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ON COLOURING ORIENTED GRAPHS OF LARGE GIRTH

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ABSTRACT. We prove that for every oriented graph D and every choice of positive integers k and ℓ, there exists an oriented graph D^* along with a surjective homomorphism $\psi\colon V(D^*) \to V(D)$ such that: (i) $\mathrm{girth}(D^*) \geq \ell$; (ii) for every oriented graph C with at most k vertices, there exists a homomorphism from D^* to C if and only if there exists a homomorphism from D to C; and (iii) for every D-pointed oriented graph C with at most k vertices and for every homomorphism $\varphi\colon V(D^*) \to V(C)$ there exists a unique homomorphism $f\colon V(D) \to V(C)$ such that $\varphi = f \circ \psi$. Determining the oriented chromatic number of an oriented graph D is equivalent to finding the smallest integer k such that D admits a homomorphism to an order-k tournament, so our main theorem yields results on the girth and oriented chromatic number of oriented graphs. While our main proof is probabilistic (hence nonconstructive), for any given $\ell \geq 3$ and $k \geq 5$, we include a construction of an oriented graph with girth ℓ and oriented chromatic number k.

1. Introduction

In 1959, Paul Erdős [4] famously proved probabilistically the existence of graphs of arbitrarily large girth and arbitrarily large chromatic number. We briefly discuss the history of this and related topics and point the reader to [5] or [6] for more details and references. As [4] and many of its descendants give strictly nonconstructive proofs, one is led to seek constructions. Other natural directions of inquiry are (1) generalizing Erdős' result and (2) developing analogues of his results for other types of graphs, specifically of interest here, directed graphs.

Both refinements and generalizations of [4] have followed in the intervening six-plus decades. In 1976, Bollobás and Sauer [2] refined Erdős' result

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by showing that for any positive integer n there are graphs of arbitrarily large girth that are 'uniquely' n-colourable. In 1996, Zhu [13], working with graph homomorphisms as a generalization of colouring, was able to carry forward the work of [2] by showing that for any 'core' H, there are uniquely H-colourable graphs of arbitrarily large girth. As complete graphs are cores, Zhu's work generalizes [4]. Zhu's main result in [13] was further generalized by Nešetřil and Zhu [9] to the notion of 'pointed' graphs. We follow a similar trajectory in the present paper.

So let us shift our attention to digraphs. Bokal et al. [1] studied the digraph circular chromatic number and showed that digraph colouring theory is similar to that of undirected graphs. For undirected graphs G, the circular chromatic number $\chi_c(G)$ is a refinement of the chromatic number $\chi(G)$ because $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ (see, e.g., [14]). Analogously, the circular chromatic number $\chi_c(D)$ of a digraph refines the chromatic number $\chi(D)$, here defined to be the minimum integer k such that $V(D)$ can be partitioned into k acyclic subsets. The former parameter is defined using 'acyclic' homomorphisms—see [1] for details—which introduced complications. For example, the authors of [6] had to use a lot of care to demonstrate that certain mappings don't fail to be acyclic homomorphisms. The fact that we consider oriented colouring here means we have no need to turn to acyclic homomorphisms. The reader might appreciate how much this simplifies our proofs in comparison with those of [6].

Subsequently to [1], a subset of the authors and their doctoral students in [5] completed work in the realm of digraphs analogous to that of Zhu for graphs in [13]. Then [6] generalized the results of [1, 5] just as Nešetřil and Zhu in [9] generalized [4, 13]. One of our successes in the present work is a similar sequence of generalizations for oriented graphs.

We delay definitions for a little longer (until Section 2) and proceed to state our main result and a couple of its consequences:

Theorem 1.1. *For every oriented graph D and every choice of positive integers k and ℓ, there exists an oriented graph D^* along with a surjective homomorphism $\psi\colon V(D^*) \to V(D)$ such that:*

- *(i) girth$(D^*) \geq \ell$;*
- *(ii) for every oriented graph C with at most k vertices, there exists a homomorphism from D^* to C if and only if there exists a homomorphism from D to C; and*
- *(iii) for every D-pointed oriented graph C with at most k vertices and for every homomorphism $\varphi\colon V(D^*) \to V(C)$ there exists a unique homomorphism $f\colon V(D) \to V(C)$ such that $\varphi = f \circ \psi$.*

An attentive reader familiar with [6] may be concerned that our Theorem 1.1 is an immediate consequence of [6, Theorem 1]. After all, the earlier result applies to digraphs in general, and oriented graphs are a specific type of digraph. Furthermore, oriented colourings are homomorphisms from oriented graphs to oriented graphs, so in particular they are acyclic homomorphisms. We can see this because preimages under a homomorphism must be independent sets, and are hence acyclic. The important difference is that in our Theorem 1.1, we are able to get an *oriented* D^* and an *oriented colouring* ψ, whereas [6] guarantees only a *digraph* D^* and an *acyclic homomorphism* ψ. Although D^* of [6] will in fact be an oriented graph when D is an oriented graph, one can readily check that the acyclic homomorphism ψ of [6] in general will not be an oriented colouring, so the earlier results do not guarantee the desired results for oriented graphs. The importance of this distinction becomes clear as we discuss two consequences of Theorem 1.1, which we now state.

Corollary 1.2. *If D and C are oriented graphs such that D is not C-colourable, then for every positive integer ℓ, there exists an oriented graph D^* of girth at least ℓ that is D-colourable but not C-colourable.*

Corollary 1.3. *For every oriented core D and every positive integer ℓ, there is an oriented graph D^* of girth at least ℓ that is uniquely D-colourable.*

To see that Theorem 1.1 implies Corollary 1.2, if we have D and C as in Corollary 1.2 with a given integer ℓ and take k to be the order of C, then (i) of Theorem 1.1 gives us a D^* of required girth such that $\psi\colon V(D^*) \to V(D)$, so D^* is D-colourable. But as D is not C-colourable, condition (ii) of Theorem 1.1 implies that D^* is not C-colourable.

To see that Theorem 1.1 implies Corollary 1.3 follows a similar argument as in [6]. We note that cores D are D-pointed. So if we are given a positive integer ℓ and a core D, we can take $k = |V(D)|$. Then Theorem 1.1 gives a D^* of girth at least ℓ and a D-colouring $\psi\colon V(D^*) \to V(D)$. We can set $C = D$ in part (iii) of Theorem 1.1, which gives us that for every D-colouring $\varphi\colon V(D^*) \to V(D)$ there is a (unique) homomorphism $f\colon V(D) \to V(D)$ such that $\varphi = f \circ \psi$. Because D is a core, f is an automorphism, so φ and ψ differ by this automorphism and D^* is indeed uniquely D-colourable.

2. Terminology and notation

We assume basic familiarity with graphs and digraphs and refer the reader to [3] for any omitted concepts. Here we consider oriented graphs and oriented colourings going forward unless indicated otherwise. An *oriented graph* D is a digraph in which for every pair of vertices u, v, at most one of uv and vu is an element of $A(D)$, the *arc set* of D. Our oriented graphs will always be finite and loopless without multiple arcs; opposite arcs are precluded by the definition of oriented graphs. It can be easier to think about an oriented graph as one obtained by assigning directions to each edge of some (undirected) graph G. Recall that a *tournament* D on n vertices is an oriented graph obtained by assigning a direction to each edge of the complete graph K_n. *Cycles* of oriented graphs are directed cycles, and the *girth* of an oriented graph D is the length of a shortest directed cycle in

D. Finally, for oriented graphs D and C, an *oriented graph homomorphism* is a map $f\colon V(D) \to V(C)$ such that whenever $xy \in A(D)$, we also have $f(x)f(y) \in A(C)$.

We are now ready to define an 'oriented colouring' of an oriented graph D. An *oriented k-colouring*, then, is a map $c\colon V(D) \to \{1, \ldots, k\}$ such that:

(1) $c(x) \neq c(y)$ for every arc $xy \in A(D)$, and
(2) $c(u) \neq c(y)$ for every two arcs $uv \in A(D)$ and $xy \in A(D)$ with $c(v) = c(x)$.

This is by now a standard definition; see, e.g., [12].

This definition of an oriented colouring is equivalent to that of a homomorphism to a tournament on k vertices. First, it is clear that a homomorphism to a tournament satisfies condition (1) of being an oriented colouring because it is a homomorphism, and condition (2) is satisfied because tournaments have no opposite arcs. On the other hand, given such a map c, we can construct an oriented graph C^* with $V(C^*) = \{1, \ldots, k\}$ and $A(C^*) = \{xy\colon x, y \in V(C^*) \text{ and } xy = c(a)c(b) \text{ for some } ab \in A(D)\}$. Then it is clear that C^* is a subgraph of a tournament C' on k vertices by property (2) of c. Furthermore, C^* was constructed so that c is a homomorphism to C^* and thus a homomorphism to C', so c is a homomorphism to a tournament on k vertices. We always consider oriented colourings to be homomorphisms to tournaments.

Having defined an oriented colouring, we now give the related definition of 'oriented chromatic number.' Given an oriented graph D, its *oriented chromatic number* $\chi_o(D)$ is the minimum number of vertices of an oriented graph C such that there exists a homomorphism of D to C. As C is always a subgraph of some tournament T, we will always consider the oriented chromatic number of D as the minimum number of vertices of a tournament T such that there exists a homomorphism of D to T.

For terminology more directly related to our theorem statements, we say that a homomorphism of oriented graphs of D to C is a *C-colouring of D*, and we say that D is *C-colourable*. We say that D is *uniquely C-colourable* if there is a homomorphism of D onto C, and for any two C-colourings ψ and φ of D, these homomorphisms 'differ by an automorphism'. That is, there is some $f \in \mathrm{Aut}(C)$ such that $\psi = f \circ \varphi$. For an oriented graph D, we say that D is a *core* if every homomorphism $f\colon V(D) \to V(D)$ is an automorphism. Finally, we say that for oriented graphs C and D, the digraph C is *D-pointed* if there do not exist two distinct C-colourings of D that agree on all but one vertex of D.

3. Setup for the proof of Theorem 1.1

For a given oriented graph D, we begin the 'construction' of the digraph D^*, and we do so by first constructing a digraph D_0, again inspired by [6].

We define $V(D_0) = V_1 \cup V_2 \cup \cdots \cup V_a$ where $V(D) = \{1, 2, \ldots, a\}$, and each $|V_i| = n$ for some fixed n large enough to satisfy necessary probabilistic inequalities. Then we define the arc set $A(D_0) = \{xy : x \in V_i, y \in V_j \text{ and } ij \in A(D)\}$. We can view each V_i simply as the preimage of a vertex $i \in V(D)$ under the natural homomorphism $\psi\colon V(D_0) \to V(D)$, mapping each V_i to i, for $i \in \{1, \ldots, a\}$.

Now we use D_0 to 'construct' an oriented graph D^* probabilistically. First we fix an ε with $0 < \varepsilon < 1/(4\ell)$ where ℓ is chosen as in the statement of Theorem 1.1. Then our random oriented graph model $\mathcal{D}(n, p)$ consists of spanning subgraphs of D_0 where arcs are chosen randomly and independently with probability $p = n^{\varepsilon - 1}$ with n sufficiently large. We now introduce three lemmas from [6].

Lemma 3.1. *(i) The expected number of cycles of length less than ℓ in a digraph $\hat{D} \in \mathcal{D}(n, p)$ is bounded above by $n^{\varepsilon\ell} n^{-\varepsilon/2}$;*
(ii) the expected number of pairs of cycles of length less than ℓ in a digraph $\hat{D} \in \mathcal{D}(n, p)$ which intersect in at least one vertex is bounded above by $n^{-1/2}$.

This is Lemma 5 of [6], except that our oriented graph model $\mathcal{D}(n, p)$ differs. In particular, our D_0 has fewer arcs than the analogue in [6], so the lemma remains true in our case. This along with Markov's Inequality shows that asymptotically almost all oriented graphs in $\mathcal{D}(n, p)$ have at most $n^{\varepsilon\ell}$ cycles of length less than ℓ which are pairwise vertex-disjoint; see, e.g., [6].

We introduce some definitions from [6] (which itself adopted these from [9]), first calling a set $\mathcal{A} \subseteq V(D_0)$ *large* if there are distinct $i, j \in \{1, \ldots, a\}$ with $ij \in A(D)$ such that $|\mathcal{A} \cap V_i| \geq n/k$ and $|\mathcal{A} \cap V_j| \geq n/k$, and calling $ij \in A(D)$ in this case a *good arc* for $\mathcal{A}$. Then given a large $\mathcal{A}$, we denote by $|\hat{D}/\mathcal{A}|$ the minimum number of arcs of a random $\hat{D}$ which lie in the set $\{xy\colon x \in \mathcal{A} \cap V_i, y \in \mathcal{A} \cap V_j\}$, taken over all instances in which ij is a good arc. Then we have:

Lemma 3.2 ([6])**.** *If $\hat{D} \in \mathcal{D}(n, p)$ and $\mathcal{A}$ is large, then $P(|\hat{D}/\mathcal{A}| \geq n) = 1 - o(1)$.*

Again the space $\mathcal{D}(n, p)$ in [6] differs from ours, but the proof still follows through unchanged because the arcs counted in $|\hat{D}/\mathcal{A}|$ in [6] are all present in the current model.

We shall need to adopt one last lemma from [6], and its validity here follows using similar arguments to those for Lemmas 3.1 and 3.2.

Lemma 3.3 ([6])**.** *For almost all digraphs in $\mathcal{D}(n, p)$, all nonempty $\mathcal{A} \subseteq V_v$ and $\mathcal{B} \subseteq V_{i_0}$ (for $v, i_0 \in \{1, \ldots, a\}$ with $vi_0 \in A(D)$) with $|\mathcal{A}| = n - |\mathcal{B}|(k-1)$ and $|\mathcal{B}| \leq n/k$ satisfy the property of $\mathcal{A} \cup \mathcal{B}$ inducing more than min$\{|\mathcal{B}|, n^{\epsilon\ell}\}$ arcs from $\mathcal{A}$ to $\mathcal{B}$.*

Now we can move on to the proof of our main theorem.

4. Proof of Theorem 1.1

Lemma 3.1 and its consequences mean that asymptotically almost all $D' \in \mathcal{D}(n,p)$ have at most $n^{\varepsilon\ell}$ pairwise-disjoint cycles of length less than ℓ. Similarly, Lemma 3.2 guarantees that asymptotically almost all $D' \in \mathcal{D}(n,p)$ have the property that all good arcs of D for large sets $\mathcal{A}$ induce at least n arcs of D'. Finally Lemma 3.3 guarantees the existence of necessary arcs as described later for almost all D'. Therefore, there exists some $D' \in \mathcal{D}(n,p)$ enjoying the three stated properties, and we select such a D'. Now we pick one arc from each of the at most $n^{\varepsilon\ell}$ cycles of length less than ℓ in D', giving an independent arc set (i.e., a matching) M, and define $D^* = D' - M = (V(D_0), A(D') \setminus M)$. It is clear then that D^* has girth at least ℓ, and that $\psi\colon V(D^*) \to V(D)$ defined by $\psi(x) = i$ if and only if $x \in V_i$ gives a surjective homomorphism, yielding (i) from Theorem 1.1. Note that since $\varepsilon < 1/(4\ell)$, the deleted arc set satisfies $|M| \le n^{\varepsilon\ell} < n^{1/4}$.

Now we work toward (ii) from Theorem 1.1. Let us fix an oriented graph C of order at most k, and assume that there is a homomorphism $\varphi\colon V(D^*) \to V(C)$. Then the Pigeonhole Principle implies that for every $i \in V(D)$ there is a vertex $x \in V(C)$ such that $|V_i \cap \varphi^{-1}(x)| \ge n/k$. Then let us define $f\colon V(D) \to V(C)$ by $f(i) = x$ for some $x \in V(C)$ such that $|V_i \cap \varphi^{-1}(x)| \ge n/k$. We must show this f is a homomorphism.

Let $ij \in A(D)$ and consider all possible $a, b \in V(D^*)$ where $a \in V_i \cap \varphi^{-1}(f(i))$ and $b \in V_j \cap \varphi^{-1}(f(j))$. If there is one such arc $ab \in A(D^*)$, this will guarantee the existence of an arc $f(i)f(j) \in A(C)$ by the existence of φ. Recall that f satisfies $|V_i \cap \varphi^{-1}(f(i))| \ge n/k$ and $|V_j \cap \varphi^{-1}(f(j))| \ge n/k$. Then $\mathcal{A} = \big(V_i \cap \varphi^{-1}(f(i))\big) \cup \big(V_j \cap \varphi^{-1}(f(j))\big)$ is large as defined for Lemma 3.2, so by our choice of D' relying on that lemma, D' has at least n arcs with endpoints in $\mathcal{A}$. Then since we have removed at most $n^{1/4}$ arcs from D' to construct D^*, there exists at least one such arc $ab \in A(D^*)$, and in fact many such arcs. So we have $\varphi(a)\varphi(b) \in A(C)$, and we have that $f(i) = \varphi(a)$ and $f(j) = \varphi(b)$ with $f(i) \neq f(j)$ because φ is a homomorphism. So $f(i)f(j) \in A(C)$, and f maps arcs to arcs and is thus a homomorphism.

Conversely, if we assume that there is a homomorphism $f\colon V(D) \to V(C)$, then we get the homomorphism $\varphi\colon V(D^*) \to V(C)$ by $\varphi = f \circ \psi$, completing our proof of (ii).

Now we turn to (iii), letting C be a D-pointed oriented graph of order at most k and $\varphi\colon V(D^*) \to V(C)$ be a homomorphism. We shall use $f\colon V(D) \to V(C)$ as in the proof of (ii). The D-pointedness of C forces for every $i \in V(D)$ the existence of a unique $x_i \in V(C)$ such that $|\varphi^{-1}(x_i) \cap V_i| \ge n/k$. We demonstrate this using an argument similar to that in [6]. If some x_i were not unique and x_i' also satisfies $|\varphi^{-1}(x_i') \cap V_i| \ge n/k$, then we could define f' by

$$f'(j) = \begin{cases} f(j) & \text{for } j \neq i \\ x_i' & \text{for } j = i, \end{cases}$$

giving another homomorphism differing at one vertex of D and contradicting the D-pointedness of C. This establishes the uniqueness of a homomorphism f chosen in this way. If we assume that $\varphi \neq f \circ \psi$, then there must be some vertex $z \in V(D^*)$ such that $\varphi(z) \neq (f \circ \psi)(z)$. So if $z \in V_j$, then $\varphi(z) \neq (f \circ \psi)(z) = f(j)$. Thus $V_j \setminus \big(\varphi^{-1}(f(j)) \cap V_j\big) \neq \varnothing$ (as it contains z), which leads to a contradiction as we proceed to show.

We begin by choosing a vertex $i_0 \in \{1, \ldots, a\}$ so that $t := |\varphi^{-1}(f(i_0)) \cap V_{i_0}|$ is minimized; the definition of f gives $t \geq n/k$ while the purported z of the preceding paragraph gives $t < n$. The last inequality shows that $\varphi^{-1}(f(i_0)) \cap V_{i_0}$ is a proper subset of V_{i_0}. Let us now choose a vertex $x \in V(C)$, distinct from $f(i_0)$, so as to maximize the size of the set $\mathcal{B} := \varphi^{-1}(x) \cap V_{i_0}$. Denoting this size by $b = |\mathcal{B}|$, we see that $b < n/k$ by the previously established uniqueness property of f (exactly one vertex of $V(C)$ satisfies '$\geq n/k$' here, and $f(i_0) \neq x$ is already that witness). Notice also that these new parameters satisfy

$$b(k-1) \geq n - t \tag{4.1}$$

because the (at most) $(k-1)$ preimages $\varphi^{-1}(y) \cap V_{i_0}$ within V_{i_0} (as y runs through $V(C) \setminus \{f(i_0)\}$) exhaust the $(n-t)$ vertices within V_{i_0} that are not mapped to $f(i_0)$ by φ.

Now we define $f' \colon V(D) \to V(C)$ by:

$$f'(i) = \begin{cases} f(i) & \text{for } i \neq i_0 \\ x & \text{for } i = i_0. \end{cases}$$

Because f and f' differ only at i_0 and C is D-pointed, f' is not a homomorphism. Thus, it fails to send arcs to arcs. So it must be for some $v \in V(D)$, distinct from i_0, either $vi_0 \in A(D)$ and $f(v)x \notin A(C)$ or $i_0v \in A(D)$ and $xf(v) \notin A(C)$. Without loss of generality, we assume that $vi_0 \in A(D)$ and

$$f(v)x \notin A(C). \tag{4.2}$$

With v being among the candidate vertices $1, \ldots, a$ during our choice of i_0, we have $|\varphi^{-1}(f(v)) \cap V_v| \geq t$, and (4.1) shows that $t \geq n - b(k-1)$; therefore, we can select a subset $\mathcal{A} \subseteq \varphi^{-1}(f(v)) \cap V_v$ with $|\mathcal{A}| = n - b(k-1)$.

Because we chose a digraph D' satisfying the likely properties articulated in Lemma 3.3, we know that there are more than $\min\{|\mathcal{B}|, n^{\epsilon\ell}\}$ arcs from $\mathcal{A}$ to $\mathcal{B}$ in D'. And because the arcs removed from D' to form D^* comprised a matching of size at most $n^{\epsilon\ell}$, no matter which entry achieves $\min\{|\mathcal{B}|, n^{\epsilon\ell}\}$, there exist vertices $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $ab \in A(D^*)$. Lastly, because φ is a homomorphism, we have $\varphi(a)\varphi(b) \in A(C)$. But $\varphi(a) = f(v)$ and $\varphi(b) = x$, so that $f(v)x \in A(C)$, which contradicts (4.2). Therefore, our assumption of the existence of a vertex $z \in V(D^*)$ such that $\varphi(z) \neq (f \circ \psi)(z)$ is incorrect, and we must have $\varphi = f \circ \psi$. Finally, we note that the surjectivity of ψ implies that such a homomorphism f is unique. □

5. Constructions

Our last natural direction of exploration from Erdős' original theorem is that of actually constructing those graphs which we have probabilistically proven to exist. These constructions are generally challenging and delicate. The common approach is to proceed by induction, constructing a (di)graph of chromatic number $n+1$ with girth ℓ using copies of a (di)graph of chromatic number n with girth ℓ. The first such construction was completed by Lovász [8] in 1968 using hypergraphs intermediately. It was not until 1989 that Křìž [7] was able to create a purely graph-theoretic construction of highly chromatic graphs without short cycles. Similarly, Severino [10] demonstrated constructions of highly chromatic digraphs without short cycles and in [11] constructed uniquely n-colourable digraphs with arbitrarily large girth.

Ideally, we would like to construct the digraph D^* with all the properties described in Theorem 1.1. We shall content ourselves with a construction of an oriented graph of a given girth and oriented chromatic number and leave the construction of such a D^* for future authors.

Theorem 5.1. *For integers $k \geq 5$ and $\ell \geq 3$, there exists an oriented graph D with oriented chromatic number k and girth ℓ.*

Remark: Some instances of (k, ℓ) with $k = 3$ or $k = 4$ are also feasible. In particular, $k = 3$ is feasible for $\ell \equiv 0 \pmod 3$, and $k = 4$ is feasible for all $\ell \geq 3$ with $\ell \neq 5$. However, we state Theorem 5.1 as such because when $\ell = 5$, of necessity our basis starts at $k = 5$. Readers may find it illustrative to convince themselves that the directed 5-cycle admits no homomorphism to a tournament on four vertices, while a directed cycle of any other order admits such a homomorphism.

Proof. We follow the common approach to which we alluded above and proceed by induction on k, so let us fix integers k and ℓ. Then we begin by considering $\overrightarrow{C_\ell}$, an oriented cycle of length ℓ (and girth ℓ). We define $V(\overrightarrow{C_\ell}) = \{v_0, v_1, \ldots, v_{\ell-1}\}$, and there is a homomorphism $c\colon V(\overrightarrow{C_\ell}) \to V(T_5)$ where $V(T_5) = \{t_0, t_1, t_2, t_3, t_4\}$ and $\{t_0t_1, t_1t_2, t_2t_3, t_3t_4, t_2t_0, t_3t_0, t_4t_0\} \subseteq A(T_5)$. Then if $\ell \equiv 0 \pmod 3$, we have $c\colon V(\overrightarrow{C_\ell}) \to V(T_5)$ defined by

$$c(v_r) = t_{r \bmod 3}.$$

If $\ell \equiv 1 \pmod 3$, then c is defined by

$$c(v_r) = \begin{cases} t_{r \bmod 3} & \text{for } r < \ell - 1 \\ t_3 & \text{for } r = \ell - 1. \end{cases}$$

And finally, if $\ell \equiv 2 \pmod 3$, then c is defined by

$$c(v_r) = \begin{cases} t_{r \bmod 3} & \text{for } r < \ell - 2 \\ t_3 & \text{for } r = \ell - 2 \\ t_4 & \text{for } r = \ell - 1. \end{cases}$$

We note that our base cases have given us an oriented graph of girth ℓ with oriented chromatic number $k \leq 5$. The verification of our induction below will then guarantee the existence of an oriented graph of girth ℓ with any given oriented chromatic number $k \geq 5$.

Having established our base cases, we now proceed with the induction. So assume we have an oriented graph D_k of girth ℓ, oriented chromatic number k, and order m, and then define $V(D_k) = \{v_0, v_1, \ldots, v_{m-1}\}$. Because D_k has oriented chromatic number k, there exists a tournament T_k with $V(T_k) = \{t_0, t_1, \ldots, t_{k-1}\}$ and a homomorphism $\varphi_k \colon V(D_k) \to V(T_k)$. Now we construct D_{k+1} and the corresponding T_{k+1}. Define the vertex set $V(D_{k+1}) = V(D_k) \cup \{v_m\}$, and define the arc set $A(D_{k+1}) = A(D_k) \cup \{v_i v_m \colon i \in \{0, 1, \ldots, m-1\}\}$. Then we construct T_{k+1} in exactly the same fashion; i.e., $V(T_{k+1}) = V(T_k) \cup \{t_k\}$ and $A(T_{k+1}) = A(T_k) \cup \{t_i t_k \colon i \in \{0, 1, \ldots, k-1\}\}$.

We now examine the girth and oriented chromatic number of D_{k+1}. First, it is immediately clear that we have created no new oriented cycles in this construction, so D_{k+1} also has girth ℓ. It is equally clear that we have a homomorphism $\varphi_{k+1} \colon V(D_{k+1}) \to V(T_{k+1})$ defined by

$$\varphi_{k+1}(v) = \begin{cases} \varphi_k(v) & \text{for } v \neq v_m \\ t_k & \text{for } v = v_m. \end{cases}$$

Therefore, $\chi_o(D_{k+1}) \leq k+1$.

To complete the proof, it remains to show that D_{k+1} admits no homomorphism to a tournament on k vertices. Assume to the contrary that for some order-k tournament T'_k the digraph D_{k+1} admits a homomorphism $\psi \colon V(D_{k+1}) \to V(T'_k)$. Let's say that $\psi(v_m) = x \in V(T'_k)$. Then because every vertex $v \in V(D_{k+1}) \setminus \{v_m\}$ forms an arc vv_m, we know that $\psi(v) \neq x$ for every $v \neq v_m$. If we let Λ be the subgraph of D_{k+1} induced by the vertex set $\{v_0, \ldots, v_{m-1}\}$, then Λ is isomorphic to D_k. Similarly, if we let Γ be the subgraph of T'_k induced by $V(T'_k) \setminus \{x\}$, then Γ is a tournament on $k-1$ vertices. But then $\psi|_{V(\Lambda)}$ gives a homomorphism from Λ to Γ, a tournament on $k-1$ vertices, contradicting the fact that D_k has oriented chromatic number k. Therefore, D_{k+1} indeed has oriented chromatic number $k+1$. □

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