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# ON A GENERALIZED BASIC SERIES AND ROGERS–RAMANUJAN TYPE IDENTITIES

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Abstract. In this paper, we give the generalization of MacMahon's type combinatorial identities. A generalized $q$-series is interpreted as the generating function of two different combinatorial objects, viz., restricted $n$-color partitions and weighted lattice paths which give entirely new Rogers–Ramanujan–MacMahon type combinatorial identities. This result yields an infinite class of 2-way combinatorial identities which further extends the work of Agarwal and Goyal. We also discuss the bijective proof of the main result. Forbye, eight particular cases are also discussed which give a combinatorial interpretation of eight entirely new Rogers–Ramanujan type identities.

## 1. Introduction

An efficient procedure for analyzing partitions lies in their graphical representations. In 1989, Agarwal and Bressoud [4] studied a novel category of weighted lattice paths and they interpreted certain basic hypergeometric series with multiple indices of summation as generating functions for weighted lattice paths. They also provided a one-to-one correspondence between weighted lattice paths of weight $\lambda$ and $n$-color partitions of $\lambda$. In recent years, the literature witnessed substantial growth in the study of graphical representation of ordinary partitions, $(n+t)$-color partitions, split $(n+t)$-color partitions, etc. To depict partitions graphically, several mathematicians use different combinatorial tools such as Ferrers graphs [9], weighted lattice paths [4], associated lattice paths [6, 8], modified lattice paths [7, 14], split lattice paths [12], etc.
The following $q$-series (1.1)–(1.3) of Slater [17] and $q$-series (1.4)–(1.6) of Rogers [15] were combinatorially explained by Agarwal [1, 2], Agarwal and

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Goyal [5, 13] respectively in terms of $n$-color partitions and weighted lattice paths:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q^2)_n(q^2;q^2)_n}=\frac{(q^2,q^8,q^{10};q^{10})_\infty(q^6,q^{14};q^{20})_\infty}{(q;q)_\infty}, \tag{1.1}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q^2)_n(q^2;q^2)_n}=\frac{(q,q^9,q^{10};q^{10})_\infty(q^8,q^{12};q^{20})_\infty}{(q;q)_\infty}, \tag{1.2}$$

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q^2)_n(q^2;q^2)_n}=\frac{(q,q^7,q^8;q^8)_\infty(q^6,q^{10};q^{16})_\infty}{(q;q)_\infty}, \tag{1.3}$$

$$\sum_{n=0}^{\infty}\frac{q^{3n^2}}{(q;q^2)_n(q^4;q^4)_n}=\frac{(-q^3,-q^5,-q^7;q^{10})_\infty}{(q^4,q^6;q^{10})_\infty}, \tag{1.4}$$

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q^2)_n(q^4;q^4)_n}=\frac{(-q^3,-q^7,-q^{11};q^{14})_\infty}{(q^2,q^6,q^8,q^{12};q^{14})_\infty}, \tag{1.5}$$

$$\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_n(q^4;q^4)_n}=\frac{(-q^5,-q^7,-q^9;q^{14})_\infty}{(q^4,q^6,q^8,q^{10};q^{14})_\infty}. \tag{1.6}$$

where

$$(a;q)_n=\prod_{r=0}^{n-1}(1-aq^r),\,(a;q)_\infty=\lim_{n\to\infty}(a;q)_n,$$

$$(a_1,a_2,\ldots,a_t;z)_\infty=\prod_{r=1}^{t}(a_r;z)_\infty \text{ and } |q|<1.$$

Agarwal and Goyal [6] also succeeded to expand the above mentioned outcomes to 3-way combinatorial identities by the use of associated lattice paths.
Let us consider the following generalized $q$-series:

$$\sum_{n=0}^{\infty}\frac{q^{\alpha n^2+\beta n}}{(q^s;q^{2s})_n(q^{2m};q^{2m})_n}, \tag{1.7}$$

where $\alpha,s,m\in\mathbb{Z}^+$ and $\beta\in\mathbb{Z}^+\cup\{0\}$.
The crux of this paper is to interpret the above generalized basic series (1.7) as the generating function of certain restricted classes of $n$-color partitions and weighted lattice paths. Hence, these results provide an infinite set of combinatorial identities and thereby provide many Rogers–Ramanujan–MacMahon type combinatorial identities. The main result in this paper is stated as:

**Theorem 1.1.** *Let $G^{(\alpha,\beta)}_{(s,m)}(\lambda)$ represent the number of $n$-color partitions of $\lambda$ in such a way that*

(1) *odd parts appear with odd subscripts and even with even, each subscript is greater than or equal to* $\alpha+\beta$ *and it is congruent to* $\alpha+\beta \pmod{s}$*,*
(2) $e \equiv i \pmod{2m}$*, provided* $e_i$ *is the least or the single part in the partition,*
(3) *the weighted difference of any two consecutive parts is greater than or equal to* $-2\beta$ *and is congruent to* $-2\beta \pmod{2m}$*.*

*Let* $H^{(\alpha,\beta)}_{(s,m)}(\lambda)$ *represent the number of lattice paths of weight* $\lambda$ *which begin from* $(0,0)$ *and*

(1) *they have no valley above height* $0$*,*
(2) *the height of every peak is greater than or equal to* $\alpha$ *and is congruent to* $\alpha \pmod{s}$*,*
(3) *there is a plain of length congruent to* $\beta \pmod{2m}$ *in the beginning of the path and the length of the other plains, if any, are congruent to* $0 \pmod{2m}$*.*

*Then, we have* $G^{(\alpha,\beta)}_{(s,m)}(\lambda) = H^{(\alpha,\beta)}_{(s,m)}(\lambda)$ *for all* $\lambda$ *and*

$$\sum_{\lambda=0}^{\infty} G^{(\alpha,\beta)}_{(s,m)}(\lambda)\, q^{\lambda} = \sum_{\lambda=0}^{\infty} H^{(\alpha,\beta)}_{(s,m)}(\lambda)\, q^{\lambda} = \sum_{n=0}^{\infty} \frac{q^{\alpha n^2+\beta n}}{(q^s;q^{2s})_n (q^{2m};q^{2m})_n}, \tag{1.8}$$

*where* $\alpha, s, m \in \mathbb{Z}^+$ *and* $\beta \in \mathbb{Z}^+ \cup \{0\}$*.*

Before we proceed further, we list some necessary definitions and terminologies from the literature.

**Definition 1.2** ([3])**.** *An n-color partition of a positive integer* $\lambda$ *is a partition where a part of size n, can appear with n different colors represented by subscripts:* $n_1, n_2, n_3, \ldots, n_n$*.*

For example, the $n$-color partitions of 3 are $3_1, 3_2, 3_3, 2_1+1_1, 2_2+1_1, 1_1+1_1+1_1$.

**Definition 1.3.** *The weighted difference of two parts* $x_i$ *and* $y_j$*,* $(x \geq y)$*, is defined by* $x-y-i-j$ *and is expressed by* $((x_i - y_j))$*.*

**Definition 1.4** ([4])**.** *All weighted lattice paths will be of finite lengths and they lie in the first quadrant. They will start on the y-axis or on the x-axis and end on the x-axis. Only three steps are allowed:*
***Northeast:*** *From* $(a,b)$ *to* $(a+1,b+1)$*.*
***Southeast:*** *From* $(a,b)$ *to* $(a+1,b-1)$*, only allowed if* $b>0$*.*
***Horizontal:*** *From* $(a,0)$ *to* $(a+1,0)$*, only allowed along x-axis.*
*Every lattice path is either empty or ends with a southeast step: from* $(a,1)$ *to* $(a+1,0)$*.*
*To illustrate the lattice paths, the following terminology is used.*
***Peak:*** *Either a vertex on the y-axis which is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.*
***Valley:*** *A vertex preceded by a southeast step and followed by a northeast*

*step. Remember that a southeast step followed by a horizontal step followed by a northeast step does not form a valley.*
***Mountain:*** *A section of the path which begins on either the $x$- or $y$-axis, which terminates on the $x$-axis, and which does not touch the $x$-axis throughout in between the end points. There is at least one peak in a mountain and the number of peaks may exceed one.*
***Plain:*** *A section of the path including only horizontal steps which begins either on the $y$-axis or at a vertex preceded by a southeast step and terminates at a vertex followed by a northeast step.*
*The height of a vertex is its $y$-coordinate, the weight of a vertex is its $x$-coordinate and the weight of a lattice path is the sum of the weights of its peaks.*

## 2. Proof of Theorem 1.1

Theorem 1.1 will be proved in three steps. Firstly, we will prove that the utmost right-hand side of (1.8) produces the $n$-color partitions enumerated by $G_{(s,m)}^{(\alpha,\beta)}(\lambda)$. Then we will illustrate that the utmost right-hand side of (1.8) also produces the weighted lattice paths enumerated by $H_{(s,m)}^{(\alpha,\beta)}(\lambda)$. In the end, we will set up a one to one correspondence between $n$-color partitions enumerated by $G_{(s,m)}^{(\alpha,\beta)}(\lambda)$ and weighted lattice paths enumerated by $H_{(s,m)}^{(\alpha,\beta)}(\lambda)$.

**Step 1**: Let $G_{(s,m)}^{(\alpha,\beta)}(\ell,\lambda)$ represent the number of partitions of $\lambda$ enumerated by $G_{(s,m)}^{(\alpha,\beta)}(\lambda)$ with the added restriction that there be exactly $\ell$ parts. First of all, we will prove the following recurrence relation:

$$\begin{aligned} G_{(s,m)}^{(\alpha,\beta)}(\ell,\lambda) &= G_{(s,m)}^{(\alpha,\beta)}(\ell,\lambda-2m\ell) + G_{(s,m)}^{(\alpha,\beta)}(\ell-1,\lambda-2\alpha\ell+\alpha-\beta) \\ &\quad + G_{(s,m)}^{(\alpha,\beta)}(\ell,\lambda-2s\ell+s) - G_{(s,m)}^{(\alpha,\beta)}(\ell,\lambda-2\ell(s+m)+s). \end{aligned}$$

To prove this, we split the partitions enumerated by $G_{(s,m)}^{(\alpha,\beta)}(\ell,\lambda)$ into three classes:
(i) those that do not contain $k_k$ as a part,
(ii) those that contain $(\alpha+\beta)_{(\alpha+\beta)}$ as a part, and
(iii) those that contain $k_k$, $k>\alpha+\beta$ as a part.
We now transform the partitions in class (i) by subtracting $2m$ from each part ignoring the subscripts. Obviously, this transformation will not disturb the inequalities between the parts and so the transformed partition will be of the type enumerated by $G_{(s,m)}^{(\alpha,\beta)}(\ell,\lambda-2m\ell)$. Next, we transform the partitions in class (ii) by deleting the part $(\alpha+\beta)_{(\alpha+\beta)}$ and then subtracting $2\alpha$ from all the remaining parts ignoring the subscripts. The transformed partition will be of the type enumerated by $G_{(s,m)}^{(\alpha,\beta)}(\ell-1,\lambda-2\alpha\ell+\alpha-\beta)$. Finally, we transform the partitions in class (iii) by replacing $k_k$ by $(k-s)_{(k-s)}$

and then subtracting $2s$ from all the remaining parts. This will produce a partition of $\lambda - 2s\ell + s$ into $\ell$ parts. It is important to note here that by this transformation we get only those partitions of $\lambda - 2s\ell + s$ into $\ell$ parts which contain a part of the form $k_k$. Therefore, the actual number of partitions which belong to class (iii) is $G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda - 2s\ell + s) - G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda - 2\ell(s+m) + s)$, where $G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda - 2\ell(s+m) + s)$ is the number of partitions of $\lambda - 2s\ell + s$ into $\ell$ parts which are free from the parts like $k_k$. The above transformations are clearly reversible and so establish a bijection between the partitions enumerated by $G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda)$ and those enumerated by

$$\begin{aligned} &G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda - 2m\ell) + G^{(\alpha,\beta)}_{(s,m)}(\ell - 1, \lambda - 2\alpha\ell + \alpha - \beta) \\ &\quad + G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda - 2s\ell + s) - G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda - 2\ell(s + m) + s). \end{aligned}$$

This generates the identity

$$\begin{aligned} G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda) &= G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda - 2m\ell) + G^{(\alpha,\beta)}_{(s,m)}(\ell - 1, \lambda - 2\alpha\ell + \alpha - \beta) \\ &\quad + G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda - 2s\ell + s) - G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda - 2\ell(s + m) + s). \end{aligned} \tag{2.1}$$

Let

$$f^{(\alpha,\beta)}_{(s,m)}(z; q) = \sum_{\lambda=0}^{\infty}\sum_{\ell=0}^{\infty} G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda) z^{\ell} q^{\lambda}. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\begin{aligned} f^{(\alpha,\beta)}_{(s,m)}(z; q) = &\sum_{\lambda=0}^{\infty}\sum_{\ell=0}^{\infty} \Big[G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda - 2m\ell) + G^{(\alpha,\beta)}_{(s,m)}(\ell - 1, \lambda - 2\alpha\ell + \alpha - \beta) \\ &+ G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda - 2s\ell + s) - G^{(\alpha,\beta)}_{(s,m)}(\ell, \lambda - 2\ell(s + m) + s)\Big] z^{\ell} q^{\lambda}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \implies f^{(\alpha,\beta)}_{(s,m)}(z; q) &= f^{(\alpha,\beta)}_{(s,m)}(zq^{2m}; q) + zq^{\alpha+\beta} f^{(\alpha,\beta)}_{(s,m)}(zq^{2\alpha}; q) + q^{-s} f^{(\alpha,\beta)}_{(s,m)}(zq^{2s}; q) \\ &\quad - q^{-s} f^{(\alpha,\beta)}_{(s,m)}(zq^{2(s+m)}; q). \end{aligned} \tag{2.4}$$

Since $f^{(\alpha,\beta)}_{(s,m)}(z; q)$ is analytic function for $|q| < 1$ and $|z| < |q|^{-1}$, we have

$$f^{(\alpha,\beta)}_{(s,m)}(z; q) = \sum_{n=0}^{\infty} \gamma_n(q) z^n. \tag{2.5}$$

Employing (2.5) into (2.4) and then comparing the coefficients of $z^n$ on each side of the resulting identity, we deduce that

$$\gamma_n(q)[(1 - q^{2mn}) - q^{-s+2sn}(1 - q^{2mn})] = q^{\beta+2\alpha n-\alpha}\gamma_{n-1}(q).$$

Therefore

$$\gamma_n(q) = \frac{q^{\alpha(2n-1)+\beta}\gamma_{n-1}(q)}{(1 - q^{2mn})(1 - q^{2s(n-1)+s})}.$$

On iterating and using $\gamma_0(q) = 1$, we obtain that

$$\gamma_n(q) = \frac{q^{\alpha n^2+\beta n}}{(q^{2m};q^{2m})_n(q^s;q^{2s})_n}.$$

Hence

$$f^{(\alpha,\beta)}_{(s,m)}(z;q) = \sum_{n=0}^{\infty} \frac{q^{\alpha n^2+\beta n}}{(q^s;q^{2s})_n(q^{2m};q^{2m})_n} z^n.$$

Therefore

$$\begin{aligned}\sum_{\lambda=0}^{\infty} G^{(\alpha,\beta)}_{(s,m)}(\lambda)q^{\lambda} &= \sum_{\lambda=0}^{\infty}\sum_{l=0}^{\infty} G^{(\alpha,\beta)}_{(s,m)}(l,\lambda)q^{\lambda} = f^{(\alpha,\beta)}_{(s,m)}(1;q)\\ &= \sum_{n=0}^{\infty} \frac{q^{\alpha n^2+\beta n}}{(q^s;q^{2s})_n(q^{2m};q^{2m})_n}.\end{aligned}$$

**Step II**: In this step, we will prove that

$$\sum_{\lambda=0}^{\infty} H^{(\alpha,\beta)}_{(s,m)}(\lambda)\; q^{\lambda} = \sum_{n=0}^{\infty} \frac{q^{\alpha n^2+\beta n}}{(q^s;q^{2s})_n(q^{2m};q^{2m})_n}. \tag{2.6}$$

In

$$\frac{q^{\alpha n^2+\beta n}}{(q^s;q^{2s})_n(q^{2m};q^{2m})_n},$$

the factor $q^{\alpha n^2+\beta n}$ generates a lattice path from $(0,0)$ to $(\beta+2\alpha n, 0)$ having $n$ peaks each of height $\alpha$ and a plain of length $\beta$ in the beginning of the path. For example, $\beta = 2, n = 4, \alpha = 3$, the path begin as

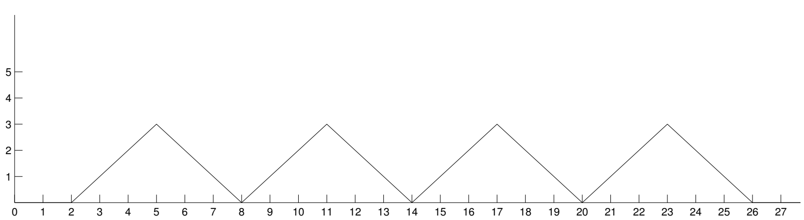



FIGURE 1. Four peaks each of height 3, three valleys each at height zero and a plain of length 2 in the beginning of the path.

In Figure 1, we take two consecutive peaks say, $jth$ and $(j+1)st$ and denote them by $A_1$ and $A_2$ respectively.

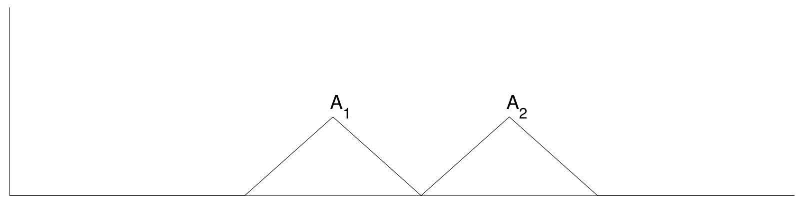



FIGURE 2. Two peaks of the same height.

Clearly, in Figure 2

$$A_1 \equiv (\beta + \alpha(2j-1), \alpha) \quad \text{and} \quad A_2 \equiv (\beta + \alpha(2j+1), \alpha).$$

The factor $1/(q^{2m}; q^{2m})_n$ generates $n$ nonnegative multiples of $2m$, say,

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$$

which are encoded by inserting $a_n$ horizontal steps in front of the first mountain and $a_j - a_{j+1}$ horizontal steps in front of the $(n-j+1)st$ mountain for $1 \leq j \leq n-1$. Thus the $x$-coordinate of the $jth$ peak is increased by $a_{n-j+1}$ and the $x$-coordinate of the $(j+1)st$ peak is increased by $a_{n-j}$. Figure 2 now turns into Figure 3.

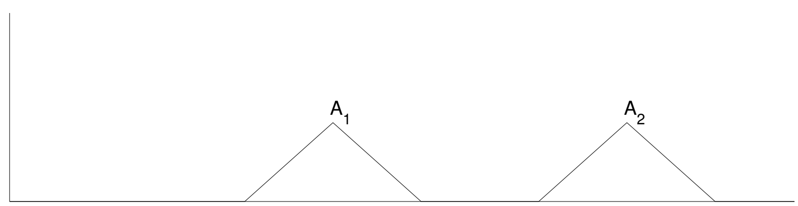



FIGURE 3. Two peaks separated by a plain of length multiple of $2m$.

Thus two consecutive peaks $A_1$ and $A_2$ becomes

$$A_1 \equiv (\beta + \alpha(2j-1) + a_{n-j+1}, \alpha) \quad \text{and} \quad A_2 \equiv (\beta + \alpha(2j+1) + a_{n-j}, \alpha).$$

The factor $1/(q^s; q^{2s})_n$ generates $n$ nonnegative odd multiples of $s$, say

$$b_1 \times s, b_2 \times 3s, b_3 \times 5s, \ldots, b_n \times (2n-1)s.$$

These can be encoded by raising the height of $jth$ peak by $sb_{n-j+1}$, $1 \leq j \leq n$. So, $jth$ peak grows to height $sb_{n-j+1}+\alpha$. Each increase by one in height of a given peak increases its weight by one and the weight of each subsequent peak by two. Figure 3 is altered to Figure 4 or Figure 5 depending on whether $b_{n-j} > b_{n-j+1}$ or $b_{n-j} < b_{n-j+1}$. When $b_{n-j} = b_{n-j+1}$, Figure 3 is not altered.

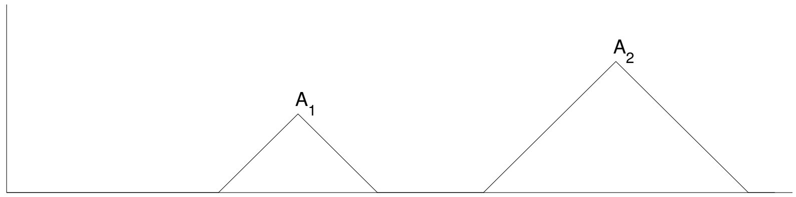



FIGURE 4. $A_2$ has more height than $A_1$ for $b_{n-j} > b_{n-j+1}$.

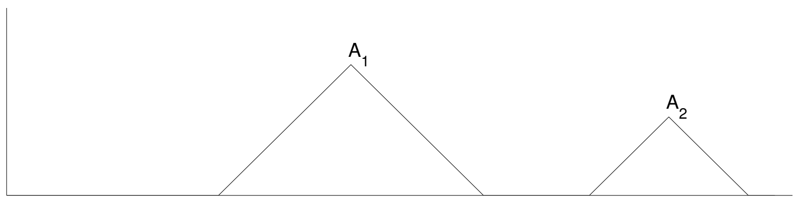



FIGURE 5. $A_1$ has more height than $A_2$ for $b_{n-j} < b_{n-j+1}$.

In this way, we can uniquely generate each lattice path enumerated by $H^{(\alpha,\beta)}_{(s,m)}(\lambda)$. This demonstrates (2.6).

**Step III**: We now establish a bijection between the lattice paths enumerated by $H^{(\alpha,\beta)}_{(s,m)}(\lambda)$ and the $n$-color partitions enumerated by $G^{(\alpha,\beta)}_{(s,m)}(\lambda)$. We do this by encoding every path as the sequence of the weights of the peaks with each weight subscripted by the height of the respective peak. Therefore, if in the final figure, we represent the $jth$ and $(j+1)st$ peak by $C_u$ and $D_v$, $(D \geq C)$, respectively, then

$$\begin{aligned}
C =&\beta + \alpha(2j-1) + a_{n-j+1} + 2s(b_n + b_{n-1} + \cdots + b_{n-j+2}) + sb_{n-j+1},\\
D =&\beta + \alpha(2j+1) + a_{n-j} + 2s(b_n + b_{n-1} + \cdots + b_{n-j+1}) + sb_{n-j},\\
u =&sb_{n-j+1} + \alpha,\\
v =&sb_{n-j} + \alpha.
\end{aligned}$$

Depending on the parity of $\alpha, \beta$ and $sb_{n-j+1}$, the following eight cases arise:

CASE 1: If $\alpha$, $\beta$ and $sb_{n-j+1}$ are odd, then $C$ is odd and $u$ is even. So, $C$ and $u$ have opposite parity.

CASE 2: If $\alpha$, $\beta$ are odd and $sb_{n-j+1}$ is even, then $C$ is even and $u$ is odd. So, $C$ and $u$ have opposite parity.

CASE 3: If $\alpha$ is even, $\beta$ and $sb_{n-j+1}$ are odd, then $C$ is even and $u$ is odd. So, $C$ and $u$ have opposite parity.

CASE 4: If $\alpha$ is even, $\beta$ is odd, $sb_{n-j+1}$ is even, then $C$ is odd and $u$ is even. So, $C$ and $u$ have opposite parity.

CASE 5: If $\alpha$, $\beta$ and $sb_{n-j+1}$ are even, then $C$ is even and $u$ is even. So, $C$ and $u$ have the same parity.
CASE 6: If $\alpha$, $\beta$ are even and $sb_{n-j+1}$ is odd, then $C$ is odd and $u$ is odd. So, $C$ and $u$ have the same parity.
CASE 7: If $\alpha$ is odd, $\beta$ is even, $sb_{n-j+1}$ is odd, then $C$ is even and $u$ is even. So, $C$ and $u$ have the same parity.
CASE 8: If $\alpha$ is odd, $\beta$ and $sb_{n-j+1}$ are even, then $C$ is odd and $u$ is odd. So, $C$ and $u$ have the same parity.
To get the same parity, we replace $C_u$ by $C_{u+\beta}$ in cases 1–4. If we replace $C_u$ by $C_{u+\beta}$ in cases 5–8, then parity remains the same. Thus, to maintain the uniformity of the result, we replace $C_u$ by $C_{u+\beta}$ in all the cases.
On similar lines, $D_v$ will be replaced by $D_{v+\beta}$. Therefore, we conclude that even parts appear with even subscripts and odd with odd. Now, let $u' = u + \beta = sb_{n-j+1} + \alpha + \beta$ and $v' = v + \beta = sb_{n-j} + \alpha + \beta$. From this, it is clear that all subscripts are greater than or equal to $\alpha + \beta$ and $u' \equiv \alpha + \beta \pmod{s}$. Now, weighted difference of $C_{u'}$ and $D_{v'}$ is equal to

$$((D_{v'} - C_{u'})) = D - C - v' - u' = -2\beta + (a_{n-j} - a_{n-j+1}).$$

Clearly, the weighted difference is greater than or equal to $-2\beta$ and it is congruent to $-2\beta \pmod{2m}$.
Next, say $C_{u'}$ is the first peak, then it will correspond to the least part in the corresponding $n$-color partition or to the singleton part if the $n$-color partition contains only one part and in both of the cases

$$C - u' = a_n \equiv 0 \pmod{2m}.$$

This gives $C \equiv u' \pmod{2m}$. To check the reverse implication, we take two $n$-color parts of a partition enumerated by $G^{(\alpha,\beta)}_{(s,m)}(\lambda)$, say $E_x$ and $F_y$. Let $B_1 \equiv (E, x - \beta)$ and $B_2 \equiv (F, y - \beta)$ be the associated peaks in the corresponding lattice path.

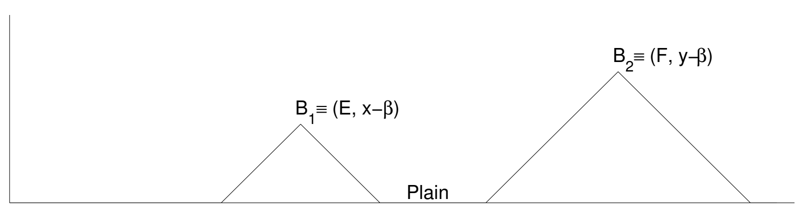



FIGURE 6. Two peaks separated by a plain.

The length of the plain between the two peaks is

$$F - E - x - y + 2\beta = ((F_y - E_x)) + 2\beta \equiv 0 \pmod{2m}.$$

Also, there cannot be a valley above height 0. This can be proved by contradiction. Let us assume a valley $V$ at height $\delta$ ($\delta > 0$) between the peaks $B_1$ and $B_2$.

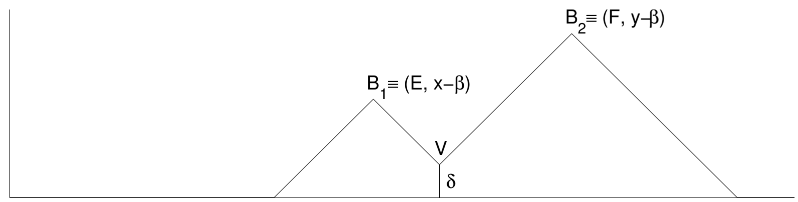



FIGURE 7. Two peaks and a Valley at height $\delta$.

Clearly, there is a descent of $x-\delta-\beta$ from $B_1$ to $V$ and an ascent of $y-\delta-\beta$ from $V$ to $B_2$. This implies

$$\begin{aligned} F = E + (x-\delta-\beta) + (y-\delta-\beta) &\implies F - E - x - y = -2\delta - 2\beta \\ &\implies ((F_y - E_x)) = -2\delta - 2\beta. \end{aligned}$$

Now,

$$((F_y - E_x)) \geq -2\beta \implies -2\delta - 2\beta \geq -2\beta \implies -2\delta \geq 0 \implies \delta = 0.$$

This confirms, there is no valley above height 0.
Now in (1.8), the extra factor $q^{\beta n}$ puts $\beta$ horizontal steps in front of the first peak. This makes the length of the plain (which is in the beginning of the path) congruent to $\beta \pmod{2m}$. This completes the proof of Theorem 1.1.

## 3. Rogers–Ramanujan–MacMahon type combinatorial identities

For some particular values of $\alpha$, $\beta$, $s$ and $m$, the generalized series (1.7) yields the following eight Rogers–Ramanujan type identities. These identities are also found in Bailey [10], Chu and Zhang [11] and Slater's compendium [17].

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^4)_n(q^2;q^2)_n} = \frac{(q^{12},q^{16},q^{28};q^{28})_\infty}{(q^2;q^2)_\infty}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^4)_n(q^4;q^4)_n} = \frac{(q^4,q^{16},q^{20};q^{20})_\infty(q^{12},q^{28};q^{40})_\infty}{(q^2;q^2)_\infty}, \tag{3.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q^4)_n(q^4;q^4)_n} = \frac{(q^2,q^{18},q^{20};q^{20})_\infty(q^{16},q^{24};q^{40})_\infty}{(q^2;q^2)_\infty}, \tag{3.3}$$

$$\sum_{n=0}^{\infty} \frac{q^{4n^2}}{(q^2;q^4)_n(q^4;q^4)_n} = \frac{(q^2,q^{14},q^{16};q^{16})_\infty(q^{12},q^{20};q^{32})_\infty}{(q^2;q^2)_\infty}, \tag{3.4}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^4)_n(q^2;q^2)_n} = \frac{(-q^2;q^2)_\infty(q^4,q^{10},q^{14};q^{14})_\infty}{(q^2;q^2)_\infty(-q^2,-q^{12};q^{14})_\infty}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} \frac{q^{4n^2}}{(q^2;q^4)_n(q^8;q^8)_n} = \frac{(-q^6,-q^{14},-q^{22};q^{28})_\infty}{(q^4,q^{12},q^{16},q^{24};q^{28})_\infty}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} \frac{q^{4n(n+1)}}{(q^2;q^4)_n(q^8;q^8)_n} = \frac{(-q^{10},-q^{14},-q^{18};q^{28})_\infty}{(q^8,q^{12},q^{16},q^{20};q^{28})_\infty}, \tag{3.7}$$

$$\sum_{n=0}^{\infty} \frac{q^{6n^2}}{(q^2;q^4)_n(q^8;q^8)_n} = \frac{(-q^6,-q^{10},-q^{14};q^{20})_\infty}{(q^8,q^{12};q^{20})_\infty}. \tag{3.8}$$

The $q$-series (3.1)–(3.8) have their combinatorial counterparts in the following theorems, respectively.

**Theorem 3.1.** *Let $X_1(\lambda)$ represent the number of ordinary partitions of $\lambda$ into parts congruent to $\pm 2, \pm 4, \pm 6, \pm 8, \pm 10, 14 \pmod{28}$. Then*

$$X_1(\lambda) = G^{(2,0)}_{(2,1)}(\lambda) = H^{(2,0)}_{(2,1)}(\lambda), \quad \textit{for all } \lambda.$$

*where $G^{(2,0)}_{(2,1)}(\lambda)$, $H^{(2,0)}_{(2,1)}(\lambda)$ are as described in Theorem 1.1.*

**Theorem 3.2.** *Let $X_2(\lambda)$ represent the number of ordinary partitions of $\lambda$ into parts congruent to $\pm 2, \pm 6, \pm 8, \pm 10, \pm 14, \pm 18 \pmod{40}$. Then*

$$X_2(\lambda) = G^{(2,0)}_{(2,2)}(\lambda) = H^{(2,0)}_{(2,2)}(\lambda), \quad \textit{for all } \lambda.$$

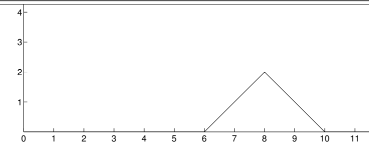
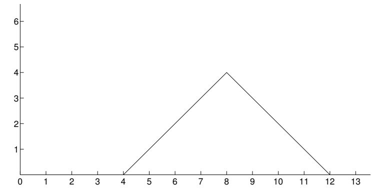
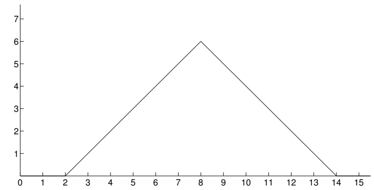
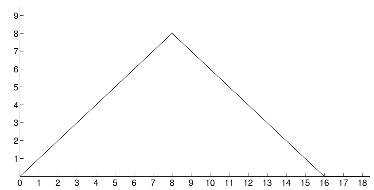
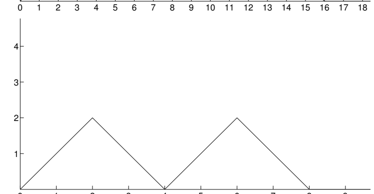
*where $G^{(2,0)}_{(2,2)}(\lambda)$, $H^{(2,0)}_{(2,2)}(\lambda)$ are as described in Theorem 1.1.*

**Theorem 3.3.** *Let $X_3(\lambda)$ represent the number of ordinary partitions of $\lambda$ into parts congruent to $\pm 4, \pm 6, \pm 8, \pm 10, \pm 12, \pm 14 \pmod{40}$. Then*

$$X_3(\lambda) = G^{(2,2)}_{(2,2)}(\lambda) = H^{(2,2)}_{(2,2)}(\lambda), \quad \textit{for all } \lambda.$$

*where $G^{(2,2)}_{(2,2)}(\lambda)$, $H^{(2,2)}_{(2,2)}(\lambda)$ are as described in Theorem 1.1.*

The below-mentioned table describes the particular case (3.1) more precisely.

| Partitions enum. by $X_1(8)$ | Partitions enum. by $G^{(2,0)}_{(2,1)}(8)$ | Lattice paths enum. by $H^{(2,0)}_{(2,1)}(8)$ |
|---|---|---|
| 8, 6+2, 4+4, 4+2+2, 2+2+2+2 | $8_2$ |  |
| | $8_4$ |  |
| | $8_6$ |  |
| | $8_8$ |  |
| | $6_2 2_2$ |  |

**Theorem 3.4.** *Let $X_4(\lambda)$ represent the number of ordinary partitions of $\lambda$ into parts congruent to $\pm 4, \pm 6, \pm 8, \pm 10 \pmod{32}$. Then*

$$X_4(\lambda) = G^{(4,0)}_{(2,2)}(\lambda) = H^{(4,0)}_{(2,2)}(\lambda), \quad \textit{for all } \lambda.$$

*where $G^{(4,0)}_{(2,2)}(\lambda)$, $H^{(4,0)}_{(2,2)}(\lambda)$ are as described in Theorem 1.1.*

**Theorem 3.5.** *Let $Y_5(\lambda)$ represent the number of ordinary partitions of $\lambda$ into parts congruent to $\pm 2, \pm 6 \pmod{14}$ and $Z_5(\lambda)$ represent the number of ordinary partitions of $\lambda$ into distinct parts congruent to $\pm 4, \pm 6, 0 \pmod{14}$. Then*

$$X_5(\lambda) = \sum_{i=0}^{\lambda} Y_5(\lambda - i) Z_5(i) = G^{(1,1)}_{(2,1)}(\lambda) = H^{(1,1)}_{(2,1)}(\lambda), \quad \textit{for all } \lambda.$$

*where $G^{(1,1)}_{(2,1)}(\lambda)$, $H^{(1,1)}_{(2,1)}(\lambda)$ are as described in Theorem 1.1.*

**Theorem 3.6.** *Let $Y_6(\lambda)$ represent the number of ordinary partitions of $\lambda$ into parts congruent to $\pm 4, \pm 12 \pmod{28}$ and $Z_6(\lambda)$ represent the number of ordinary partitions of $\lambda$ into distinct parts congruent to $\pm 6, 14 \pmod{28}$. Then*

$$X_6(\lambda) = \sum_{i=0}^{\lambda} Y_6(\lambda - i) Z_6(i) = G_{(2,4)}^{(4,0)}(\lambda) = H_{(2,4)}^{(4,0)}(\lambda), \quad for\ all\ \lambda.$$

*where $G_{(2,4)}^{(4,0)}(\lambda)$, $H_{(2,4)}^{(4,0)}(\lambda)$ are as described in Theorem 1.1*

**Theorem 3.7.** *Let $Y_7(\lambda)$ represent the number of ordinary partitions of $\lambda$ into parts congruent to $\pm 8, \pm 12 \pmod{28}$ and $Z_7(\lambda)$ represent the number of ordinary partitions of $\lambda$ into distinct parts congruent to $\pm 10, 14 \pmod{28}$. Then*

$$X_7(\lambda) = \sum_{i=0}^{\lambda} Y_7(\lambda - i) Z_7(i) = G_{(2,4)}^{(4,4)}(\lambda) = H_{(2,4)}^{(4,4)}(\lambda), \quad for\ all\ \lambda.$$

*where $G_{(2,4)}^{(4,4)}(\lambda)$, $H_{(2,4)}^{(4,4)}(\lambda)$ are as described in Theorem 1.1*

**Theorem 3.8.** *Let $Y_8(\lambda)$ represent the number of ordinary partitions of $\lambda$ into parts congruent to $\pm 8 \pmod{20}$ and $Z_8(\lambda)$ represent the number of ordinary partitions of $\lambda$ into distinct parts congruent to $\pm 6, 10 \pmod{20}$. Then*

$$X_8(\lambda) = \sum_{i=0}^{\lambda} Y_8(\lambda - i) Z_8(i) = G_{(2,4)}^{(6,0)}(\lambda) = H_{(2,4)}^{(6,0)}(\lambda), \quad for\ all\ \lambda.$$

*where $G_{(2,4)}^{(6,0)}(\lambda)$, $H_{(2,4)}^{(6,0)}(\lambda)$ are as described in Theorem 1.1.*

## 4. Conclusion

We have provided an infinite class of combinatorial identities by interpreting a generalized $q$-series in terms of $n$-color partitions and weighted lattice paths. Our results not only generalized the results we found in the literature (Agarwal [1, 2], Agarwal and Goyal [5, 13], Sareen and Rana [16]), but also provide entirely new Rogers–Ramanujan–MacMahon type combinatorial identities. So, the obvious question that arises here is, can we further explore this technique to study the results found in literature in a more generalized form and also to establish new results.

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