Contributions to Discrete Mathematics

Volume 18, Number 2, Pages 129–145
ISSN 1715-0868

EXTENDED INVERSE THEOREMS FOR h-FOLD SUMSETS IN INTEGERS

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ABSTRACT. Let $h \geq 2$, $k \geq 5$ be integers and A be a nonempty finite set of k integers. Very recently, Tang and Xing studied extended inverse theorems for $hk - h + 1 < |hA| \leq hk + 2h - 3$. In this paper, we extend the work of Tang and Xing and study all possible inverse theorems for $hk - h + 1 < |hA| \leq hk + 2h + 1$. Furthermore, we give a range of $|hA|$ for which inverse problems are not possible.

1. INTRODUCTION

Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a nonempty finite set of integers and h be a positive integer. The *h-fold sumset* hA, is defined by

$$hA := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \mathbb{N} \cup \{0\} \text{ for } i = 0, 1, \ldots, k-1 \text{ with } \sum_{i=0}^{k-1} \lambda_i = h \right\}.$$

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $|A|$ denotes the cardinality of the set A. For integers α and β, let

$$\alpha * A = \{\alpha a : a \in A\},$$
$$A + \beta = \{a + \beta : a \in A\}.$$

For $\alpha \leq \beta$, we let $[\alpha, \beta] = \{\alpha, \alpha + 1, \ldots, \beta\}$. The greatest common divisor of the integers $x_1, x_2, \ldots, x_k$ is denoted by $(x_1, x_2, \ldots, x_k)$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers with $a_0 < a_1 < \cdots < a_{k-1}$. Then, we define

$$d(A) := (a_1 - a_0, a_2 - a_0, \ldots, a_{k-1} - a_0),$$

Received by the editors July 21, 2021, and In revised form March 1, 2022.

2000 *Mathematics Subject Classification.* 11P70, 11B75, 11B13.

Key words and phrases. Sumset; h-fold sumset; inverse problem; extended inverse problem.

The first author would like to thank to the Council of Scientific and Industrial Research (CSIR), India for providing the grant to carry out the research with Grant No. 09/143(0925)/2018-EMR-I and the second author wishes to thank to the Science and Engineering Research Board (SERB), India for providing the grant with Grant No. MTR/2019/000299.

and

$$A^{(N)} := \left\{ a_i' = \frac{a_i - a_0}{d(A)} : a_i \in A \right\}.$$

The set $A^{(N)}$ is called the normal form of A. If the set A is such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$, then A is said to be in "normal form". We have $d(A^{(N)}) = 1$. The h-fold sumset hA is translation and dilation invariant of set A, that is,

$$h\left((\alpha * A) + \beta\right) = (\alpha * (hA)) + h\beta.$$

Various types of sumsets in groups are being studied from more than two centuries. The study of minimum cardinality of sumset is called the *direct problem* and the characterization of the underlying set for the minimum cardinality of sumset is known as the *inverse problem* in the area of additive number theory or additive combinatorics. Further, characterization of the underlying set for small deviation from the minimum size of the sumset is called the *extended inverse theorem*. The following are classical results for the minimum cardinality of h-fold sumset hA and for the associated inverse problems.

Theorem 1.1. [4, Theorem 1.4, Theorem 1.6] *Let $h \geq 1$ and A be a nonempty finite set of integers. Then*

$$|hA| \geq h\,|A| - h + 1.$$

This lower bound is best possible. Furthermore, if $|hA|$ attains this lower bound with $h \geq 2$, then A is an arithmetic progression.

Freiman [1, 2] proved some direct and extended inverse theorem for the 2-fold sumset $2A$ which are mentioned below in Theorem 1.2 and Theorem 1.3, respectively.

Theorem 1.2. [2, Theorem 1.10] *Let $k \geq 3$ and $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ with $d(A) = 1$.*

(a) *If $a_{k-1} \leq 2k - 3$, then $|2A| \geq k + a_{k-1}$.*
(b) *If $a_{k-1} \geq 2k - 2$, then $|2A| \geq 3k - 3$.*

Theorem 1.3. [2, Theorem 1.9] *Let A be a finite set of $k \geq 3$ integers. If $|2A| = 2k - 1 + b < 3k - 3$, then A is a subset of an arithmetic progression of length at most $k + b$.*

Lev's [3] generalization of Theorem 1.2 for the h-fold sumset hA is as follows.

Theorem 1.4. [3, Theorem 1] *Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$. Then, for integers h, $k \geq 2$,*

$$|hA| \geq |(h-1)A| + \min\{a_{k-1}, h(k-2) + 1\}.$$

Recently, Tang and Xing [5] proved some extended inverse theorems for hA, where $h \geq 2$. Some other generalizations of Theorem 1.2, can also be seen in [5]. The two main extended inverse theorems of Tang and Xing [5] are mentioned below.

Theorem 1.5. [5, Theorem 1.1] *Let $h \geq 2$ and $k \geq 5$ be integers. Let A be a set of integers with $|A| = k$. If $hk - h + 1 < |hA| \leq hk + h - 2$, then*

$$A^{(N)} = [0, k] \setminus \{x\} \text{ for } 1 \leq x \leq k - 1.$$

Moreover, $|hA| = hk$ for $x = 1$ or $k-1$, and $|hA| = hk+1$ for $2 \leq x \leq k-2$.

Theorem 1.6. [5, Theorem 1.2] *Let $h \geq 2$ and $k \geq 5$ be integers. Let A be a set of integers with $|A| = k$. If $hk + h - 2 < |hA| \leq hk + 2h - 3$, then*

$$A^{(N)} = [0, k + 1] \setminus \{x, y\}, 1 \leq x < y \leq k + 1.$$

Moreover, we have

(a) *$|hA| = hk + h - 1$ for $\{x, y\} = \{1, 2\}, \{k - 1, k\}, \{1, k\}, \{1, 3\}, \{k - 2, k\}$;*
(b) *$|hA| = hk + h$ for $x = 1$ and $4 \leq y \leq k - 1$ when $h \geq 2$; or $2 \leq x \leq k-3$ and $y = k$ when $h \geq 2$, or $\{x, y\} = \{2, 3\}, \{k-2, k-1\}$ when $h = 2$;*
(c) *$|hA| = hk + h + 1$ for $2 \leq x < y \leq k - 1$, except for $\{x, y\} = \{2, 3\}, \{k - 2, k - 1\}$ when $h = 2$.*

Conclusions from Theorem 1.5 and Theorem 1.6 are the following:

(1) $|hA| = hk$ if and only if $A^{(N)} = \{0\} \cup [2, k]$ or $[0, k - 2] \cup \{k\}$.
(2) For $h \geq 3$, $|hA| = hk + 1$ if and only if $A^{(N)} = [0, k] \setminus \{x\}$, where $2 \leq x \leq k - 2$.
(3) $|2A| = 2k + 1$ if and only if $A^{(N)} = [0, k + 1] \setminus \{x, y\}$, where $2 \leq x \leq k - 2, y = k + 1$ or $\{x, y\} = \{1, 2\}, \{k - 1, k\}, \{1, k\}, \{1, 3\}, \{k - 2, k\}$.
(4) For $h \geq 3$, $|hA| = hk + h - 1$ if and only if $A^{(N)} = [0, k + 1] \setminus \{x, y\}$, where $\{x, y\} = \{1, 2\}, \{k - 1, k\}, \{1, k\}, \{1, 3\}, \{k - 2, k\}$.
(5) For $h \geq 3$, $|hA| = hk + h$ if and only if $A^{(N)} = [0, k + 1] \setminus \{x, y\}$, where $x = 1$ and $4 \leq y \leq k - 1$; or $2 \leq x \leq k - 3$ and $y = k$.
(6) For $h \geq 4$, $|hA| = hk + h + 1$ if and only if $A^{(N)} = [0, k + 1] \setminus \{x, y\}$, where $2 \leq x < y \leq k - 1$.

In this paper, we study possible inverse problems, for $hk + h \leq |hA| \leq hk + 2h + 1$. Our main result, Theorem 1.7, extends the work of Tang and Xing [5].

Theorem 1.7. *Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of k integers with $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$. Let $h \geq 2$ and $k \geq 5$ be integers. Then*

(1) *for $k \geq 6$, $|2A| = 2k + 2$ if and only if $A = [0, k + 2] \setminus \{x, y, z\}$ or $A = (k + 2) - ([0, k + 2] \setminus \{x, y, z\})$, where $\{x, y, z\}$ is any one of the sets $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, k + 1\}, \{1, 2, 5\}, \{1, 3, 5\}, \{1, 3, k + 1\}, \{2, 3, k + 2\}, \{1, y, k + 2\}$, where $4 \leq y \leq k - 1$,*

(2) $|3A| = 3k + 4$ *if and only if* $A = [0, k + 2] \setminus \{x, y, z\}$ *or* $A = (k + 2) - ([0, k + 2] \setminus \{x, y, z\})$, *where* $\{x, y, z\}$ *is any one of the sets* $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, k+1\}, \{1, 2, 5\}, \{1, 3, 5\}, \{1, 3, k+1\}, \{x, y, k+2\}$, *where* $2 \leq x < y \leq k - 1$,

(3) *for* $h \geq 4$, $|hA| = hk + 2h - 2$ *if and only if* $A = [0, k + 2] \setminus \{x, y, z\}$ *or* $A = (k + 2) - ([0, k + 2] \setminus \{x, y, z\})$, *where* $\{x, y, z\}$ *is any one of the sets* $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, k + 1\}, \{1, 2, 5\}, \{1, 3, 5\}$, *or* $\{1, 3, k + 1\}$,

(4) *for* $h \geq 4$ *and* $k \geq 5$ *or* $h \geq 3$ *and* $k \geq 6$, $|hA| = hk + 2h - 1$ *if and only if* $A = [0, k+2] \setminus \{x, y, z\}$ *or* $A = (k+2) - ([0, k+2] \setminus \{x, y, z\})$, *where* $\{x, y, z\}$ *is any one of the sets* $\{k - 2, k - 1, k + 1\}$, $\{1, 2, z\}$, *where* $6 \leq z \leq k$, $\{1, 3, z\}$, *where* $6 \leq z \leq k$, $\{1, y, k + 1\}$, *where* $4 \leq y \leq k - 2$,

(5) *for* $h \geq 4$ *and* $k \geq 6$ *or* $h \geq 5$ *and* $k \geq 5$, $|hA| = hk + 2h$ *if and only if* $A = [0, k + 2] \setminus \{x, y, z\}$ *or* $A = (k + 2) - ([0, k + 2] \setminus \{x, y, z\})$, *where* $\{x, y, z\}$ *is any one of the sets* $\{2, 3, k + 1\}$, $\{x, x + 1, k + 1\}$, *where* $3 \leq x \leq k - 4$, $\{1, 3, z\}$, *where* $6 \leq z \leq k$, $\{x, x + 2, k + 1\}$, *where* $2 \leq x \leq k - 4$, $\{1, y, z\}$, *where* $4 \leq y \leq z - 3 \leq k - 3$,

(6) *for* $h \geq 5$ *and* $k \geq 6$ *or* $h \geq 6$ *and* $k \geq 5$, $|hA| = hk + 2h + 1$ *if and only if* $A = [0, k+2] \setminus \{x, y, z\}$ *or* $A = (k+2) - ([0, k+2] \setminus \{x, y, z\})$, *where* $2 \leq x < y < z \leq k$.

To prove Theorem 1.7, we first prove some lemmas (Section 2) and Propositions (Section 3). In Section 4, we prove Theorem 1.7 and give some concluding remarks about the cases where the extended inverse theorems are not possible in Section 5.

2. Lemmas

Note that, if $h = i + 4$ with $i \geq -2$, then $hk + 2h + i = hk + 3h - 4$. So, while studying inverse problem for $|hA| = hk + 2h + i$ where $i \in [-2, h - 4]$ with $h \geq 2$ and $k \geq 6$, it is sufficient to study the inverse problem for $|hA| = hk + 3h - 4$ with $h \geq 2$ and $k \geq 6$.

Lemma 2.1. *Let* h *and* k *be positive integers. Let* A *be a set in normal form having* k *integers. Then the following statements are true.*

(a) *Let* $h \geq 2$, $k \geq 6$ *and* $|hA| = hk + 3h - 4$. *Then* $A \subseteq [0, k + 2]$.

(b) *Let* $h \geq 3$, $|hA| = hk + 2h + i$ *with* $i \in [-2, h - 5]$. *If* $h \geq i + 5$ *and* $k \geq 5$, *then* $A \subseteq [0, k + 2]$.

Proof. By Theorem 1.4, we have

$$\begin{aligned} |hA| &\geq |(h-1)A| + \min\{a_{k-1}, h(k-2)+1\} \\ &\geq |(h-2)A| + \min\{a_{k-1}, h(k-2)+1\} \qquad (2.1)\\ &\qquad + \min\{a_{k-1}, (h-1)(k-2)+1\} \\ &\ \ \vdots \\ &\geq |A| + \min\{a_{k-1}, h(k-2)+1\} + \cdots + \min\{a_{k-1}, 2(k-2)+1\}. \qquad (2.2) \end{aligned}$$

Let $h \geq 2$, $k \geq 6$ and $|hA| = hk + 3h - 4$. If $a_{k-1} \geq 2k-2$, then from (2.1), we have

$$|hA| \geq k + (h-2)(2k-2) + 2k - 3 > hk + 3h - 4,$$

which is not possible. Therefore $a_{k-1} \leq 2k-3$ and again from (2.1),

$$hk + 3h - 4 = |hA| \geq k + (h-1)a_{k-1},$$

hence $a_{k-1} \leq k+2$. This proves (a). Similarly, if $a_{k-1} \geq 2k-2$, then from (2.1), we have

$$|hA| \geq k + (h-2)(2k-2) + 2k - 3 > hk + 2h + i,$$

which is not possible because $h \geq i+5$ and $k \geq 5$. Therefore $a_{k-1} \leq 2k-3$ and again from (2.1),

$$hk + 2h + i = |hA| \geq k + (h-1)a_{k-1},$$

hence $a_{k-1} \leq k+2$. □

Lemma 2.2. *Let $A = [0, x-1] \cup [x+r, y]$, where $x \geq 2$, $r \geq 0$, $x+r \leq y-1$, and $|A| \geq 4$. If $h \geq r+1$, then $hA = [0, hy]$.*

Proof. Clearly, $hA \supseteq [0, h(x-1)] \cup [h(x+r), hy]$. Let $A_1 = \{x-2, x-1\}$ and $A_2 = \{x+r, x+r+1\}$ be subsets of A. Then $h(A_1 \cup A_2) \subseteq hA$. We have

$$\begin{aligned} h(A_1 \cup A_2) &= \bigcup_{l=0}^{h} ((h-l)\,A_1 + lA_2) \\ &= \bigcup_{l=0}^{h} ([(h-l)(x-2), (h-l)(x-1)] + [l(x+r), l(x+r+1)]) \\ &= \bigcup_{l=0}^{h} [hx - 2h + l(r+2), hx - h + l(r+2)]. \end{aligned}$$

If $h \geq r+1$, then $hx - 2h + (l+1)(r+2) \leq hx - h + l(r+2) + 1$. So

$$h(A_1 \cup A_2) = \bigcup_{l=0}^{h} [hx - 2h + l(r+2), hx - h + l(r+2)] = [hx - 2h, hx + h(r+1)].$$

Hence, $hA = [0, hy]$. □

Now we generalize Lemma 2.2 in Lemma 2.3.

Lemma 2.3. *Let* $A = [0, x-1] \cup [x+r, y]$*, where* $x \geq t \geq 2$*,* $r \geq 0$*,* $x + r \leq y - t + 1$ *and* $|A| \geq 2t$*. If* $h \geq (r+t-1)/(t-1)$*, then* $hA = [0, hy]$*.*

Proof. Clearly, $hA \supseteq [0, h(x-1)] \cup [h(x+r), hy]$. Let $A_1 = \{x-t, \ldots, x-2, x-1\}$ and $A_2 = \{x+r, x+r+1, \ldots, x+r+t-1\}$ be subsets of A. Then $h(A_1 \cup A_2) \subseteq hA$, where

$$\begin{aligned} &h(A_1 \cup A_2) \\ &= \bigcup_{l=0}^{h} ((h-l)\, A_1 + lA_2) \\ &= \bigcup_{l=0}^{h} ([(h-l)(x-t), (h-l)(x-1)] + [l(x+r), l(x+r+t-1)]) \\ &= \bigcup_{l=0}^{h} [hx - ht + l(r+t), hx - h + l(r+t)]. \end{aligned}$$

If $h \geq (r+t-1)(t-1)$, then $hx - ht + (l+1)(r+t) \leq hx - h + l(r+t) + 1$. So,

$$h(A_1 \cup A_2) = \bigcup_{l=0}^{h} [hx - ht + l(r+t), hx - h + l(r+t)] = [hx - ht, hx + h(r+t-1)].$$

Hence $hA = [0, hy]$. □

Remark: Putting $t = 2$ in Lemma 2.3 we get Lemma 2.2.

Lemma 2.5. *Let* $A = \{a_0, a_1, \ldots, a_{k-1}\}$ *be a set of integers with* $0 = a_0 < a_1 < \cdots < a_{k-1}$*. If* $0 < a_t \leq m \leq ha_t - 1$ *and* r_m *is the least nonnegative residue of* m *modulo* a_t *then* $m \in hA$*, provided* $a_t + r_m \in 2A$*.*

Proof. Since $0 < a_t \leq m \leq ha_t - 1$, we have $1 \leq (m - r_m)(a_t) \leq h - 1$. Also, $a_t + r_m = a_i + a_j \in 2A$, for some i, j. So, we can write m as

$$m = \underbrace{a_t + \cdots + a_t}_{\frac{m-r_m}{a_t} - 1 \text{ times}} + a_i + a_j + \underbrace{0 + \cdots + 0}_{h - 1 - \frac{m - r_m}{a_t} \text{ times}}.$$

Hence, we get the lemma. □

3. Propositions

Proposition 3.1. (1) *If* $A = \{0\} \cup [x, y]$ *with* $y \geq 2x - 1$*, then* $hA = \{0\} \cup [x, hy]$*.*
(2) *If* $A = [0, y - x] \cup \{y\}$ *with* $y \geq 2x - 1$*, then* $hA = [0, hy - x] \cup \{hy\}$*.*

Now, we find $|hA|$ when $A = [0, k+2] \setminus \{x, y, z\}$ with $|A| = k \geq 5$. If $z = k + 2$, then $A = [0, k+1] \setminus \{x, y\}$ and this was already studied in [5]. So we can assume $a_{k-1} = k + 2$ and $1 \leq x < y < z \leq k + 1$.

Proposition 3.2. *Let $h \geq 2, k \geq 5$ be positive integers and*

$$A = \{a_0, a_1, \ldots, a_{k-1}\} = [0, k+2] \setminus \{x, x+1, x+2\}$$

with $0 = a_0 < a_1 < \cdots < a_{k-1} = k+2$ and $1 \leq x \leq k-1$. Then

(1) *If $x = 1$ or $k-1$, then $|hA| = h(k+2) - 2$.*
(2) *If $x = 2$ or $k-2$, then*

$$|hA| = \begin{cases} h(k+2)+1 & \text{if } h \geq 4 \text{ and } k \geq 5 \\ 3(k+2) & \text{if } h = 3 \text{ and } k \geq 5 \\ 2k+3 & \text{if } h = 2 \text{ and } k \geq 6 \\ 12 & \text{if } h = 2 \text{ and } k = 5. \end{cases}$$

(3) *If $3 \leq x \leq k-3$, then*

$$|hA| = \begin{cases} h(k+2)+1 & \text{if } h \geq 3 \text{ and } k \geq 6 \\ 2(k+2) & \text{if } h = 2,\ x = 3 \text{ or } k-3 \text{ and } k \geq 6 \\ 2k+5 & \text{if } h = 2,\ 4 \leq x \leq k-4 \text{ and } k \geq 8. \end{cases}$$

Proof. (1) Let $x = 1$. Then $A = \{0\} \cup [4, k+2]$. So, by Proposition 3.1, we have

$$hA = \{0\} \cup [4, h(k+2)],$$

which gives

$$|hA| = h(k+2) - 2.$$

Let $x = k-1$ then $A = \{k+2\} - (\{0\} \cup [4, k+2])$. Hence,

$$|hA| = |h(\{k+2\} - (\{0\} \cup [4, k+2]))| = |h(\{0\} \cup [4, k+2])| = h(k+2) - 2.$$

(2) Let $x = 2$. Then $A = [0,1] \cup [5, k+2]$. Let $h \geq 3$. Clearly, $hA \supseteq [0, h] \cup [5h, h(k+2)]$. Let $5 \leq m \leq 5h-1$ and r_m be the least nonnegative residue of m modulo 5. If $r_m \in \{0, 1, 2, 3\}$, then $5 + r_m \in 2A$. Hence, by Lemma 2.5, $m \in hA$.
Let $r_m = 4$. If $\frac{m-r_m}{5} \geq 2$, then

$$m = \underbrace{5 + \cdots + 5}_{\frac{m-r_m}{5}-2 \text{ times}} + 7 + 7 + \underbrace{0 + \cdots + 0}_{h-\frac{m-r_m}{5} \text{ times}};$$

and if $(m - r_m)/5 = 1$, then m= 9, where

$$9 = 7 + 1 + 1 \in hA \text{ for all } h \geq 3, k \geq 5,$$

also

$$9 = 8 + 1 \in hA \text{ for all } h \geq 2, k \geq 6.$$

Hence $hA = [0, h] \cup [5, h(k+2)]$ for all $h \geq 3$ and $k \geq 5$, and

$$2A = \begin{cases} [0,2] \cup [5, 2(k+2)] & \text{if } k \geq 6 \\ [0,2] \cup [5,8] \cup [10,14] & \text{if } k = 5. \end{cases}$$

Hence

$$hA = \begin{cases} [0, h(k+2)] & \text{if } h \geq 4 \text{ and } k \geq 5 \\ [0,3] \cup [5, 3(k+2)] & \text{if } h = 3 \text{ and } k \geq 5 \\ [0,2] \cup [5, 2(k+2)] & \text{if } h = 2 \text{ an } k \geq 6 \\ [0,2] \cup [5,8] \cup [10,14] & \text{if } h = 2 \text{ and } k = 5, \end{cases}$$

and

$$|hA| = \begin{cases} h(k+2)+1 & \text{if } h \geq 4 \text{ and } k \geq 5 \\ 3(k+2) & \text{if } h = 3 \text{ and } k \geq 5 \\ 2k+3 & \text{if } h = 2 \text{ and } k \geq 6 \\ 12 & \text{if } h = 2 \text{ and } k = 5. \end{cases}$$

If $x = k-2$, then we can write $A = (k+2) - ([0,1] \cup [5, k+2])$, which is a translation of $[0,1] \cup [5, k+2]$, the set for which we have the result. So, we have the result for $x = k-2$ also, as $|hA|$ is translation invariant.

(3) Let $3 \leq x \leq k-3$. Then $A = [0, x-1] \cup [x+3, k+2]$ and $k \geq 6$. If $h \geq 3$ then by Lemma 2.3 (for $t = r = 3$), we get $hA = [0, h(k+2)]$. Let $h = 2$. If $x = 3$, then $A = [0,2] \cup [6, k+2]$ and if $x = k-3$, then $A = (k+2) - ([0,2] \cup [6, k+2])$. For $A = [0,2] \cup [6, k+2]$, we have $2A = [0,4] \cup [6, 2(k+2)]$ and hence $|2A| = 2(k+2)$. Similarly, if $A = (k+2) - ([0,2] \cup [6, k+2])$, then $|2A| = 2(k+2)$. If $4 \leq x \leq k-4$ then $k \geq 8$ and using Lemma 2.3 for $r = 3, t = 4$, and $A = [0, x-1] \cup [x+3, k+2]$, we have $2A = [0, 2(k+2)]$. Hence

$$|hA| = \begin{cases} h(k+2)+1 & \text{if } h \geq 3 \text{ and } k \geq 6 \\ 2(k+2) & \text{if } h = 2,\ x = 3 \text{ or } k-3 \text{ and } k \geq 6 \\ 2k+5 & \text{if } h = 2,\ 4 \leq x \leq k-4 \text{ and } k \geq 8. \end{cases}$$

This completes the proof of the proposition. □

Now, if exactly two of x, y, z are consecutive, then it is sufficient to assume x and y are consecutive and for the case y, z are consecutive, we take $A = (k+2) - ([0, k+2] \setminus \{x, x+1, z\})$.

Proposition 3.3. *Let $h \geq 2, k \geq 5$ be positive integers and*

$$A = \{a_0, a_1, \ldots, a_{k-1}\} = [0, k+2] \setminus \{x, x+1, z\}$$

with $0 = a_0 < a_1 < \cdots < a_{k-1} = k+2$ and $1 \leq x \leq z-3 \leq k-2$.

(1) *If $\{x, y, z\} = \{1, 2, 4\}$, then $|hA| = h(k+2) - 2$.*
(2) *If $\{x, y, z\} = \{1, 2, k+1\}$, then $|hA| = h(k+2) - 2$.*
(3) *If $\{x, y, z\} = \{1, 2, z\}$, where $5 \leq z \leq k$, then*

$$|hA| = \begin{cases} h(k+2) - 2 & \textit{if } z = 5 \\ h(k+2) - 1 & \textit{if } z \neq 5. \end{cases}$$

(4) *If* $\{x, y, z\} = \{2, 3, 5\}$ *or* $\{x, y, z\} = \{k-3, k-1, k\}$, *then*

$$|hA| = \begin{cases} h(k+2)+1 & \text{if } h \geq 3 \text{ and } k \geq 5 \\ h(k+2) & \text{if } h = 2 \text{ and } k \geq 6 \\ 13 & \text{if } h = 2 \text{ and } k = 5. \end{cases}$$

(5) *If* $\{x, y, z\} = \{2, 3, k+1\}$, *then*

$$|hA| = \begin{cases} h(k+2) & \text{if } h \geq 3 \text{ and } k \geq 5 \\ 2k+3 & \text{if } h = 2 \text{ and } k \geq 5. \end{cases}$$

(6) *If* $\{x, y, z\} = \{2, 3, z\}$ *where* $6 \leq z \leq k$, *then*

$$|hA| = \begin{cases} h(k+2)+1 & \text{if } h \geq 3 \text{ and } k \geq 6 \\ 2k+4 & \text{if } h = 2 \text{ and } k \geq 6. \end{cases}$$

(7) *If* $\{x, y, z\} = \{k-2, k-1, k+1\}$, *then* $|hA| = h(k+2) - 1$.
(8) *If* $\{x, y, z\} = \{x, x+1, k+1\}$ *where* $3 \leq x \leq k-3$, *then* $|hA| = h(k+2)$.
(9) *If* $\{x, y, z\} = \{x, x+1, x+3\}$ *where* $x \geq 3$ *and* $x + 3 \leq k$, *then* $|hA| = h(k+2) + 1$.
(10) *If* $\{x, y, z\} = \{x, x+1, z\}$ *where* $3 \leq x \leq k-4$ *and* $k \geq z \geq x+4$, *then* $|hA| = h(k+2) + 1$.

Proof. (1) Let $\{x, y, z\} = \{1, 2, 4\}$. Then $A = \{0, 3\} \cup [5, k+2]$. So,

$$hA \supseteq [5h, h(k+2)].$$

Let $5 \leq m \leq 5h - 1$ and r_m be the least nonnegative residue of m modulo 5. Then clearly $5 + r_m \in 2A$. Hence by Lemma 2.5, $m \in hA$. So

$$hA = \{0, 3\} \cup [5, h(k+2)],$$

and

$$|hA| = h(k+2) - 2 \text{ for } h \geq 2 \text{ and } k \geq 5.$$

(2) Let $\{x, y, z\} = \{1, 2, k+1\}$. Then $A = \{0\} \cup [3, k] \cup \{k+2\}$. Clearly $hA \supseteq \{0\} \cup [3h, hk] \cup \{h(k+2)\}$. Since $k \geq 5$, by Proposition 3.1, we have

$$h(\{0\} \cup [3, k]) = \{0\} \cup [3, hk] \subseteq hA.$$

Furthermore, $h([3, k] \cup \{k+2\}) \subseteq hA$, where

$$\begin{aligned} h([3, k] \cup \{k+2\}) &= h((k+2) - (\{0\} \cup [2, k-1])) \\ &= h(k+2) - h(\{0\} \cup [2, k-1]) \\ &= h(k+2) - (\{0\} \cup [2, h(k-1)]) \text{ (by Proposition 3.1)} \\ &= [3h, h(k+2) - 2] \cup \{h(k+2)\}. \end{aligned}$$

Hence

$$hA = \{0\} \cup [3, h(k+2) - 2] \cup \{h(k+2)\},$$

and

$$|hA| = h(k+2) - 2, \text{ for all } h \geq 2 \text{ and } k \geq 5.$$

(3) Let $\{x, y, z\} = \{1, 2, z\}$, where $5 \leq z \leq k$. Then $A = \{0\} \cup [3, z-1] \cup [z+1, k+2]$. We have

$$2A = \begin{cases} \{0,3,4\} \cup [6, 2(k+2)] & \text{if } z = 5 \\ \{0\} \cup [3, 2(k+2)] & \text{if } z \geq 6. \end{cases}$$

Let $h \geq 3$. Then

$$\begin{aligned} h([3, z-1] \cup [z+1, k+2]) &= h(([0, z-4] \cup [z-2, k-1]) + 3) \\ &= h([0, z-4] \cup [z-2, k-1]) + 3h \\ &= [0, h(k-1)] + 3h \text{ (by Lemma 2.3)} \\ &= [3h, h(k+2)]. \end{aligned}$$

Let $6 \leq m \leq 3h-1$ and r_m be the least nonnegative residue of m modulo 3. Then $2 \leq (m - r_m)/3 \leq h-1$.
If $r_m = 0$ or 1, then $3 + r_m \in 2A$ and hence by Lemma 2.5, $m \in hA$ for $h \geq 2$.
If $r_m = 2$, then writing

$$m = \underbrace{3 + \cdots + 3}_{\frac{m-r_m}{3} - 2 \text{ times}} + 4 + 4 + \underbrace{0 + \cdots + 0}_{h - \frac{m-r_m}{3} \text{ times}},$$

we get $m \in hA$. Hence

$$hA = \begin{cases} \{0\} \cup [3,4] \cup [6, h(k+2)] & \text{if } z = 5 \\ \{0\} \cup [3, h(k+2)] & \text{if } z \geq 6, \end{cases}$$

and

$$|hA| = \begin{cases} h(k+2) - 2 & \text{if } z = 5 \\ h(k+2) - 1 & \text{if } z \geq 6. \end{cases}$$

(4) Let $\{x, y, z\} = \{2, 3, 5\}$. Then $A = \{0, 1, 4\} \cup [6, k+2]$. We have

$$\begin{aligned} h(\{4\} \cup [6, k+2]) &= h((\{0\} \cup [2, k-2]) + 4) \\ &= h(\{0\} \cup [2, k-2]) + 4h \\ &= (\{0\} \cup [2, h(k-2)]) + 4h \text{ (by Proposition 3.1)} \\ &= \{4h\} \cup [4h+2, h(k+2)]. \end{aligned}$$

For $h \geq 3$,

$$4h + 1 = 4(h-3) + 6 + 6 + 1 \in hA.$$

Let $4 \leq m \leq 4h-1$ and r_m be the least nonnegative residue of m modulo 4. Then $4 + r_m \in 2A$. So, by Lemma 2.5, $m \in hA$. Hence

$$hA = \begin{cases} [0, h(k+2)] & \text{if } h \geq 3 \text{ and } k \geq 5 \\ [0,2] \cup [4, h(k+2)] & \text{if } h = 2 \text{ and } k \geq 6 \\ [0,2] \cup [4,8] \cup [10,14] & \text{if } h = 2 \text{ and } k = 5, \end{cases}$$

and

$$|hA| = \begin{cases} h(k+2)+1 & \text{if } h \geq 3 \text{ and } k \geq 5 \\ h(k+2) & \text{if } h = 2 \text{ and } k \geq 6 \\ 13 & \text{if } h = 2 \text{ and } k = 5. \end{cases}$$

(5) Let $\{x, y, z\} = \{2, 3, k+1\}$. Then $A = [0,1] \cup [4,k] \cup \{k+2\}$. For $h \geq 3$, $h([0,1] \cup [4,k]) = [0, hk]$ (by Lemma 2.3) and

$$\begin{aligned} h([4,k] \cup \{k+2\}) &= h((k+2) - (\{0\} \cup [2, k-2])) \\ &= h(k+2) - h(\{0\} \cup [2, k-2]) \\ &= h(k+2) - (\{0\} \cup [2, h(k-2)]) \text{ (by Proposition 3.1)} \\ &= [4h, h(k+2)-2] \cup \{h(k+2)\}. \end{aligned}$$

So,

$$hA = \begin{cases} [0, h(k+2)-2] \cup \{h(k+2)\} & \text{if } h \geq 3 \text{ and } k \geq 5 \\ [0,2] \cup [4, 2k+2] \cup \{2k+4\} & \text{if } h = 2 \text{ and } k \geq 5, \end{cases}$$

and

$$|hA| = \begin{cases} h(k+2) & \text{if } h \geq 3 \text{ and } k \geq 5 \\ 2k+3 & \text{if } h = 2 \text{ and } k \geq 5. \end{cases}$$

(6) Let $\{x, y, z\} = \{2, 3, z\}$ where $6 \leq z \leq k$. Then $A = [0,1] \cup [4, z-1] \cup [z+1, k+2]$. By Lemma 2.3, we have

$$hA = \begin{cases} [0, h(k+2)] & \text{if } h \geq 3 \text{ and } k \geq 6 \\ [0,2] \cup [4, 2k+4] & \text{if } h = 2 \text{ and } k \geq 6, \end{cases}$$

and

$$|hA| = \begin{cases} h(k+2)+1 & \text{if } h \geq 3 \text{ and } k \geq 6 \\ 2k+4 & \text{if } h = 2 \text{ and } k \geq 6. \end{cases}$$

(7) Let $\{x, y, z\} = \{k-2, k-1, k+1\}$. Then $A = [0, k-3] \cup \{k, k+2\}$. We have

$$2A = [0, 2k] \cup \{2k+2, 2k+4\}.$$

Let $h \geq 3$, $k+2 \leq m \leq h(k+2)-1$ and r_m be the least nonnegative residue of m modulo $(k+2)$. If $r_m \in \{0, 1, \ldots, k-3, k\}$, then by Lemma 2.5, $m \in hA$. Let $k+2 \leq m \leq (h-1)(k+2)-1$. Then

$1 \le (m - r_m)/(k+2) \le h-2$. If $r_m = k-2$ or $k-1$ or $k+1$, then writing m respectively by

$$m = \underbrace{(k+2)+\cdots+(k+2)}_{\frac{m-r_m}{k+2} \text{ times}} + (k-3) + 1 + \underbrace{0+\cdots+0}_{h-2-\frac{m-r_m}{k+2} \text{ times}}$$

$$m = \underbrace{(k+2)+\cdots+(k+2)}_{\frac{m-r_m}{k+2} \text{ times}} + (k-3) + 2 + \underbrace{0+\cdots+0}_{h-2-\frac{m-r_m}{k+2} \text{ times}}$$

$$m = \underbrace{(k+2)+\cdots+(k+2)}_{\frac{m-r_m}{k+2} \text{ times}} + k + 1 + \underbrace{0+\cdots+0}_{h-2-\frac{m-r_m}{k+2} \text{ times}},$$

we see that $m \in hA$. Furthermore, $(h-1)(k+2), (h-1)(k+2)+1, \ldots, h(k+2)-5$ are all in hA. We also have

$$h(k+2) - 4 = (h-2)(k+2) + 2k \in hA.$$

Hence

$$hA = [0, h(k+2)-4] \cup \{h(k+2)-2, h(k+2)\} \text{ for } h \ge 3 \text{ and } k \ge 5,$$

and so

$$|hA| = h(k+2) - 1 \text{ for } h \ge 2 \text{ and } k \ge 5.$$

(8) Let $\{x, y, z\} = \{x, x+1, k+1\}$, where $3 \le x \le k-3$. Then $A = [0, x-1] \cup [x+2, k] \cup \{k+2\}$. By Lemma 2.3, we have

$$h([0, x-1] \cup [x+2, k]) = [0, hk] \text{ for } h \ge 3.$$

Furthermore, $h([x+2, k] \cup \{k+2\}) = [h(x+2), h(k+2)-2] \cup \{h(k+2)\}$ by Proposition 3.1. Hence $hA = [0, h(k+2)-2] \cup \{h(k+2)\}$ for $h \ge 3$. Now, we also have $2A = [0, 2k+2] \cup \{2k+4\}$. So,

$$|hA| = h(k+2) \text{ for } h \ge 2 \text{ and } k \ge 6.$$

(9) Let $\{x, y, z\} = \{x, x+1, x+3\}$, where $x \ge 3$ and $x+3 \le k$. Then $A = [0, x-1] \cup \{x+2\} \cup [x+4, k+2]$. We have

$$\begin{aligned} h(\{x+2\} \cup [x+4, k+2]) &= h((\{0\} \cup [2, k-x]) + x + 2) \\ &= h(\{0\} \cup [2, k-x]) + h(x+2) \\ \text{(by Proposition 3.1)} &= \{h(x+2)\} \cup [h(x+2)+2, h(k+2)] \end{aligned}$$

and

$$h(x+2) + 1 = (h-3)(x+2) + (x+4) + (x+4) + (x-1) \in hA, \text{ for } h \ge 3.$$

Furthermore,

$$\begin{aligned} h([0, x-1] \cup \{x+2\}) &= h(\{x+2\} - (\{0\} \cup [3, x+2])) \\ &= h(x+2) - h(\{0\} \cup [3, x+2]) \\ &= h(x+2) - (\{0\} \cup [3, h(x+2)]) \\ &= [0, h(x+2)-3] \cup \{h(x+2)\}, \end{aligned}$$

$$h(x+2) - 2 = (h-3)(x+2) + (x-2) + (x+2) + (x+4) \in hA$$

and

$$h(x+2)-1=(h-3)(x+3)+(x-1)+(x+2)+(x+4)\in hA, \text{ for } h\geq 3.$$

We also have, $2A=[0,2k+4]$. Hence, for $h\geq 2, k\geq 5$, $hA=[0,h(k+2)]$ and $|hA|=h(k+2)+1$.

(10) Let $\{x,y,z\}=\{x,x+1,z\}$, where $3\leq x\leq k-4$ and $x+4\leq z\leq k$. Then $A=[0,x-1]\cup[x+2,z-1]\cup[z+1,k+2]$. So, $hA=[0,h(k+2)]$ for $h\geq 2$ (by Lemma 2.2). Hence $|hA|=h(k+2)+1$.

This completes the proof of the proposition. □

Proposition 3.4. *Let $h\geq 2, k\geq 5$ be positive integers and*

$$A=\{a_0,a_1,\ldots,a_{k-1}\}=[0,k+2]\setminus\{x,y,z\}$$

with $0=a_0<a_1<\cdots<a_{k-1}=k+2$, $y-x\geq 2$ and $z-y\geq 2$. Then the following statements are true.

(1) *If $\{x,y,z\}=\{1,3,5\}$ or $\{k-3,k-1,k+1\}$, then $|hA|=h(k+2)-2$.*
(2) *If $\{x,y,z\}=\{1,k-1,k+1\}$ or $\{1,3,k+1\}$, then $|hA|=h(k+2)-2$.*
(3) *If $\{x,y,z\}=\{1,3,z\}$, where $6\leq z\leq k$ or $\{x,k-1,k+1\}$, where $2\leq x\leq k-4$, then $|hA|=h(k+2)$.*
(4) *If $\{x,y,z\}=\{x,x+2,x+4\}$, where $2\leq x\leq k-4$, then $|hA|=h(k+2)+1$.*
(5) *If $\{x,y,z\}=\{x,x+2,k+1\}$, where $2\leq x\leq k-4$ or $\{x,y,z\}=\{1,y,y+2\}$, where $4\leq y\leq k-2$, then $|hA|=h(k+2)$.*
(6) *If $\{x,y,z\}=\{x,x+2,z\}$, where $2\leq x\leq z-5\leq k-5$ or $\{x,y,z\}=\{x,y,y+2\}$, where $2\leq x\leq y-3\leq k-5$, then $|hA|=h(k+2)+1$.*
(7) *If $\{x,y,z\}=\{1,y,k+1\}$, where $4\leq y\leq k-2$, then $|hA|=h(k+2)-1$.*
(8) *If $\{x,y,z\}=\{1,y,z\}$, where $4\leq y\leq k-3$ and $y+3\leq z\leq k$ or $\{x,y,z\}=\{x,y,k+1\}$, where $2\leq x\leq y-3\leq k-5$, then $|hA|=h(k+2)$.*
(9) *If $2\leq x\leq y-3\leq z-6\leq k-6$, then $|hA|=h(k+2)+1$.*

Proof. (1) Let $x=1,y=3,z=5$. Then $A=\{0,2,4\}\cup[6,k+2]$. We have

$$\begin{aligned}h(\{4\}\cup[6,k+2])&=h(\{0\}\cup[2,k-2]+4)\\&=h(\{0\}\cup[2,k-2])+4h\\&=\{4h\}\cup[4h+2,h(k+2)].\end{aligned}$$

Furthermore,

$$4h+1=\underbrace{4+\cdots+4}_{h-2\text{ times}}+7+2\in hA \text{ for all } h\geq 2.$$

Let $4\leq m\leq 4h-1$ and r_m be the least nonnegative residue of m modulo 4. If $r_m\in\{0,2,3\}$, then $4+r_m\in 2A$, so by Lemma 2.5,

$m \in hA$.

If $r_m = 1$ and $\frac{m-r_m}{4} \geq 2$, then

$$m = \underbrace{4 + \cdots + 4}_{\frac{m-r_m}{4}-2 \text{ times}} +7 + 2 \in hA.$$

If $\frac{m-r_m}{4} = 1$, then $m = 5 \notin hA$. Hence

$$hA = \{0, 2, 4\} \cup [6, h(k + 2)],$$

and

$$|hA| = h(k + 2) - 2.$$

Now, let $x = k - 3, y = k - 1, z = k + 1$. Then $A = [0, k - 4] \cup \{k - 2, k, k + 2\} = (k + 2) - (\{0, 2, 4\} \cup [6, k + 2])$. Hence

$$|hA| = h(k + 2) - 2.$$

(2) Let $\{x, y, z\} = \{1, k-1, k+1\}$. Then $A = \{0\} \cup [2, k-2] \cup \{k, k+2\}$. Let $h \geq 3$, $k + 2 \leq m \leq (h - 1)(k + 2) - 1$ and r_m be the least nonnegative residue of m modulo $k + 2$. Then $1 \leq \frac{m-r_m}{k+2} \leq h - 2$. If $r_m \in \{0, 1, 2, \ldots, k - 2, k\}$, then $(k + 2) + r_m \in 2A$, so by Lemma 2.5, $m \in hA$. If $r_m = k - 1, k + 1$, then write m respectively, by

$$m = \underbrace{(k + 2) + \cdots + (k + 2)}_{\frac{m-r_m}{k+2}-1 \text{ times}} +k + (k - 2) + 3 + \underbrace{0 + \cdots + 0}_{h-\frac{m-r_m}{k+2}-2 \text{ times}} \in hA,$$

$$m = \underbrace{(k + 2) + \cdots + (k + 2)}_{\frac{m-r_m}{k+2}-1 \text{ times}} +(k + 2) + (k - 2) + 3 + \underbrace{0 + \cdots + 0}_{h-\frac{m-r_m}{k+2}-2 \text{ times}} \in hA.$$

Furthermore, $(h-1)(k+2), (h-1)(k+2)+1, \ldots, h(k+2)-4, h(k+2)-2, h(k+2)$, all are in hA. If $i \geq 2$, then $(h-i)(k+2)+ik \leq h(k+2)-4$. So, $h(k + 2) - 3$ and $h(k + 2) - 1$ are not in hA. We also have, $2A = \{0\} \cup [2, 2k] \cup \{2k + 2, 2k + 4\}$. Hence

$$hA = \{0\} \cup [2, h(k + 2) - 4] \cup \{h(k + 2) - 2, h(k + 2)\},$$

and

$$|hA| = h(k + 2) - 2.$$

Now, let $x = 1, y = k - 1, z = k + 1$. Then $A = (k + 2) - (\{0\} \cup [2, k - 2] \cup \{k, k + 2\})$. Hence

$$|hA| = h(k + 2) - 2.$$

(3) Let $\{x, y, z\} = \{1, 3, z\}$, where $6 \leq z \leq k$. Then $A = \{0, 2\} \cup [4, z - 1] \cup [z + 1, k + 2]$. We have,

$$\begin{aligned} h([4, z - 1] \cup [z + 1, k + 2]) &= h([0, z - 5] \cup [z - 3, k - 2]) + 4h \\ &= [4h, h(k + 2)], \text{ (by Lemma 2.2)}. \end{aligned}$$

Let $4 \leq m \leq 4h - 1$ and r_m be the least nonnegative residue of m modulo 4. Then $4 + r_m \in 2A$, so by Lemma 2.5, $m \in hA$. Hence

$$hA = \{0, 2\} \cup [4, h(k + 2)],$$

and

$$|hA| = h(k+2).$$

Let $\{x,y,z\} = \{x, k-1, k+1\}$, where $2 \leq x \leq k-4$. Then $A = (k+2) - (\{0,2\} \cup [4, z-1] \cup [z+1, k+2])$, where $6 \leq z = k+2-x \leq k$. Hence

$$|hA| = h(k+2).$$

(4) Let $\{x,y,z\} = \{x, x+2, x+4\}$, where $2 \leq x \leq k-4$. Then $A = [0, x-1] \cup \{x+1, x+3\} \cup [x+5, k+2]$.
We have

$$\begin{aligned} h(\{x+3\} \cup [x+5, k+2]) &= h(\{0\} \cup [2, k-x-1] + (x+3)) \\ &= h(\{0\} \cup [2, k-x-1]) + h(x+3) \\ \text{(by Proposition 3.1)} &= (\{0\} \cup [2, h(k-x-1)]) + h(x+3) \\ &= \{h(x+3)\} \cup [h(x+3)+2, h(k+2)]. \end{aligned}$$

Moreover, $h(x+3)+1 = (h-2)(x+3)+(x+6)+(x+1) \in hA$. Let $x+3 \leq m \leq h(x+3)-1$ and r_m be the least nonnegative integer residue of m modulo $x+3$. If $r_m \in \{0, 1, \ldots, x-1, x+1\}$, then $(x+3)+r_m \in 2A$. If $r_m = x$ or $x+2$, then we write

$$(x+3)+x = (x+5)+(x-2)$$

and

$$(x+3)+(x+2) = (x+6)+(x-1).$$

So, by Lemma 2.5, $m \in hA$. Hence

$$hA = [0, h(k+2)]$$

and

$$|hA| = h(k+2)+1 \text{ for } h \geq 2 \text{ and } k \geq 6.$$

(5) Let $\{x,y,z\} = \{x, x+2, k+1\}$, where $2 \leq x \leq k-4$. Then $A = [0, x-1] \cup \{x+1\} \cup [x+3, k] \cup \{k+2\}$. We have

$$\begin{aligned} h([x+3, k] \cup \{k+2\}) &= h((k+2) - (\{0\} \cup [2, k-x-1])) \\ &= h(k+2) - (\{0\} \cup [2, h(k-x-1)]) \\ &= [h(x+3), h(k+2)-2] \cup \{h(k+2)\} \subseteq hA. \end{aligned}$$

Let $x+3 \leq m \leq h(x+3)-1$ and r_m be the least nonnegative integer residue of m modulo $x+3$. If $r_m \in \{0, 1, \ldots, x-1, x+1\}$, then $(x+3)+r_m \in 2A$ and if $r_m = x$ or $x+2$, then we write

$$(x+3)+x = (x+4)+(x-1)$$

$$(x+3)+(x+2) = (x+4)+(x+1).$$

So, by Lemma 2.5, $m \in hA$. Hence

$$hA = [0, h(k+2)-2] \cup \{h(k+2)\}$$

and

$$|hA| = h(k+2).$$

Now, let $\{x,y,z\} = \{1,y,y+2\}$, where $4 \le y \le k-2$. Then $A = \{k+2\} - ([0,k+2] \setminus \{x,x+2,k+1\})$ with $2 \le x = k-y \le k-4$. Hence

$$|hA| = h(k+2).$$

(6) Let $\{x,y,z\} = \{x,x+2,z\}$, where $2 \le x \le z-5 \le k-5$. Then $A = [0,x-1] \cup \{x+1\} \cup [x+3,z-1] \cup [z+1,k+2]$. Since $x+3 \le z-2$, then by Lemma 2.2, $h([x+3,z-1] \cup [z+1,k+2]) = [h(x+3), h(k+2)]$. The arguments similar to the ones in case 5, show that $m \in hA$, if $x+3 \le m \le h(x+3)-1$. Hence $hA = [0, h(k+2)]$ and $|hA| = h(k+2)+1$.

(7) Let $\{x,y,z\} = \{1,y,k+1\}$, where $4 \le y \le k-2$. Then $A = \{0\} \cup [2,y-1] \cup [y+1,k] \cup \{k+2\}$. We have,

$$h(\{0\} \cup [2,y-1]) = \{0\} \cup [2, h(y-1)] \text{ (by Proposition 3.1)},$$

$$h([2,y-1] \cup [y+1,k]) = [2h, 2k] \text{ (by Lemma 2.2)}$$

and

$$h([y+1,k] \cup \{k+2\}) = h(k+2) - h(\{0\} \cup [2,k+1-y])$$
$$\text{(by Proposition 3.1)} = [h(y+1), h(k+2)-2] \cup \{hk+2\}.$$

Hence $hA = \{0\} \cup [2, h(k+2)-2] \cup \{h(k+2)\}$ and $|hA| = h(k+2)-1$.

(8) Let $\{x,y,z\} = \{1,y,z\}$, where $4 \le y \le k-3$ and $y+3 \le z \le k$. Then $A = \{0\} \cup [2,y-1] \cup [y+1,z-1] \cup [z+1,k+2]$. The arguments in this case are similar to the one in case 7. Hence $hA = \{0\} \cup [2, h(k+2)]$ and $|hA| = h(k+2)$. If $\{x,y,z\} = \{x,y,k+1\}$, where $2 \le x \le y-3 \le k-5$ then it is a translation of $\{0\} \cup [2,y-1] \cup [y+1,z-1] \cup [z+1,k+2]$, where $4 \le y \le k-3$ and $y+3 \le z \le k$. Hence $|hA| = h(k+2)$.

(9) Let $2 \le x \le y-3 \le z-6 \le k-6$. Then $A = [0,x-1] \cup [x+1,y-1] \cup [y+1,z-1] \cup [z+1,k+2]$. So by Lemma 2.2, $hA = [0, h(k+2)]$, hence $|hA| = h(k+2)+1$.

□

4. Proof of Theorem 1.7

Proof. Let $k \ge 6$ and $|2A| = 2k+2$. Then $|2A| = 2k+2 \le 3k-4$ and by Theorem 1.3, $A \subseteq [0,k+2]$. In rest of the cases, $A \subseteq [0,k+2]$ due to Lemma 2.1. Now, Theorem 1.6, Proposition 3.2, 3.3 and 3.4 give the structure of A. This completes the proof of Theorem 1.7. □

5. Conclusion

We know that, $|3A| = 3k-2$ if and only if $|2A| = 2k-1$, the reason being A is an arithmetic progression. Theorem 1.4 gives a relation between the sizes of hA and $(h-1)A$. Let $k \ge 6$. Then by Theorem 1.5, 1.6 and 1.7, we have the following:

(1) $3k+1 \le |3A| \le 3k+2$ if and only if $|2A| = 2k+1$.
(2) If $|3A| = 3k+3$, then $|2A| = 2k+2$.

(3) If $|3A| = 3k + 4$, then $2k + 2 \leq |2A| \leq 2k + 3$.

A remark of Tang and Xing ([5], Remark 1.3), states that there is no set A such that $|3A| = 3|A| - 1$. Similar to this remark, we have the following observation.

(1) If $h \geq 3$ and $k \geq 5$, then there is no set A such that $hk - h + 2 \leq |hA| \leq hk - 1$ (See Lemma 2.1 and Proposition 3.1 in [5]).
(2) If $h \geq 4$ and $k \geq 5$, then there is no set A such that $hk + 2 \leq |hA| \leq hk + h - 2$ (See Lemma 2.1 and Proposition 3.1 in [5]).
(3) If $h \geq 5$ and $k \geq 5$, then there is no set A such that $hk + h + 2 \leq |hA| \leq hk + 2h - 3$ (See Lemma 2.1 and Proposition 3.2 - 3.4 in [5]).
(4) If $h \geq i + 4$ where $i \in [2, h - 4]$ and $k \geq 5$, then there is no set A such that $hk + 2h + 2 \leq |hA| = hk + 2h + i \leq hk + 3h - 4$ (See Lemma 2.1 and Proposition 3.2 – 3.4).

Acknowledgment

The authors thank the referees for providing useful comments/corrections on the paper.

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