ON SOME PARTITION THEOREMS OF M. V. SUBBARAO

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Abstract. M.V. Subbarao proved that the number of partitions of \( n \) in which parts occur with multiplicities 2, 3 and 5 is equal to the number of partitions of \( n \) in which parts are congruent to \( \pm 2, \pm 3, 6 \) (mod 12), and generalized this result. In this paper, we give a new generalization of this identity and also present a new partition theorem in the spirit of Subbarao’s generalization of the identity.

1. Introduction

A partition of a positive integer \( n \) is a representation \( \lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_r \) where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1 \) and

\[
\sum_{i=1}^{r} \lambda_i = n.
\]

The integer \( n \) is called the weight of \( \lambda \) which is denoted by \( |\lambda| \) and \( \lambda_i \)’s are called parts. Other alternative notations include \((\lambda_1, \lambda_2, \ldots, \lambda_r)\) or \((\mu_1^{\mu_1}, \mu_2^{\mu_2}, \ldots)\) where \( \mu_1 > \mu_2 > \cdots \) and \( m_i \) is the multiplicity of \( \mu_i \).

The partition \( 14 + 14 + 10 + 10 + 7 + 7 + 7 + 1 + 1 + 1 + 1 \) can be written as \((14, 14, 10, 10, 7, 7, 7, 1, 1, 1, 1)\) or \((14^2, 10^2, 7^3, 1^4)\). The union of two partitions \( \lambda \) and \( \beta \) is simply the multiset union \( \lambda \cup \beta \) where \( \lambda \) and \( \beta \) are treated as multisets. For example, \((14^2, 10^2, 7^3, 1^4) \cup (13, 10^3, 6, 1^4) = (14^2, 13, 10^5, 7^3, 6, 1^8)\).

The number of all partitions of \( n \) is called the (unrestricted) partition function. If restrictions are imposed on the parts of partitions, the corresponding enumerating function is then called a restricted partition function. One such example is the number of partitions of \( n \) into distinct parts. It turns out that for a fixed weight, partitions into distinct parts are related to partitions into odd parts. The following theorem demonstrates the relationship.

Theorem 1.1 (Euler). The number of partitions of \( n \) into distinct parts is equal to the number of partitions of \( n \) into odd parts.
Theorem 1.1 was extended to a more general setting. This extension is due to J. W. L Glaisher (see [2]).

**Theorem 1.2** (Glaisher). Let \( k > 1 \). The number of partitions of \( n \) wherein parts are not divisible by \( k \) is equal to the number of partitions of \( n \) in which parts occur at most \( k - 1 \) times.

The theorem above has an interesting bijective proof which we recall from the literature. Let \( \lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_r^{m_r}) \) be a partition of \( n \) whose parts are not divisible by \( k \). Note that the notation for \( \lambda \) implies \( \lambda_1 > \lambda_2 > \cdots \) are parts with multiplicities \( m_1, m_2, \ldots \), respectively. Now, write \( m_i \)'s in \( k \)-ary expansion, i.e.

\[
m_i = \sum_{j=0}^{l_i} a_{ij} k^j \quad \text{where} \quad 0 \leq a_{ij} \leq k - 1.
\]

We map \( \lambda_i^{m_i} \) to \( \bigcup_{j=0}^{l_i} (k^j \lambda_i)^{a_{ij}} \), where now \( k^j \lambda_i \) is a part with multiplicity \( a_{ij} \). The image of \( \lambda \) which we shall denote by \( \phi(\lambda) \), is given by

\[
\bigcup_{i=1}^r \bigcup_{j=0}^{l_i} (k^j \lambda_i)^{a_{ij}}.
\]

Clearly, this is a partition of \( n \) with parts occurring not more than \( k - 1 \) times.

On the other hand, assume that \( \mu = (\mu_1^{f_1}, \mu_2^{f_2}, \ldots) \) is a partition of \( n \) into parts occurring not more than \( k - 1 \) times. Write \( \mu_i = k^{r_i} a_i \) where \( k \) does not divide \( a_i \) and then map \( \mu_i^{f_i} \) to \( (a_i)^{k^{r_i} f_i} \) for each \( i \), where now \( a_i \) is a part with multiplicity \( k^{r_i} f_i \). The inverse of \( \phi \) is then given by

\[
\phi^{-1}(\mu) = \bigcup_{i \geq 1} (a_i)^{k^{r_i} f_i}.
\]

In the resulting partition, it is also clear that the parts are not divisible by \( k \). M.V Subbarao proved the following theorem.

**Theorem 1.3** (Subbarao, [5]). The number of partitions of \( n \) in which parts occur with multiplicities 2, 3 and 5 is equal to the number of partitions of \( n \) in which parts are congruent to \( \pm 2, \pm 3, 6 \pmod{12} \).

A generalization of the above theorem was also given as follows.

**Theorem 1.4** (Subbarao, [5]). Let \( m > 1, r \geq 0 \) be integers, and let \( A_{m,r}(n) \) denote the number of partitions of \( n \) such that all even multiplicities of the parts are less than \( 2m \), and all odd multiplicities are at least \( 2r + 1 \) and at most \( 2(m + r) - 1 \). Let \( B_{m,r}(n) \) be the number of partitions of \( n \) in which parts are either odd and congruent to \( 2r + 1 \pmod{4r + 2} \) or even and not congruent to \( 0 \pmod{2m} \). Then \( A_{m,r}(n) = B_{m,r}(n) \).

Indeed Theorem 1.3 is a special case of Theorem 1.4 ( \( m = 2 \) and \( r = 1 \)). It is worth mentioning that Theorem 1.3 has been generalized in different
directions (see [5], [1] and the references therein). To avoid ambiguity, any partition identity in which one side describes a restriction on the multiplicity of parts and the other side describes parts being in certain residue classes shall be called an identity of Subbarao type. The partitions involved will be called Subbarao type partitions.

In this paper, we provide a new generalization of Theorem 1.3 and deduce the parity for a partition function of Subbarao type. This is done in Section 2. In Section 3, we state a new partition theorem in the spirit of Theorem 1.4 and give its bijective proof.

2. Generalization of Theorem 1.3

Our goal in this section is to give a simple extension of Theorem 1.3. We start by formulating the following definition.

**Definition 2.1.** Let $k \geq 2$ and $a_1, a_2, \ldots, a_r \geq 1$ be integers. The tuple $(a_1, a_2, \ldots, a_r)$ is called $k$-admissible if

1. $\gcd(a_i, a_j) = 1 \ \forall \ i \neq j$;
2. $\sum_{i=1}^{r} a_ia_i = \sum_{i=1}^{r} \beta_i a_i$ and $0 \leq \alpha_i, \beta_i \leq k-1$ for all $i = 1, 2, \ldots, r$ \Rightarrow $\alpha_i = \beta_i \ \forall \ i = 1, \ldots, r$;
3. $A_i \cap A_j = \emptyset$ for $1 \leq i \neq j \leq r$ where
   
   \[
   A_i = \left\{ x \in \mathbb{Z}_{\geq 1} : x \equiv -(s + km)a_i \left( \mod k \prod_{i=1}^{r} a_i \right), \right. \\
   \left. 1 \leq s \leq k - 1, \ m = 0, 1, \ldots, \left( \prod_{j=1}^{r} a_j \right) - 1 \right\}.
   \]

For instance, consider the tuple $(2, 5)$ and let $k = 5$. Observe that $\gcd(2, 5) = 1$ so that condition (i) of Definition 2.1 is satisfied. Note that $0 \cdot 2 + 0 \cdot 5 = 0, 0 \cdot 2 + 2 \cdot 5 = 10, 0 \cdot 2 + 3 \cdot 5 = 15, \ldots$. Since none of the linear combinations above evaluates to the same value, condition (ii) of Definition 2.1 is satisfied. Furthermore, $A_1 = \{ x \in \mathbb{Z}_{\geq 1} : x \equiv 2i \ (\mod 50), i \in \{1, 2, 3, \ldots, 24\} \setminus \{5, 10, 15, 20\}\}$, and $A_2 = \{ x \in \mathbb{Z}_{\geq 1} : x \equiv 5, 10, 15, 20, 30, 35, 40, 45 \ (\mod 50)\}$. 
Clearly, $A_1 \cap A_2 = \emptyset$ and thus condition (iii) of Definition 2.1 is satisfied. Indeed, $(2, 5)$ is 5-admissible.

Let $(a_1, a_2, a_3, \ldots, a_r)$ be a $k$-admissible tuple where $k \geq 2$. Denote by $B(a_1, a_2, \ldots, a_r, k, n)$, the number of partitions of $n$ in which parts occur with multiplicities $\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_r a_r$ where $0 \leq \alpha_i \leq k - 1$. Then we have the following theorem.

**Theorem 2.2.** Let $C(a_1, a_2, \ldots, a_r, k, n)$ be the number of partitions of $n$ into parts congruent to $-(s + mk)a_j \pmod{ka_1a_2a_3\ldots a_r}$ where $s = 1, 2, \ldots, k - 1$, $j = 1, 2, \ldots, r$ and

\[
m = 0, 1, 2, \ldots, \left( \prod_{i=1}^{r} a_i \right) - 1.
\]

Then

\[B(a_1, a_2, \ldots, a_r, k, n) = C(a_1, a_2, \ldots, a_r, k, n).\]
Proof. Note that

\[
\sum_{n=0}^{\infty} B(a_1, a_2, \ldots, a_r, n) q^n
= \prod_{n=1}^{\infty} \left( 1 + q^{a_1 n} + q^{2a_1 n} + \ldots + q^{(k-1)a_1 n} + q^{a_2 n} + q^{2a_2 n} + \ldots + q^{(k-1)a_2 n} + q^{a_1 a_2 n} + q^{a_1 + 2a_2 n} + \ldots + q^{(k-1)a_1 + (k-1)a_2 + \ldots + (k-1)a_r n} \right)
\]

\[
= \prod_{n=1}^{\infty} \sum_{a_1=0}^{k-1} \sum_{a_2=0}^{k-1} \sum_{a_3=0}^{k-1} \ldots \sum_{a_r=0}^{k-1} q^{(a_1 a_2 + a_2 a_3 + \ldots + a_r) n}
\]

\[
= \prod_{n=1}^{\infty} \left( \sum_{a_1=0}^{k-1} q^{a_1 n} \right) \left( \sum_{a_2=0}^{k-1} q^{a_2 n} \right) \left( \sum_{a_3=0}^{k-1} q^{a_3 n} \right) \ldots \left( \sum_{a_r=0}^{k-1} q^{a_r n} \right)
\]

\[
= \prod_{n=1}^{\infty} \frac{1 - q^{a_1 n}}{1 - q^{a_1 n}} \cdot \frac{1 - q^{a_2 n}}{1 - q^{a_2 n}} \cdot \frac{1 - q^{a_3 n}}{1 - q^{a_3 n}} \ldots \frac{1 - q^{a_r n}}{1 - q^{a_r n}}
\]

\[
= \prod_{j=1}^{r} \prod_{n=1}^{\infty} \frac{1 - q^{a_j n}}{1 - q^{a_j n}}
\]

\[
= \prod_{j=1}^{r} \prod_{n=1}^{\infty} \frac{1}{1 - q^{a_j n}} \cdot \frac{1}{1 - q^{a_j n - 2a_j}} \cdot \frac{1}{1 - q^{a_j n - 3a_j}} \ldots \frac{1}{1 - q^{a_j n - (k-1)a_j}}
\]
\[\prod_{j=1}^{r} \prod_{s=1}^{k-1} \prod_{n=1}^{\infty} \left( \frac{1}{1 - q^{ka_j n - sa_j}} \right)\]

\[= \prod_{j=1}^{r} \prod_{s=1}^{k-1} \prod_{n=1}^{\infty} \left( \frac{1}{1 - q^{ka_j n - sa_j}} \right) \left( \frac{1}{1 - q^{ka_j n - ka_j}} \right) \left( \frac{1}{1 - q^{ka_j n - 2ka_j}} \right) \left( \frac{1}{1 - q^{ka_j n - 3ka_j}} \right) \]

\[= \prod_{j=1}^{r} \prod_{s=1}^{k-1} \prod_{m=0}^{\infty} \prod_{n=1}^{\infty} \left( \frac{1}{1 - q^{ka_j n - sa_j - mka_j}} \right)\]

\[= \sum_{n=0}^{\infty} C(a_1, a_2, \ldots, a_r, k, n)q^n.\]

The second to last equal sign is due to the fact that the exponents of \(q\) in the denominator are of the form \(k \prod_{i=1}^{r} a_i n - sa_j - mka_j\) where

\[m = 0, 1, 2, \ldots, \left( \prod_{i=1}^{k} a_i - k \right) - 1.\]

\[\Box\]

**Lemma 2.3.** Let \(k \in \mathbb{Z}_{>1}\). If \(a_1 \not\equiv a_2 \pmod{k}\), then \(A_1 \cap A_2 = \emptyset\) where

\[A_1 = \{ x \in \mathbb{Z}_{\geq 1} : x \equiv -(1 + km)a_1 \pmod{ka_1 a_2} : m = 0, \ldots, a_2 - 1 \},\]

\[A_2 = \{ x \in \mathbb{Z}_{\geq 1} : x \equiv -(1 + k\alpha)a_2 \pmod{ka_1 a_2} : \alpha = 0, \ldots, a_1 - 1 \}.\]

**Proof.** Suppose \(a_1 \not\equiv a_2 \pmod{k}\). If \(A_1 \cap A_2 \neq \emptyset\), then there is \(x \in \mathbb{Z}_{\geq 1}\) such that \(x \equiv -(1 + km)a_1 \pmod{ka_1 a_2}\) and \(x \equiv -(1 + k\alpha)a_2 \pmod{ka_1 a_2}\) for some \(m \in \{0, 1, \ldots, a_2 - 1\}\) and \(\alpha \in \{0, 1, \ldots, a_1 - 1\}\). Thus

\[-(1 + km)a_1 + (1 + k\alpha)a_2 \equiv 0 \pmod{ka_1 a_2}\]
so that
\[(a_2 - a_1) + k(\alpha a_2 - ma_1) \equiv 0 \pmod{ka_1 a_2}.\]

From this, it is clear that \(a_2 - a_1\) is congruent to 0 \(\pmod{k}\), which is a contradiction. Thus we must have \(A_1 \cap A_2 = \emptyset\). \(\square\)

**Corollary 2.4.** If \(a_1 \not\equiv a_2 \pmod{2}\) and \(\gcd(a_1, a_2) = 1\), then the number of partitions of \(n\) in which parts occur with multiplicities \(a_1, a_2, a_1 + a_2\) is equal to the number of partitions of \(n\) into parts congruent to \(-(1 + 2m)a_j \pmod{2a_1 a_2}\) where \(j = 1, 2\) and
\[m = 0, 1, \ldots, \left(\prod_{i=1 \atop i \neq j}^{2} a_i\right) - 1.\]

**Proof.** By Lemma 2.3 with \(k = 2\), we have \(A_1 \cap A_2 = \emptyset\). Since \(\gcd(a_1, a_2) = 1\) and \(a_1, a_2, a_1 + a_2\) are all different, by Definition 2.1, we conclude that \((a_1, a_2)\) is a 2-admissible tuple. Setting \(k = r = 2\) in Theorem 2.2 yields the result. \(\square\)

**Remark:** Corollary 2.4 is a generalization of Subbarao’s partition theorem, Theorem 1.3, where \(a_1 = 2\) and \(a_2 = 3\). Of course Theorem 2.2 is a more general extension.

**Bijective proof of Theorem 2.2.** Let \(\lambda\) be enumerated by \(B(a_1, a_2, \ldots, a_r, k, n)\). Write each multiplicity \(m\) of a part \(p\) as a linear combination of \(a_1, a_2, \ldots, a_r\), i.e.
\[m = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_r a_r.\]

For clarity sake, we shall call \(\alpha_i\) the coefficient of \(a_i\) in the multiplicity \(m\) of \(p\). Now construct partitions \(\beta_j\)'s in this way:
\[\beta_j = \left((a_j p_1)^{\alpha_{1j}}, (a_j p_2)^{\alpha_{2j}}, \ldots\right), \quad 1 \leq j \leq r\]
where \(p_1 > p_2 > \ldots\) are the distinct parts of \(\lambda\) and \(\alpha_{ij}\) is the coefficient of \(a_j\) in the multiplicity of the part \(p_i\) of \(\lambda\). Then divide each part in \(\beta_j\) by \(a_j\) and then apply the Glaisher map \(\phi\) (see Theorem 1.2). In other words, compute
\[\phi^{-1}\left(p_1^{\alpha_{1j}}, p_2^{\alpha_{2j}}, \ldots\right).\]

Multiply each part in (2.1) by \(a_j\) to get \(a_j \phi^{-1}(p_1^{\alpha_{1j}}, p_2^{\alpha_{2j}}, \ldots\) and the image of \(\lambda\) is given by taking the union over all \(j\)'s, i.e.
\[\bigcup_{j=1}^{r} a_j \phi^{-1}(p_1^{\alpha_{1j}}, p_2^{\alpha_{2j}}, \ldots).\]

To invert the process, suppose \(\mu\) is enumerated by \(C(a_1, a_2, \ldots, a_r, k, n)\). Decompose \(\mu\) into \((\mu_1, \mu_2, \ldots, \mu_r)\) where \(u_j\) is a sub-partition consisting of
parts congruent to 
\[-(s + mk) a_j \pmod{k \prod_{i=1}^{r} a_i}.
\]

For each \(\mu_j\), perform the following steps:

(a) Divide each part by \(a_j\).
(b) Apply \(\phi\) to the result in (a).
(c) Repeat each part in (b), \(a_j\) times, and call the resulting partition \(\bar{\mu}_j\).

Then the image of \(\mu\) is given by 
\[
\bigcup_{j=1}^{r} \bar{\mu}_j.
\]

We include the following example to illustrate our bijection.

**Example 2.6.** Let \(\lambda = (27^3, 22^6, 19^3, 15^5, 12^6, 10^8, 7^{11}, 5^{10}, 4^{13}, 2^{16})\) and \(k = 3, a_1 = 3, a_2 = 5\).

Clearly, \(\lambda\) is enumerated by \(B(3, 5, 5, 708)\). We have 
\(\beta_1 = (81, 66^2, 57, 36^2, 30, 21^2, 12, 6^2)\), \(\beta_2 = (75, 50, 35, 25^2, 20^2, 10^2)\)
so that 
\[
\frac{\beta_1}{a_1} = (27, 22^2, 19, 12^2, 10, 7^2, 4^2) \text{ and } \frac{\beta_2}{a_2} = (15, 10, 7, 5^2, 4^2, 2^2).
\]

We now apply the Glaisher map to \(\beta_1/a_1\) and \(\beta_2/a_2\) and obtain
\[
\phi^{-1} \left( \frac{\beta_1}{a_1} \right) = (22^2, 19, 10, 7^2, 4^2, 2^2, 1^{27}),
\]
\[
\phi^{-1} \left( \frac{\beta_2}{a_2} \right) = (10, 7, 5^2, 4^2, 2^2).
\]

Multiplying each part of \(\phi^{-1}(\beta_1/a_1)\) by \(a_1\) and each part of \(\phi^{-1}(\beta_2/a_2)\) by \(a_2\), and then taking the union results in the image of \(\lambda\) as
\((66^2, 57, 50, 35, 25^2, 21^2, 20^2, 12, 10^2, 6^2, 3^{27})\)
a partition enumerated by \(C(3, 5, 5, 708)\).

We now invert the process. The residue set is
\[
\{3, 5, 6, 10, 12, 15, 20, 21, 24, 25, 30, 33, 35, 39, 40, 42\}.
\]

Thus
\[
\mu_1 = (66^2, 57, 30, 21^2, 12, 6^2, 3^{27}),
\]
\[
\mu_2 = (50, 35, 25, 20^2, 10^2).
\]

Applying the Glaisher map \(\phi\) with
\[
1 = 1 \cdot 3^0, \quad 2 = 2 \cdot 3^0, \quad 5 = 2 \cdot 3^0 + 1 \cdot 3^1, \quad 7 = 1 \cdot 3^0 + 2 \cdot 3^1 \text{ and } 27 = 3^3,
\]
we obtain
\[
\phi \left( \frac{\mu_1}{a_1} \right) = (27, 22^2, 19, 10, 7^2, 4, 2^2),
\]
\[
\phi \left( \frac{\mu_2}{a_2} \right) = (10, 7, 5^2, 4^2, 2^2).
\]

Thus repeating each part in \(\phi(\mu_1/a_1)\) \(a_1\) times, and each part in \(\phi(\mu_2/a_2)\) \(a_2\) times, and taking the union leads us to the following:
\[
\lambda = (27^3, 22^6, 19^3, 15^5, 12^6, 10^8, 7^{11}, 5^{10}, 4^{13}, 2^{16}).
\]

More generally, consider a function in which even parts and odd parts satisfy the ‘Subbarao’ condition separately. Let \(g(m_1, m_2, a_1, a_2, n)\) denote the number of partitions of \(n\) in which even parts occur with multiplicities \(m_1, m_2, m_1 + n_2\) and odd parts occur with multiplicities \(a_1, a_2, a_1 + a_2\). It is clear that \(g(2, 3, 2, 3, n)\) enumerates partitions considered in Theorem 1.3. In the next section, we look at a special case of \(g(m_1, m_2, a_1, a_2, n)\).

3. On the Function \(g(m_1, m_2, 2m_1, 2m_2, n)\)

First, we recall the following \(q\)-series notation. In all cases, \(|q| < 1\),

\[
(a; q)_n = \prod_{i=1}^{n} (1 - aq^i),
\]

\[
(a; q)_\infty = \lim_{n \to \infty} \prod_{i=1}^{n} (1 - aq^i).
\]

From the definition of \(g(m_1, m_2, a_1, a_2, n)\), it is not difficult to note that
\[
\sum_{n=0}^{\infty} g(m_1, m_2, a_1, a_2, n)q^n = \frac{(-q^{2m_1}; q^{2m_1})_\infty (-q^{2m_2}; q^{2m_2})_\infty}{(-q^{a_1}; q^{a_1})_\infty (-q^{a_2}; q^{a_2})_\infty}.
\]

Thus
\[
\sum_{n=0}^{\infty} g(m_1, m_2, 2m_1, 2m_2, n)q^n = \frac{(-q^{2m_1}; q^{2m_1})_\infty (-q^{2m_2}; q^{2m_2})_\infty}{(-q^{2m_1}; q^{2m_1})_\infty (-q^{2m_2}; q^{2m_2})_\infty}
\]
\[
\equiv 1 + 0q + 0q^2 + 0q^3 + \cdots \pmod{2}.
\]

This yields
\[
g(m_1, m_2, 2m_1, 2m_2, n) \equiv 0 \pmod{2} \; \forall \; n \geq 1.
\]

Denote by \(G(m_1, m_2, 2m_1, 2m_2, n)\) the set of partitions enumerated by \(g(m_1, m_2, 2m_1, 2m_2, n)\). Let \(h(m_1, m_2, 2m_1, 2m_2, n)\) be the number of partitions in \(G(m_1, m_2, 2m_1, 2m_2, n)\) satisfying the following property:

All parts are congruent to 0 \(\pmod{4}\) or else for every odd part \(t\) occurring with multiplicity \(2j\), there is an even part \(2t\) having multiplicity \(j\), and for every even part \(s\) congruent to 2 \(\pmod{4}\) occurring with multiplicity \(l\), there is an odd part \(\frac{s}{2}\) which has multiplicity \(2l\).

Then we have the following result.
Theorem 3.1.

\[ h(m_1, m_2, 2m_1, 2m_2, n) \equiv 0 \pmod{2} \quad \forall n \geq 1. \]

Proof. Define the map

\[ \psi : G(m_1, m_2, 2m_1, 2m_2, n) \to G(m_1, m_2, 2m_1, 2m_2, n) \]

as follows:

For \( \lambda = (\lambda_1^{f_1}, \lambda_2^{f_2}, \ldots, \lambda_l^{f_l}) \in G(m_1, m_2, 2m_1, 2m_2, n) \),

\[ \lambda_i^{f_i} \mapsto \left\{ \begin{array}{ll}
(2\lambda_i)^{f_i} / 2, & \text{if } \lambda_i \equiv 1 \pmod{2}; \\
\left(\frac{\lambda_i}{2}\right)^{2f_i}, & \text{if } \lambda_i \equiv 2 \pmod{4}; \\
\lambda_i^{f_i}, & \text{otherwise.}
\end{array} \right. \]

Then

\[ \psi(\lambda) = \bigcup_{i=1}^l \psi(\lambda_i^{f_i}). \]

We claim that \( \psi \) is an involution. To see this, proceed as follows.

If \( \lambda_i \equiv 0 \pmod{4} \), then

\[ \psi(\psi(\lambda_i^{f_i})) = \psi(\lambda_i^{f_i}) = \lambda_i^{f_i}. \]

If \( \lambda_i \equiv 1 \pmod{2} \), then

\[ \psi(\psi(\lambda_i^{f_i})) = \psi((2\lambda_i)^{f_i} / 2) = \left(\frac{2\lambda_i}{2}\right)^{2(f_i / 2)} = \lambda_i^{f_i} \quad \text{(since } 2\lambda_i \equiv 2 \pmod{4}). \]

We have that \( \psi(\psi(\lambda_i^{f_i})) = \lambda_i^{f_i} \).

If \( \lambda_i \equiv 2 \pmod{4} \), then

\[ \psi(\psi(\lambda_i^{f_i})) = \psi \left(\left(\frac{\lambda_i}{2}\right)^{2f_i} \right) = \left(2 \left(\frac{\lambda_i}{2}\right)\right)^{2f_i} = \lambda_i^{f_i} \quad \text{(since } \frac{\lambda_i}{2} \text{ is odd).} \]

Thus \( \psi(\psi(\lambda_i^{f_i})) = \lambda_i^{f_i} \).

Therefore \( \psi(\psi(\lambda)) = \lambda \). Indeed \( \psi \) is an involution on

\[ G(m_1, m_2, 2m_1, 2m_2, n). \]

A careful look into the set reveals that the

\[ h(m_1, m_2, 2m_1, 2m_2, n) \]-partitions

are precisely those partitions fixed under the involution. For this reason, their number must have the same parity as the number of all partitions in \( G(m_1, m_2, 2m_1, 2m_2, n) \). \qed
4. A new partition theorem of Subbarao type

Bijections for Theorem 1.4 have been given in [1, 3, 4]. In the spirit of Theorem 1.4, we state the following new theorem.

**Theorem 4.1.** For \( r \in \mathbb{Z}_{\geq 1} \), let \( E_r(n) \) denote the set of partitions of \( n \) wherein parts appear \( 5, 10, 15, 5r + 2, 5r + 7, 5r + 12, 5r + 17, 10r + 4, 10r + 9, 10r + 14, 15r + 6, 15r + 11, 15r + 21, 20r + 8, 20r + 13, 20r + 18 \) or \( 20r + 23 \) times and \( F_r(n) \) be the set of partitions of \( n \) wherein parts are congruent to \( \pm 5, 10 \pmod{20} \) or \( \pm(5r + 2), \pm(10r + 4) \pmod{25r + 10} \). Then \( |E_r(n)| = |F_r(n)| \).

Although our goal is to exhibit a bijection for this theorem, observe that

\[
\sum_{n \geq 0} |E_r(n)|q^n = \prod_{n=1}^\infty \left( 1 + q^{5n} + q^{10n} + q^{15n} + q^{(5r+2)n} + q^{(5r+7)n} + q^{(5r+12)n} + q^{(5r+17)n} + q^{(10r+4)n} + q^{(10r+9)n} + q^{(10r+14)n} + q^{(15r+6)n} + q^{(15r+11)n} + q^{(15r+16)n} + q^{(20r+8)n} + q^{(20r+13)n} + q^{(20r+18)n} + q^{(20r+23)n} \right)
\]

\[
= \prod_{n=1}^\infty \sum_{j=0}^4 \sum_{i=0}^3 q^{(5r+2)nj} q^{5ni} = \prod_{n=1}^\infty \left( \frac{1 - q^{5(5r+2)n}}{(1 - q^{5n})} \frac{1 - q^{20n}}{(1 - q^{5n})} \right)
\]

\[
= \prod_{s=1}^4 \prod_{n=1}^\infty \frac{1}{1 - q^{(5r+2)(5n-s)}} \prod_{j \equiv 0 \pmod{5}, j \not\equiv 0 \pmod{20}}^\infty \frac{1}{1 - q^j}
\]

\[
= \prod_{s=1}^4 \prod_{n=1}^\infty \frac{1}{1 - q^{(25r+10)n-s(5r+2)}} \prod_{j \equiv 0 \pmod{5}, j \not\equiv 0 \pmod{20}}^\infty \frac{1}{1 - q^j}
\]

\[
= \sum_{n=0}^\infty |F_r(n)|q^n.
\]

We now describe a bijective proof.

**Proof.** Let \( \lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots) \in E_r(n) \). For an integer \( t > 1 \), denote the order of \( a \) with respect to \( t \) by \( \text{ord}_t(a) \) which we define as

\[
\text{ord}_t(a) = \max\{\alpha \in \mathbb{Z}_{\geq 0} : t^\alpha | a\}
\]

and define a map \( \tau : E_r(n) \to F_r(n) \) as follows.
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Case I: \( m_i \equiv 0 \pmod{5} \)

\[ \lambda_i^{m_i} \mapsto \left( \frac{5\lambda_i}{4^s} \right)^{\frac{m_i}{5^t}}, \text{ where } s = \text{ord}_4(\lambda_i). \]

Case II: \( m_i \equiv 1 \pmod{5} \)

\[ \left\{ \begin{array}{l} \left( (5r+2)\lambda_i, \lambda_i^{m_i-15r-6} \right), \quad \text{if } \lambda_i \not\equiv 0 \pmod{5} \\ \left( \left( \frac{(5r+2)\lambda_i}{5^t} \right)^{3-5^t}, \lambda_i^{m_i-15r-6} \right), \quad \text{if } \lambda_i \equiv 0 \pmod{5}. \end{array} \right. \]

Case III: \( m_i \equiv 2 \pmod{5} \)

\[ \lambda_i^{m_i} \mapsto \left\{ \begin{array}{l} \left( (5r+2)\lambda_i, \lambda_i^{m_i-5r-2} \right), \quad \text{if } \lambda_i \not\equiv 0 \pmod{5} \\ \left( \left( \frac{(5r+2)\lambda_i}{5^t} \right)^{5^t}, \lambda_i^{m_i-5r-2} \right), \quad \text{if } \lambda_i \equiv 0 \pmod{5}. \end{array} \right. \]

Case IV: \( m_i \equiv 3 \pmod{5} \)

\[ \lambda_i^{m_i} \mapsto \left\{ \begin{array}{l} \left( (5r+2)\lambda_i, \lambda_i^{m_i-20r-8} \right), \quad \text{if } \lambda_i \not\equiv 0 \pmod{5} \\ \left( \left( \frac{(5r+2)\lambda_i}{4^t} \right)^{4-5^t}, \lambda_i^{m_i-20r-8} \right), \quad \text{if } \lambda_i \equiv 0 \pmod{5}. \end{array} \right. \]

Case V: \( m_i \equiv 4 \pmod{5} \)

\[ \lambda_i^{m_i} \mapsto \left\{ \begin{array}{l} \left( ((5r+2)\lambda_i)^2, \lambda_i^{m_i-10r-4} \right), \quad \text{if } \lambda_i \not\equiv 0 \pmod{5} \\ \left( \left( \frac{(5r+2)\lambda_i}{5^t} \right)^{2-5^t}, \lambda_i^{m_i-10r-4} \right), \quad \text{if } \lambda_i \equiv 0 \pmod{5}. \end{array} \right. \]

where \( t = \text{ord}_5(\lambda_i) \) in Cases II, III, IV and V.

In Cases II, III, IV and V, if \( m_i-15r-6 > 0, m_i-5r-2 > 0, m_i-20r-8 > 0, m_i-10r-4 > 0 \), then apply Case I to the sub-partitions \( \lambda_i^{m_i-15r-6}, \lambda_i^{m_i-5r-2}, \lambda_i^{m_i-20r-8}, \lambda_i^{m_i-10r-4} \), respectively.
The image is then defined as

\[ \tau(\lambda) = \bigcup_{i \geq 1} \tau(\lambda_i^{m_i}). \]

\[ \square \]

Example 4.2. Let \( n = 42 \) and \( r = 4 \). Then

\[ E_4(42) = \{(4^5, 1^{22}), (3^5, 1^{27}), (2^{10}, 1^{22}), (2^5, 1^{32})\}. \]
The sub-partitions 2⁵ and 1₃₂ of (2⁵, 1₃₂) have multiplicities congruent to 0 (mod 5) and 2 (mod 5), respectively. Applying the map τ gives

\[ 2^5 \mapsto 10 \]
\[ 1^{32} \mapsto (22, 5^2) \]

Hence, taking the union of the image parts we obtain that (2⁵, 1₃₂) \mapsto (2², 5⁴). Similarly, applying τ to the remaining partitions gives

\[ (4^5, 1^{22}) \mapsto (2², 5^4) \]
\[ (3^5, 1^{27}) \mapsto (2², 15, 5) \]
\[ (2^{10}, 1^{22}) \mapsto (2², 10, 5²) \]

which are partitions in \( F_4(42) \).

We now describe the inverse of τ. Let \( \mu = (\mu^{\omega_1}_1, \mu^{\omega_2}_2, \ldots, \mu^{\omega_t}_t) \in F_r(n) \). Define a map \( \tau^{-1} : F_r(n) \to E_r(n) \) as follows.

**Case I:** \( \mu_i \not\equiv 0 \pmod{5^r+2} \).

Using the base 4 representation, we write \( \omega_i \) as

\[ \omega_i = 4^p_1 + 4^p_2 + 4^p_3 + \ldots + 4^p_l + \eta \]

where \( p_1 \geq p_2 \geq \ldots \geq p_l \geq 1 \) and \( 0 \leq \eta < 4 \). This representation of \( \omega_i \) in (4.1) comes from its unique quaternary expansion. Just for illustration at this stage, if \( \omega_i = 1 + 2 \cdot 4 + 2 \cdot 4^2 + 3 \cdot 4^3 \), we rewrite \( \omega_i \) as \( 1 + (4 + 4) + (4^2 + 4^2) + (4^3 + 4^3 + 4^3) \) so that \( \eta = 1, p_1 = p_2 = p_3 = 4, p_4 = p_5 = 2, p_6 = p_7 = 1 \).

Having identified, \( p_1, p_2, \ldots, p_l \) and \( \eta \) from (4.1), construct a partition

\[ x_i = \left( 4^{p_1} \times \frac{\mu_i}{5} \right)^5 \cup \left( 4^{p_2} \times \frac{\mu_i}{5} \right)^5 \cup \ldots \cup \left( 4^{p_l} \times \frac{\mu_i}{5} \right)^5 \]

Thus

\[ \mu^{\omega_i}_i \mapsto x_i \cup \left( \frac{\mu_i}{5} \right)^{5\eta} \]

**Case II:** \( \mu_i \equiv 0 \pmod{5^r+2} \).

Using the base 5 representation, we write \( \omega_i \) as

\[ \omega_i = 5^p_1 + 5^p_2 + 5^p_3 + \ldots + 5^p_l + \zeta \]

where \( p_1 \geq p_2 \geq \ldots \geq p_l > 0 \) and \( 0 \leq \zeta < 5 \) (just as in (4.1)).

Construct a partition

\[ y_i = \left( 5^{p_1} \times \frac{\mu_i}{5^r+2} \right)^{5r+2} \cup \left( 5^{p_2} \times \frac{\mu_i}{5^r+2} \right)^{5r+2} \]
\[ \cup \left( 5^{p_3} \times \frac{\mu_i}{5^r+2} \right)^{5r+2} \cup \ldots \cup \left( 5^{p_l} \times \frac{\mu_i}{5^r+2} \right)^{5r+2} \]
Thus
\[ \mu_i^{\omega_i} \mapsto y_i \cup \left( \frac{\mu_i}{5r + 2} \right)^{(5r+2)} \).

The image is then defined as
\[ \tau^{-1}(\mu) = \bigcup_{i \geq 1} \tau^{-1}(\mu_i^{\omega_i}). \]

**Example 4.3.** Let \( n = 42 \) and \( r = 4 \). Then
\[ F_4(42) = \{(22, 5^4), (22, 15, 5), (22, 10^2), (22, 10, 5^2)\}. \]

The sub-partitions 22 and 5\(^4\) of \((22, 5^4)\) have parts congruent to 0 (mod 22) and 5 (mod 22), respectively. Applying the map \( \tau^{-1} \) gives
\[ 22 \mapsto 1^{22} \]
\[ 5^4 \mapsto 4^5. \]

Hence, taking the union of the image parts we obtain that \((22, 5^4) \mapsto (4^5, 1^{22})\). Similarly, applying \( \tau^{-1} \) to the remaining partitions gives
\[ (22, 15, 5) \mapsto (3^5, 1^{27}) \]
\[ (22, 10^2) \mapsto (2^{10}, 1^{22}) \]
\[ (22, 10, 5^2) \mapsto (2^5, 1^{32}) \]

which are partitions in \( E_4(42) \).

**References**


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