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ON SIGNS OF CERTAIN TOEPLITZ–HESSENBERG DETERMINANTS WHOSE ELEMENTS INVOLVE BERNOULLI NUMBERS

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Dedicated to my granddaughter, Taylor Xi-Ke Qi, who was born in February 2023

Abstract. In the paper, by virtue of Wronski's formula and Kaluza's theorem related to a power series and its reciprocal, by means of Cahill and Narayan's recursive relation, and with the aid of the logarithmic convexity of the sequence of the Bernoulli numbers, the author presents the signs of certain Toeplitz–Hessenberg determinants whose elements involve the Bernoulli numbers and combinatorial numbers. Moreover, with the help of a derivative formula for the ratio of two differentiable functions, the author provides an alternative proof of Wronski's formula.

1. Motivations

A lower (respectively upper) Hessenberg matrix is an $n\times n$ matrix $H_n=(h_{ij})_{1\le i,j\le n}$, where $h_{ij}=0$ for all pairs (i,j) such that $i+1<j$ (respectively $j+1<i$). See [14, Chapter 10]. A Toeplitz matrix is an $n\times n$ matrix $T_n=(t_{k,j})_{0\le k,j\le n-1}$, where $t_{k,j}=t_{k-j}$, that is, a matrix of the form

$$\begin{pmatrix}
t_0 & t_{-1} & t_{-2} & \cdots & t_{-n} & t_{-n+1}\\
t_1 & t_0 & t_{-1} & \cdots & t_{-n+1} & t_{-n+2}\\
t_2 & t_1 & t_0 & \cdots & t_{-n+2} & t_{-n+3}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
t_{n-2} & t_{n-3} & t_{n-4} & \cdots & t_0 & t_{-1}\\
t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 & t_0
\end{pmatrix}.$$

See [5]. For our convenience, we call the determinants $|H_n|$ and $|T_n|$ the Hessenberg determinant and the Toeplitz determinant, respectively. If an

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$n\times n$ matrix M_n is both a Hessenberg matrix and a Toeplitz matrix, we call its determinant $|M_n|$ the Toeplitz–Hessenberg determinant.

The Bernoulli numbers B_r for $r\in\mathbb{N}_0=\{0,1,2,\dots\}$ are defined by

$$\frac{z}{e^z-1}=\sum_{r=0}^{\infty}B_r\frac{z^r}{r!}=1-\frac{z}{2}+\sum_{r=1}^{\infty}B_{2r}\frac{z^{2r}}{(2r)!},\quad |z|<2\pi.$$

The Bernoulli numbers B_r for $r\in\mathbb{N}=\{1,2,\dots\}$ can be expressed in terms of a Toeplitz–Hessenberg determinant as

$$\begin{vmatrix}
\frac{1}{2!} & 1 & 0 & \cdots & 0 & 0 & 0\\
\frac{1}{3!} & \frac{1}{2!} & 1 & \cdots & 0 & 0 & 0\\
\frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \frac{1}{(r-3)!} & \cdots & \frac{1}{2!} & 1 & 0\\
\frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \cdots & \frac{1}{3!} & \frac{1}{2!} & 1\\
\frac{1}{(r+1)!} & \frac{1}{r!} & \frac{1}{(r-1)!} & \cdots & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!}
\end{vmatrix}=(-1)^r\frac{B_r}{r!},\quad r\in\mathbb{N}. \tag{1.1}$$

The expression (1.1) is a reformulation of a determinantal expression of B_r for $r\in\mathbb{N}_0$ in the papers [1, p. 42, (2.6)], [6, pp. 351–352], [8, Section 21.5], [9, p. 1], and [13, p. 3].

Replacing the elements 1 by the Bernoulli number B_2 and substituting the Bernoulli numbers B_{2k} for the elements $\frac{1}{k!}$ for $2\le k\le r+1$ in the Toeplitz–Hessenberg determinant on the left hand side of the equation (1.1), we construct a new Toeplitz–Hessenberg determinant

$$D_r=\begin{vmatrix}
B_4 & B_2 & 0 & \cdots & 0 & 0 & 0\\
B_6 & B_4 & B_2 & \cdots & 0 & 0 & 0\\
B_8 & B_6 & B_4 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
B_{2(r-1)} & B_{2(r-2)} & B_{2(r-3)} & \cdots & B_4 & B_2 & 0\\
B_{2r} & B_{2(r-1)} & B_{2(r-2)} & \cdots & B_6 & B_4 & B_2\\
B_{2(r+1)} & B_{2r} & B_{2(r-1)} & \cdots & B_8 & B_6 & B_4
\end{vmatrix},\quad r\in\mathbb{N}.$$

Substituting the Bernoulli number B_0 for the elements 1 and replacing the elements $\frac{1}{k!}$ for $2\le k\le r+1$ by $B_{2(k-1)}$ in the Toeplitz–Hessenberg determinant on the left hand side of the equation (1.1), we create another new Toeplitz–Hessenberg determinant

$$\mathfrak{D}_r=\begin{vmatrix}
B_2 & B_0 & 0 & \cdots & 0 & 0 & 0\\
B_4 & B_2 & B_0 & \cdots & 0 & 0 & 0\\
B_6 & B_4 & B_2 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
B_{2(r-2)} & B_{2(r-3)} & B_{2(r-4)} & \cdots & B_2 & B_0 & 0\\
B_{2(r-1)} & B_{2(r-2)} & B_{2(r-3)} & \cdots & B_4 & B_2 & B_0\\
B_{2r} & B_{2(r-1)} & B_{2(r-2)} & \cdots & B_6 & B_4 & B_2
\end{vmatrix},\quad r\in\mathbb{N}.$$

Straightforward computation gives

$$
\begin{aligned}
&D_1=-\frac{1}{30}, \quad D_2=-\frac{1}{350}, \quad D_3=-\frac{11}{15750}, \quad D_4=-\frac{5189}{18191250}, \\
&\mathfrak{D}_1=\frac{1}{6}, \qquad \mathfrak{D}_2=\frac{11}{180}, \qquad \mathfrak{D}_3=\frac{299}{7560}, \qquad \mathfrak{D}_4=\frac{10417}{226800}.
\end{aligned} \tag{1.2}
$$

Therefore, we guess that, for $r \in \mathbb{N}$,

$$D_r < 0, \tag{1.3}$$

and

$$\mathfrak{D}_r > 0. \tag{1.4}$$

In this paper, among other things, we will confirm that these two guesses are both true.

2. Wronski's formula and Kaluza's theorem

For the main theorems of this paper, we require results of Wronski and Kaluza, as stated below.

Lemma 2.1 (Wronski's formula [4, p. 17, Theorem 1.3])**.** *If $a_0 \ne 0$ and*

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots, \tag{2.1}$$

is a formal series, then the coefficients of the reciprocal series

$$\frac{1}{P(x)} = b_0 + b_1x + b_2x^2 + \cdots \tag{2.2}$$

are given by

$$
b_r = \frac{(-1)^r}{a_0^{r+1}}
\begin{vmatrix}
a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
a_2 & a_1 & a_0 & 0 & \cdots & 0 & 0 & 0 \\
a_3 & a_2 & a_1 & a_0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
a_{r-2} & a_{r-3} & a_{r-4} & a_{r-5} & \cdots & a_1 & a_0 & 0 \\
a_{r-1} & a_{r-2} & a_{r-3} & a_{r-4} & \cdots & a_2 & a_1 & a_0 \\
a_r & a_{r-1} & a_{r-2} & a_{r-3} & \cdots & a_3 & a_2 & a_1
\end{vmatrix}, \quad r \in \mathbb{N}. \tag{2.3}
$$

The formula (2.3) can also be found in [6, p. 347] and [12, Lemma 2.4].

We also remark that, in [2, p. 40, Entry 5], there exists a general derivative formula

$$(2.4)\quad \frac{\mathrm{d}^r}{\mathrm{d}\,t^r}\left[\frac{p(x)}{q(x)}\right]=\frac{(-1)^r}{q^{r+1}(x)}$$
$$\times\begin{vmatrix} p(x) & q(x) & 0 & \cdots & 0 & 0\\ p'(x) & q'(x) & q(x) & \cdots & 0 & 0\\ p''(x) & q''(x) & \binom{2}{1}q'(x) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ p^{(r-2)}(x) & q^{(r-2)}(x) & \binom{r-2}{1}q^{(r-3)}(x) & \cdots & q(x) & 0\\ p^{(r-1)}(x) & q^{(r-1)}(x) & \binom{r-1}{1}q^{(r-2)}(x) & \cdots & \binom{r-1}{r-2}q'(x) & q(x)\\ p^{(r)}(x) & q^{(r)}(x) & \binom{r}{1}q^{(r-1)}(x) & \cdots & \binom{r}{r-2}q''(x) & \binom{r}{r-1}q'(x)\end{vmatrix},$$

for $r\in\mathbb{N}_0$. The derivative formula (2.4) for the ratio of two differentiable functions can also be found in [12, p. 94] and [13, Lemma 1]. We observe that Lemma 2.1 can be proved from the formula (2.4) alternatively. See Section 5 in this paper. This is a proof of Lemma 2.1 that differs from those in [4, 12].

Lemma 2.2 (Kaluza's theorem [4, p. 13, Problem 6] and [7])**.** *Let*

$$P(x)=a_0+a_1x+a_2x^2+\cdots,$$

be a formal power series over the field of real numbers such that $a_r>0$ and

$$a_{r+2}a_r-a_{r+1}^2>0, \tag{2.5}$$

for $r=0,1,2,\dots$. If

$$\frac{1}{P(x)}=b_0+b_1x+b_2x^2+\cdots,$$

then $b_r<0$ for $r=1,2,\dots$.

The inequality (2.5) means that the sequence a_r for $r\in\mathbb{N}_0$ is strictly logarithmically convex.

3. Negativity of certain Toeplitz–Hessenberg determinants

In this section, we present the negativity of certain Toeplitz–Hessenberg determinants whose elements involve the products of the Bernoulli numbers B_{2r} and combinatorial numbers $\binom{2r+\ell}{2r}$. As a consequence, we verify that the first guess (1.3) is true.

Theorem 3.1. *For $m, r \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$, we have*

$$(-1)^{(m+1)r}\begin{vmatrix} a_{m+1,\ell} & a_{m,\ell} & 0 & \cdots & 0 & 0 \\ a_{m+2,\ell} & a_{m+1,\ell} & a_{m,\ell} & \cdots & 0 & 0 \\ a_{m+3,\ell} & a_{m+2,\ell} & a_{m+1,\ell} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m+r-2,\ell} & a_{m+r-3,\ell} & a_{m+r-4,\ell} & \cdots & a_{m,\ell} & 0 \\ a_{m+r-1,\ell} & a_{m+r-2,\ell} & a_{m+r-3,\ell} & \cdots & a_{m+1,\ell} & a_{m,\ell} \\ a_{m+r,\ell} & a_{m+r-1,\ell} & a_{m+r-2,\ell} & \cdots & a_{m+2,\ell} & a_{m+1,\ell} \end{vmatrix} < 0, \tag{3.1}$$

where

$$a_{i,j} = \binom{2i+j}{2i} B_{2i}, \quad i \in \mathbb{N}, \quad j \in \mathbb{N}_0. \tag{3.2}$$

In particular, for $\ell \in \mathbb{N}_0$ and $r \in \mathbb{N}$, we have

$$\begin{vmatrix} a_{2,\ell} & a_{1,\ell} & 0 & \cdots & 0 & 0 & 0 \\ a_{3,\ell} & a_{2,\ell} & a_{1,\ell} & \cdots & 0 & 0 & 0 \\ a_{4,\ell} & a_{3,\ell} & a_{2,\ell} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{r-1,\ell} & a_{r-2,\ell} & a_{r-3,\ell} & \cdots & a_{2,\ell} & a_{1,\ell} & 0 \\ a_{r,\ell} & a_{r-1,\ell} & a_{r-2,\ell} & \cdots & a_{3,\ell} & a_{2,\ell} & a_{1,\ell} \\ a_{r+1,\ell} & a_{r,\ell} & a_{r-1,\ell} & \cdots & a_{4,\ell} & a_{3,\ell} & a_{2,\ell} \end{vmatrix} < 0, \tag{3.3}$$

and for $\ell = 0$ we obtain the negativity $D_r < 0$ in (1.3).

Proof. From [15, Theorems 1.1 and 1.2], we conclude that, for fixed $\ell \in \mathbb{N}_0$, the sequence

$$\frac{(2r+\ell)!}{(2r)!}|B_{2r}|,$$

is logarithmically convex in $r \in \mathbb{N}$. Let

$$a_r = \frac{[2(r+m)+\ell]!}{[2(m+r)]!}|B_{2(m+r)}|, \quad m \in \mathbb{N}, \quad \ell, r \in \mathbb{N}_0. \tag{3.4}$$

Then the sequence a_r for $r \in \mathbb{N}_0$ is logarithmically convex, that is, the inequality (2.5) is valid.

By virtue of Wronski's formula (2.3) in Lemma 2.1, we obtain

$$b_r = \frac{(-1)^r}{a_0^{r+1}}\begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{r-2} & a_{r-3} & a_{r-4} & \cdots & a_1 & a_0 & 0 \\ a_{r-1} & a_{r-2} & a_{r-3} & \cdots & a_2 & a_1 & a_0 \\ a_r & a_{r-1} & a_{r-2} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}$$

$$= \frac{(-1)^r(\ell!)^r}{a_0^{r+1}} \begin{vmatrix} c_1 & c_0 & 0 & \cdots & 0 & 0 & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 & 0 & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{r-2} & c_{r-3} & c_{r-4} & \cdots & c_1 & c_0 & 0 \\ c_{r-1} & c_{r-2} & c_{r-3} & \cdots & c_2 & c_1 & c_0 \\ c_r & c_{r-1} & c_{r-2} & \cdots & c_3 & c_2 & c_1 \end{vmatrix},$$

for $m, r \in \mathbb{N}$, where

$$c_r = \binom{2(m+r)+\ell}{2(m+r)} |B_{2(m+r)}| = \frac{a_r}{\ell!}, \quad m \in \mathbb{N}, \quad \ell, r \in \mathbb{N}_0.$$

Utilizing Lemma 2.2 and logarithmic convexity of the sequence (3.4), we arrive at the negativity of the sequence b_r, that is,

$$(-1)^r \begin{vmatrix} c_1 & c_0 & 0 & \cdots & 0 & 0 & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 & 0 & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{r-2} & c_{r-3} & c_{r-4} & \cdots & c_1 & c_0 & 0 \\ c_{r-1} & c_{r-2} & c_{r-3} & \cdots & c_2 & c_1 & c_0 \\ c_r & c_{r-1} & c_{r-2} & \cdots & c_3 & c_2 & c_1 \end{vmatrix} < 0,$$

for $m, r \in \mathbb{N}$. Employing the relation $|B_{2r}| = (-1)^{r+1} B_{2r}$ for $r \in \mathbb{N}$, and utilizing a basic property of determinants lead to the formula (3.1).

When taking $m = 1$ in (3.1), we readily derive (3.3). When taking $m = 1$ and $\ell = 0$ in (3.1) or taking $\ell = 0$ in (3.3), we immediately derive $D_r < 0$. The proof of Theorem 3.1 is complete. □

Theorem 3.2. *For $\ell \in \mathbb{N}_0$ and $r \in \mathbb{N}$, if $\alpha > \frac{5(\ell+2)!}{(\ell+3)(\ell+4)}$, we have*

$$(-1)^r \begin{vmatrix} a_{1,\ell} & -\frac{\alpha}{\ell!} & 0 & \cdots & 0 & 0 & 0 \\ a_{2,\ell} & a_{1,\ell} & -\frac{\alpha}{\ell!} & \cdots & 0 & 0 & 0 \\ a_{3,\ell} & a_{2,\ell} & a_{1,\ell} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{r-2,\ell} & a_{r-3,\ell} & a_{r-4,\ell} & \cdots & a_{1,\ell} & -\frac{\alpha}{\ell!} & 0 \\ a_{r-1,\ell} & a_{r-2,\ell} & a_{r-3,\ell} & \cdots & a_{2,\ell} & a_{1,\ell} & -\frac{\alpha}{\ell!} \\ a_{r,\ell} & a_{r-1,\ell} & a_{r-2,\ell} & \cdots & a_{3,\ell} & a_{2,\ell} & a_{1,\ell} \end{vmatrix} < 0, \tag{3.5}$$

where $a_{i,j}$ is defined by (3.2). In particular, for $r \in \mathbb{N}$ and $\alpha > \frac{5}{6}$, we have

$$(-1)^r \begin{vmatrix} B_2 & -\alpha & 0 & \cdots & 0 & 0 & 0 \\ B_4 & B_2 & -\alpha & \cdots & 0 & 0 & 0 \\ B_6 & B_4 & B_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ B_{2r-4} & B_{2r-6} & B_{2r-8} & \cdots & B_2 & -\alpha & 0 \\ B_{2r-2} & B_{2r-4} & B_{2r-6} & \cdots & B_4 & B_2 & -\alpha \\ B_{2r} & B_{2r-2} & B_{2r-4} & \cdots & B_6 & B_4 & B_2 \end{vmatrix} < 0, \tag{3.6}$$

and

$$(-1)^r \begin{vmatrix} B_2 & -B_0 & 0 & \cdots & 0 & 0 & 0 \\ B_4 & B_2 & -B_0 & \cdots & 0 & 0 & 0 \\ B_6 & B_4 & B_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ B_{2r-4} & B_{2r-6} & B_{2r-8} & \cdots & B_2 & -B_0 & 0 \\ B_{2r-2} & B_{2r-4} & B_{2r-6} & \cdots & B_4 & B_2 & -B_0 \\ B_{2r} & B_{2r-2} & B_{2r-4} & \cdots & B_6 & B_4 & B_2 \end{vmatrix} < 0. \tag{3.7}$$

Proof. Let

$$a_r = \frac{(2r+\ell)!}{(2r)!}|B_{2r}|, \quad r \ge 1.$$

Since

$$a_2\alpha - a_1^2 = \frac{(4+\ell)!}{4!}|B_4|\alpha - \left[\frac{(2+\ell)!}{2!}|B_2|\right]^2 > 0,$$

is equivalent to

$$\alpha > \frac{\left[\frac{(2+\ell)!}{2!}|B_2|\right]^2}{\frac{(4+\ell)!}{4!}|B_4|} = \frac{5(\ell+2)!}{(\ell+3)(\ell+4)},$$

the sequence a_r for $r \in \mathbb{N}_0$ with $a_0 = \alpha$ is strictly logarithmically convex.

By virtue of Wronski's formula (2.3) in Lemma 2.1, we obtain

$$b_r = \frac{(-1)^r}{\alpha^{r+1}} \begin{vmatrix} a_1 & \alpha & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_1 & \alpha & 0 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & \alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{r-2} & a_{r-3} & a_{r-4} & a_{r-5} & \cdots & a_1 & \alpha & 0 \\ a_{r-1} & a_{r-2} & a_{r-3} & a_{r-4} & \cdots & a_2 & a_1 & \alpha \\ a_r & a_{r-1} & a_{r-2} & a_{r-3} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}$$

$$
= \frac{(-1)^r(\ell!)^r}{\alpha^{r+1}} \begin{vmatrix} c_1 & \frac{\alpha}{\ell!} & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_2 & c_1 & \frac{\alpha}{\ell!} & 0 & \cdots & 0 & 0 & 0 \\ c_3 & c_2 & c_1 & \frac{\alpha}{\ell!} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{r-2} & c_{r-3} & c_{r-4} & c_{r-5} & \cdots & c_1 & \frac{\alpha}{\ell!} & 0 \\ c_{r-1} & c_{r-2} & c_{r-3} & c_{r-4} & \cdots & c_2 & c_1 & \frac{\alpha}{\ell!} \\ c_r & c_{r-1} & c_{r-2} & c_{r-3} & \cdots & c_3 & c_2 & c_1 \end{vmatrix}
$$

$$
= \frac{(-1)^r(\ell!)^r}{\alpha^{r+1}} \begin{vmatrix} a_{1,\ell} & -\frac{\alpha}{\ell!} & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_{2,\ell} & a_{1,\ell} & -\frac{\alpha}{\ell!} & 0 & \cdots & 0 & 0 & 0 \\ a_{3,\ell} & a_{2,\ell} & a_{1,\ell} & -\frac{\alpha}{\ell!} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{r-2,\ell} & a_{r-3,\ell} & a_{r-4,\ell} & a_{r-5,\ell} & \cdots & a_{1,\ell} & -\frac{\alpha}{\ell!} & 0 \\ a_{r-1,\ell} & a_{r-2,\ell} & a_{r-3,\ell} & a_{r-4,\ell} & \cdots & a_{2,\ell} & a_{1,\ell} & -\frac{\alpha}{\ell!} \\ a_{r,\ell} & a_{r-1,\ell} & a_{r-2,\ell} & a_{r-3,\ell} & \cdots & a_{3,\ell} & a_{2,\ell} & a_{1,\ell} \end{vmatrix},
$$

where c_r is defined by

$$
c_r = \binom{2r+\ell}{2r}|B_{2r}|, \quad \ell, r \in \mathbb{N}_0,
$$

and $a_{i,\ell}$ for $1 \le r \le r$ are defined by (3.2). Making use of Lemma 2.2 and logarithmic convexity of the sequence a_r for $r \in \mathbb{N}_0$ reveals the negativity of the sequence b_r, that is, the inequality (3.5) is valid.

Letting $\ell = 0$ in (3.5) results in (3.6). When letting $\alpha = 1$ in (3.6) and considering $B_0 = 1 > \frac{5}{6}$, we deduce (3.7).

The negativity in (3.7) can also be proved by directly combining Wronski's formula in Lemma 2.1 and Kaluza's theorem in Lemma 2.2 with the result in [15, Theorem 1.1], which reads that the sequence $|B_{2n}|$ for $n \ge 0$ is strictly logarithmically convex. The proof of Theorem 3.2 is complete. □

4. Positivity of the Toeplitz–Hessenberg determinant $\mathfrak{D}_r$

In this section, by induction, and with the recursive relation of Cahill and Narayan [3, p. 222, Theorem] for general Hessenberg determinants, we confirm that the second guess (1.4) is also true.

Theorem 4.1. *The sequence of the Toeplitz–Hessenberg determinants $\mathfrak{D}_r$ for $r \in \mathbb{N}$ satisfies the recursive relation*

$$
\mathfrak{D}_r = \sum_{\ell=1}^{r} |B_{2(r-\ell+1)}|\mathfrak{D}_{\ell-1}, \quad r \ge 2, \tag{4.1}
$$

where $\mathfrak{D}_0 = 1$. Consequently, the positivity $\mathfrak{D}_r > 0$ for $r \in \mathbb{N}$ in (1.4) is true.

Proof. Let $H_0=1$ and

$$H_r=\begin{vmatrix}
h_{1,1} & h_{1,2} & 0 & \dots & 0 & 0\\
h_{2,1} & h_{2,2} & h_{2,3} & \dots & 0 & 0\\
h_{3,1} & h_{3,2} & h_{3,3} & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
h_{r-2,1} & h_{r-2,2} & h_{r-2,3} & \dots & h_{r-2,r-1} & 0\\
h_{r-1,1} & h_{r-1,2} & h_{r-1,3} & \dots & h_{r-1,r-1} & h_{r-1,r}\\
h_{r,1} & h_{r,2} & h_{r,3} & \dots & h_{r,r-1} & h_{r,r}
\end{vmatrix},$$

for $r\in\mathbb{N}$. The recursive relation of Cahill and Narayan [3, p. 222, Theorem] states that the sequence H_r for $r\in\mathbb{N}_0$, with $H_1=h_{1,1}$, satisfies

$$H_r=\sum_{\ell=1}^{r}(-1)^{r-\ell}h_{r,\ell}\left(\prod_{j=\ell}^{r-1}h_{j,j+1}\right)H_{\ell-1}, \tag{4.2}$$

for $r\ge 2$, where the empty product is understood to be 1. Setting

$$h_{\ell,m}=B_{2(\ell-m+1)}, \quad -1\le \ell-m\le r-1,$$

in (4.2) yields

$$\mathfrak{D}_r=\sum_{\ell=1}^{r}(-1)^{r-\ell}B_{2(r-\ell+1)}\left(\prod_{j=\ell}^{r-1}B_0\right)\mathfrak{D}_{\ell-1}=\sum_{\ell=1}^{r}|B_{2(r-\ell+1)}|\mathfrak{D}_{\ell-1},$$

for $r\ge 2$, where we used the facts that $B_0=1$ and $(-1)^{r+1}B_{2r}>0$ for $r\in\mathbb{N}$. The recursive relation (4.1) is thus proved.

Since $\mathfrak{D}_0=1$ and $\mathfrak{D}_\ell>0$ for $1\le\ell\le4$, see the second line in (1.2), from the recursive relation (4.1), by induction, we readily conclude that $\mathfrak{D}_r>0$ for $r\in\mathbb{N}$. The proof of Theorem 4.1 is complete. □

By virtue of (4.2), we can deduce that the sequence of the Toeplitz–Hessenberg determinants D_r for $r\in\mathbb{N}$ satisfies the recursive relation

$$D_r=-\sum_{\ell=1}^{r}|B_{2(r-\ell+2)}|B_2^{r-\ell}D_{\ell-1}, \tag{4.3}$$

where $D_0=1$. From (4.3), we cannot see the negativity $D_r<0$ for $r\in\mathbb{N}$ immediately.

The recursive relation (4.2) reveals that all the Hessenberg determinants whose elements satisfy $h_{j,j+1}>0$ for $1\le j\le r-1$ and $(-1)^{\ell-m}h_{\ell,m}>0$ for $\ell-m\ge0$ are positive.

5. An alternative proof of Wronski's formula

In this section, making use of the derivative formula (2.4) for the ratio of two differentiable functions, we provide an alternative proof, which is different from those in [4, 12], of Wronski's formula [4, p. 17, Theorem 1.3] recited in Lemma 2.1 as follows.

The formal series (2.1) means that

$$P^{(r)}(x)=\sum_{\ell=0}^{\infty}a_\ell\langle\ell\rangle_r x^{\ell-r}\to\langle r\rangle_r a_r=r!a_r,\quad r\in\mathbb{N}_0,$$

as $x\to0$, where

$$\langle z\rangle_\ell=\prod_{m=0}^{\ell-1}(z-m)=\begin{cases}z(z-1)\cdots(z-\ell+1), & \ell\ge1\\ 1, & \ell=0\end{cases}$$

is called the ℓ-th falling factorial of $z\in\mathbb{C}$. The formal series (2.2) means that

$$b_r=\frac{1}{r!}\lim_{x\to0}\biggl[\frac{1}{P(x)}\biggr]^{(r)},\quad r\in\mathbb{N}_0. \tag{5.1}$$

Letting $p(x)=1$ and $q(x)=P(x)$ in the derivative formula (2.4) yields

$$\biggl[\frac{1}{P(x)}\biggr]^{(r)}=\frac{(-1)^r}{P^{r+1}(x)}$$

$$\times\begin{vmatrix}1 & P(x) & 0 & \cdots & 0 & 0\\ 0 & P'(x) & P(x) & \cdots & 0 & 0\\ 0 & P''(x) & \binom{2}{1}P'(x) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & P^{(r-2)}(x) & \binom{r-2}{1}P^{(r-3)}(x) & \cdots & P(x) & 0\\ 0 & P^{(r-1)}(x) & \binom{r-1}{1}P^{(r-2)}(x) & \cdots & \binom{r-1}{r-2}P'(x) & P(x)\\ 0 & P^{(r)}(x) & \binom{r}{1}P^{(r-1)}(x) & \cdots & \binom{r}{r-2}P''(x) & \binom{r}{r-1}P'(x)\end{vmatrix}$$

$$\to\frac{(-1)^r}{a_0^{r+1}}\begin{vmatrix}1 & a_0 & 0 & \cdots & 0 & 0\\ 0 & a_1 & a_0 & \cdots & 0 & 0\\ 0 & 2!a_2 & \binom{2}{1}a_1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & (r-2)!a_{r-2} & \binom{r-2}{1}(r-3)!a_{r-3} & \cdots & a_0 & 0\\ 0 & (r-1)!a_{r-1} & \binom{r-1}{1}(r-2)!a_{r-2} & \cdots & \binom{r-1}{r-2}a_1 & a_0\\ 0 & r!a_r & \binom{r}{1}(r-1)!a_{r-1} & \cdots & \binom{r}{r-2}2!a_2 & \binom{r}{r-1}a_1\end{vmatrix}$$

$$=\frac{(-1)^r}{a_0^{r+1}}\begin{vmatrix}a_1 & a_0 & 0 & \cdots & 0 & 0\\ 2!a_2 & \frac{2!}{1!}a_1 & a_0 & \cdots & 0 & 0\\ 3!a_3 & \frac{3!}{1!}a_2 & \frac{3!}{2!}a_1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ (r-2)!a_{r-2} & \frac{(r-2)!}{1!}a_{r-3} & \frac{(r-2)!}{2!}a_{r-4} & \cdots & a_0 & 0\\ (r-1)!a_{r-1} & \frac{(r-1)!}{1!}a_{r-2} & \frac{(r-1)!}{2!}a_{r-3} & \cdots & \frac{(r-1)!}{(r-2)!}a_1 & a_0\\ r!a_r & \frac{r!}{1!}a_{r-1} & \frac{r!}{2!}a_{r-2} & \cdots & \frac{r!}{(r-2)!}a_2 & \frac{r!}{(r-1)!}a_1\end{vmatrix}$$

$$
= \frac{(-1)^r r!}{a_0^{r+1}} \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{r-2} & a_{r-3} & a_{r-4} & \cdots & a_0 & 0 \\ a_{r-1} & a_{r-2} & a_{r-3} & \cdots & a_1 & a_0 \\ a_r & a_{r-1} & a_{r-2} & \cdots & a_2 & a_1 \end{vmatrix},
$$

as $x \to 0$. Substituting this result into (5.1) gives the formula (2.3). The proof of Wronski's formula [4, p. 17, Theorem 1.3] recited in Lemma 2.1 is complete.

By the way, this paper is a companion of the articles [10, 11].

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