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ON SOME PARTITION THEOREMS OF M. V. SUBBARAO

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Abstract. M.V. Subbarao proved that the number of partitions of n in which parts occur with multiplicities 2, 3 and 5 is equal to the number of partitions of n in which parts are congruent to $\pm 2, \pm 3, 6 \pmod{12}$, and generalized this result. In this paper, we give a new generalization of this identity and also present a new partition theorem in the spirit of Subbarao's generalization of the identity.

1. Introduction

A partition of a positive integer n is a representation $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_r$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$ and

$$\sum_{i=1}^{r} \lambda_i = n.$$

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The integer n is called the weight of λ which is denoted by $|\lambda|$ and λ_i's are called parts. Other alternative notations include $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ or $(\mu_1^{m_1}, \mu_2^{m_2}, \ldots)$ where $\mu_1 > \mu_2 > \cdots$ and m_i is the multiplicity of μ_i. The partition $14 + 14 + 10 + 10 + 7 + 7 + 7 + 1 + 1 + 1 + 1$ can be written as $(14, 14, 10, 10, 7, 7, 7, 1, 1, 1, 1)$ or $(14^2, 10^2, 7^3, 1^4)$. The union of two partitions λ and β is simply the multiset union $\lambda \cup \beta$ where λ and β are treated as multisets. For example, $(14^2, 10^2, 7^3, 1^4) \cup (13, 10^3, 6, 1^4) = (14^2, 13, 10^5, 7^3, 6, 1^8)$. The number of all partitions of n is called the (unrestricted) partition function. If restrictions are imposed on the parts of partitions, the corresponding enumerating function is then called a restricted partition function. One such example is the number of partitions of n into distinct parts. It turns out that for a fixed weight, partitions into distinct parts are related to partitions into odd parts. The following theorem demonstrates the relationship.

Theorem 1.1 (Euler)**.** *The number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.*

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Theorem 1.1 was extended to a more general setting. This extension is due to J. W. L Glaisher (see [2]).

Theorem 1.2 (Glaisher). *Let $k > 1$. The number of partitions of n wherein parts are not divisible by k is equal to the number of partitions of n in which parts occur at most $k-1$ times.*

The theorem above has an interesting bijective proof which we recall from the literature. Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_r^{m_r})$ be a partition of n whose parts are not divisible by k. Note that the notation for λ implies $\lambda_1 > \lambda_2 > \cdots$ are parts with multiplicities $m_1, m_2, \ldots$, respectively. Now, write m_i's in k-ary expansion, i.e.

$$m_i = \sum_{j=0}^{l_i} a_{ij} k^j \quad \text{where} \quad 0 \leq a_{ij} \leq k-1.$$

We map $\lambda_i^{m_i}$ to $\bigcup_{j=0}^{l_i} (k^j \lambda_i)^{a_{ij}}$, where now $k^j \lambda_i$ is a part with multiplicity a_{ij}. The image of λ which we shall denote by $\phi(\lambda)$, is given by

$$\bigcup_{i=1}^{r} \bigcup_{j=0}^{l_i} (k^j \lambda_i)^{a_{ij}}.$$

Clearly, this is a partition of n with parts occurring not more than $k-1$ times.

On the other hand, assume that $\mu = (\mu_1^{f_1}, \mu_2^{f_2}, \ldots)$ is a partition of n into parts occurring not more than $k-1$ times. Write $\mu_i = k^{r_i} a_i$ where k does not divide a_i and then map $\mu_i^{f_i}$ to $(a_i)^{k^{r_i} f_i}$ for each i, where now a_i is a part with multiplicity $k^{r_i} f_i$. The inverse of ϕ is then given by

$$\phi^{-1}(\mu) = \bigcup_{i \geq 1} (a_i)^{k^{r_i} f_i}.$$

In the resulting partition, it is also clear that the parts are not divisible by k. M.V Subbarao proved the following theorem.

Theorem 1.3 (Subbarao, [5]). *The number of partitions of n in which parts occur with multiplicities 2, 3 and 5 is equal to the number of partitions of n in which parts are congruent to $\pm 2, \pm 3, 6 \pmod{12}$.*

A generalization of the above theorem was also given as follows.

Theorem 1.4 (Subbarao, [5]). *Let $m > 1, r \geq 0$ be integers, and let $A_{m,r}(n)$ denote the number of partitions of n such that all even multiplicities of the parts are less than $2m$, and all odd multiplicities are at least $2r+1$ and at most $2(m+r)-1$. Let $B_{m,r}(n)$ be the number of partitions of n in which parts are either odd and congruent to $2r+1 \pmod{4r+2}$ or even and not congruent to $0 \pmod{2m}$. Then $A_{m,r}(n) = B_{m,r}(n)$.*

Indeed Theorem 1.3 is a special case of Theorem 1.4 ($m = 2$ and $r = 1$). It is worth mentioning that Theorem 1.3 has been generalized in different directions (see [5], [1] and the references therein). To avoid ambiguity, any partition identity in which one side describes a restriction on the multiplicity of parts and the other side describes parts being in certain residue classes shall be called an identity of *Subbarao type*. The partitions involved will be called Subbarao type partitions.

In this paper, we provide a new generalization of Theorem 1.3 and deduce the parity for a partition function of Subbarao type. This is done in Section 2. In Section 3, we state a new partition theorem in the spirit of Theorem 1.4 and give its bijective proof.

2. Generalization of Theorem 1.3

Our goal in this section is to give a simple extension of Theorem 1.3. We start by formulating the following definition.

Definition 2.1. *Let $k \geq 2$ and $a_1, a_2, \ldots, a_r \geq 1$ be integers. The tuple $(a_1, a_2, \ldots, a_r)$ is called k-admissible if*

i. $\gcd(a_i, a_j) = 1 \ \forall\, i \neq j$;
ii. $\sum\limits_{i=1}^{r} \alpha_i a_i = \sum\limits_{i=1}^{r} \beta_i a_i$ *and* $0 \leq \alpha_i, \beta_i \leq k-1$ *for all* $i = 1, 2, \ldots, r \ \Rightarrow$ $\alpha_i = \beta_i \ \forall\, i = 1, \ldots, r$;
iii. $A_i \cap A_j = \emptyset$ *for* $1 \leq i \neq j \leq r$ *where*

$$A_i = \left\{ x \in \mathbb{Z}_{\geq 1} \ : \ x \equiv -(s+km)a_i \pmod{k \prod_{i=1}^{r} a_i}, \right.$$
$$\left. 1 \leq s \leq k-1, \ \ m = 0, 1, \ldots, \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} a_j \right) - 1 \right\}.$$

For instance, consider the tuple $(2, 5)$ and let $k = 5$. Observe that $\gcd(2, 5) = 1$ so that condition (i) of Definition 2.1 is satisfied. Note that

$0 \cdot 2 + 0 \cdot 5 = 0, 0 \cdot 2 + 1 \cdot 5 = 5, 0 \cdot 2 + 2 \cdot 5 = 10, 0 \cdot 2 + 3 \cdot 5 = 15,$

$0 \cdot 2 + 4 \cdot 5 = 20, 1 \cdot 2 + 0 \cdot 5 = 2, 1 \cdot 2 + 1 \cdot 5 = 7, 1 \cdot 2 + 2 \cdot 5 = 12,$

$1 \cdot 2 + 3 \cdot 5 = 17, 1 \cdot 2 + 4 \cdot 5 = 22, 2 \cdot 2 + 0 \cdot 5 = 4, 2 \cdot 2 + 1 \cdot 5 = 9,$

$2 \cdot 2 + 2 \cdot 5 = 14, 2 \cdot 2 + 3 \cdot 5 = 19, 2 \cdot 2 + 4 \cdot 5 = 24, 3 \cdot 2 + 0 \cdot 5 = 6,$

$3 \cdot 2 + 1 \cdot 5 = 11, 3 \cdot 2 + 2 \cdot 5 = 16, 3 \cdot 2 + 3 \cdot 5 = 21, 3 \cdot 2 + 4 \cdot 5 = 26,$

$4 \cdot 2 + 0 \cdot 5 = 8, 4 \cdot 2 + 1 \cdot 5 = 13, 4 \cdot 2 + 2 \cdot 5 = 18, 4 \cdot 2 + 3 \cdot 5 = 23,$

$4 \cdot 2 + 4 \cdot 5 = 28.$

Since none of the linear combinations above evaluates to the same value, condition (ii) of Definition 2.1 is satisfied. Furthermore,

$$A_1 = \{x \in \mathbb{Z}_{\geq 1} : x \equiv 2i \pmod{50}, i \in \{1, 2, 3 \ldots, 24\} \setminus \{5, 10, 15, 20\}\},$$

and

$$A_2 = \{x \in \mathbb{Z}_{\geq 1} : x \equiv 5, 10, 15, 20, 30, 35, 40, 45 \pmod{50}\}.$$

Clearly, $A_1 \cap A_2 = \emptyset$ and thus condition (iii) of Definition 2.1 is satisfied. Indeed, $(2, 5)$ is 5-admissible.

Let $(a_1, a_2, a_3, \ldots, a_r)$ be a k-admissible tuple where $k \geq 2$. Denote by $B(a_1, a_2, \ldots, a_r, k, n)$, the number of partitions of n in which parts occur with multiplicities $\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_r a_r$ where $0 \leq \alpha_i \leq k-1$. Then we have the following theorem.

Theorem 2.2. *Let $C(a_1, a_2, \ldots, a_r, k, n)$ be the number of partitions of n into parts congruent to $-(s+mk)a_j \pmod{ka_1a_2a_3\ldots a_r}$ where $s = 1, 2, \ldots, k-1$, $j = 1, 2, \ldots, r$ and*

$$m = 0, 1, 2, \ldots, \left(\prod_{\substack{i=1 \\ i \neq j}}^{r} a_i\right) - 1.$$

Then

$$B(a_1, a_2, \ldots, a_r, k, n) = C(a_1, a_2, \ldots, a_r, k, n).$$

Proof. Note that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} B(a_1, a_2, \ldots, a_r, n) q^n \\
&= \prod_{n=1}^{\infty} \Big(1 + q^{a_1 n} + q^{2a_1 n} + \ldots + q^{(k-1)a_1 n} + q^{a_2 n} + q^{2a_2 n} + \ldots + q^{(k-1)a_2 n} \\
&\qquad\qquad + \ q^{(a_1+a_2)n} + q^{(a_1+2a_2)n} + \ldots + q^{((k-1)a_1+(k-1)a_2+\ldots+(k-1)a_r)n} \Big) \\
&= \prod_{n=1}^{\infty} \sum_{\alpha_1=0}^{k-1} \sum_{\alpha_2=0}^{k-1} \sum_{\alpha_3=0}^{k-1} \cdots \sum_{\alpha_r=0}^{k-1} q^{(\alpha_1 a_1 + \alpha_2 a_2 + \ldots \alpha_r a_r)n} \\
&= \prod_{n=1}^{\infty} \left(\sum_{\alpha_1=0}^{k-1} q^{\alpha_1 a_1 n} \right) \left(\sum_{\alpha_2=0}^{k-1} q^{\alpha_2 a_2 n} \right) \left(\sum_{\alpha_3=0}^{k-1} q^{\alpha_3 a_3 n} \right) \cdots \left(\sum_{\alpha_r=0}^{k-1} q^{\alpha_r a_r n} \right) \\
&= \prod_{n=1}^{\infty} \left(\frac{1-q^{a_1 nk}}{1-q^{a_1 n}} \right) \left(\frac{1-q^{a_2 nk}}{1-q^{a_2 n}} \right) \left(\frac{1-q^{a_3 nk}}{1-q^{a_3 n}} \right) \cdots \left(\frac{1-q^{a_r nk}}{1-q^{a_r n}} \right) \\
&= \prod_{j=1}^{r} \prod_{n=1}^{\infty} \frac{(1-q^{a_j nk})}{(1-q^{a_j n})} \\
&= \prod_{j=1}^{r} \prod_{n=1}^{\infty} \left(\frac{1}{1-q^{ka_j n - a_j}} \right) \left(\frac{1}{1-q^{ka_j n - 2a_j}} \right) \left(\frac{1}{1-q^{ka_j n - 3a_j}} \right) \cdots \\
&\qquad\qquad \left(\frac{1}{1-q^{ka_j n - (k-1)a_j}} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{j=1}^{r} \prod_{s=1}^{k-1} \prod_{n=1}^{\infty} \left(\frac{1}{1 - q^{ka_j n - sa_j}} \right) \\
&= \prod_{j=1}^{r} \prod_{s=1}^{k-1} \prod_{n=1}^{\infty} \left(\frac{1}{1 - q^{k \prod_{i=1}^{r} a_i n - sa_j}} \right) \left(\frac{1}{1 - q^{k \prod_{i=1}^{r} a_i n - sa_j - ka_j}} \right) \\
&\qquad \cdot \left(\frac{1}{1 - q^{k \prod_{i=1}^{r} a_i n - sa_j - 2ka_j}} \right) \left(\frac{1}{1 - q^{k \prod_{i=1}^{k} a_i n - sa_j - 3ka_j}} \right) \\
&\qquad \cdots \left(\frac{1}{1 - q^{k \prod_{i=1}^{r} a_i n - sa_j - \left(\left(\left(\prod_{\substack{i=1 \\ i \neq j}}^{k} a_i \right) - 1 \right) ka_j}} \right) \\
&= \prod_{j=1}^{r} \prod_{s=1}^{k-1} \prod_{m=0}^{-1 + \prod_{\substack{i=1 \\ i \neq j}}^{k} a_i} \prod_{n=1}^{\infty} \left(\frac{1}{1 - q^{k \prod_{i=1}^{r} a_i n - sa_j - mka_j}} \right) \\
&= \sum_{n=0}^{\infty} C(a_1, a_2, \ldots, a_r, k, n) q^n.
\end{aligned}
$$

The second to last equal sign is due to the fact that the exponents of q in the denominator are of the form $k \prod_{i=1}^{r} a_i n - sa_j - mka_j$ where

$$m = 0, 1, 2, \ldots, \left(\prod_{\substack{i=1 \\ i \neq j}}^{k} a_i \right) - 1.$$

□

Lemma 2.3. *Let* $k \in \mathbb{Z}_{>1}$. *If* $a_1 \not\equiv a_2 \pmod{k}$, *then* $A_1 \cap A_2 = \emptyset$ *where*

$$A_1 = \{x \in \mathbb{Z}_{\geq 1} : x \equiv -(1 + km)a_1 \pmod{ka_1 a_2} : m = 0, \ldots, a_2 - 1\},$$

$$A_2 = \{x \in \mathbb{Z}_{\geq 1} : x \equiv -(1 + k\alpha)a_2 \pmod{ka_1 a_2} : \alpha = 0, \ldots, a_1 - 1\}.$$

Proof. Suppose $a_1 \not\equiv a_2 \pmod{k}$. If $A_1 \cap A_2 \neq \emptyset$, then there is $x \in \mathbb{Z}_{\geq 1}$ such that $x \equiv -(1+km)a_1 \pmod{ka_1 a_2}$ and $x \equiv -(1+k\alpha)a_2 \pmod{ka_1 a_2}$ for some $m \in \{0, 1, \ldots, a_2 - 1\}$ and $\alpha \in \{0, 1, \ldots, a_1 - 1\}$. Thus

$$-(1 + km)a_1 + (1 + k\alpha)a_2 \equiv 0 \pmod{ka_1 a_2}$$

so that

$$(a_2 - a_1) + k(\alpha a_2 - m a_1) \equiv 0 \pmod{k a_1 a_2}.$$

From this, it is clear that $a_2 - a_1$ is congruent to $0 \pmod{k}$, which is a contradiction. Thus we must have $A_1 \cap A_2 = \emptyset$. □

Corollary 2.4. *If $a_1 \not\equiv a_2 \pmod{2}$ and $\gcd(a_1, a_2) = 1$, then the number of partitions of n in which parts occur with multiplicities $a_1, a_2, a_1 + a_2$ is equal to the number of partitions of n into parts congruent to $-(1 + 2m)a_j \pmod{2a_1 a_2}$ where $j = 1, 2$ and*

$$m = 0, 1, \ldots, \left(\prod_{\substack{i=1 \\ i \neq j}}^{2} a_i\right) - 1.$$

Proof. By Lemma 2.3 with $k = 2$, we have $A_1 \cap A_2 = \emptyset$. Since $\gcd(a_1, a_2) = 1$ and $a_1, a_2, a_1 + a_2$ are all different, by Definition 2.1, we conclude that (a_1, a_2) is a 2-admissible tuple. Setting $k = r = 2$ in Theorem 2.2 yields the result. □

Remark: Corollary 2.4 is a generalization of Subbarao's partition theorem, Theorem 1.3, where $a_1 = 2$ and $a_2 = 3$. Of course Theorem 2.2 is a more general extension.

Bijective proof of Theorem 2.2. Let λ be enumerated by $B(a_1, a_2, \ldots, a_r, k, n)$. Write each multiplicity m of a part p as a linear combination of $a_1, a_2, \ldots, a_r$, i.e.

$$m = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_r a_r.$$

For clarity sake, we shall call α_i the coefficient of a_i in the multiplicity m of p. Now construct partitions β_j's in this way:

$$\beta_j = ((a_j p_1)^{\alpha_{1j}}, (a_j p_2)^{\alpha_{2j}}, \ldots), \ \ 1 \leq j \leq r$$

where $p_1 > p_2 > \ldots$ are the distinct parts of λ and α_{ij} is the coefficient of a_j in the multiplicity of the part p_i of λ. Then divide each part in β_j by a_j and then apply the Glaisher map ϕ (see Theorem 1.2). In other words, compute

$$\phi^{-1}(p_1^{\alpha_{1j}}, p_2^{\alpha_{2j}}, \ldots). \tag{2.1}$$

Multiply each part in (2.1) by a_j to get $a_j \phi^{-1}(p_1^{\alpha_{1j}}, p_2^{\alpha_{2j}}, \ldots)$ and the image of λ is given by taking the union over all j's, i.e.

$$\bigcup_{j=1}^{r} a_j \phi^{-1}(p_1^{\alpha_{1j}}, p_2^{\alpha_{2j}}, \ldots).$$

To invert the process, suppose μ is enumerated by $C(a_1, a_2, \ldots, a_r, k, n)$. Decompose μ into $(\mu_1, \mu_2, \ldots, \mu_r)$ where u_j is a sub-partition consisting of parts congruent to

$$-(s+mk)a_j \left(\bmod\ k\prod_{i=1}^{r} a_i\right).$$

For each μ_j, perform the following steps:

(a) Divide each part by a_j.
(b) Apply ϕ to the result in (a).
(c) Repeat each part in (b), a_j times, and call the resulting partition $\bar{\mu}_j$.

Then the image of μ is given by

$$\bigcup_{j=1}^{r} \bar{\mu}_j.$$

We include the following example to illustrate our bijection.

Example 2.6. *Let* $\lambda = (27^3, 22^6, 19^3, 15^5, 12^6, 10^8, 7^{11}, 5^{10}, 4^{13}, 2^{16})$ *and* $k = 3$, $a_1 = 3$, $a_2 = 5$.

Clearly, λ is enumerated by $B(3,5,5,708)$. We have

$$\beta_1 = (81, 66^2, 57, 36^2, 30, 21^2, 12, 6^2),\ \ \beta_2 = (75, 50, 35, 25^2, 20^2, 10^2)$$

so that

$$\frac{\beta_1}{a_1} = (27, 22^2, 19, 12^2, 10, 7^2, 4, 2^2) \ \text{ and } \ \frac{\beta_2}{a_2} = (15, 10, 7, 5^2, 4^2, 2^2).$$

We now apply the Glaisher map to β_1/a_1 and β_2/a_2 and obtain

$$\phi^{-1}\left(\frac{\beta_1}{a_1}\right) = (22^2, 19, 10, 7^2, 4^7, 2^2, 1^{27}),$$
$$\phi^{-1}\left(\frac{\beta_2}{a_2}\right) = (10, 7, 5^5, 4^2, 2^2).$$

Multiplying each part of $\phi^{-1}(\beta_1/a_1)$ by a_1 and each part of $\phi^{-1}(\beta_2/a_2)$ by a_2, and then taking the union results in the image of λ as

$$(66^2, 57, 50, 35, 30, 25^5, 21^2, 20^2, 12, 10^2, 6^2, 3^{27})$$

a partition enumerated by $C(3,5,5,708)$.

We now invert the process. The residue set is

$$\{3, 5, 6, 10, 12, 15, 20, 21, 24, 25, 30, 33, 35, 39, 40, 42\}.$$

Thus

$$\mu_1 = (66^2, 57, 30, 21^2, 12, 6^2, 3^{27}),$$
$$\mu_2 = (50, 35, 25, 20^2, 10^2).$$

Applying the Glaisher map ϕ with

$$1 = 1\cdot 3^0,\ \ 2 = 2\cdot 3^0,\ \ 5 = 2\cdot 3^0 + 1\cdot 3^1,\ \ 7 = 1\cdot 3^0 + 2\cdot 3^1 \text{ and } 27 = 3^3,$$

we obtain

$$\phi\left(\frac{\mu_1}{a_1}\right) = (27, 22^2, 19, 10, 7^2, 4, 2^2),$$
$$\phi\left(\frac{\mu_2}{a_2}\right) = (10, 7, 5^2, 4^2, 2^2).$$

Thus repeating each part in $\phi(\mu_1/a_1)$ a_1 times, and each part in $\phi(\frac{\mu_2}{a_2})$ a_2 times, and taking the union leads us to the following:

$$\lambda = (27^3, 22^6, 19^3, 15^5, 12^6, 10^8, 7^{11}, 5^{10}, 4^{13}, 2^{16}).$$

More generally, consider a function in which even parts and odd parts satisfy the 'Subbarao' condition separately. Let $g(m_1, m_2, a_1, a_2, n)$ denote the number of partitions of n in which even parts occur with multiplicities $m_1, m_2, m_1 + m_2$ and odd parts occur with multiplicities $a_1, a_2, a_1 + a_2$. It is clear that $g(2, 3, 2, 3, n)$ enumerates partitions considered in Theorem 1.3. In the next section, we look at a special case of $g(m_1, m_2, a_1, a_2, n)$.

3. On the function $g(m_1, m_2, 2m_1, 2m_2, n)$

First, we recall the following q-series notation. In all cases, $|q| < 1$,

$$(a; q)_n = \prod_{i=1}^{n}(1 - aq^i),$$
$$(a; q)_\infty = \lim_{n\to\infty}\prod_{i=1}^{n}(1 - aq^i).$$

From the definition of $g(m_1, m_2, a_1, a_2, n)$, it is not difficult to note that

$$\sum_{n=0}^{\infty} g(m_1, m_2, a_1, a_2, n)q^n = \frac{(-q^{2m_1}; q^{2m_1})_\infty(-q^{2m_2}; q^{2m_2})_\infty}{(-q^{a_1}; q^{a_1})_\infty(-q^{a_2}; q^{a_2})_\infty}.$$

Thus

$$\begin{aligned}\sum_{n=0}^{\infty} g(m_1, m_2, 2m_1, 2m_2, n)q^n &= \frac{(-q^{2m_1}; q^{2m_1})_\infty(-q^{2m_2}; q^{2m_2})_\infty}{(-q^{2m_1}; q^{2m_1})_\infty(-q^{2m_2}; q^{2m_2})_\infty} \\ &\equiv 1 + 0q + 0q^2 + 0q^3 + \cdots \pmod 2 .\end{aligned}$$

This yields

$$g(m_1, m_2, 2m_1, 2m_2, n) \equiv 0 \pmod 2 \ \forall\ n \geq 1. \tag{3.1}$$

Denote by $G(m_1, m_2, 2m_1, 2m_2, n)$ the set of partitions enumerated by $g(m_1, m_2, 2m_1, 2m_2, n)$. Let $h(m_1, m_2, 2m_1, 2m_2, n)$ be the number of partitions in $G(m_1, m_2, 2m_1, 2m_2, n)$ satisfying the following property:

All parts are congruent to $0 \pmod 4$ *or else for every odd part* t *occurring with multiplicity* $2j$*, there is an even part* $2t$ *having multiplicity* j*, and for every even part* s *congruent to* $2 \pmod 4$ *occurring with multiplicity* l*, there is an odd part* $\frac{s}{2}$ *which has multiplicity* $2l$.

Then we have the following result.

Theorem 3.1.

$$h(m_1, m_2, 2m_1, 2m_2, n) \equiv 0 \pmod 2 \ \forall\, n \geq 1. \tag{3.2}$$

Proof. Define the map

$$\psi : G(m_1, m_2, 2m_1, 2m_2, n) \to G(m_1, m_2, 2m_1, 2m_2, n)$$

as follows:

For $\lambda = (\lambda_1^{f_1}, \lambda_2^{f_2}, \ldots, \lambda_l^{f_l}) \in G(m_1, m_2, 2m_1, 2m_2, n)$,

$$\lambda_i^{f_i} \mapsto \begin{cases} (2\lambda_i)^{\frac{f_i}{2}}, & \text{if } \lambda_i \equiv 1 \pmod 2\,; \\ \left(\dfrac{\lambda_i}{2}\right)^{2f_i}, & \text{if } \lambda_i \equiv 2 \pmod 4\,; \\ \lambda_i^{f_i}, & \text{otherwise.} \end{cases}$$

Then

$$\psi(\lambda) = \bigcup_{i=1}^{l} \psi(\lambda_i^{f_i}).$$

We claim that ψ is an involution. To see this, proceed as follows.

If $\lambda_i \equiv 0 \pmod 4$, then

$$\psi(\psi(\lambda_i^{f_i})) = \psi(\lambda_i^{f_i}) = \lambda_i^{f_i}.$$

If $\lambda_i \equiv 1 \pmod 2$, then

$$\psi(\psi(\lambda_i^{f_i})) = \psi((2\lambda_i)^{\frac{f_i}{2}}) = \left(\frac{2\lambda_i}{2}\right)^{2(\frac{f_i}{2})} \quad (\text{since } 2\lambda_i \equiv 2 \pmod 4).$$

We have that $\psi(\psi(\lambda_i^{f_i})) = \lambda_i^{f_i}$.

If $\lambda_i \equiv 2 \pmod 4$, then

$$\psi(\psi(\lambda_i^{f_i})) = \psi\left(\left(\frac{\lambda_i}{2}\right)^{2f_i}\right) = \left(2\left(\frac{\lambda_i}{2}\right)\right)^{\frac{2f_i}{2}} \ (\text{since } \frac{\lambda_i}{2} \text{ is odd}).$$

Thus $\psi(\psi(\lambda_i^{f_i})) = \lambda_i^{f_i}$.

Therefore $\psi(\psi(\lambda)) = \lambda$. Indeed ψ is an involution on

$$G(m_1, m_2, 2m_1, 2m_2, n).$$

A careful look into the set reveals that the

$$h(m_1, m_2, 2m_1, 2m_2, n)\text{-partitions}$$

are precisely those partitions fixed under the involution. For this reason, their number must have the same parity as the number of all partitions in $G(m_1, m_2, 2m_1, 2m_2, n)$. □

4. A new partition theorem of Subbarao type

Bijections for Theorem 1.4 have been given in [1, 3, 4]. In the spirit of Theorem 1.4, we state the following new theorem.

Theorem 4.1. *For $r \in \mathbb{Z}_{\geq 1}$, let $E_r(n)$ denote the set of partitions of n wherein parts appear* $5, 10, 15, 5r+2, 5r+7, 5r+12, 5r+17, 10r+4, 10r+9, 10r+14, 10r+19, 15r+6, 15r+11, 15r+16, 15r+21, 20r+8, 20r+13, 20r+18$ *or* $20r+23$ *times and $F_r(n)$ be the set of partitions of n wherein parts are congruent to* $\pm 5, 10 \pmod{20}$ *or* $\pm(5r+2), \pm(10r+4) \pmod{25r+10}$. *Then* $|E_r(n)| = |F_r(n)|$.

Although our goal is to exhibit a bijection for this theorem, observe that

$$\begin{aligned}
\sum_{n\geq 0} |E_r(n)|q^n &= \prod_{n=1}^{\infty} \Big(1 + q^{5n} + q^{10n} + q^{15n} + q^{(5r+2)n} + q^{(5r+7)n} + q^{(5r+12)n} \\
&\qquad + q^{(5r+17)n} + q^{(10r+4)n} + q^{(10r+9)n} + q^{(10r+14)n} \\
&\qquad + q^{(10r+19)n} + q^{(15r+6)n} + q^{(15r+11)n} + q^{(15r+16)n} \\
&\qquad + q^{(15r+21)n} + q^{(20r+8)n} + q^{(20r+13)n} \\
&\qquad + q^{(20r+18)n} + q^{(20r+23)n}\Big) \\
&= \prod_{n=1}^{\infty} \sum_{j=0}^{4} q^{(5r+2)nj} \sum_{i=0}^{3} q^{5ni} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{5(5r+2)n})(1-q^{20n})}{(1-q^{(5r+2)n})(1-q^{5n})} \\
&= \prod_{s=1}^{4} \prod_{n=1}^{\infty} \frac{1}{1-q^{(5r+2)(5n-s)}} \prod_{\substack{j\equiv 0 \pmod{5},\\ j\not\equiv 0 \pmod{20}}} \frac{1}{1-q^j} \\
&= \prod_{s=1}^{4} \prod_{n=1}^{\infty} \frac{1}{1-q^{(25r+10)n-s(5r+2)}} \prod_{\substack{j\equiv 0 \pmod{5},\\ j\not\equiv 0 \pmod{20}}} \frac{1}{1-q^j} \\
&= \sum_{n=0}^{\infty} |F_r(n)|q^n.
\end{aligned}$$

We now describe a bijective proof.

Proof. Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_1}, \ldots) \in E_r(n)$. For an integer $t > 1$, denote the order of a with respect to t by $\mathrm{ord}_t(a)$ which we define as

$$\mathrm{ord}_t(a) = \max\{\alpha \in \mathbb{Z}_{\geq 0} : t^{\alpha} | a\}$$

and define a map $\tau : E_r(n) \to F_r(n)$ as follows.

Case I: $m_i \equiv 0 \pmod 5$

$$\lambda_i^{m_i} \mapsto \left(\frac{5\lambda_i}{4^s}\right)^{\frac{m_i}{5}4^s}, \text{ where } s = \mathrm{ord}_4(\lambda_i).$$

Case II: $m_i \equiv 1 \pmod 5$

$$\begin{cases} \left(((5r+2)\lambda_i)^3, \lambda_i^{m_i-15r-6}\right), & \text{if } \lambda_i \not\equiv 0 \pmod 5 \\ \left(\left(\frac{(5r+2)\lambda_i}{5^t}\right)^{3\cdot 5^t}, \lambda_i^{m_i-15r-6}\right), & \text{if } \lambda_i \equiv 0 \pmod 5. \end{cases}$$

Case III: $m_i \equiv 2 \pmod 5$

$$\lambda_i^{m_i} \mapsto \begin{cases} \left((5r+2)\lambda_i, \lambda_i^{m_i-5r-2}\right), & \text{if } \lambda_i \not\equiv 0 \pmod 5 \\ \left(\left(\frac{(5r+2)\lambda_i}{5^t}\right)^{5^t}, \lambda_i^{m_i-5r-2}\right), & \text{if } \lambda_i \equiv 0 \pmod 5. \end{cases}$$

Case IV: $m_i \equiv 3 \pmod 5$

$$\lambda_i^{m_i} \mapsto \begin{cases} \left(((5r+2)\lambda_i)^4, \lambda_i^{m_i-20r-8}\right), & \text{if } \lambda_i \not\equiv 0 \pmod 5 \\ \left(\left(\frac{(5r+2)\lambda_i}{4^t}\right)^{4\cdot 5^t}, \lambda_i^{m_i-20r-8}\right), & \text{if } \lambda_i \equiv 0 \pmod 5. \end{cases}$$

Case V: $m_i \equiv 4 \pmod 5$

$$\lambda_i^{m_i} \mapsto \begin{cases} \left(((5r+2)\lambda_i)^2, \lambda_i^{m_i-10r-4}\right), & \text{if } \lambda_i \not\equiv 0 \pmod 5 \\ \left(\left(\frac{(5r+2)\lambda_i}{5^t}\right)^{2\cdot 5^t}, \lambda_i^{m_i-10r-4}\right), & \text{if } \lambda_i \equiv 0 \pmod 5. \end{cases}$$

where $t = \mathrm{ord}_5(\lambda_i)$ in Cases II, III, IV and V.

In Cases II, III, IV and V, if $m_i-15r-6 > 0$, $m_i-5r-2 > 0$, $m_i-20r-8 > 0$, $m_i - 10r - 4 > 0$, then apply Case I to the sub-partitions $\lambda_i^{m_i-15r-6}$, $\lambda_i^{m_i-5r-2}$, $\lambda_i^{m_i-20r-8}$, and $\lambda_i^{m_i-10r-4}$, respectively.
The image is then defined as

$$\tau(\lambda) = \bigcup_{i\geq 1} \tau(\lambda_i^{m_i}).$$

□

Example 4.2. *Let $n = 42$ and $r = 4$. Then*

$$E_4(42) = \{(4^5, 1^{22}), (3^5, 1^{27}), (2^{10}, 1^{22}), (2^5, 1^{32})\}.$$

The sub-partitions 2^5 and 1^{32} of $(2^5, 1^{32})$ have multiplicities congruent to $0 \pmod 5$ and $2 \pmod 5$, respectively. Applying the map τ gives

$$2^5 \mapsto 10$$
$$1^{32} \mapsto (22, 5^2).$$

Hence, taking the union of the image parts we obtain that $(2^5, 1^{32}) \mapsto (22, 10, 5^2)$. Similarly, applying τ to the remaining partitions gives

$$(4^5, 1^{22}) \mapsto (22, 5^4)$$
$$(3^5, 1^{27}) \mapsto (22, 15, 5)$$
$$(2^{10}, 1^{22}) \mapsto (22, 10^2)$$

which are partitions in $F_4(42)$.

We now describe the inverse of τ. Let $\mu = (\mu_1^{\omega_1}, \mu_2^{\omega_2}, \ldots, \mu_t^{\omega_t}) \in F_r(n)$. Define a map $\tau^{-1} : F_r(n) \to E_r(n)$ as follows.

Case I: $\mu_i \not\equiv 0 \pmod{5r+2}$.

Using the base 4 representation, we write ω_i as

$$\omega_i = 4^{\rho_1} + 4^{\rho_2} + 4^{\rho_3} + \cdots + 4^{\rho_l} + \eta \tag{4.1}$$

where $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_l \geq 1$ and $0 \leq \eta < 4$. This representation of ω_i in (4.1) comes from its unique quaternary expansion. Just for illustration at this stage, if $\omega_i = 1 + 2 \cdot 4 + 2 \cdot 4^2 + 3 \cdot 4^4$, we rewrite ω_i as $1 + (4+4) + (4^2+4^2) + (4^4+4^4+4^4)$ so that $\eta = 1, p_1 = p_2 = p_3 = 4, p_4 = p_5 = 2, p_6 = p_7 = 1$.

Having identified, $p_1, p_2, \ldots, p_l$ and η from (4.1), construct a partition

$$x_i = \left(4^{\rho_1} \times \frac{\mu_i}{5}\right)^5 \cup \left(4^{\rho_2} \times \frac{\mu_i}{5}\right)^5 \cup \cdots \cup \left(4^{\rho_l} \times \frac{\mu_i}{5}\right)^5.$$

Thus

$$\mu_i^{\omega_i} \mapsto x_i \cup \left(\frac{\mu_i}{5}\right)^{5\eta}.$$

Case II: $\mu_i \equiv 0 \pmod{5r+2}$.

Using the base 5 representation, we write ω_i as

$$\omega_i = 5^{\rho_1} + 5^{\rho_2} + 5^{\rho_3} + \cdots + 5^{\rho_l} + \zeta$$

where $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_l > 0$ and $0 \leq \zeta < 5$ (just as in (4.1)). Construct a partition

$$y_i = \left(5^{\rho_1} \times \frac{\mu_i}{5r+2}\right)^{5r+2} \cup \left(5^{\rho_2} \times \frac{\mu_i}{5r+2}\right)^{5r+2} \cup \left(5^{\rho_3} \times \frac{\mu_i}{5r+2}\right)^{5r+2} \cup \cdots \cup \left(5^{\rho_l} \times \frac{\mu_i}{5r+2}\right)^{5r+2}.$$

Thus

$$\mu_i^{\omega_i} \mapsto y_i \cup \left(\frac{\mu_i}{5r+2}\right)^{(5r+2)\zeta}.$$

The image is then defined as

$$\tau^{-1}(\mu) = \bigcup_{i\geq 1} \tau^{-1}(\mu_i^{\omega_i}).$$

Example 4.3. *Let $n = 42$ and $r = 4$. Then*

$$F_4(42) = \{(22, 5^4), (22, 15, 5), (22, 10^2), (22, 10, 5^2)\}.$$

The sub-partitions 22 and 5^4 of $(22, 5^4)$ have parts congruent to $0 \pmod{22}$ and $5 \pmod{22}$, respectively. Applying the map τ^{-1} gives

$$22 \mapsto 1^{22}$$
$$5^4 \mapsto 4^5.$$

Hence, taking the union of the image parts we obtain that $(22, 5^4) \mapsto (4^5, 1^{22})$. Similarly, applying τ^{-1} to the remaining partitions gives

$$(22, 15, 5) \mapsto (3^5, 1^{27})$$
$$(22, 10^2) \mapsto (2^{10}, 1^{22})$$
$$(22, 10, 5^2) \mapsto (2^5, 1^{32})$$

which are partitions in $E_4(42)$.

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