\} : k \in \{0, \dots, t-3\}\} \cup \{\{t-2, t+1\}\}.) \end{aligned}$$

Similarly, we have

$$\begin{aligned} &E(\mathbb{P}(\sigma - \{\varphi{-}1(j-1), t, 2n+1\})) \\ &= \{\{\varphi^{-1}(k), \varphi^{-1}(k+1)\} : k \in \{0, \dots, j-3\} \cup \{j+1, \dots, 2n-1\}\} \\ &\quad \cup \{\{\varphi^{-1}(j-2), \varphi^{-1}(j+1)\}\}. \end{aligned} \tag{B.19}$$

(Note that if $j = 2n-1$, then

$$\begin{aligned} &E(\mathbb{P}(\sigma - \{\varphi{-}1(j-1), t, 2n+1\})) \\ &\quad = \{\{\varphi^{-1}(k), \varphi^{-1}(k+1)\} : k \in \{0, \dots, j-3\}\} \\ &\qquad \cup \{\{\varphi^{-1}(j-2), \varphi^{-1}(j+1)\}\}.) \end{aligned} \tag{B.20}$$

Since (B.15) holds, we have

$$E(\mathbb{P}(\sigma - \{t-1, t, 2n+1\})) = E(\mathbb{P}(\sigma - \{\varphi^{-1}(j-1), t, 2n+1\}). \tag{B.21}$$

We show that $\varphi^{-1}(j-2) \neq t-2$. Otherwise, we have $\varphi^{-1}(j-2) = t-2$. By proceeding by induction, it follows from (B.18), (B.19), and (B.21) that $\varphi^{-1}(j-k) = t-k$ for $k \in \{1, \dots, \min(t, j)\}$. It follows that $t = j$. Hence, $\varphi(l) = l$ for $l \in \{0, \dots, t-1\}$. Analogously, by proceeding by induction, we obtain $\varphi(l) = l$ for $l \in \{t+1, \dots, 2n\}$. Thus, $\{t, 2n+1\}$ is a module of σ, which contradicts the fact that σ is prime. It follows that

$$\varphi^{-1}(j-2) \neq t-2.$$

Since $\varphi^{-1}(j+1)=t+1$ by (B.15), it follows from (B.18) and (B.19) that

$$\begin{cases} t\le 2n-2, \\ j\le 2n-2, \\ \varphi^{-1}(j+2)=t-2, \\ \text{and} \\ \varphi^{-1}(j-2)=t+2. \end{cases}$$

By proceeding by induction, it follows from (B.18), (B.19), and (B.21) that $\varphi^{-1}(j+k)=t-k$ for $k\in\{2,\dots,\min(t,2n-j)\}$. Since $t,j\in\{n,\dots,2n-2\}$, we have $\min(t,2n-j)=2n-j$. For $k=2n-j$, we obtain $\varphi^{-1}(2n)=t-2n+j$. Therefore, $t-2n+j=0$ or $2n$. Since $t\le 2n-2$ and $j\le 2n-2$, we have $t+j=2n$. Since $t\ge n$ and $j\ge n$, we obtain

$$t=j \text{ and } t=n.$$

It follows that for $k\in\{2,\dots,n\}$, we have

$$\begin{cases} \varphi^{-1}(n+k)=n-k \\ \text{and (similarly)} \\ \varphi^{-1}(n-k)=n+k. \end{cases} \tag{B.22}$$

For a contradiction with (B.17), suppose that n is even. We obtain

$$\begin{aligned} (0,1)_\sigma &= (\varphi^{-1}(2n),\varphi^{-1}(2n-1))_\sigma && \text{by (B.22)} \\ &= (\varphi^{-1}(n+1),\varphi^{-1}(n-1))_\sigma && \text{by (B.11)} \\ &= (n+1,n-1)_\sigma && \text{by (B.15) (because } t=j \text{ and } t=n) \\ &= (1,0)_\sigma && \text{by (B.8)}, \end{aligned}$$

which contradicts (B.7). It follows that n is odd. It follows from (B.22) that $\varphi(l)=2n-l$ for each $l\in\{0,\dots,n-2\}\cup\{n+2,\dots,2n\}$. Hence, (B.17) holds. Set

$$\psi=\pi_{2n+1}\circ\varphi.$$

Clearly, ψ is another isomorphism from $\mathbb{P}(\sigma-t)$ onto P_{2n+1}. As previously for φ, we obtain that for any $p,q\in\{0,\dots,2n\}$ such that $p<q$,

$$[\psi^{-1}(p),\psi^{-1}(q)]_\sigma = \begin{cases} [\psi^{-1}(0),\psi^{-1}(2)]_\sigma \text{ if } p \text{ and } q \text{ are even}, \\ [\psi^{-1}(0),\psi^{-1}(1)]_\sigma \text{ otherwise}. \end{cases} \tag{B.23}$$

Since (B.17) holds, we obtain

$$\psi(l)=l \text{ for each } l\in\{0,\dots,n-2\}\cup\{n+2,\dots,2n\}. \tag{B.24}$$

Since $n\ge 3$, we have $[0,1]_\sigma=[\psi^{-1}(0),\psi^{-1}(1)]_\sigma$. Furthermore, we have

$$\begin{aligned} [0,2]_\sigma &= [0,2n]_\sigma && \text{by (B.8)} \\ &= [\psi^{-1}(0),\psi^{-1}(2n)]_\sigma && \text{by (B.24)} \\ &= [\psi^{-1}(0),\psi^{-1}(2)]_\sigma && \text{by (B.23)}. \end{aligned}$$

It follows from (B.8) and (B.23) that

$$[p,q]_\sigma = [\psi^{-1}(p), \psi^{-1}(q)]_\sigma$$

for any $p,q \in \{0,\dots,2n\}$ such that $p < q$. Therefore, ψ^{-1} is an isomorphism from $\sigma - (2n+1)$ onto $\sigma - t$. Finally, since $t = j$ and $t = n$, it follows from (B.15) that

$$\psi(n-1) = n+1 \text{ and } \psi(n+1) = n-1. \tag{B.25}$$

It follows from (B.24) and (B.25) that ψ^{-1} satisfies (5.24).

Lastly, suppose that (B.16) holds. Since (B.16) holds, we have $\sigma - \{t, t+1, 2n+1\} = \sigma - \{t, \varphi^{-1}(j-1), 2n+1\}$. Thus, we have

$$E(\mathbb{P}((\sigma-(2n+1)) - \{t,t+1\}) = E(\mathbb{P}((\sigma-t) - \{\varphi^{-1}(j-1), 2n+1\}). \tag{B.26}$$

Set

$$\mu = \sigma - \{t, t+1, 2n+1\}.$$

To conclude, we distinguish the following two cases.

Case 1: $t = 2n-1$.

It follows from Lemma 4.40 applied to $(\sigma - (2n+1)) - \{2n-1, 2n\}$ that

$$E(\mathbb{P}(\mu)) = \{\{k,k+1\} : k \in \{0,\dots,2n-3\}\}. \tag{B.27}$$

Thus,

$$N_{\mathbb{P}(\mu)}(2n-2) = \{2n-3\}. \tag{B.28}$$

If $j < 2n-1$, then it follows from (B.19) that

$$N_{\mathbb{P}(\mu)}(\varphi^{-1}(j+1)) = \{\varphi^{-1}(j-2), \varphi^{-1}(j+2)\},$$

which contradicts (B.28) because $\varphi^{-1}(j+1) = 2n-2$ by (B.16). Therefore, $j = 2n-1$. It follows from (B.20) that

$$N_{\mathbb{P}(\mu)}(\varphi^{-1}(2n)) = \{\varphi^{-1}(2n-3)\}.$$

Since (B.16) holds, we have $\varphi^{-1}(2n) = 2n-2$. It follows that

$$\varphi^{-1}(2n-3) = 2n-3.$$

By proceeding by induction, it follows from (B.26), (B.27), and (B.28) that

$$\varphi^{-1}(2n-k) = 2n-k$$

for each $k \in \{3,\dots,2n\}$. We obtain that for each $k \in \{0,\dots,2n-3\}$,

$$\varphi(k) = k. \tag{B.29}$$

Since $n \geq 3$, we obtain

$$\begin{cases} [0,1]_\sigma = [\varphi^{-1}(0), \varphi^{-1}(1)]_\sigma \\ \text{and} \\ [0,2]_\sigma = [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma. \end{cases}$$

It follows from (B.8) and (B.11) that

$$[p,q]_\sigma = (\varphi^{-1}(p), \varphi^{-1}(q)]_\sigma$$

for any $p, q \in \{0, \dots, 2n\}$ such that $p < q$. Therefore, φ^{-1} is an isomorphism from $\sigma - (2n+1)$ onto $\sigma - t$. Since $t = 2n-1$, we have $N_{\mathbb{P}(\sigma-(2n+1))}(t) = \{2n-2, 2n\}$. It follows from (B.16) that $\varphi^{-1}(2n) = 2n-2$ and $\varphi^{-1}(2n-2) = 2n$. Consequently, φ^{-1} satisfies (5.24).

CASE 2: $t < 2n-1$.

Recall that $\mathbb{P}(\sigma - (2n+1)) = P_{2n+1}$. It follows from Lemma 4.39 applied to $(\sigma - (2n+1)) - \{t, t+1\}$ that

$$E(\mathbb{P}(\mu)) = \{\{k, k+1\} : k \in \{0, \dots, t-2\} \cup \{t+2, \dots, 2n-1\}\} \cup \{\{t-1, t+2\}\}. \tag{B.30}$$

We obtain

$$N_{\mathbb{P}(\mu)}(t-1) = \{t-2, t+2\}.$$

Since (B.16) holds, we have

$$\varphi^{-1}(j+1) = t-1.$$

If $j = 2n-1$, then it follows from (B.20) that

$$N_{\mathbb{P}(\mu)}(\varphi^{-1}(j+1)) = \{\varphi^{-1}(j-2)\}.$$

Therefore, we have

$$j < 2n-1.$$

It follows from (B.19) that

$$N_{\mathbb{P}(\mu)}(\varphi^{-1}(j+1)) = \{\varphi^{-1}(j-2), \varphi^{-1}(j+2)\}.$$

Therefore, we have

$$\varphi^{-1}(j-2) = t-2 \text{ and } \varphi^{-1}(j+2) = t+2 \tag{B.31}$$

or

$$\varphi^{-1}(j-2) = t+2 \text{ and } \varphi^{-1}(j+2) = t-2. \tag{B.32}$$

For a contradiction, suppose that (B.32) holds. By proceeding by induction, it follows from (B.30), (B.19), and (B.26) that

$$\varphi^{-1}(j+k) = t-k$$

for each $k \in \{1, \dots, \min(t, 2n-j)\}$. Since $t, j \in \{n, \dots, 2n\}$, we have $\min(t, 2n-j) = 2n-j$. Thus, for $k = 2n-j$, we obtain $\varphi^{-1}(2n) = t+j-2n$. It follows that $t+j = 2n$ or $4n$. Since $n \leq t < 2n-1$ and $n \leq j \leq 2n$, we obtain $t+j = 2n$, and hence, $t = n$ and $j = n$. Therefore, for each $l \in \{0, \dots, n-1\}$, we have $\varphi^{-1}(2n-l) = l$. Symmetrically, we obtain $\varphi^{-1}(l) = 2n-l$ for $l \in \{0, \dots, n-1\}$. It follows that for each $p \in \{0, \dots, n-1\} \cup \{n+1, \dots, 2n\}$, we have

$$\varphi^{-1}(2n-p) = p. \tag{B.33}$$

Set

$$\psi = \pi_{2n+1} \circ \varphi.$$

Clearly, ψ is another isomorphism from $\mathbb{P}(\sigma - t)$ onto P_{2n+1}. By Proposition 4.27, for any $p, q \in \{0, \ldots, 2n\}$ such that $p < q$, we have

$$[\psi^{-1}(p), \psi^{-1}(q)]_\sigma = \begin{cases} [\psi^{-1}(0), \psi^{-1}(2)]_\sigma \text{ if } p \text{ and } q \text{ are even,} \\ [\psi^{-1}(0), \psi^{-1}(1)]_\sigma \text{ otherwise.} \end{cases} \tag{B.34}$$

It follows from (B.33) that for each $p \in \{0, \ldots, n-1\} \cup \{n+1, \ldots, 2n\}$, we have

$$\psi^{-1}(p) = p. \tag{B.35}$$

Since $n \geq 3$, we have $[0,1]_\sigma = [\psi^{-1}(0), \psi^{-1}(1)]_\sigma$ and $[0,2]_\sigma = [\psi^{-1}(0), \psi^{-1}(2)]_\sigma$. It follows from (B.8) and (B.34) that for each $p \in \{0, \ldots, n-1\} \cup \{n+1, \ldots, 2n\}$, we have

$$[p, t]_\sigma = [\psi^{-1}(p), 2n+1]_\sigma.$$

We obtain that $\{t, 2n+1\}$ is a module of σ, which contradicts the fact that σ is prime. Consequently, (B.32) does not hold. Hence, (B.31) holds. By proceeding by induction, it follows from (B.30), (B.19), and (B.26) that $\varphi^{-1}(j-k) = t-k$ for each $k \in \{2, \ldots, \min(t,j)\}$. Therefore, $t = j$. Symmetrically, we obtain $\varphi^{-1}(t+k) = t+k$ for each $k \in \{2, \ldots, 2n-t\}$. It follows that for $l \in \{0, \ldots, t-2\} \cup \{t+2, \ldots, 2n\}$, we have

$$\varphi^{-1}(l) = l. \tag{B.36}$$

For a contradiction, suppose that t is even. Since $n \geq 3$ and $t \geq n$, we have $t \geq 3$. By (B.36), we have

$$\begin{cases} \varphi^{-1}(0) = 0 \\ \text{and} \\ \varphi^{-1}(1) = 1. \end{cases} \tag{B.37}$$

We obtain

$$\begin{aligned} (0,1)_\sigma &= (t-1, t+1)_\sigma && \text{by (B.8)} \\ &= (\varphi^{-1}(t+1), \varphi^{-1}(t-1))_\sigma && \text{by (B.16) (because } t = j) \\ &= (\varphi^{-1}(1), \varphi^{-1}(0))_\sigma && \text{by (B.11)} \\ &= (1,0)_\sigma && \text{by (B.37),} \end{aligned}$$

which contradicts (B.7). It follows that t is odd. By (B.36),

$$[0,1]_\sigma = [\varphi^{-1}(0), \varphi^{-1}(1)]_\sigma.$$

Furthermore, we have

$$\begin{aligned} [0,2]_\sigma &= [0, 2n]_\sigma && \text{by (B.8)} \\ &= [\varphi^{-1}(0), \varphi^{-1}(2n)]_\sigma && \text{by (B.36)} \\ &= [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma && \text{by (B.11).} \end{aligned}$$

It follows from (B.8) and (B.11) that

$$[p,q]_\sigma = [\varphi^{-1}(p), \varphi^{-1}(q)]_\sigma$$

for any $p, q \in \{0, \dots, 2n\}$ such that $p < q$. Therefore, φ^{-1} is an isomorphism from $\sigma - s$ onto $\sigma - t$. By (B.16), we have $\varphi^{-1}(t+1) = t-1$ and $\varphi^{-1}(t-1) = t+1$. It follows from (B.36) that φ^{-1} satisfies (5.24).

To conclude, we verify that $\mathscr{S}_c(\sigma) = \{t, 2n+1\}$. As shown above, (5.24) holds. For a contradiction, suppose that there exists $t' \in \mathscr{S}_c(\sigma) \smallsetminus \{t, 2n+1\}$. By what precedes, (5.24) holds also when t is replaced by t'. It follows that $\{t, t'\}$ is a module of $\sigma - (2n+1)$, which contradicts the fact that $\sigma - (2n+1)$ is prime. □

Remark B.1. Let σ be a prime 2-structure with $v(\sigma) \geq 7$. Suppose that there exist distinct $s, t \in \mathscr{S}_c(\sigma)$. Suppose also that $\mathbb{P}(\sigma - s) \simeq P_{2n+1}$. By Proposition 5.28,

$$\mathscr{S}_c(\sigma) = \{s, t\}.$$

Moreover, consider an isomorphism φ_s from $\mathbb{P}(\sigma - s)$ onto P_{2n+1}. It follows from Proposition 5.28 that

$$\begin{cases} \varphi_s(t) \text{ is odd} \\ \text{and} \\ (\varphi_s(0), \varphi_s(2))_\sigma = (\varphi_s(2), \varphi_s(0))_\sigma. \end{cases}$$

Proof of Proposition 5.29. Consider $s \in \mathscr{S}_c(\sigma)$ such that $\mathbb{P}(\sigma - s) \simeq P_{2n}$. Therefore, $v(\sigma) = 2n+1$. Since $v(\sigma) \geq 7$, we obtain

$$n \geq 3.$$

Suppose that there exists $t \in \mathscr{S}_c(\sigma) \smallsetminus \{s\}$. We verify that we can assume that (5.25) holds. Let φ_s be an isomorphism from $\mathbb{P}(\sigma - s)$ onto P_{2n}. Since $\pi_{2n} \in \mathrm{Aut}(P_{2n})$ (see Notation 4.21), we can assume that

$$n \leq \varphi_s(t) \leq 2n-1.$$

Denote by τ_s the unique 2-structure defined on $\{0, \dots, 2n-1\}$ such that φ_s is an isomorphism from $\sigma - s$ onto τ_s. Since φ_s is an isomorphism from $\mathbb{P}(\sigma - s)$ onto P_{2n}, τ_s is critical and $\mathbb{P}(\tau_s) = P_{2n}$. Here, we can assume that $V(\sigma) = \{0, \dots, 2n\}$, $s = 2n$, and $\varphi_s = \mathrm{Id}_{\{0,\dots,2n-1\}}$. Thus, we have $t \in \{n, \dots, 2n-1\}$ and $\mathbb{P}(\sigma - (2n)) = P_{2n}$, so (5.25) holds. Furthermore, note that $\tau_s = \sigma - (2n)$.

It follows from Proposition 4.15 that

(B.38) $$\langle 0,1 \rangle_\sigma \neq \langle 0,2 \rangle_\sigma \text{ (see Notation 1.1).}$$

Moreover, for any $p, q \in \{0, \dots, 2n-1\}$ such that $p < q$, we have

(B.39) $$[p,q]_\sigma = \begin{cases} [0,1]_\sigma \text{ if } p \text{ is even and } q \text{ is odd,} \\ [0,2]_\sigma \text{ otherwise.} \end{cases}$$

It follows from Corollary 5.25 that

$$N_{\mathbb{P}(\sigma-(2n))}(t) = N_{\mathbb{P}(\sigma-t)}(2n). \tag{B.40}$$

Since $v(\sigma-(2n)) = 2n$, it follows from Corollary 4.6 that $\mathbb{P}(\sigma-t) \simeq P_{2n}$. As above, there exists an isomorphism φ from $\sigma-t$ onto τ, where τ is a critical 2-structure such that $\mathbb{P}(\tau) = P_{2n}$. It follows from Proposition 4.15 that

$$\prec\varphi^{-1}(0),\varphi^{-1}(1)\succ_\sigma \neq \prec\varphi^{-1}(0),\varphi^{-1}(2)\succ_\sigma . \tag{B.41}$$

Furthermore, for any $p,q \in \{0,\dots,2n-1\}$ such that $p<q$, we have

$$[\varphi^{-1}(p),\varphi^{-1}(q)]_\sigma = \begin{cases} [\varphi^{-1}(0),\varphi^{-1}(1)]_\sigma & \text{if } p \text{ is even and } q \text{ is odd,} \\ [\varphi^{-1}(0),\varphi^{-1}(2)]_\sigma & \text{otherwise.} \end{cases} \tag{B.42}$$

Similarly, we can assume that $n \le \varphi(2n) \le 2n-1$.

To begin with case 1, suppose that

$$d_{\mathbb{P}(\sigma-(2n))}(t) = 1.$$

Since $n \le t \le 2n-1$, we have

$$t = 2n-1.$$

By (B.40), $d_{\mathbb{P}(\sigma-t)}(2n) = 1$. Similarly, we have

$$\varphi(2n) = 2n-1.$$

Moreover, it follows from (B.40) that

$$\varphi^{-1}(2n-2) = 2n-2. \tag{B.43}$$

It follows from Lemma 4.40 that $(\sigma-(2n))-\{2n-2,2n-1\}$ is critical and

$$\begin{aligned} &E(\mathbb{P}((\sigma-(2n))-\{2n-2,2n-1\})) \\ &\quad = E(\mathbb{P}(\sigma-s)) \smallsetminus \{\{k,k+1\} : k \in \{2n-3,2n-2\}\} \\ &\quad = \{\{k,k+1\} : k \in \{0,\dots,2n-4\}\}. \end{aligned}$$

Observe that $\mathbb{P}(\sigma-\{2n-2,2n-1,2n\}) = P_{2n-2}$. Clearly, $(\varphi^{-1})_{\restriction\{0,\dots,2n-3\}}$ is an isomorphism from P_{2n-2} onto $\mathbb{P}(\sigma-\{\varphi^{-1}(2n-2),2n-1,2n\})$. By (B.43), we have

$$\sigma-\{\varphi^{-1}(2n-2),2n-1,2n\} = \sigma-\{2n-2,2n-1,2n\}.$$

It follows that

$$(\varphi^{-1})_{\restriction\{0,\dots,2n-3\}} \in \mathrm{Aut}(P_{2n-2}).$$

Therefore, we obtain

$$\varphi_{\restriction\{0,\dots,2n-3\}} = \mathrm{Id}_{\{0,\dots,2n-3\}} \text{ or } \pi_{2n-2} \text{ (see Notation 4.21).} \tag{B.44}$$

For a contradiction, suppose that

$$\varphi_{\restriction\{0,\dots,2n-3\}} = \mathrm{Id}_{\{0,\dots,2n-3\}}. \tag{B.45}$$

By (B.43), we have

$$\varphi(k) = k \tag{B.46}$$

for each $k \in \{0, \ldots, 2n-2\}$. We verify that $\{2n-1, 2n\}$ is a module of σ. Let $v \in \{0, \ldots, 2n-2\}$. For instance, assume that v is even. We obtain

$$\begin{aligned}
[v, 2n-1]_\sigma &= [0,1]_\sigma && \text{by (B.39)} \\
&= [\varphi^{-1}(0), \varphi^{-1}(1)]_\sigma && \text{by (B.46)} \\
&= [\varphi^{-1}(v), \varphi^{-1}(2n-1)]_\sigma && \text{by (B.42)} \\
&= [v, 2n]_\sigma && \text{because } \varphi(v) = v \text{ by (B.46),} \\
& && \text{and } \varphi(2n) = 2n-1.
\end{aligned}$$

Similarly, we have $[v, 2n-1]_\sigma = [v, 2n]_\sigma$ when v is odd. It follows that $\{2n-1, 2n\}$ is a module of σ, which contradicts the fact that σ is prime. Consequently, (B.45) does not hold. By (B.44), we have

$$(\varphi^{-1})_{\restriction\{0,\ldots,2n-3\}}) = \pi_{2n-2}.$$

We obtain

$$\varphi^{-1}(k) = 2n-3-k. \tag{B.47}$$

for each $k \in \{0, \ldots, 2n-3\}$. Therefore, we have

$$\begin{aligned}
(0,2)_\sigma &= (\varphi^{-1}(2n-3), \varphi^{-1}(2n-5))_\sigma && \text{by (B.47)} \\
&= (\varphi^{-1}(2n-2), \varphi^{-1}(2n-5))_\sigma && \text{by (B.42)} \\
&= (2n-2, 2)_\sigma && \text{by (B.43) and (B.47)} \\
&= (2,0)_\sigma && \text{by (B.39).}
\end{aligned}$$

Moreover, set

$$\psi = \pi_{2n} \circ \varphi.$$

Clearly, ψ is another isomorphism from $\mathbb{P}(\sigma - t)$ onto P_{2n}. As previously for φ, we obtain that for any $p, q \in \{0, \ldots, 2n\}$ such that $p < q$,

$$[\psi^{-1}(p), \psi^{-1}(q)]_\sigma = \begin{cases} [\psi^{-1}(0), \psi^{-1}(1)]_\sigma \text{ if } p \text{ is even and } q \text{ is odd,} \\ [\psi^{-1}(0), \psi^{-1}(2)]_\sigma \text{ otherwise.} \end{cases} \tag{B.48}$$

Since $\varphi^{-1}(2n-1) = 2n$, it follows from (B.43) and (B.47) that ψ^{-1} is defined by

$$\begin{array}{ccc}
\{0, \ldots, 2n-1\} & \longrightarrow & \{0, \ldots, 2n-2\} \cup \{2n\} \\
0 & \longmapsto & 2n, \\
1 & \longmapsto & 2n-2, \\
2 \le k \le 2n-1 & \longmapsto & k-2.
\end{array} \tag{B.49}$$

We obtain

$$\begin{aligned}[\psi^{-1}(0),\psi^{-1}(1)]_\sigma &= [2n,2n-2]_\sigma && \text{by (B.49)}\\ &= [\varphi^{-1}(2n-1),\varphi^{-1}(2n-2)]_\sigma \\ &= [\varphi^{-1}(2n-3),\varphi^{-1}(2n-4)]_\sigma && \text{by (B.42)}\\ &= [0,1]_\sigma && \text{by (B.47).}\end{aligned}$$

Similarly, we have

$$\begin{aligned}[\psi^{-1}(0),\psi^{-1}(2)]_\sigma &= [2n,0]_\sigma && \text{by (B.49)}\\ &= [\varphi^{-1}(2n-1),\varphi^{-1}(2n-3)]_\sigma && \text{by (B.47)}\\ &= [\varphi^{-1}(2n-3),\varphi^{-1}(2n-5)]_\sigma && \text{by (B.42)}\\ &= [0,2]_\sigma && \text{by (B.47).}\end{aligned}$$

It follows from (B.48) and (B.39) that

$$[\psi^{-1}(p),\psi^{-1}(q)]_\sigma = [p,q]_\sigma$$

for any $p,q \in \{0,\dots,2n-1\}$ such that $p < q$. Consequently, ψ^{-1} is an isomorphism from $\sigma-(2n)$ onto $\sigma-(2n-1)$. Hence, (5.26) holds.

To continue with case 2, suppose that

$$d_{\mathbb{P}(\sigma-(2n))}(t) = 2.$$

Since $n \le t \le 2n-1$, we have $n \le t \le 2n-2$. Thus, $N_{\mathbb{P}(\sigma-(2n))}(t) = \{t-1,t+1\}$. Set

$$j = \varphi(2n).$$

Recall that $n \le j \le 2n-1$. By (B.40), $N_{\mathbb{P}(\sigma-t)}(2n) = \{t-1,t+1\}$. Hence, $n \le j \le 2n-2$ and

$$\{\varphi^{-1}(j-1),\varphi^{-1}(j+1)\} = \{t-1,t+1\}.$$

It follows that

$$\varphi^{-1}(j-1) = t-1 \text{ and } \varphi^{-1}(j+1) = t+1 \tag{B.50}$$

or

$$\varphi^{-1}(j-1) = t+1 \text{ and } \varphi^{-1}(j+1) = t-1. \tag{B.51}$$

For a contradiction, suppose that (B.50) holds. Recall that

$$E(\mathbb{P}(\sigma-(2n))) = \{\{k,k+1\} : k \in \{0,\dots,2n-2\}\}.$$

By Lemma 4.39,

$$\begin{aligned}&E(\mathbb{P}((\sigma-(2n))-\{t-1,t\}))\\ &\quad= \{\{k,k+1\} : k \in \{0,\dots,t-3\} \cup \{t+1,\dots,2n-2\}\}\\ &\qquad \cup \{\{t-2,t+1\}\}.\end{aligned} \tag{B.52}$$

Similarly, we have

$$\begin{aligned}
&E(\mathbb{P}(\sigma - \{t, \varphi^{-1}(j-1), 2n\})) \\
&= \{\{\varphi^{-1}(k), \varphi^{-1}(k+1)\} : k \in \{0, \dots, j-3\} \cup \{j+1, \dots, 2n-2\}\} \\
&\quad \cup \{\{\varphi_t^{-1}(j-2), \varphi^{-1}(j+1)\}\}.
\end{aligned} \tag{B.53}$$

Since (B.50) holds, we have

$$E(\mathbb{P}(\sigma - \{t-1, t, 2n\})) = E(\mathbb{P}(\sigma - \{t, \varphi^{-1}(j-1), 2n\}). \tag{B.54}$$

We distinguish the following two cases. Both lead us to a contradiction.

CASE 1: $\varphi^{-1}(j-2) = t-2$.

By proceeding by induction, we obtain $\varphi^{-1}(j-k) = t-k$ for $k \in \{2, \dots, \min(j,t)\}$. It follows that $j = t$. Similarly, we obtain that $\varphi^{-1}(l) = l$ for $k \in \{t+2, \dots, 2n-1\}$. Since (B.50) holds, we obtain $(\varphi_t)^{-1}(l) = l$ for $l \in \{0, \dots, 2n-1\} \smallsetminus \{t\}$. It follows from (B.39) and (B.42) that $\{t, 2n\}$ is a module of σ, which contradicts the fact that σ is prime.

CASE 2: $\varphi^{-1}(j-2) \neq t-2$.

Since $\varphi^{-1}(j+1) = t+1$, it follows from (B.52), (B.53), and (B.54) that $t \leq 2n-3$, $j \leq 2n-3$, and

$$\varphi^{-1}(j-2) = t+2 \text{ and } \varphi^{-1}(j+2) = t-2.$$

By proceeding by induction, we obtain $\varphi^{-1}(j-k) = t+k$ for $k \in \{2, \dots, \min(j, 2n-t-1)\}$. Since $t, j \in \{n, \dots, 2n-3\}$, we have

$$\min(j, 2n-t-1) = 2n-t-1.$$

For $k = 2n-t-1$, we obtain $j+t-2n+1 = 0$ or $2n-1$, which is impossible because $j, t \in \{n, \dots, 2n-3\}$.

Consequently, (B.50) does not hold. Therefore, (B.51) holds. Recall that

$$\begin{aligned}
&E(\mathbb{P}(\sigma - \{t, \varphi^{-1}(j-1), 2n\})) \\
&= \{\{\varphi^{-1}(k), \varphi^{-1}(k+1)\} : k \in \{0, \dots, j-3\} \cup \{j+1, \dots, 2n-2\}\} \\
&\quad \cup \{\{\varphi^{-1}(j-2), \varphi^{-1}(j+1)\}\}.
\end{aligned} \tag{B.55}$$

Furthermore, by Lemma 4.39, we have

$$\begin{aligned}
&E(\mathbb{P}((\sigma - (2n)) - \{t, t+1\})) \\
&= \{\{k, k+1\} : k \in \{0, \dots, t-2\} \cup \{t+2, \dots, 2n-2\}\} \\
&\quad \cup \{\{t-1, t+2\}\}.
\end{aligned} \tag{B.56}$$

Since (B.51) holds, we obtain

$$E(\mathbb{P}(\sigma - \{t, t+1, 2n\})) = E(\mathbb{P}(\sigma - \{t, \varphi^{-1}(j-1), 2n\}). \tag{B.57}$$

Set

$$\mu = \sigma - \{t, t+1, 2n\}.$$

Since $\varphi^{-1}(j+1) = t-1$, we obtain

$$N_{\mathbb{P}(\mu)}(t-1) = N_{\mathbb{P}(\mu)}(\varphi^{-1}(j+1)). \tag{B.58}$$

For a contradiction, suppose that

$$\varphi^{-1}(j-2) = t+2.$$

Since $d_{\mathbb{P}(\mu)}(\varphi^{-1}(j-2)) = 2$, we have $d_{\mathbb{P}(\mu)}(t+2) = 2$, so $t \leq 2n-4$. By proceeding by induction, we obtain $\varphi^{-1}(j-k) = t+k$ for $k \in \{2, \dots, \min(j, 2n-t-1)\}$. Since $j \geq n$ and $t \geq n$, we have $\min(j, 2n-t-1) = 2n-t-1$. For $k = 2n-t-1$, we obtain $\varphi^{-1}(j+t+1-2n) = 2n-1$. Thus, $j+t+1-2n = 0$ or $2n-1$, which is impossible because $j, t \in \{n, \dots, 2n-2\}$. Consequently, we have

$$\varphi^{-1}(j-2) = t-2. \tag{B.59}$$

By proceeding by induction, we obtain $\varphi^{-1}(j-k) = t-k$ for $k \in \{2, \dots, \min(j,t)\}$. It follows that $j = t$. We obtain

$$\varphi^{-1}(l) = l \text{ for } l \in \{0, \dots, t-2\}. \tag{B.60}$$

Since $t \geq n$ and $n \geq 3$, we obtain

$$[\varphi^{-1}(0), \varphi^{-1}(1)]_\sigma = [0,1]_\sigma.$$

Similarly, if $t \geq 4$, then $[\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma = [0,2]_\sigma$. Hence, suppose that $t = 3$. We obtain $t \leq 2n-3$ because $n \geq 3$. It follows from (B.58) that

$$\{t-2, t+2\} = \{\varphi^{-1}(j-2), \varphi^{-1}(j+2)\}. \tag{B.61}$$

Since $\varphi^{-1}(j-2) = t-2$ by (B.59), we have $\varphi^{-1}(j+2) = t+2$. Recall that $j = t$. By proceeding by induction, we obtain

$$\varphi^{-1}(l) = l \text{ for } l \in \{t+2, \dots, 2n-1\}. \tag{B.62}$$

We have

$$\begin{aligned}
[\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma &= [\varphi^{-1}(1), \varphi^{-1}(2n-1)]_\sigma && \text{by (B.42)} \\
&= [1, 2n-1]_\sigma && \text{by (B.60) and (B.62)} \\
&= [0,2]_\sigma && \text{by (B.39).}
\end{aligned}$$

Therefore, we have

$$[\varphi^{-1}(0), \varphi^{-1}(1)]_\sigma = [0,1]_\sigma \text{ and } [\varphi^{-1}(0), \varphi^{-1}(2)]_\sigma = [0,2]_\sigma.$$

It follows from (B.39) and (B.42) that

$$[\varphi^{-1}(p), \varphi^{-1}(q)]_\sigma = [p,q]_\sigma$$

for any $p, q \in \{0, \dots, 2n-1\}$ such that $p < q$. Consequently, φ^{-1} is an isomorphism from $\sigma - (2n)$ onto $\sigma - t$. Moreover, φ^{-1} is defined by

$$\begin{array}{ccll}
\{0, \dots, 2n-1\} & \longrightarrow & \{0, \dots, 2n\} \setminus \{t\} & \\
t & \longmapsto & 2n & \text{because } t = j \text{ and } \varphi(2n) = j, \\
t-1 & \longmapsto & t+1 & \text{by (B.51),} \\
t+1 & \longmapsto & t-1 & \text{by (B.51),} \\
v \in V(\sigma) \setminus \{t-1, t, t+1, 2n\} & \longmapsto & v & \text{by (B.60) and (B.62).}
\end{array}$$

Consequently, (5.27) holds.

We conclude as follows. For a contradiction, suppose that there exists $u \in \mathscr{S}_c(\sigma) \setminus \{t, 2n\}$. We distinguish the following two cases.

CASE 1: $d_{\mathbb{P}(\sigma-(2n))}(t) = d_{\mathbb{P}(\sigma-(2n))}(u)$.

First, suppose that

$$d_{\mathbb{P}(\sigma-(2n))}(t) = 1.$$

Thus, (5.26) holds. In particular, we have $t = 2n-1$. It follows that $u = 0$. Since $\{0,1\} \in E(\mathbb{P}(\sigma-(2n)))$, $(\sigma-(2n)) - \{0,1\}$ is prime. Set

$$X = V(\sigma) \setminus \{0,1,2n\}.$$

It follows from (5.26) that $2n \in \mathrm{Ext}_\sigma(X)$. Hence $\sigma - \{0,1\}$ is prime, which contradicts $0 \in \mathscr{S}_c(\sigma)$.

Second, suppose that $d_{\mathbb{P}(\sigma-(2n))}(t) = 2$. We have $t, u \in \{1, \dots, 2n-2\}$. For instance, assume that $t < u$. We obtain that (5.27) holds, but also (5.27) holds after replacing t by u. Precisely, the function

$$\begin{array}{llll} \theta_t : & \{0,\dots,2n-1\} & \longrightarrow & \{0,\dots,2n\} \setminus \{t\} \\ & t & \longmapsto & 2n, \\ & t-1 & \longmapsto & t+1, \\ & t+1 & \longmapsto & t-1, \\ & v \in V(\sigma) \setminus \{t-1,t,t+1,2n\} & \longmapsto & v, \end{array}$$

is an isomorphism from $\sigma-(2n)$ onto $\sigma - t$. Similarly, the function

$$\begin{array}{llll} \theta_u : & \{0,\dots,2n-1\} & \longrightarrow & \{0,\dots,2n\} \setminus \{u\} \\ & u & \longmapsto & 2n, \\ & u-1 & \longmapsto & u+1, \\ & u+1 & \longmapsto & u-1, \\ & v \in V(\sigma) \setminus \{u-1,u,u+1,2n\} & \longmapsto & v, \end{array}$$

is an isomorphism from $\sigma-(2n)$ onto $\sigma-u$. We distinguish the following three subcases.

Subcase a: $t \le u-3$.

Since $N_{\mathbb{P}(\sigma-(2n))}(u) = \{u-1, u+1\}$, it follows from Lemma 4.4 that $\{u-1,u+1\}$ is a module of $(\sigma-(2n)) - u$. In particular, we have

$$[t, u-1]_\sigma = [t, u+1]_\sigma.$$

Moreover, we have

$$\begin{aligned} [t,u-1]_\sigma &= [2n, u-1]_\sigma && \text{by applying } \theta_t \\ &= [u, u+1]_\sigma && \text{by applying } (\theta_u)^{-1}, \end{aligned}$$

and

$$\begin{aligned} [t,u+1]_\sigma &= [2n, u+1]_\sigma && \text{by applying } \theta_t \\ &= [u, u-1]_\sigma && \text{by applying } (\theta_u)^{-1}. \end{aligned}$$

It follows that

$$[u, u-1]_\sigma = [u, u+1]_\sigma.$$

Since $\{u-1,u+1\}$ is a module of $(\sigma-(2n))-u$, $\{u-1,u+1\}$ is a module of $\sigma-(2n)$, which contradicts $2n\in\mathscr{S}(\sigma)$.

Subcase b: $t=u-2$.

Since $N_{\mathbb{P}(\sigma-(2n))}(t+1)=\{t,t+2\}$, it follows from Lemma 4.4 that $\{t,t+2\}$ is a module of $(\sigma-(2n))-(t+1)$. Furthermore, we have

$$\begin{aligned}[t+1,t+2]_\sigma &= [t-1,t+2]_\sigma && \text{by applying } (\theta_t)^{-1}\\ &= [t-1,2n]_\sigma && \text{by applying } \theta_u\\ &= [t+1,t]_\sigma && \text{by applying } (\theta_t)^{-1}\\ &= [t+3,t]_\sigma && \text{by applying } \theta_u\\ &= [t+1,t]_\sigma && \text{by (B.39).}\end{aligned}$$

Therefore, $\{t,t+2\}$ is a module of $(\sigma-(2n))$, which contradicts $2n\in\mathscr{S}(\sigma)$.

Subcase c: $t=u-1$.

First, suppose that t is even. We obtain

$$\begin{aligned}[0,1]_\sigma &= [t,t+1]_\sigma && \text{by (B.39)}\\ &= [2n,t-1]_\sigma && \text{by applying } \theta_t\\ &= [t+1,t-1]_\sigma && \text{by applying } (\theta_u)^{-1}\\ &= [2,0]_\sigma && \text{by (B.39),}\end{aligned}$$

which contradicts (B.38). Second, suppose that t is odd. We have $1\le t\le 2n-3$. If $t\le 2n-5$, then $t+4\le 2n-1$ and we obtain

$$\begin{aligned}[0,1]_\sigma &= [t+1,t+4]_\sigma && \text{by (B.39)}\\ &= [2n,t+4]_\sigma && \text{by applying } \theta_u\\ &= [t,t+4]_\sigma && \text{by applying } (\theta_t)^{-1}\\ &= [0,2]_\sigma && \text{by (B.39),}\end{aligned}$$

which contradicts (B.38). If $t\ge 2n-4$, then $t=2n-3$, $u=2n-2$, and we obtain

$$\begin{aligned}[0,1]_\sigma &= [0,t]_\sigma && \text{by (B.39)}\\ &= [0,2n]_\sigma && \text{by applying } \theta_t\\ &= [0,t+1]_\sigma && \text{by applying } (\theta_u)^{-1}\\ &= [0,2]_\sigma && \text{by (B.39),}\end{aligned}$$

which contradicts (B.38).

Case 2: $d_{\mathbb{P}(\sigma-(2n))}(t)\neq d_{\mathbb{P}(\sigma-(2n))}(u)$.

For instance, assume that

$$d_{\mathbb{P}(\sigma-(2n))}(t)=1 \text{ and } d_{\mathbb{P}(\sigma-(2n))}(u)=2.$$

We have $t = 2n-1$ and $1 \leq u \leq 2n-2$. We obtain that (5.26) holds, and (5.27) holds after replacing t by u. Precisely, the function

$$\begin{array}{llll} \theta_t: & \{0,\dots,2n-1\} & \longrightarrow & \{0,\dots,2n-2\} \cup \{2n\} \\ & 0 & \longmapsto & 2n, \\ & 1 & \longmapsto & 2n-2, \\ & 2 \leq k \leq 2n-1 & \longmapsto & k-2, \end{array}$$

is an isomorphism from $\sigma - (2n)$ onto $\sigma - t$. Furthermore, the function

$$\begin{array}{llll} \theta_u: & \{0,\dots,2n-1\} & \longrightarrow & \{0,\dots,2n\} \smallsetminus \{u\} \\ & u & \longmapsto & 2n, \\ & u-1 & \longmapsto & u+1, \\ & u+1 & \longmapsto & u-1, \\ & v \in V(\sigma) \smallsetminus \{u-1,u,u+1,2n\} & \longmapsto & v, \end{array}$$

is an isomorphism from $\sigma - (2n)$ onto $\sigma - u$. We distinguish the following three subcases.

Subcase a: $u \leq 2n-4$.

Since $N_{\mathbb{P}(\sigma-(2n))}(u) = \{u-1, u+1\}$, it follows from Lemma 4.4 that $\{u-1,u+1\}$ is a module of $(\sigma-(2n))-u$. Furthermore, we have

$$\begin{aligned} [2n, u-1]_\sigma &= [0, u+1]_\sigma && \text{by applying } (\theta_t)^{-1} \\ &= [0, u+3]_\sigma && \text{by (B.39)} \\ &= [2n, u+1]_\sigma && \text{by applying } \theta_t. \end{aligned}$$

Therefore, $\{u-1,u+1\}$ is a module of $\sigma - u$, which contradicts $u \in \mathscr{S}(\sigma)$.

Subcase b: $u = 2n-3$.

We obtain

$$\begin{aligned} [0,1]_\sigma &= [0,u]_\sigma && \text{by (B.39)} \\ &= [0,2n]_\sigma && \text{by applying } \theta_u \\ &= [2,0]_\sigma && \text{by applying } (\theta_t)^{-1}, \end{aligned}$$

which contradicts (B.38).

Subcase c: $u = 2n-2$.

We obtain

$$\begin{aligned} [0,1]_\sigma &= [0,3]_\sigma && \text{by (B.39)} \\ &= [2n,1]_\sigma && \text{by applying } \theta_t \\ &= [2n-2,1]_\sigma && \text{by applying } (\theta_u)^{-1} \\ &= [2,0]_\sigma && \text{by (B.39)}, \end{aligned}$$

which contradicts (B.38).

Both cases above lead us to a contradiction. Consequently, $\mathscr{S}_c(\sigma) = \{t, 2n\}$. □

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Index

Author Index

Figure Index

Symbol Index

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