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q-ANALOGUES OF π-SERIES BY APPLYING CARLITZ INVERSIONS TO q-PFAFF-SAALSCHÜTZ THEOREM

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Abstract. By applying multiplicate forms of the Carlitz inverse series relations to the q-Pfaff-Saalschtz summation theorem, we establish twenty five nonterminating q-series identities with several of them serving as q-analogues of infinite series expressions for π and $1/\pi$, including some typical ones discovered by Ramanujan (1914) and Guillera.

1. Introduction and Motivation

Let $\mathbb{N}$ and $\mathbb{N}_0$ be the sets of natural numbers and non-negative integers, respectively. For an indeterminate x, the Pochhammer symbol is defined by

$$(x)_0 \equiv 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}$$

with the following shortened multiparameter notation

$$\begin{bmatrix} \alpha, \beta, \cdots, \gamma \\ A, B, \cdots, C \end{bmatrix}_n = \frac{(\alpha)_n(\beta)_n\cdots(\gamma)_n}{(A)_n(B)_n\cdots(C)_n}.$$

Analogously, the rising and falling factorials with base q are given by $(x;q)_0 = \langle x;q\rangle_0 \equiv 1$ and

$$(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x),$$
$$\langle x;q\rangle_n = (1-x)(1-q^{-1}x)\cdots(1-q^{1-n}x).$$

Then the Gaussian binomial coefficient can be expressed as

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{(q;q)_m}{(q;q)_n(q;q)_{m-n}} = \frac{(q^{m-n+1};q)_n}{(q;q)_n} \quad \text{where} \quad m,\ n \in \mathbb{N}.$$

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When $|q| < 1$, the infinite product $(x;q)_\infty$ is well-defined. We have hence the q-gamma function [12, §1.10]

$$\Gamma_q(x) = (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty} \quad \text{and} \quad \lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x).$$

For the sake of brevity, the product and quotient of the q-shifted factorials will be abbreviated respectively to

$$\begin{aligned}[\alpha,\beta,\cdots,\gamma;q]_n &= (\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n,\\ \left[\begin{matrix}\alpha,\beta,\cdots,\gamma\\ A,B,\cdots,C\end{matrix}\middle|q\right]_n &= \frac{(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n}{(A;q)_n(B;q)_n\cdots(C;q)_n}.\end{aligned}$$

Following Bailey [2] and Gasper–Rahman [12], we define the basic q-series below:

$${}_{\ell+1}\phi_\ell\left[\begin{matrix}a_0,a_1,\cdots,a_\ell\\ b_1,\cdots,b_\ell\end{matrix}\middle|q;z\right] = \sum_{n=0}^{\infty}\left[\begin{matrix}a_0,a_1,\cdots,a_\ell\\ q,b_1,\cdots,b_\ell\end{matrix}\middle|q\right]_n z^n.$$

This series is well-defined when none of the denominator parameters has the form q^{-m} with $m \in \mathbb{N}_0$. If one of the numerator parameters has the form q^{-m} with $m \in \mathbb{N}_0$, the series is terminating (in that case, it is a polynomial of z). Otherwise, the series is said to be nonterminating, where we assume that $0 < |q| < 1$.

As the q-analogues of the Gould–Hsu [13] inversions, Carlitz [4] found, in 1973, two well–known pairs of inverse series relations, which can be reproduced as follows. Let $\{a_k\}_{k\ge 0}$ and $\{b_k\}_{k\ge 0}$ be two sequences such that the φ-polynomials defined by

$$\varphi(x;0)\equiv 1 \quad \text{and} \quad \varphi(x;n) = \prod_{k=0}^{n-1}(a_k + xb_k) \quad \text{for} \quad n = 1,2,\ldots$$

differ from zero at $x = q^{-m}$ for $m \in \mathbb{N}_0$. Then the first pair of inverse series relations discovered by Carlitz can equivalently be restated, under the replacement

$$g(k) \to q^{-\binom{k}{2}}g(k),$$

as follows.

Theorem 1.1 (Carlitz [4, Theorem 2])**.**

$$f(n) = \sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}\varphi(q^{-k};n)g(k), \tag{1.1}$$

$$g(n) = \sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}\frac{a_k + q^{-k}b_k}{\varphi(q^{-n};k+1)}f(k). \tag{1.2}$$

Alternatively, if the φ-polynomials differ from zero at $x = q^m$ for $m \in \mathbb{N}_0$, Carlitz deduced, under the base change $q \to q^{-1}$, another equivalent pair.

We reproduce it under the replacement

$$f(k) \to q^{-\binom{k}{2}} f(k),$$

as another theorem.

Theorem 1.2 (Carlitz [4, Theorem 4]).

$$f(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \varphi(q^k; n) g(k), \tag{1.3}$$

$$g(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{a_k + q^k b_k}{\varphi(q^n; k+1)} f(k). \tag{1.4}$$

These inversion theorems have been shown by Chu [6–8] to be very useful in proving terminating q-series identities. Among numerous q-series identities, the following q-Pfaff–Saalschütz theorem (cf. [12, II-12]) for the terminating balanced series is fundamental.

Theorem 1.3. *For $n \in \mathbb{N}_0$, we have the identity*

$$ {}_3\phi_2 \left[\begin{matrix} q^{-n}, a, b \\ c, q^{1-n} ab/c \end{matrix} \middle| q; q \right] = \left[\begin{matrix} c/a, c/b \\ c, c/ab \end{matrix} \middle| q \right]_n . \tag{1.5}$$

As a warm–up, we illustrate how to derive the q-Dougall sum by making use of Carlitz' inversions. Observe that (1.5) is equivalent to

$$ {}_3\phi_2 \left[\begin{matrix} q^{-n}, q^n a, qa/bd \\ qa/b, qa/d \end{matrix} \middle| q; q \right] = \left(\frac{qa}{bd} \right)^n \left[\begin{matrix} b,\ d \\ qa/b, qa/d \end{matrix} \middle| q \right]_n $$

which can be rewritten as a q-binomial sum

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (q^k a; q)_n \left[\begin{matrix} a, qa/bd \\ qa/b, qa/d \end{matrix} \middle| q \right]_k = \left(\frac{qa}{bd} \right)^n \left[\begin{matrix} a,\ b,\ d \\ qa/b, qa/d \end{matrix} \middle| q \right]_n q^{\binom{n}{2}}.$$

This matches exactly (1.3) under the specifications

$$f(n) = \left(\frac{qa}{bd} \right)^n \left[\begin{matrix} a,\ b,\ d \\ qa/b, qa/d \end{matrix} \middle| q \right]_n q^{\binom{n}{2}},$$

$$g(k) = \left[\begin{matrix} a,\ qa/bd \\ qa/b, qa/d \end{matrix} \middle| q \right]_k \quad \text{and} \quad \varphi(x; n) = (ax; q)_n.$$

Then the dual relation corresponding to (1.4) reads as

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1 - q^{2k} a}{(q^n a; q)_{k+1}} \left(\frac{qa}{bd} \right)^k \left[\begin{matrix} a,\ b,\ d \\ qa/b, qa/d \end{matrix} \middle| q \right]_k q^{\binom{k}{2}} = \left[\begin{matrix} a,\ qa/bd \\ qa/b, qa/d \end{matrix} \middle| q \right]_n .$$

This is equivalent to the q-Dougall sum (cf. [12, II-21]):

$$ {}_6\phi_5 \left[\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ d,\ q^{-n} \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/d, q^{n+1} a \end{matrix} \middle| q; \frac{q^{n+1} a}{bd} \right] = \left[\begin{matrix} qa, qa/bd \\ qa/b, qa/d \end{matrix} \middle| q \right]_n . \tag{1.6}$$

For $a=b=d=q^{1/2}$, the limiting case $n\to\infty$ of equation (1.6) becomes

$$\frac{1}{\Gamma_q^2(\frac{1}{2})}=\sum_{k=0}^{\infty}(-1)^k\frac{(q^{1/2};q)_k^3}{(q;q)_k^3}\frac{1-q^{2k+\frac{1}{2}}}{1-q}q^{\frac{k^2}{2}}$$

which reduces, for $q\to 1^-$, to the following infinite series expression for π

$$\frac{2}{\pi}=\sum_{k=0}^{\infty}(-1)^k\frac{(\frac{1}{2})_k^3}{(1)_k^3}\{1+4k\}$$

as recorded in one of Ramanujan's letters to Hardy [21]. More difficult formulae for $1/\pi$ were subsequently discovered by Ramanujan [23, 1914], where 17 similar series representations were announced. Three of them are reproduced as follows:

$$\frac{4}{\pi}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\1,\ 1,\ 1\end{bmatrix}_k\frac{1+6k}{4^k}. \tag{1.7}$$

$$\frac{8}{\pi}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{2},\frac{1}{4},\frac{3}{4}\\1,\ 1,\ 1\end{bmatrix}_k\frac{3+20k}{(-4)^k}. \tag{1.8}$$

$$\frac{16}{\pi}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\1,\ 1,\ 1\end{bmatrix}_k\frac{5+42k}{64^k}. \tag{1.9}$$

For their proofs and recent developments, the reader can consult the papers by Baruah-Berndt-Chan [3], Guillera [14–16] and Chu *et al* [9–11].

Recently, there has been a growing interest in finding q-analogues of Ramanujan–like series (cf. [5, 10, 17–20]). Following the procedure just described, the aim of this paper is to show systematically q-analogues of π-related series by applying the multiplicate form of Carlitz inverse series relations to the q-Pfaff–Saalschütz summation theorem. In the next section, we shall derive, by employing the duplicate inversions, twenty q-series identities including q-analogues of the identities in (1.7–1.9). Then in section 3, the triplicate inversions will be utilized to establish five q-series identities. By applying the bisection series method to two resulting series, q-analogues are established also for the following two remarkable series discovered by Guillera [14, 15]:

$$\frac{2\sqrt{2}}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{8}\Big)^k\begin{bmatrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\1,\ 1,\ 1\end{bmatrix}_k\{1+6k\}.$$

$$\frac{32\sqrt{2}}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{-3}{8}\Big)^{3k}\begin{bmatrix}\frac{1}{2},\frac{1}{6},\frac{5}{6}\\1,\ 1,\ 1\end{bmatrix}_k\{15+154k\}.$$

2. Duplicate Inverse Series Relations

For $x \in \mathbb{R}$ (the set of real numbers), we denote by $\lfloor x \rfloor$ the nearest integer less than or equal to x. Then for all $n \in \mathbb{N}_0$, there holds the equality

$$n = \lfloor \tfrac{n}{2} \rfloor + \lfloor \tfrac{1+n}{2} \rfloor . \tag{2.1}$$

Using (2.1), we shall reformulate (1.5) in three different ways. Their dual relations will lead us to q-series counterparts for several remarkable infinite series expressions of π and $1/\pi$.

2.1. **First version.** According to the q-Pfaff–Saalschütz formula (1.5), it is not hard to verify that

$${}_3\phi_2\left[\begin{matrix} q^{-n},\ a,\ c \\ q^{-\lfloor \frac{n}{2} \rfloor}ae, q^{1-\lfloor \frac{n+1}{2} \rfloor}c/e \end{matrix}\middle| q;q\right] = \left[\begin{matrix} q^{-\lfloor \frac{n}{2} \rfloor}e, q^{-\lfloor \frac{n}{2} \rfloor}ae/c \\ q^{-\lfloor \frac{n}{2} \rfloor}ae, q^{-\lfloor \frac{n}{2} \rfloor}e/c \end{matrix}\middle| q\right]_n$$

which is equivalent to the q-binomial sum

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (q^{1-k}/ae;q)_{\lfloor \frac{n}{2} \rfloor}(q^{-k}e/c;q)_{\lfloor \frac{n+1}{2} \rfloor} \left[\begin{matrix} a,\ c \\ ae, qc/e \end{matrix}\middle| q\right]_k q^{\binom{k+1}{2}}$$
$$= \left[\begin{matrix} e, ae/c \\ ae \end{matrix}\middle| q\right]_{\lfloor \frac{n+1}{2} \rfloor} \left[\begin{matrix} q/e, qc/ae \\ qc/e \end{matrix}\middle| q\right]_{\lfloor \frac{n}{2} \rfloor}.$$

Observing that this equation matches exactly to (1.1) specified by

$$f(k) = \left[\begin{matrix} e, ae/c \\ ae \end{matrix}\middle| q\right]_{\lfloor \frac{k+1}{2} \rfloor} \left[\begin{matrix} q/e, qc/ae \\ qc/e \end{matrix}\middle| q\right]_{\lfloor \frac{k}{2} \rfloor},$$
$$g(k) = \left[\begin{matrix} a,\ c \\ ae, qc/e \end{matrix}\middle| q\right]_k q^{\binom{k+1}{2}},$$
$$\varphi(x;n) = (qx/ae;q)_{\lfloor \frac{n}{2} \rfloor}(ex/c;q)_{\lfloor \frac{n+1}{2} \rfloor};$$

we may state the dual relation corresponding to (1.2) as the proposition.

Proposition 2.1 (Terminating reciprocal relation)**.**

$$\left[\begin{matrix} a,\ c \\ ae, qc/e \end{matrix}\middle| q\right]_n$$
$$= \sum_{k\geq 0} \begin{bmatrix} n \\ 2k \end{bmatrix} \frac{(1-q^{-k}e/c)q^{(1+2k)(k-n)}}{(q^{1-n}/ae;q)_k(q^{-n}e/c;q)_{k+1}} \left[\begin{matrix} e, ae/c \\ ae \end{matrix}\middle| q\right]_k \left[\begin{matrix} q/e, qc/ae \\ qc/e \end{matrix}\middle| q\right]_k$$
$$- \sum_{k\geq 0} \begin{bmatrix} n \\ 2k+1 \end{bmatrix} \frac{(1-q^{-k}/ae)q^{(1+k)(1+2k-2n)}}{(q^{1-n}/ae;q)_{k+1}(q^{-n}e/c;q)_{k+1}} \left[\begin{matrix} e, ae/c \\ ae \end{matrix}\middle| q\right]_{k+1} \left[\begin{matrix} q/e, qc/ae \\ qc/e \end{matrix}\middle| q\right]_k .$$

The two sums just displayed are, in fact, balanced ${}_8\phi_7$-series, which do not admit closed forms. However their combination does have a closed form. That is the reason why we call the last relation reciprocal.

Letting $n \to \infty$ in Proposition 2.1 and then applying the Weierstrass M-test (cf. Stromberg [24, §3.106]), we get the limiting relation:

$$\begin{aligned}
&\text{(2.2)} && \left[\begin{matrix} a,\ c \\ ae, qc/e \end{matrix}\Big| q\right]_\infty \\
&\text{(2.3)} && = \sum_{k\geq 0} \frac{1-q^k c/e}{(q;q)_{2k}} \left[\begin{matrix} e, ae/c \\ ae \end{matrix}\Big| q\right]_k \left[\begin{matrix} q/e, qc/ae \\ qc/e \end{matrix}\Big| q\right]_k q^{k^2-k}(ac)^k \\
&\text{(2.4)} && + \frac{c}{e} \sum_{k\geq 0} \frac{1-ae/q}{(q;q)_{2k+1}} \left[\begin{matrix} e, ae/c \\ ae/q \end{matrix}\Big| q\right]_{k+1} \left[\begin{matrix} q/e, qc/ae \\ qc/e \end{matrix}\Big| q\right]_k q^{k^2}(ac)^k.
\end{aligned}$$

Combining the two sums in (2.3) and (2.4) together, we obtain the following theorem.

Theorem 2.2 (Nonterminating series identity)**.**

$$\left[\begin{matrix} a,\ c \\ ae, c/e \end{matrix}\Big| q\right]_\infty = \sum_{k=0}^{\infty} \frac{(ac)^k}{(q;q)_{2k}} \left[\begin{matrix} e, ae/c \\ ae \end{matrix}\Big| q\right]_k \left[\begin{matrix} q/e, qc/ae \\ c/e \end{matrix}\Big| q\right]_k q^{k^2-k}$$
$$\times \left\{1 + q^k \frac{c(1-q^k e)(1-q^k ae/c)}{e(1-q^{1+2k})(1-q^k c/e)}\right\}.$$

We highlight two important corollaries about reciprocal product of q-gamma functions. Their limiting cases as $q \to 1^-$ yield infinite series for π and $1/\pi$.

Corollary 2.3. *For $\lambda \in \mathbb{R}$, the following identity holds:*

$$\frac{1}{\Gamma_q(1+\lambda)\Gamma_q(2-\lambda)} = \sum_{k=0}^{\infty} \frac{\left[q^\lambda, q^{1+\lambda}, q^{1-\lambda}, q^{2-\lambda}; q\right]_k}{(q;q)_k^2 (q^2;q)_{2k}} q^{k^2+k}$$
$$\times \left\{1 - \frac{(1-q^{-k})(1-q^{1+2k})}{(1-q^{\lambda+k})(1-q^{1-\lambda+k})}\right\}.$$

Proof. By inverting the fraction inside the braces $\{\cdots\}$ and then absorbing the factors involving k in the factorial quotients, we can equivalently reformulate the equation in Theorem 2.2 as

$$\left[\begin{matrix} qa,\ c \\ ae, qc/e \end{matrix}\Big| q\right]_\infty \frac{e(1-q)(1-a)}{c(1-e)(1-ae/c)}$$
$$= \sum_{k=0}^{\infty} \frac{q^{k^2}(ac)^k}{(q^2;q)_{2k}} \left[\begin{matrix} qe, qae/c \\ ae \end{matrix}\Big| q\right]_k \left[\begin{matrix} q/e, qc/ae \\ qc/e \end{matrix}\Big| q\right]_k$$
$$\times \left\{1 + q^{-k} \frac{e(1-q^{1+2k})(1-q^k c/e)}{c(1-q^k e)(1-q^k ae/c)}\right\}.$$

The formula in Corollary 2.3 follows by specifying $a = q^\lambda$ and $c = e = q^{1-\lambda}$ in the above equation. □

Corollary 2.4. *For $\lambda \in \mathbb{R}$, the following identity holds:*

$$\Gamma_q(\lambda)\Gamma_q(1-\lambda) = \sum_{k=0}^{\infty} q^{k^2+k} \frac{(q^{\lambda};q)_k(q^{1-\lambda};q)_k}{(q^2;q)_{2k}} \left\{ \frac{1-q^{1+2k}}{1-q^{\lambda+k}} - \frac{1-q^{\lambda+k}}{1-q^{\lambda-1-k}} \right\}.$$

Proof. This result simply follows from Theorem 2.2 with $a = c = q$ and $e = q^{\lambda}$. □

Remark. Letting $q \to 1^-$ on both sides of Corollary 2.4 and then using Euler's reflection formula (cf. [22, §17])

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

we obtain the following infinite series identity

$$\frac{\pi}{\sin(\pi\lambda)} = \sum_{k=0}^{\infty} \frac{(\lambda)_k(1-\lambda)_k}{(2k+1)!} \left\{ \frac{2k+1}{\lambda+k} - \frac{\lambda+k}{\lambda-k-1} \right\}. \tag{2.5}$$

This series (2.5) for $1/\sin(\pi z)$ is analogous to the well–known partial fraction decomposition for $\cot(\pi z)$ that can be obtained by using logarithmic differentiation of the Weierstrass factorization theorem for $\sin(\pi z)$.

By properly choosing special values of a, c and e, we find ten interesting q-series identities, that correspond to the classical series with the same convergence rate of $1/4$. Here the convergence rate for a series

$$\sum_{k=0}^{\infty} a_k$$

is defined by $\lim_{k\to\infty} a_{k+1}/a_k$, if this limit exists.

A1. For the series discovered by Ramanujan [23]

$$\frac{4}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{bmatrix}_k \frac{1+6k}{4^k},$$

we recover, by letting $\lambda = 1/2$ in Corollary 2.3, the following q-analogue (cf. Chen–Chu [5, Example 38] and Guo [18, Equation 1.6]):

$$\frac{1}{\Gamma_q^2(\frac{1}{2})} = \sum_{k=0}^{\infty} q^{k^2} \frac{(q^{1/2};q)_k^4}{(q;q)_k^2(q;q)_{2k}} \frac{1+q^{k+1/2}-2q^{2k+1/2}}{(1-q)(1+q^{k+1/2})}.$$

A different, but simpler q-analogue can be found in Guo–Liu [19, Equation 3] and Chen–Chu [5, Example 4]:

$$\sum_{k=0}^{\infty} \frac{1-q^{6k+1}}{1-q^4} \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3} q^{k^2} = \frac{1}{\Gamma_{q^4}^2(\frac{1}{2})}.$$

A2. For $\lambda = 1/3$, we get, from Corollary 2.3, the following identity

$$\frac{1}{\Gamma_q(\frac{4}{3})\Gamma_q(\frac{5}{3})} = \sum_{k=0}^{\infty} q^{k^2+k} \frac{\big[q^{1/3}, q^{2/3}, q^{4/3}, q^{5/3}; q\big]_k}{(q;q)_k^2 (q^2;q)_{2k}} \bigg\{1 - \frac{(1-q^{-k})(1-q^{2k+1})}{(1-q^{k+\frac{1}{3}})(1-q^{k+\frac{2}{3}})}\bigg\}$$

which gives a q-analogue of the series

$$\frac{9\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \\ 1, 1, 1, \frac{3}{2} \end{bmatrix}_k \frac{2+18k+27k^2}{4^k}.$$

A3. For $\lambda = 1/4$, we have, from Corollary 2.3, the following identity due to Guo and Zudilin [20, Equation 1.6]

$$\frac{1}{\Gamma_q(\frac{5}{4})\Gamma_q(\frac{7}{4})} = \sum_{k=0}^{\infty} q^{k^2+k} \frac{\Big[q^{\frac{1}{4}}, q^{\frac{3}{4}}, q^{\frac{5}{4}}, q^{\frac{7}{4}}; q\Big]_k}{(q;q)_k^2 (q^2;q)_{2k}} \bigg\{1 - \frac{(1-q^{-k})(1-q^{2k+1})}{(1-q^{k+\frac{1}{4}})(1-q^{k+\frac{3}{4}})}\bigg\}$$

which offers a q-analogue of the series

$$\frac{8\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \\ 1, 1, 1, \frac{3}{2} \end{bmatrix}_k \frac{3+32k+48k^2}{4^k}.$$

A4. For $\lambda = 1/6$, we find, from Corollary 2.3, the following identity

$$\frac{1}{\Gamma_q(\frac{7}{6})\Gamma_q(\frac{11}{6})} = \sum_{k=0}^{\infty} q^{k^2+k} \frac{\big[q^{1/6}, q^{5/6}, q^{7/6}, q^{11/6}; q\big]_k}{(q;q)_k^2 (q^2;q)_{2k}} \bigg\{1 - \frac{(1-q^{-k})(1-q^{2k+1})}{(1-q^{k+\frac{1}{6}})(1-q^{k+\frac{5}{6}})}\bigg\}$$

which provides a q-analogue of the series

$$\frac{18}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ 1, 1, 1, \frac{3}{2} \end{bmatrix}_k \frac{5+72k+108k^2}{4^k}.$$

A5. Letting $\lambda = 1/2$ in Corollary 2.4, we get the following identity

$$\Gamma_q^2\Big(\frac{1}{2}\Big) = \sum_{k=0}^{\infty} q^{k^2+k} \frac{(q^{1/2};q)_k^2}{(q^2;q)_{2k}} (1+2q^{k+1/2})$$

which is a q-analogue of the series

$$\frac{\pi}{3} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2} \end{bmatrix}_k \Big(\frac{1}{4}\Big)^k.$$

A6. Letting $\lambda = 1/3$ in Corollary 2.4, we deduce the following identity

$$\Gamma_q\Big(\frac{1}{3}\Big)\Gamma_q\Big(\frac{2}{3}\Big) = \sum_{k=0}^{\infty} q^{k^2+k} \frac{(q^{1/3};q)_k (q^{2/3};q)_k}{(q^2;q)_{2k}} \bigg\{\frac{1-q^{1+2k}}{1-q^{k+\frac{1}{3}}} - \frac{1-q^{k+\frac{1}{3}}}{1-q^{-k-\frac{2}{3}}}\bigg\}$$

which gives a q-analogue of the series

$$\frac{4\pi}{\sqrt{3}} = \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \\ 1, \frac{3}{2}, \frac{4}{3}, \frac{5}{3} \end{matrix}\right]_k \frac{7+27k+27k^2}{4^k}.$$

A7. Letting $\lambda = 1/6$ in Corollary 2.4, we obtain the following identity

$$\Gamma_q\Big(\frac{1}{6}\Big)\Gamma_q\Big(\frac{5}{6}\Big) = \sum_{k=0}^{\infty} q^{k^2+k} \frac{(q^{1/6};q)_k (q^{5/6};q)_k}{(q^2;q)_{2k}} \left\{ \frac{1-q^{1+2k}}{1-q^{k+\frac{1}{6}}} - \frac{1-q^{k+\frac{1}{6}}}{1-q^{-k-\frac{5}{6}}} \right\}$$

which results in a q-analogue of the series

$$10\pi = \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ 1, \frac{3}{2}, \frac{7}{6}, \frac{11}{6} \end{matrix}\right]_k \frac{31+108k+108k^2}{4^k}.$$

A8. By specifying $a = q,\ c = q^{2/3}$ and $e = q^{1/3}$ in Theorem 2.2, we find

$$\frac{\Gamma_q^2(\frac{1}{3})}{\Gamma_q(\frac{2}{3})} = \sum_{k=0}^{\infty} q^{k^2+\frac{2k}{3}} \frac{(q^{1/3};q)_k (q^{2/3};q)_k^2}{(q^{4/3};q)_k (q^2;q)_{2k}} \frac{1+q^{k+\frac{1}{3}}-2q^{2k+1}}{1-q^{\frac{1}{3}}}$$

which corresponds to the identity

$$\frac{\sqrt{3}\,\Gamma^3(\frac{1}{3})}{2\pi} = \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \\ 1, \frac{3}{2}, \frac{4}{3} \end{matrix}\right]_k \frac{5+9k}{4^k}.$$

A9. By specifying $a = c = q^{1/4}$ and $e = q^{1/2}$ in Theorem 2.2, we have

$$\frac{\Gamma_q^2(\frac{3}{4})}{\Gamma_q^2(\frac{1}{4})} = \sum_{k=0}^{\infty} q^{k(k-\frac{1}{2})} \frac{(q^{1/2};q)_k^3 (q^{3/2};q)_k}{(q^{3/4};q)_k^2 (q^2;q)_{2k}} \frac{1+q^{k+\frac{1}{2}}-2q^{2k+\frac{1}{4}}}{(1-q)(1+q^{\frac{1}{2}})}$$

which corresponds to the identity

$$\frac{2\Gamma^2(\frac{3}{4})}{3\Gamma^2(\frac{1}{4})} = \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{4}, \frac{3}{4} \end{matrix}\right]_k \frac{k}{4^k} \iff \frac{12\Gamma^2(\frac{3}{4})}{\Gamma^2(\frac{1}{4})} = \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \\ 1, \frac{7}{4}, \frac{7}{4} \end{matrix}\right]_k \frac{1}{4^k}.$$

A10. By specifying $a = c = q^{3/4}$ and $e = q^{1/2}$ in Theorem 2.2, we find

$$\frac{\Gamma_q^2(\frac{1}{4})}{\Gamma_q^2(\frac{3}{4})} = \sum_{k=0}^{\infty} q^{k(k+\frac{1}{2})} \frac{(q^{1/2};q)_k^3 (q^{3/2};q)_k}{(q^{5/4};q)_k^2 (q^2;q)_{2k}} \frac{(1+q^{\frac{1}{4}})(1+q^{k+\frac{1}{2}}-2q^{2k+\frac{3}{4}})}{1-q^{\frac{1}{4}}}$$

which corresponds to the identity

$$\frac{\Gamma^2(\frac{1}{4})}{8\Gamma^2(\frac{3}{4})} = \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{5}{4}, \frac{5}{4} \end{matrix}\right]_k \frac{1+3k}{4^k}.$$

2.2. **Second version.** According to (1.5), it is routine to check that

$$ {}_3\phi_2\left[\begin{matrix} q^{-n},\ q^{\lfloor\frac{n}{2}\rfloor}a,\ c \\ ae, q^{1-\lfloor\frac{n+1}{2}\rfloor}c/e \end{matrix}\Big| q;q\right] = \left[\begin{matrix} q^{-\lfloor\frac{n}{2}\rfloor}e, ae/c \\ q^{-\lfloor\frac{n}{2}\rfloor}e/c, ae \end{matrix}\Big| q\right]_n. \tag{2.6} $$

By making use of the factorial expression

$$ (q^{-k}y;q)_{\lfloor\frac{n+1}{2}\rfloor} q^{\lfloor\frac{n+1}{2}\rfloor k} = \langle q^k/y;q\rangle_{\lfloor\frac{n+1}{2}\rfloor}(-y)^{\lfloor\frac{n+1}{2}\rfloor} q^{\binom{\lfloor\frac{n+1}{2}\rfloor}{2}}, $$

we can reformulate (2.6) as the q-binomial identity:

$$ \begin{aligned} &\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (q^k a;q)_{\lfloor\frac{n}{2}\rfloor}\langle q^k c/e;q\rangle_{\lfloor\frac{n+1}{2}\rfloor} \left[\begin{matrix} a,\ c \\ ae, qc/e \end{matrix}\Big| q\right]_k \\ =&(-1)^{\lfloor\frac{n+1}{2}\rfloor} q^{\binom{n}{2}-\binom{\lfloor\frac{n+1}{2}\rfloor}{2}} c^n \frac{(e;q)_{\lfloor\frac{n+1}{2}\rfloor}}{e^{\lfloor\frac{n+1}{2}\rfloor}} \left[\begin{matrix} ae/c \\ ae \end{matrix}\Big| q\right]_n \left[\begin{matrix} q/e, a \\ qc/e \end{matrix}\Big| q\right]_{\lfloor\frac{n}{2}\rfloor}. \end{aligned} $$

Since the last equation matches exactly to (1.3) specified by

$$ \begin{aligned} f(k) &= (-1)^{\lfloor\frac{k+1}{2}\rfloor} q^{\binom{k}{2}-\binom{\lfloor\frac{k+1}{2}\rfloor}{2}} c^k \frac{(e;q)_{\lfloor\frac{k+1}{2}\rfloor}}{e^{\lfloor\frac{k+1}{2}\rfloor}} \left[\begin{matrix} ae/c \\ ae \end{matrix}\Big| q\right]_k \left[\begin{matrix} q/e, a \\ qc/e \end{matrix}\Big| q\right]_{\lfloor\frac{k}{2}\rfloor}, \\ g(k) &= \left[\begin{matrix} a,\ c \\ ae, qc/e \end{matrix}\Big| q\right]_k \quad \text{and} \quad \varphi(x;n) = (ax;q)_{\lfloor\frac{n}{2}\rfloor}\langle cx/e;q\rangle_{\lfloor\frac{n+1}{2}\rfloor}; \end{aligned} $$

the dual relation corresponding to (1.4) is given in the proposition.

Proposition 2.5 (Terminating reciprocal relation)**.**

$$ \begin{aligned} &\left[\begin{matrix} a,\ c \\ ae, qc/e \end{matrix}\Big| q\right]_n \\ =&\sum_{k\geq 0} q^{\frac{3k^2-k}{2}} \begin{bmatrix} n \\ 2k \end{bmatrix} \frac{(1-q^k c/e)(-1)^k c^{2k}}{(q^n a;q)_k\langle q^n c/e;q\rangle_{k+1}} \frac{(e;q)_k}{e^k} \left[\begin{matrix} ae/c \\ ae \end{matrix}\Big| q\right]_{2k} \left[\begin{matrix} q/e, a \\ qc/e \end{matrix}\Big| q\right]_k \\ &+\sum_{k\geq 0}\left(q^{\frac{3k^2+k}{2}} \begin{bmatrix} n \\ 2k+1 \end{bmatrix} \frac{(1-aq^{3k+1})(-1)^k c^{2k+1}}{(q^n a;q)_{k+1}\langle q^n c/e;q\rangle_{k+1}} \frac{(e;q)_{k+1}}{e^{k+1}} \right. \\ &\left. \times \left[\begin{matrix} ae/c \\ ae \end{matrix}\Big| q\right]_{2k+1} \left[\begin{matrix} q/e, a \\ qc/e \end{matrix}\Big| q\right]_k \right). \end{aligned} $$

Both sums just displayed can be expressed as terminating q-series, which do not have closed forms. However their combination does have a closed form.

Letting $n \to \infty$ in Proposition 2.5 and then applying the Weierstrass M-test, we get the limiting relation:

$$\left[\begin{matrix} a,\ c \\ ae, qc/e \end{matrix}\middle| q\right]_\infty \tag{2.7}$$

$$= \sum_{k\geq 0}(-1)^k q^{\frac{3k^2-k}{2}} \frac{(1-q^k c/e)}{(q;q)_{2k}} \frac{c^{2k}(e;q)_k}{e^k} \left[\begin{matrix} ae/c \\ ae \end{matrix}\middle| q\right]_{2k} \left[\begin{matrix} q/e, a \\ qc/e \end{matrix}\middle| q\right]_k \tag{2.8}$$

$$+ \frac{c}{e}\sum_{k\geq 0}(-1)^k q^{\frac{3k^2+k}{2}} \frac{(1-aq^{3k+1})}{(q;q)_{2k+1}} \frac{c^{2k}(e;q)_{k+1}}{e^k} \left[\begin{matrix} ae/c \\ ae \end{matrix}\middle| q\right]_{2k+1} \left[\begin{matrix} q/e, a \\ qc/e \end{matrix}\middle| q\right]_k. \tag{2.9}$$

Combining the two sums in (2.8) and (2.9), we derive the following theorem.

Theorem 2.6 (Nonterminating series identity)**.**

$$\begin{aligned} &\left[\begin{matrix} a,\ c \\ ae, c/e \end{matrix}\middle| q\right]_\infty \\ &= \sum_{k=0}^{\infty} \frac{(-c^2/e)^k}{(q;q)_{2k}} \frac{(ae/c;q)_{2k}}{(ae;q)_{2k}} \left[\begin{matrix} a, e, q/e \\ c/e \end{matrix}\middle| q\right]_k q^{\frac{3k^2-k}{2}} \\ &\times \left\{1 + q^k \frac{c(1-aq^{3k+1})(1-q^k e)(1-q^{2k}ae/c)}{e(1-q^{1+2k})(1-q^k c/e)(1-aeq^{2k})}\right\}. \end{aligned}$$

Two implications are given below about reciprocal product of q-gamma functions.

Corollary 2.7. *For $\lambda \in \mathbb{R}$, we have the infinite series identity*

$$\begin{aligned} &\frac{1}{\Gamma_q(1+\lambda)\Gamma_q(2-\lambda)} \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{(q^{1+\lambda};q)_{2k}}{(q^2;q)_{2k}^2} \left[\begin{matrix} q^\lambda, q^\lambda, q^{2-\lambda} \\ q \end{matrix}\middle| q\right]_k q^{\frac{k}{2}(3+3k-2\lambda)} \\ &\times \frac{1-q^{1+\lambda+3k}}{1-q} \left\{1 + \frac{q^{-k}(1-q^k)(1-q^{1+2k})(1-q^{1+2k})}{(1-q^{1-\lambda+k})(1-q^{\lambda+2k})(1-q^{1+\lambda+3k})}\right\}. \end{aligned}$$

Proof. The formula is confirmed by reformulating the equality displayed in Theorem 2.6 in an analogous manner as that for the proof of Corollary 2.3 and then letting $a = q^\lambda$ and $c = e = q^{1-\lambda}$ in the resulting equation. □

Corollary 2.8. *For $\lambda \in \mathbb{R}$, we have the infinite series identity*

$$\Gamma_q(1+\lambda)\Gamma_q(1-\lambda) = \sum_{k=0}^{\infty}(-1)^k \frac{[q, q^\lambda; q]_k}{(q;q)_{2k}} \frac{(q^\lambda;q)_{2k}}{(q^{1+\lambda};q)_{2k}} q^{\frac{k}{2}(3+3k-2\lambda)} \times \left\{1 + \frac{q^{1+k-\lambda}(1-q^{2+3k})(1-q^{\lambda+k})(1-q^{\lambda+2k})}{(1-q^{1+2k})(1-q^{1-\lambda+k})(1-q^{1+\lambda+2k})}\right\}.$$

Proof. Specifying $a = c = q$ and $e = q^\lambda$ in Theorem 2.6, we get the desired result. □

Five q-series as well as their counterparts of classical series are exemplified as follows.

B1. For Ramanujan's series [23]

$$\frac{8}{\pi} = \sum_{k=0}^{\infty}\left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ 1, 1, 1 \end{matrix}\right]_k \{3+20k\},$$

we recover, by letting $\lambda = 1/2$ in Corollary 2.7, the following q-analogue (cf. Chen and Chu [5, Example 39])

$$\frac{1}{\Gamma_q^2(\frac{1}{2})} = \sum_{k=0}^{\infty}(-1)^k \frac{(q^{1/2};q)_k^3(q^{1/2};q)_{2k}}{(q;q)_k(q;q)_{2k}^2} q^{3k^2/2} \times \left\{\frac{(1+q^{k+1/2})^2(1-q^{3k+1/2}) - q^{2k+1/2}(1-q^{2k+1/2})}{(1-q)(1+q^{k+1/2})^2}\right\}.$$

Guo and Zudilin [20, Equation 1.4] derived, by means of the *WZ* machinery, another q-analogue

$$\frac{1}{\Gamma_q^2(\frac{1}{2})} = \sum_{k=0}^{\infty}(-1)^k \frac{(q^{1/2};q)_k^2(q^{1/4};q^{1/2})_{2k}}{(q;q)_k^2(q;q)_{2k}} q^{k^2/2} \times \left\{\frac{1-q^{2k+1/4}}{1-q} + \frac{q^{k+1/4}(1-q^{k+1/4})}{(1-q)(1+q^{k+1/2})}\right\}.$$

This is another example (apart from A1) that there may exist different q-analogues for the same classical series.

B2. For $\lambda = 1/2$, we get, from Corollary 2.8, the identity

$$\Gamma_q\left(\frac{1}{2}\right)\Gamma_q\left(\frac{3}{2}\right) = \sum_{k=0}^{\infty}(-1)^k q^{\frac{3k^2}{2}+k} \frac{(q;q)_k(q^{1/2};q)_k(q^{1/2};q)_{2k}}{(q^{3/2};q)_{2k}(q;q)_{2k}} \times \left\{1 + \frac{q^{k+\frac{1}{2}}(1-q^{3k+2})(1-q^{2k+\frac{1}{2}})}{(1-q^{2k+1})(1-q^{2k+\frac{3}{2}})}\right\} \tag{2.10}$$

which can also be obtained from Chu [10, Proposition 14: $x = y^2 = q$]. Identity (2.10) is a q-analogue of the classical series

$$\frac{3\pi}{2} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ \frac{3}{2}, \frac{5}{4}, \frac{7}{4} \end{matrix}\right]_k \{5 + 21k + 20k^2\},$$

which is equivalent to a formula of BBP-type due to Adamchik and Wagon [1].

B3. For $\lambda = 1/3$, we have, from Corollary 2.8, the identity

$$\Gamma_q\left(\frac{1}{3}\right)\Gamma_q\left(\frac{2}{3}\right) = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{(q;q)_{k+1}(q^{1/3};q)_k(q^{4/3};q)_{2k}}{(q^2;q)_{2k}(q^{4/3};q)_{2k+1}}q^{\frac{3k^2}{2}+\frac{19k}{6}+1} \times \left\{1 + \frac{(1+q^{k+\frac{2}{3}})(1-q^{-2k-1})(1-q^{3k+1})}{(1-q^{k+1})(1-q^{2k+\frac{1}{3}})}\right\}$$

which offers a q-analogue of the series

$$\frac{8\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{6} \\ \frac{3}{2}, \frac{5}{3}, \frac{7}{6} \end{matrix}\right]_k \{5 + 23k + 30k^2\}.$$

B4. For $\lambda = 2/3$, we obtain, from Corollary 2.8, the identity

$$\Gamma_q\left(\frac{1}{3}\right)\Gamma_q\left(\frac{5}{3}\right) = \sum_{k=0}^{\infty}(-1)^k\frac{(q;q)_k(q^{2/3};q)_k(q^{2/3};q)_{2k}}{(q;q)_{2k}(q^{5/3};q)_{2k}}q^{\frac{3k^2}{2}+\frac{5k}{6}} \times \left\{1 + \frac{q^{k+\frac{1}{3}}(1+q^{k+\frac{1}{3}})(1-q^{k+\frac{2}{3}})(1-q^{3k+2})}{(1-q^{2k+1})(1-q^{2k+\frac{5}{3}})}\right\}$$

which provides a q-analogue of the series

$$\frac{20\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{5}{6} \\ \frac{3}{2}, \frac{4}{3}, \frac{11}{6} \end{matrix}\right]_k \{13 + 40k + 30k^2\}.$$

B5. In addition, by specifying $a = q^{2/3},\ c = q^{1/3}$ and $e = q^{1/6}$ in Theorem 2.6, we find the following strange looking identity

$$\frac{\Gamma_q(\frac{1}{6})\Gamma_q(\frac{5}{6})}{\Gamma_q(\frac{1}{3})\Gamma_q(\frac{2}{3})} = \sum_{k=0}^{\infty}(-1)^k\frac{\left[q^{\frac{2}{3}}, q^{\frac{5}{6}}; q\right]_k}{(q;q)_{2k}}\frac{(q^{\frac{1}{2}};q)_{2k}}{(q^{\frac{5}{6}};q)_{2k}}q^{\frac{3k^2}{2}} \times \left\{1 + q^{k+\frac{1}{6}}\frac{(1-q^{\frac{1}{2}+2k})(1-q^{\frac{5}{3}+3k})}{(1-q^{1+2k})(1-q^{\frac{5}{6}+2k})}\right\}$$

which turns out to be a q-analogue of the series

$$5\sqrt{3} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{5}{6} \\ 1, \frac{3}{2}, \frac{11}{12}, \frac{17}{12} \end{matrix}\right]_k \{10 + 51k + 60k^2\}.$$

2.3. **Third version.** According to (1.5), it is not difficult to show that

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, q^{\lfloor\frac{n}{2}\rfloor}a, q^{\lfloor\frac{n+1}{2}\rfloor}c \\ ae, \quad qc/e \end{matrix}\Big| q;q\right] = \left[\begin{matrix} q^{-\lfloor\frac{n}{2}\rfloor}e, q^{-\lfloor\frac{n+1}{2}\rfloor}ae/c \\ q^{-n}e/c, \quad ae \end{matrix}\Big| q\right]_n, $$

which can be rewritten as the following q-binomial sum

$$ \begin{aligned} &\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k\end{bmatrix} q^{\binom{n-k}{2}}(q^k a;q)_{\lfloor\frac{n}{2}\rfloor}(q^k c;q)_{\lfloor\frac{n+1}{2}\rfloor}\left[\begin{matrix} a,\ c \\ ae, qc/e\end{matrix}\Big| q\right]_k \\ &= q^{\lfloor\frac{3n^2-2n}{4}\rfloor}a^{\lfloor\frac{n+1}{2}\rfloor}c^{\lfloor\frac{n}{2}\rfloor}\frac{[a,q/e,ae/c;q]_{\lfloor\frac{n}{2}\rfloor}[c,e,qc/ae;q]_{\lfloor\frac{n+1}{2}\rfloor}}{[ae,qc/e;q]_n}. \end{aligned} \tag{2.11} $$

The identity in (2.11) is equivalent to (1.3) with

$$ f(k) = q^{\lfloor\frac{3k^2-2k}{4}\rfloor}a^{\lfloor\frac{k+1}{2}\rfloor}c^{\lfloor\frac{k}{2}\rfloor}\frac{[a,q/e,ae/c;q]_{\lfloor\frac{k}{2}\rfloor}[c,e,qc/ae;q]_{\lfloor\frac{k+1}{2}\rfloor}}{[ae,qc/e;q]_k}, $$

$$ g(k) = \left[\begin{matrix} a,\ c \\ ae, qc/e\end{matrix}\Big| q\right]_k \quad \text{and} \quad \varphi(x;n) = (ax;q)_{\lfloor\frac{n}{2}\rfloor}(cx;q)_{\lfloor\frac{n+1}{2}\rfloor}. $$

Thus, we have the dual relation corresponding to (1.4) which is given below.

Proposition 2.9 (Terminating reciprocal relation)**.**

$$ \begin{aligned} &\left[\begin{matrix} a,\ c \\ ae, qc/e\end{matrix}\Big| q\right]_n \\ &= \sum_{k\geq 0}\begin{bmatrix} n \\ 2k\end{bmatrix}\frac{(1-q^{3k}c)q^{3k^2-k}(ac)^k}{(q^n a;q)_k(q^n c;q)_{k+1}}\frac{[a,q/e,ae/c;q]_k\,[c,e,qc/ae;q]_k}{[ae,qc/e;q]_{2k}} \\ &\quad - a\sum_{k\geq 0}\begin{bmatrix} n \\ 2k+1\end{bmatrix}\frac{(1-q^{3k+1}a)q^{3k^2+2k}(ac)^k}{(q^n a;q)_{k+1}(q^n c;q)_{k+1}}\frac{[a,q/e,ae/c;q]_k\,[c,e,qc/ae;q]_{k+1}}{[ae,qc/e;q]_{2k+1}}. \end{aligned} $$

The two sums on the right-hand side of Proposition 2.9 are terminating q-series and neither of them admit closed forms. Nevertheless, their combination does have an unexpected closed form.

Letting $n \to \infty$ in Proposition 2.9 and then applying the Weierstrass M-test, we get the limiting relation:

$$ \left[\begin{matrix} a, c \\ ae, qc/e\end{matrix}\Big| q\right]_\infty \tag{2.12} $$

(2.13)

$$ = \sum_{k\geq 0}\frac{(1-q^{3k}c)q^{3k^2-k}(ac)^k}{(q;q)_{2k}}\frac{[a,q/e,ae/c;q]_k\,[c,e,qc/ae;q]_k}{[ae,qc/e;q]_{2k}} $$

$$ - a\sum_{k\geq 0}\frac{(1-q^{3k+1}a)q^{3k^2+2k}(ac)^k}{(q;q)_{2k+1}}\frac{[a,q/e,ae/c;q]_k\,[c,e,qc/ae;q]_{k+1}}{[ae,qc/e;q]_{2k+1}}. \tag{2.14} $$

Combining the two sums in (2.13) and (2.14), we establish the following theorem.

Theorem 2.10 (Nonterminating series identity)**.**

$$\begin{aligned}&\left[\begin{matrix}a,\ qc\\ ae, qc/e\end{matrix}\Big| q\right]_\infty\\ &=\sum_{k=0}^{\infty}\left(\frac{1-q^{3k}c}{1-c}\right)\frac{[a,c,e,q/e,ae/c,qc/ae;q]_k}{[q,ae,qc/e;q]_{2k}}q^{3k^2-k}(ac)^k\\ &\times\left\{1-q^{3k}\frac{a(1-q^{3k+1}a)(1-q^kc)(1-q^ke)(1-q^{1+k}c/ae)}{(1-q^{3k}c)(1-q^{1+2k})(1-q^{2k}ae)(1-q^{1+2k}c/e)}\right\}.\end{aligned}$$

Below we record two special cases of Theorem 2.10 which can be utilized to obtain q-analogues of classical series for π and $1/\pi$.

Corollary 2.11. *For $\lambda \in \mathbb{R}$, the identity below holds true*

$$\begin{aligned}&\frac{1}{\Gamma_q(\lambda)\Gamma_q(1-\lambda)}\\ &=\sum_{k=0}^{\infty}q^{3k^2}\frac{(q^\lambda;q)_k^3(q^{1-\lambda};q)_k^3}{(q;q)_{2k}^3}\frac{1-q^{3k+1-\lambda}}{1-q}\\ &\times\left\{1-\frac{q^{3k+\lambda}(1-q^{3k+1+\lambda})(1-q^{k+1-\lambda})^3}{(1-q^{3k+1-\lambda})(1-q^{2k+1})^3}\right\}.\end{aligned}$$

Proof. The identity in this corollary is deduced directly by specifying $a = q^\lambda$ and $c = e = q^{1-\lambda}$ in Theorem 2.10. □

Corollary 2.12. *For $\lambda \in \mathbb{R}$, the identity below holds true*

$$\begin{aligned}&\Gamma_q(1+\lambda)\Gamma_q(2-\lambda)\\ &=\sum_{k=0}^{\infty}\frac{1-q^{3k+1}}{1-q}\frac{[q,q,q^\lambda,q^{1-\lambda},q^\lambda,q^{1-\lambda};q]_k}{[q,q^{1+\lambda},q^{2-\lambda};q]_{2k}}q^{3k^2+k}\\ &\times\left\{1-\frac{q^{1+3k}(1-q^{2+3k})(1-q^{1+k})(1-q^{\lambda+k})(1-q^{1-\lambda+k})}{(1-q^{1+3k})(1-q^{1+2k})(1-q^{1+\lambda+2k})(1-q^{2-\lambda+2k})}\right\}.\end{aligned}$$

Proof. The result follows straightforwardly from Theorem 2.10 with $a = c = q$ and $e = q^\lambda$. □

From these two corollaries, we also obtain the following five q-series identities which are q-analogues of some classical identities.

C1. Recall the following series of Ramanujan [23]:

$$\frac{16}{\pi}=\sum_{k=0}^{\infty}\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,\ 1,\ 1\end{matrix}\right]_k\frac{5+42k}{64^k}.$$

By letting $\lambda = 1/2$ in Corollary 2.11, we recover its q-analogue (cf. Chen and Chu [5, Example 40]) as follows

$$\frac{1}{\Gamma_q^2(\frac{1}{2})} = \sum_{k=0}^{\infty} q^{3k^2} \frac{(q^{1/2};q)_k^6}{(q;q)_{2k}^3} \frac{1-q^{3k+1/2}}{1-q} \Big\{ 1 - \frac{q^{3k+1/2}(1-q^{3k+3/2})}{(1+q^{k+1/2})^3(1-q^{3k+1/2})} \Big\}.$$

C2. For $\lambda = 1/4$, we get, from Corollary 2.11, the q-series identity

$$\begin{aligned}\frac{1}{\Gamma_q(\frac{1}{4})\Gamma_q(\frac{3}{4})} =& \sum_{k=0}^{\infty} \frac{1-q^{3k+\frac{3}{4}}}{1-q} \frac{(q^{\frac{1}{4}};q)_k^3 (q^{\frac{3}{4}};q)_k^3}{(q;q)_{2k}^3} q^{3k^2} \\ & \times \Big\{ 1 - \frac{q^{3k+\frac{1}{4}}(1-q^{3k+\frac{5}{4}})(1-q^{k+\frac{3}{4}})^3}{(1-q^{3k+\frac{3}{4}})(1-q^{2k+1})^3} \Big\}.\end{aligned} \tag{2.15}$$

The right-hand side of (2.15) can further be simplified. To do so, consider the series defined by

$$\sum_{k=0}^{\infty} \Lambda(k), \quad \text{where} \quad \Lambda(k) := (-1)^k \frac{1-q^{\frac{1+6k}{4}}}{1-q} \frac{(q^{\frac{1}{4}};q^{\frac{1}{2}})_k^3}{(q;q)_k^3} q^{\frac{3}{4}k^2}.$$

Then its *bisection series* can be reformulated as

$$\begin{aligned}\sum_{k=0}^{\infty} \Lambda(k) =& \sum_{k=0}^{\infty} \big\{ \Lambda(2k) + \Lambda(2k+1) \big\} \\ =& \sum_{k=0}^{\infty} \Lambda(2k) \Big\{ 1 + \frac{\Lambda(2k+1)}{\Lambda(2k)} \Big\} \\ =& \sum_{k=0}^{\infty} \Big(\frac{1-q^{3k+\frac{1}{4}}}{1-q} \Big) \frac{(q^{\frac{1}{4}};q^{\frac{1}{2}})_{2k}^3}{(q;q)_{2k}^3} q^{3k^2} \\ & \times \Big\{ 1 - \frac{q^{3k+\frac{3}{4}}(1-q^{3k+\frac{7}{4}})(1-q^{k+\frac{1}{4}})^3}{(1-q^{3k+\frac{1}{4}})(1-q^{2k+1})^3} \Big\}.\end{aligned}$$

Now it is not hard to check that

$$\begin{aligned}& \frac{1-q^{3k+\frac{1}{4}}}{1-q} \Big\{ 1 - \frac{q^{3k+\frac{3}{4}}(1-q^{3k+\frac{7}{4}})(1-q^{k+\frac{1}{4}})^3}{(1-q^{3k+\frac{1}{4}})(1-q^{2k+1})^3} \Big\} \\ =& \frac{1-q^{3k+\frac{3}{4}}}{1-q} \Big\{ 1 - \frac{q^{3k+\frac{1}{4}}(1-q^{3k+\frac{5}{4}})(1-q^{k+\frac{3}{4}})^3}{(1-q^{3k+\frac{3}{4}})(1-q^{2k+1})^3} \Big\}.\end{aligned}$$

We therefore find the following simpler series (see Chen–Chu [5, Example 5] and Guo–Liu [19, Equation 4])

$$\frac{1}{\Gamma_q(\frac{1}{4})\Gamma_q(\frac{3}{4})} = \sum_{k=0}^{\infty} (-1)^k \frac{1-q^{\frac{1+6k}{4}}}{1-q} \frac{(q^{\frac{1}{4}};q^{\frac{1}{2}})_k^3}{(q;q)_k^3} q^{\frac{3}{4}k^2}.$$

Evidently, this is a q-analogue of the classical identity due to Guillera [15]

$$\frac{2\sqrt{2}}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{8}\Big)^k\left[\begin{matrix}\frac12,\frac12,\frac12\\ 1,\ 1,\ 1\end{matrix}\right]_k\big\{1+6k\big\}.$$

C3. For $\lambda=1/2$, we have, from Corollary 2.12, the q-series identity

$$\Gamma_q^2\Big(\frac32\Big)=\sum_{k=0}^{\infty}q^{3k^2+k}\frac{1-q^{3k+1}}{1-q}\frac{(q;q)_k^2(q^{\frac12};q)_k^4}{(q^{\frac32};q)_{2k}^2(q;q)_{2k}}\times\left\{1-\frac{q^{3k+1}(1-q^{k+\frac12})(1-q^{k+1})(1-q^{3k+2})}{(1+q^{k+\frac12})(1-q^{2k+\frac32})^2(1-q^{3k+1})}\right\}$$

which gives a q-analogue of the following series

$$\frac{9\pi}{4}=\sum_{k=0}^{\infty}\left[\begin{matrix}1,\frac12,\frac12,\frac12\\ \frac54,\frac54,\frac74,\frac74\end{matrix}\right]_k\frac{7+42k+75k^2+42k^3}{64^k}.$$

We remark that the above q-series can also be derived by letting $x=y^2=q$ in Chu [10, Proposition 15].

C4. Letting $a=c=e=q^{1/4}$ in Theorem 2.10, we get the q-series identity

$$\frac{\Gamma_q(\frac12)}{\Gamma_q^2(\frac14)}=\sum_{k=0}^{\infty}q^{3k^2-\frac{k}{2}}\frac{1-q^{3k+\frac14}}{1-q}\frac{(q^{\frac14};q)_k^4(q^{\frac34};q)_k^2}{(q^{\frac12};q)_{2k}(q;q)_{2k}^2}\times\left\{1-\frac{q^{3k+\frac14}(1-q^{k+\frac14})(1-q^{k+\frac34})(1-q^{3k+\frac54})}{(1+q^{k+\frac14})(1-q^{2k+1})^2(1-q^{3k+\frac14})}\right\}$$

which provides a q-analogue of the following series

$$\frac{128\sqrt{\pi}}{\Gamma^2(\frac14)}=\sum_{k=0}^{\infty}\left[\begin{matrix}\frac14,\frac14,\frac14,\frac34\\ 1,1,\ \frac32,\frac32\end{matrix}\right]_k\frac{17+396k+1392k^2+1344k^3}{64^k}.$$

C5. Letting $a=c=e=q^{3/4}$ in Theorem 2.10, we derive the q-series identity

$$\frac{\Gamma_q(\frac32)}{\Gamma_q^2(\frac34)}=\sum_{k=0}^{\infty}q^{3k^2+\frac{k}{2}}\frac{1-q^{3k+\frac34}}{1-q}\frac{(q^{\frac14};q)_k^2(q^{\frac34};q)_k^4}{(q^{\frac32};q)_{2k}(q;q)_{2k}^2}\times\left\{1-\frac{q^{3k+\frac34}(1-q^{k+\frac14})(1-q^{k+\frac34})(1-q^{3k+\frac74})}{(1+q^{k+\frac34})(1-q^{2k+1})^2(1-q^{3k+\frac34})}\right\}$$

which serves as a q-analogue of the series

$$\frac{64\sqrt{\pi}}{\Gamma^2(\frac34)}=\sum_{k=0}^{\infty}\left[\begin{matrix}\frac14,\frac34,\frac34,\frac34\\ 1,1,\ \frac32,\frac32\end{matrix}\right]_k\frac{(12k+5)(28k+15)}{64^k}.$$

3. Triplicate Inverse Series Relations

For all $n \in \mathbb{N}_0$, we have the two equalities

$$n = \left\lfloor \tfrac{1+n}{3} \right\rfloor + \left\lfloor \tfrac{1+2n}{3} \right\rfloor = \left\lfloor \tfrac{n}{3} \right\rfloor + \left\lfloor \tfrac{1+n}{3} \right\rfloor + \left\lfloor \tfrac{2+n}{3} \right\rfloor. \tag{3.1}$$

Then six dual relations can be established from (1.5). However, only two of them give some interesting q-series identities. Five examples are illustrated in this section without reproducing the whole inversion procedure.

3.1. **First version.** Starting from the following form of the q-Pfaff –Saalschütz theorem (1.5)

$${}_3\phi_2\left[\begin{matrix} q^{-n}, \ a, \ c \\ q^{-\lfloor\frac{1+n}{3}\rfloor}ae, q^{1-\lfloor\frac{2n+1}{3}\rfloor}c/e \end{matrix}\Big| q;q\right] = \left[\begin{matrix} q^{-\lfloor\frac{1+n}{3}\rfloor}e, q^{-\lfloor\frac{1+n}{3}\rfloor}ae/c \\ q^{-\lfloor\frac{1+n}{3}\rfloor}ae, q^{-\lfloor\frac{1+n}{3}\rfloor}e/c \end{matrix}\Big| q\right]_n$$

we can derive three q-series identities corresponding to the classical series of convergence rate 4/27.

D1. For $a = q^{1/3}$ and $c = e = q^{2/3}$, we have the corresponding identity

$$\begin{aligned}&\frac{1}{\Gamma_q(\frac{1}{3})\Gamma_q(\frac{2}{3})}\\ =&\sum_{k=0}^{\infty}\frac{q^{2k^2+k}}{1-q}\frac{\left[q^{\frac{1}{3}},q^{\frac{2}{3}};q\right]_k\left[q^{\frac{1}{3}},q^{\frac{2}{3}};q\right]_{2k+1}}{(q;q)_k(q;q)_{2k}(q;q)_{3k+1}}\\ &\times\left\{1-\frac{(1-q^{-k})(1-q^{3k+1})}{(1-q^{2k+\frac{1}{3}})(1-q^{2k+\frac{2}{3}})}+\frac{q^{2k+1}(1-q^{k+\frac{1}{3}})(1-q^{k+\frac{2}{3}})}{(1-q^{2k+1})(1-q^{3k+2})}\right\}\end{aligned}$$

which gives a q-analogue of the classical series

$$\frac{81\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty}\left(\frac{4}{27}\right)^k\left[\begin{matrix}\frac{1}{3},\frac{2}{3},\frac{1}{6},\frac{5}{6}\\ 1,1,1,\ \frac{3}{2}\end{matrix}\right]_k\left\{20+243k+414k^2\right\}.$$

D2. For $a = c = q$ and $e = q^{1/3}$, we get the corresponding identity

$$\begin{aligned}&\Gamma_q\left(\frac{4}{3}\right)\Gamma_q\left(\frac{5}{3}\right)\\ =&\sum_{k=0}^{\infty}\frac{q^{(k+1)(2k+2/3)}}{1-q}\frac{(q^{\frac{2}{3}};q)_k^2(q^{\frac{1}{3}};q)_{2k+1}^2}{(q^{\frac{5}{3}};q)_k(q^{\frac{4}{3}};q)_{2k}(q;q)_{3k+1}}\\ &\times\left\{1-\frac{(1-q^{-k-\frac{2}{3}})(1-q^{3k+1})}{(1-q^{2k+\frac{1}{3}})^2}+\frac{q^{2k+\frac{4}{3}}(1-q^{k+\frac{2}{3}})^2}{(1-q^{2k+\frac{4}{3}})(1-q^{3k+2})}\right\}\end{aligned}$$

which is a q-analogue of the following series

$$8\pi\sqrt{3} = \sum_{k=0}^{\infty}\left(\frac{4}{27}\right)^k\left[\begin{matrix}\frac{2}{3},\frac{2}{3},\frac{1}{6},\frac{1}{6}\\ 1,\frac{4}{3},\frac{5}{3},\frac{7}{6}\end{matrix}\right]_k\left\{43+246k+414k^2\right\}.$$

D3. For $a=c=q$ and $e=q^{2/3}$, we find the corresponding identity

$$\Gamma_q\Big(\frac{4}{3}\Big)\Gamma_q\Big(\frac{5}{3}\Big) =\sum_{k=0}^{\infty}\frac{q^{(k+1)(2k+1/3)}}{1-q}\frac{(q^{\frac13};q)_k^2(q^{\frac23};q)_{2k+1}^2}{(q^{\frac43};q)_k(q^{\frac53};q)_{2k}(q;q)_{3k+1}} \times\Big\{1-\frac{(1-q^{-k-\frac13})(1-q^{3k+1})}{(1-q^{2k+\frac23})^2}+\frac{q^{2k+\frac53}(1-q^{k+\frac13})^2}{(1-q^{2k+\frac53})(1-q^{3k+2})}\Big\}$$

which results in a q-analogue of the classical series

$$40\pi\sqrt{3}=\sum_{k=0}^{\infty}\Big(\frac{4}{27}\Big)^k\left[\begin{matrix}\frac13,\frac13,\frac56,\frac56\\1,\frac43,\frac53,\frac{11}{6}\end{matrix}\right]_k\big\{214+591k+414k^2\big\}.$$

3.2. **Second version.** Rewriting the q-Pfaff–Saalschütz theorem (1.5) as

$${}_3\phi_2\left[\begin{matrix}q^{-n},q^{\lfloor\frac{n}{3}\rfloor}a,q^{\lfloor\frac{1+n}{3}\rfloor}c\\ae,q^{1-\lfloor\frac{2+n}{3}\rfloor}c/e\end{matrix}\Big|q;q\right]=\left[\begin{matrix}q^{-\lfloor\frac{n}{3}\rfloor}e,q^{-\lfloor\frac{1+n}{3}\rfloor}ae/c\\ae,\quad q^{-\lfloor\frac{2n}{3}\rfloor}e/c\end{matrix}\Big|q\right]_n$$

we obtain two further q-series identities.

D4. For $a=q^{1/3}$ and $c=e=q^{2/3}$, the corresponding identity reads as

$$\frac{1}{\Gamma_q(\frac13)\Gamma_q(\frac23)}=\sum_{k=0}^{\infty}\frac{1-q^{4k+\frac53}}{1-q}\frac{(q^{\frac13};q)_k^2(q^{\frac23};q)_k^2\left[q^{\frac13},q^{\frac23};q\right]_{2k+1}}{(q;q)_{2k}(q;q)_{3k+1}^2}q^{5k^2+2k} \times\Big\{1-\frac{(1-q^{-2k})(1-q^{3k+1})^2}{(1-q^{2k+\frac13})(1-q^{2k+\frac23})(1-q^{4k+\frac53})} -q^{4k+\frac43}\frac{(1-q^{2k+\frac53})(1-q^{k+\frac23})^2(1-q^{4k+\frac73})}{(1-q^{2k+1})(1-q^{3k+2})^2(1-q^{4k+\frac53})}\Big\}$$

which provides a q-analogue of the series

$$\frac{729\sqrt{3}}{4\pi}=\sum_{k=0}^{\infty}\Big(\frac{4}{729}\Big)^k\left[\begin{matrix}\frac13,\frac23,\frac16,\frac56\\1,1,1,\frac32\end{matrix}\right]_k\big\{100+1521k+2610k^2\big\}.$$

D5. For $a=c=q$ and $e=q^{1/2}$, the corresponding identity can be stated as

$$\Gamma_q^2\Big(\frac32\Big)=\sum_{k=0}^{\infty}\frac{q^{5k^2+\frac{3k}{2}}}{1+q^{\frac12}}\frac{(q^{\frac12};q)_k^2(q;q)_k^2(q^{\frac12};q)_{2k}}{(q^{\frac32};q)_{3k}(q;q)_{3k}} \times\Big\{1+q^{2k+\frac12}\frac{(1-q^{2k+\frac12})(1-q^{4k+2})}{(1-q^{3k+1})(1-q^{3k+\frac32})} -q^{6k+\frac52}\frac{(1-q^{k+\frac12})(1-q^{k+1})(1-q^{2k+\frac12})(1-q^{4k+3})}{(1-q^{3k+1})(1-q^{3k+\frac32})(1-q^{3k+2})(1-q^{3k+\frac52})}\Big\}. \tag{3.2}$$

By carrying out the same procedure as done in the case of C2, we can show that series in the right-hand side of (3.2) is, in fact, the *bisection series* of the following one

$$\Gamma_q^2\Big(\frac{1}{2}\Big)=\sum_{k=0}^{\infty}q^{\frac{k}{4}(3+5k)}\frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})_k^2(q^{\frac{1}{2}};q)_k}{(q^{\frac{3}{2}};q^{\frac{1}{2}})_{3k}}\frac{1+q^{\frac{1}{2}+k}-q^{1+\frac{3k}{2}}-q^{1+2k}}{1-q^{\frac{1}{2}}}.$$

This is in turn the q-analogue of the classical series (cf. Zhang [25, Example 8]):

$$\pi=\sum_{k=0}^{\infty}\Big(\frac{2}{27}\Big)^k\begin{bmatrix}1,\ \frac{1}{2}\\ \frac{4}{3},\ \frac{5}{3}\end{bmatrix}_k(3+5k)=\sum_{k=0}^{\infty}\frac{6+10k}{2^k\binom{3k+2}{k+1}(k+1)(2k+1)}.$$

4. Conclusive Comments

We have shown that the inversion technique is efficient for obtaining q-series identities whose limiting cases result in interesting infinite series for π. The examples presented in this paper are far from exhaustive. For instance, if we start with the quadruplicate form of the q-Pfaff–Saalschütz theorem (1.5)

$${}_3\phi_2\left[\begin{matrix}q^{-n},q^{\lfloor\frac{1+n}{4}\rfloor}a,q^{\lfloor\frac{3+n}{4}\rfloor}c\\ ae,q^{1-\lfloor\frac{n}{2}\rfloor}c/e\end{matrix}\Big|q;q\right]=\left[\begin{matrix}q^{-\lfloor\frac{1+n}{4}\rfloor}e,q^{-\lfloor\frac{3+n}{4}\rfloor}ae/c\\ ae,\quad q^{-\lfloor\frac{1+n}{2}\rfloor}e/c\end{matrix}\Big|q\right]_n,$$

then its dual series will give rise to the *bisection series* of the following q-series

$$\text{(4.1)}\quad \begin{aligned}\frac{1}{\Gamma_q(\frac{1}{4})\Gamma_q(\frac{3}{4})}&=\sum_{k=0}^{\infty}(-1)^k\frac{(q^{\frac{1}{4}};q^{\frac{1}{2}})_k^2(q^{\frac{1}{4}};q^{\frac{1}{2}})_{3k}}{(q;q)_k(q;q)_{2k}^2}q^{\frac{7}{4}k^2}\\&\times\left\{\frac{1-q^{\frac{1}{4}+\frac{5k}{2}}}{1-q}-\frac{q^{\frac{3}{4}+\frac{5k}{2}}(1-q^{\frac{1}{4}+\frac{3k}{2}})}{(1-q)(1+q^{\frac{1}{4}+\frac{k}{2}})^2(1+q^{\frac{1}{2}+k})^2}\right\}\end{aligned}$$

which turns out to be a q-analogue of the elegant series for $\sqrt{2}/\pi$ with convergence rate $-27/512$ discovered by Guillera [14]:

$$\frac{32\sqrt{2}}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{-3}{8}\Big)^{3k}\begin{bmatrix}\frac{1}{2},\ \frac{1}{6},\ \frac{5}{6}\\ 1,\ 1,\ 1\end{bmatrix}_k\big\{15+154k\big\}.$$

We remark that the fractions in the braces of (4.1) is slightly simpler than that obtained recently by Guillera [17] through a totally different approach – “the WZ-method”.

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