Contributions to Discrete Mathematics

Volume 18, Number 1, Pages 53–65
ISSN 1715-0868

BECK-TYPE COMPANION IDENTITIES FOR FRANKLIN'S IDENTITY

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ABSTRACT. In 2017, Beck conjectured that the difference in the number of parts in all partitions of n into odd parts and the number of parts in all strict partitions of n is equal to the number of partitions of n whose set of even parts has one element, and also to the number of partitions of n with exactly one part repeated. Andrews proved the conjecture using generating functions. Beck's identity is a companion identity to Euler's identity. The theorem has been generalized (with a combinatorial proof) by Yang to a companion identity to Glaisher's identity. Franklin generalized Glaisher's identity, and in this article, we provide a Beck-type companion identity to Franklin's identity and prove it both analytically and combinatorially. Andrews' and Yang's respective theorems fit naturally into this very general frame. We also discuss how Franklin's identity and the companion Beck-type identities can be further generalized to Euler pairs of any order.

1. INTRODUCTION

Let n be a non-negative integer. A *partition* λ of n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ that add up to n, i.e.,

$$\sum_{i=1}^{\ell} \lambda_i = n.$$

The numbers λ_i are called the *parts* of λ and n is called the *size* of λ. The number of parts of the partition is called the *length* of λ and is denoted by $\ell(\lambda)$.

We will also use the exponential notation for parts in a partition. The exponent of a part is the multiplicity of the part in the partition. For example, $(5^2, 4, 3^3, 1^2)$ denotes the partition $(5, 5, 4, 3, 3, 3, 1, 1)$. Mostly, we will use the exponential notation when referring to rectangular partitions,

Received by the editors June 8, 2021, and in revised form October 24, 2021.
2000 *Mathematics Subject Classification.* 05A17, 11P83.
Key words and phrases. partitions, Franklin's identity, Beck-type identities, parts in partitions.

i.e., partitions in which all parts are equal. Thus, we write (m^i) for the partition consisting of i parts equal to m.

It is customary to denote by $p(n)$ the number of partitions of n. We denote by $p(n \mid X)$ the number of partitions of n satisfying condition X. Partition identities are statements asserting that, for all non-negative integers n,

$$p(n \mid X) = p(n \mid Y). \tag{1.1}$$

A Beck-type identity for (1.1) is a companion identity asserting that the difference between the number of parts in all partitions of n satisfying condition X and the number of parts in all partitions of n satisfying condition Y equals $c\,p(n \mid X')$ and also $c\,p(n \mid Y')$, where c is some constant related to the original identity, and X', respectively Y', is a condition that is a small relaxation of condition X, respectively Y. This idea appeared first in [1], where George Beck conjectured companion identities to Euler's identity.

In the remainder of the introduction, we give a brief history of the development of Beck-type identities in recent years and introduce companion identities to Franklin's identity. Before we continue, we note that in the introduction and in the notation we use, there is substantial overlap with [7]. However, the essential difference between the results of that article and those of the current article is that the method of proof used here can be generalized to Euler pairs while the method used in [7] gives a modular refinement of the identity that is very interesting from a number theoretic point of view but does not have a natural generalization to Euler pairs.

Next, we introduce notation used throughout the article.

We denote by $\mathcal{O}_{j,r}(n)$, respectively $\mathcal{O}_{\le j,r}(n)$, the set of partitions of n with exactly j, respectively at most j, different parts (possibly repeated) congruent to $0 \pmod r$ and by $\mathcal{D}_{j,r}(n)$, respectively $\mathcal{D}_{\le j,r}(n)$, the set of partitions of n in which exactly j, respectively at most j, different parts are repeated at least r times and all other parts appear no more than $r-1$ times.

Euler's partition identity states that for all non-negative integers n,

$$|\mathcal{O}_{0,2}(n)| = |\mathcal{D}_{0,2}(n)|. \tag{1.2}$$

Glaisher's identity generalizes Euler's identity and states that for all non-negative integers n and all integers $r \ge 2$,

$$|\mathcal{O}_{0,r}(n)| = |\mathcal{D}_{0,r}(n)|. \tag{1.3}$$

In 1883, Franklin [9] proved the generalization of Glaisher's identity: for all non-negative integers n, j and all integers $r \ge 2$,

$$|\mathcal{O}_{j,r}(n)| = |\mathcal{D}_{j,r}(n)|. \tag{1.4}$$

George Beck conjectured a companion identity to Euler's partition identity (1.2), namely

$$|\mathcal{O}_{1,2}(n)| = |\mathcal{D}_{1,2}(n)| = b(n), \tag{1.5}$$

where $b(n)$ is the difference between the number of parts in all partitions in $\mathcal{O}_{0,2}(n)$ and the number of parts in all partitions in $\mathcal{D}_{0,2}(n)$. Andrews proved these identities in [3] using generating functions. Since then, in a fairly short time, many articles appeared giving generalizations of this result as well as combinatorial proofs in many cases. See for example [10, 16, 5, 11, 12, 13, 4, 6, 8]. Some authors have started referring to these companion identities as Beck-type identities. Some of the earlier generalizations [16] gave companion identities to Glaisher's identity (1.3). Let $b_{j,r}(n)$ denote the difference between the number of parts in all partitions in $\mathcal{O}_{j,r}(n)$ and the number of parts in all partitions in $\mathcal{D}_{j,r}(n)$. Then the Beck-type identity accompanying (1.3) introduced in [16] is

$$|\mathcal{O}_{1,r}(n)| = |\mathcal{D}_{1,r}(n)| = \frac{1}{r-1} b_{0,r}(n). \tag{1.6}$$

The main result of this article gives a Beck-type companion identity for Franklin's identity (1.4). Let $b_{\le j,r}(n)$ denote the difference between the number of parts in all partitions in $\mathcal{O}_{\le j,r}(n)$ and the number of parts in all partitions in $\mathcal{D}_{\le j,r}(n)$, i.e.,

$$b_{\le j,r}(n) = \sum_{\lambda \in \mathcal{O}_{\le j,r}(n)} \ell(\lambda) - \sum_{\lambda \in \mathcal{D}_{\le j,r}(n)} \ell(\lambda).$$

Theorem 1.1. *Let n, j, r be non-negative integers with $r \ge 2$. Then,*

$$\frac{1}{r-1} b_{\le j,r}(n) = (j+1)|\mathcal{O}_{j+1,r}(n)| = (j+1)|\mathcal{D}_{j+1,r}(n)|.$$

The case $j = 0$ gives (1.6). Theorem 1.1 is obtained from the repeated application of the next theorem for which we give both analytic and combinatorial proofs in Section 2.

Theorem 1.2. *For all non-negative integers n, j and all integers $r \ge 2$, we have*

$$\frac{1}{r-1} b_{j,r}(n) = (j+1)|\mathcal{O}_{j+1,r}(n)| - j|\mathcal{O}_{j,r}(n)| \tag{1.7}$$

$$= (j+1)|\mathcal{D}_{j+1,r}(n)| - j|\mathcal{D}_{j,r}(n)|. \tag{1.8}$$

Note that Theorem 1.2 itself can be viewed as a generalization of (1.6). However, based on numerical evidence, it appears that for $j \ge 1$, the right hand side of (1.7), and thus also of (1.8), is non-positive.

We remark that while [12] gives another generalization of (1.6) involving the number of parts in $\mathcal{O}_{j,r}(n)$ (but not in $\mathcal{D}_{j,r}(n)$), their result does not lead to a natural generalization as in Theorem 1.1 nor to the further generalization to Euler pairs described in Section 4.

In our combinatorial proofs, we use the following two operations on partitions. Given partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)})$, the partition $\lambda \cup \mu$ is the partition whose parts are precisely the parts of λ and μ, i.e., $\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}, \mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}$, arranged in non-increasing order. The partition $\lambda \setminus \mu$ is defined only if all parts of μ (considered with multiplicity) are also parts of λ. Then, $\lambda \setminus \mu$ is the partition obtained from λ by removing all parts of μ (with multiplicity).

2. Proof of Theorem 1.2

2.1. **Analytic proof.** For the remainder of this section, fix an integer $r \geq 2$. Denote by $\mathcal{O}_{j,r}(m,n)$, respectively $\mathcal{D}_{j,r}(m,n)$, the subset of partitions in $\mathcal{O}_{j,r}(n)$, respectively $\mathcal{D}_{j,r}(n)$, with m parts. We start with the trivariate generating functions for the sequences $\{|\mathcal{O}_{j,r}(m,n)|\}$ and $\{|\mathcal{D}_{j,r}(m,n)|\}$. Let z, w, and q be complex variables of modulus less than 1 so that all series converge absolutely. We define

$$\mathcal{O}_r(z,w,q) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} |\mathcal{O}_{j,r}(m,n)| z^m w^j q^n,$$

and

$$\mathcal{D}_r(z,w,q) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} |\mathcal{D}_{j,r}(m,n)| z^m w^j q^n.$$

We have

$$\begin{aligned}
&\mathcal{O}_r(z,w,q)\\
&= \prod_{n=1}^{\infty}\left(1 + wzq^{rn} + wz^2q^{2(rn)} + wz^3q^{3(rn)} + \cdots\right)\cdot \prod_{\substack{n=1\\ n\not\equiv 0 \pmod r}}^{\infty} \frac{1}{1-zq^n}\\
&= \prod_{n=1}^{\infty}\left(1 + \frac{wzq^{rn}}{1-zq^{rn}}\right)\cdot \prod_{\substack{n=1\\ n\not\equiv 0 \pmod r}}^{\infty} \frac{1}{1-zq^n},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{D}_r(z,w,q)\\
&= \prod_{n=1}^{\infty}\left(1 + zq^n + z^2q^{2n} + \cdots + z^{r-1}q^{(r-1)n} + wz^rq^{rn} + wz^{r+1}q^{(r+1)n} + \cdots\right)\\
&= \prod_{n=1}^{\infty}\left(1 + zq^n + z^2q^{2n} + \cdots + z^{r-1}q^{(r-1)n}\right)\left(1 + wz^rq^{rn} + wz^{2r}q^{(2r)n} + \cdots\right)\\
&= \prod_{n=1}^{\infty}\left(1 + \frac{wz^rq^{rn}}{1-z^rq^{rn}}\right)\cdot \prod_{n=1}^{\infty}\frac{1-z^rq^{rn}}{1-zq^n}.
\end{aligned}$$

Clearly,

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} b_{j,r}(n) w^j q^n = \frac{\partial}{\partial z}\bigg|_{z=1}\left(\mathcal{O}_r(z,w,q) - \mathcal{D}_r(z,w,q)\right).$$

Using logarithmic differentiation,

$$\begin{aligned}
&\frac{\partial}{\partial z}\Big|_{z=1} \mathcal{O}_r(z,w,q)\\
&= \prod_{n=1}^{\infty}\left(1+\frac{wq^{rn}}{1-q^{rn}}\right)\cdot \prod_{\substack{n=1\\ n\not\equiv 0 \pmod{r}}}^{\infty} \frac{1}{1-q^n}\\
&\quad \cdot\left(\sum_{m=1}^{\infty}\frac{wq^{mr}}{(1-q^{mr})(1-(1-w)q^{mr})} + \sum_{\substack{m=1\\ m\not\equiv 0 \pmod{r}}}^{\infty}\frac{q^m}{1-q^m}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial z}\Big|_{z=1} \mathcal{D}_r(z,w,q)\\
&= \prod_{n=1}^{\infty}\left(1+\frac{wq^{rn}}{1-q^{rn}}\right)\cdot \prod_{n=1}^{\infty} \frac{1-q^{rn}}{1-q^n}\\
&\quad \cdot\left(\sum_{m=1}^{\infty}\frac{wrq^{mr}}{(1-q^{mr})(1-(1-w)q^{mr})} + \sum_{m=1}^{\infty}\frac{(r-1)q^{(r+1)m}+q^m-rq^{rm}}{(1-q^m)(1-q^{rm})}\right)\\
&= \prod_{n=1}^{\infty}\left(1+\frac{wq^{rn}}{1-q^{rn}}\right)\cdot \prod_{n=1}^{\infty} \frac{1-q^{rn}}{1-q^n}\\
&\quad \cdot\left(\sum_{m=1}^{\infty}\frac{wrq^{mr}}{(1-q^{mr})(1-(1-w)q^{mr})} + \frac{q^m}{1-q^m} - \frac{q^{rm}}{1-q^{rm}} - \frac{(r-1)q^{rm}}{1-q^{rm}}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} b_{j,r}(n)w^j q^n = (1-r)\prod_{n=1}^{\infty}\left(1+\frac{wq^{rn}}{1-q^{rn}}\right) \prod_{\substack{n=1\\ n\not\equiv 0 \pmod{r}}}^{\infty} \frac{1}{1-q^n}\\
\cdot\left(\sum_{m=1}^{\infty}\left(\frac{wq^{mr}}{(1-q^{mr})(1-(1-w)q^{mr})} - \frac{q^{rm}}{1-q^{rm}}\right)\right).
\end{aligned}$$

Since

$$\frac{wq^{mr}}{(1-q^{mr})(1-(1-w)q^{mr})} = \frac{1}{1-q^{rm}} - \frac{1}{1-(1-w)q^{rm}},$$

we have

$$\sum_{m=1}^{\infty}\left(\frac{wq^{mr}}{(1-q^{mr})(1-(1-w)q^{mr})} - \frac{q^{rm}}{1-q^{rm}}\right) = \sum_{m=1}^{\infty}\frac{-(1-w)q^{mr}}{1-(1-w)q^{mr}}.$$

Thus,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} b_{j,r}(n)w^j q^n\\
&= (r-1)\prod_{n=1}^{\infty}\left(1+\frac{wq^{rn}}{1-q^{rn}}\right)\prod_{\substack{n=1\\ n\not\equiv 0 \pmod r}}^{\infty}\frac{1}{1-q^n}\left(\sum_{m=1}^{\infty}\frac{(1-w)q^{mr}}{1-(1-w)q^{mr}}\right)\\
&= (r-1)(1-w)\sum_{m=1}^{\infty}\left(\frac{q^{mr}}{1-q^{mr}}\prod_{\substack{n=1\\ n\neq m}}^{\infty}\left(1+\frac{wq^{rn}}{1-q^{rn}}\right)\cdot\prod_{\substack{n=1\\ n\not\equiv 0 \pmod r}}^{\infty}\frac{1}{1-q^n}\right)\\
&= (r-1)(1-w)\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}(j+1)|\mathcal{O}_{j+1,r}(n)|w^j q^n\\
&= (r-1)\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}((j+1)|\mathcal{O}_{j+1,r}(n)|-j|\mathcal{O}_{j,r}(n)|)w^j q^n.
\end{aligned}$$

To see the second to last equality above, notice that the exponent of q coming from the term

$$\frac{q^{mr}}{1-q^{mr}},$$

keeps track of the part mr (with multiplicity) of a partition λ in $\mathcal{O}_{j+1,r}(n)$ and the exponent of q coming from the first product keeps track of the other j different parts of λ divisible by r and it is weighted by w^j. The exponent of q in the second product keeps track of the parts of λ not divisible by r. Then, λ contributes $j+1$ to the coefficient of $w^j q^n$.

2.2. **Combinatorial proof.** To begin, we briefly recall Franklin's bijective proof of (1.4). Denote Glaisher's bijection by $\psi : \mathcal{O}_{0,r}(n) \to \mathcal{D}_{0,r}(n)$. Then Franklin's bijection $\varphi : \mathcal{O}_{j,r}(n) \to \mathcal{D}_{j,r}(n)$ is defined as follows. Let $\lambda \in \mathcal{O}_{j,r}(n)$. Suppose the parts of λ divisible by r are $(m_i r)^{k_i}$ with $m_i, k_i > 0$ for $1 \le i \le j$, and the m_i are distinct. Let

$$\overline{\lambda} = \lambda \setminus \bigcup_{i=1}^{j}(m_i r)^{k_i},$$

be the partition obtained from λ by removing all parts equal to $m_i r$ for $1 \le i \le j$. Then

$$\overline{\lambda} \in \mathcal{O}_{0,r}\left(n - \sum_{i=1}^{j} k_i m_i r\right).$$

Let

$$\overline{\mu} = \psi(\overline{\lambda}) \in \mathcal{D}_{0,r}\left(n - \sum_{i=1}^{j} k_i m_i r\right),$$

be the image of $\overline{\lambda}$ under Glaisher's bijection. Finally, let $\mu = \overline{\mu} \cup ((m_1)^{k_1 r}, (m_2)^{k_2 r}, \ldots, (m_j)^{k_j r})$. Since the parts m_i, $1 \leq i \leq j$, are all distinct and they are the only parts repeated at least r times, we have that $\mu \in \mathcal{D}_{j,r}(n)$. Set $\varphi(\lambda) = \mu$.

To describe the inverse mapping, let $\mu \in \mathcal{D}_{j,r}(n)$. Suppose the j different parts that are repeated at least r times are m_i, $1 \leq i \leq j$, and each m_i has multiplicity a_i in μ. For each $1 \leq i \leq j$, write $a_i = k_i r + d_i$ with $0 \leq d_i \leq r-1$, and remove $k_i r$ parts equal to m_i from μ to obtain a partition

$$\overline{\mu} \in \mathcal{D}_{0,r}\left(n - \sum_{i=1}^{j} k_i m_i r\right).$$

Let

$$\overline{\lambda} = \psi^{-1}(\overline{\mu}) \in \mathcal{O}_{0,r}\left(n - \sum_{i=1}^{j} k_i m_i r\right),$$

be the image of $\overline{\mu}$ under the inverse of Glaisher's bijection. Let

$$\lambda = \overline{\lambda} \cup ((m_1 r)^{k_1}, (m_2 r)^{k_2}, \ldots, (m_j r)^{k_j}).$$

Clearly, $\lambda \in \mathcal{O}_{j,r}(n)$. Then $\varphi^{-1}(\mu) = \lambda$.

Recently, Xiong and Keith [15] introduced a new beautiful bijection to prove Glaisher's identity. Clearly, it could have been used above instead of Glaisher's bijection.

In [8], we used the Xiong–Keith bijection to give a combinatorial proof for (1.6), which is the case $j = 0$ of (1.7). Note that combinatorial proofs of (1.6) using ideas similar to Glaisher's bijection are given in [16] and [6]. In this subsection, we use the combinatorial proof of (1.6) to give a combinatorial proof of (1.7). This, combined with Franklin's bijection, gives a combinatorial proof of (1.8). At this time, we do not know how to prove (1.8) combinatorially directly using the Xiong–Keith bijection, without using Franklin's bijection.

To count the difference in the number of parts in all partitions in $\mathcal{O}_{j,r}(n)$ and the number of parts in all partitions in $\mathcal{D}_{j,r}(n)$, we use Franklin's bijection $\varphi : \mathcal{O}_{j,r}(n) \to \mathcal{D}_{j,r}(n)$ described above. Then,

$$b_{j,r}(n) = \sum_{\lambda \in \mathcal{O}_{j,r}(n)} \ell(\lambda) - \sum_{\lambda \in \mathcal{D}_{j,r}(n)} \ell(\lambda) = \sum_{\lambda \in \mathcal{O}_{j,r}(n)} (\ell(\lambda) - \ell(\varphi(\lambda))). \tag{2.1}$$

Let λ be a partition in $\mathcal{O}_{j,r}(n)$. Suppose the j different parts of λ that are congruent to 0 mod r are $(m_i r)^{k_i}$ with $m_i, k_i > 0$, for $i = 1, 2, \ldots, j$, and m_i distinct. When

$$\lambda = \overline{\lambda} \bigcup_{i=1}^{j} (m_i r)^{k_i},$$

is mapped to

$$\varphi(\lambda) = \psi(\overline{\lambda}) \bigcup_{i=1}^{j} (m_i)^{k_i r},$$

we obtain a contribution of

$$(r-1)\left(-\sum_{i=1}^{j} k_i\right),$$

to $\ell(\lambda) - \ell(\varphi(\lambda))$ from mapping $(m_i r)^{k_i}$ to $(m_i)^{k_i r}$, $i = 1, 2, \ldots, j$, and a contribution of $(\ell(\overline{\lambda}) - \ell(\psi(\overline{\lambda})))$. Summing this over all possible $\lambda \in \mathcal{O}_{j,r}(n)$ whose parts congruent to 0 mod r are precisely $(m_i r)^{k_i}$, for $i = 1, 2, \ldots, j$, results in a contribution of

$$(r-1)\left(-\sum_{i=1}^{j} k_i\right)\left|\mathcal{O}_{0,r}\left(n - \sum_{i=1}^{j} r m_i k_i\right)\right| + b_{0,r}\left(n - \sum_{i=1}^{j} r m_i k_i\right).$$

Using (1.6), which has several combinatorial proofs, the above is equal to

$$(r-1)\left(-\sum_{i=1}^{j} k_i\right)\left|\mathcal{O}_{0,r}\left(n - \sum_{i=1}^{j} r m_i k_i\right)\right| + (r-1)\left|\mathcal{O}_{1,r}\left(n - \sum_{i=1}^{j} r m_i k_i\right)\right|.$$

Next, we reinterpret

$$\mathcal{O}_{0,r}\left(n - \sum_{i=1}^{j} r m_i k_i\right),$$

and

$$\mathcal{O}_{1,r}\left(n - \sum_{i=1}^{j} r m_i k_i\right).$$

First, note that

$$\zeta : \mu \mapsto \mu \bigcup_{i=1}^{j} (m_i r)^{k_i},$$

gives a bijection between

$$\mathcal{O}_{0,r}\left(n - \sum_{i=1}^{j} r m_i k_i\right),$$

and the subset of partitions in $\mathcal{O}_{j,r}(n)$ whose parts congruent to $0 \pmod r$ are precisely $(m_i r)^{k_i}$, $1 \le i \le j$.

Next, we consider ζ as a mapping on

$$\mathcal{O}_{1,r}\left(n - \sum_{i=1}^{j} r m_i k_i\right).$$

If

$$\mu \in \mathcal{O}_{1,r}\left(n - \sum_{i=1}^{j} r m_i k_i\right),$$

then $\zeta(\mu)$ fits in exactly one of the following two cases.

(i) If $m_t r$ is a part of μ for some $1 \leq t \leq j$, then $\zeta(\mu) \in \mathcal{O}_{j,r}(n)$ and it contains each part $m_i r$, $i \neq t$, with multiplicity k_i, and part $m_t r$ with multiplicity larger than k_t, or

(ii) If μ does not contain any $m_i r$ as a part, then $\zeta(\mu) \in \mathcal{O}_{j+1,r}(n)$ and it contains each part $m_i r$, $1 \leq i \leq j$ with multiplicity k_i.

We obtain a bijection between

$$\mathcal{O}_{1,r}\left(n - \sum_{i=1}^{j} r m_i k_i\right),$$

and the union of the subsets of $\mathcal{O}_{j,r}(n)$, respectively $\mathcal{O}_{j+1,r}(n)$, described in (i), respectively (ii), above.

To obtain the contribution of all partitions $\lambda \in \mathcal{O}_{j,r}(n)$ to (2.1), we consider this process for all possible j-tuples of positive integers $\mathbf{m} = (m_1, m_2, \ldots, m_j)$ with m_i distinct, and $\mathbf{k} = (k_1, k_2, \ldots, k_j)$.

For simplicity, we write $b_{j,r}(n) = A + B$, where

$$A := (r-1) \sum_{\mathbf{m},\mathbf{k}} \left(-\sum_{i=1}^{j} k_i\right) \left|\mathcal{O}_{0,r}\left(n - \sum_{i=1}^{j} r m_i k_i\right)\right|,$$

and

$$B := (r-1) \sum_{\mathbf{m},\mathbf{k}} \left|\mathcal{O}_{1,r}\left(n - \sum_{i=1}^{j} r m_i k_i\right)\right|.$$

We determine the contribution of each partition $\eta \in \mathcal{O}_{j,r}(n) \cup \mathcal{O}_{j+1,r}(n)$ to $A + B$.

Case 1: $\eta \in \mathcal{O}_{j,r}(n)$.

Suppose $(m_i r)^{k_i}$, $1 \leq i \leq j$, are the parts congruent to $0 \pmod r$ in η and let $m(\eta)$ be the total number of parts congruent to $0 \pmod r$ in η, i.e.,

$$m(\eta) = \sum_{i=1}^{j} k_i.$$

Then η contributes $-(r-1)m(\eta)$ to A.

The contribution of η to B comes from (i) above. For each choice of t, $1 \leq t \leq j$, and h_t, $1 \leq h_t \leq k_t - 1$, the partition η can be written as

$$\eta = \mu \bigcup_{i \neq t} (m_i r)^{k_i} \cup ((m_t r)^{h_t}),$$

with

$$\mu \in \mathcal{O}_{1,r}\left(n - rm_t h_t - \sum_{i \neq t} rm_i k_i\right),$$

and $m_t r$ is a part of μ with multiplicity $k_t - h_t$. Thus, η contributes

$$(r-1)\sum_{i=1}^{j}(k_i - 1) = (r-1)(m(\eta) - j),$$

to B.

CASE 2: $\eta \in \mathcal{O}_{j+1,r}(n)$.

Suppose $(m_i r)^{k_i}$, $1 \leq i \leq j+1$, are the parts congruent to $0 \pmod r$ in η. The partition η does not contribute to A. The contribution of η to B comes from (ii) above. For each choice of t, $1 \leq t \leq j+1$, the partition η can be written as

$$\eta = \mu \bigcup_{i \neq t} (m_i r)^{k_i},$$

with

$$\mu \in \mathcal{O}_{1,r}(n - \sum_{i \neq t} rm_i k_i),$$

where $m_t r$ is a part of μ with multiplicity k_t and none of $m_i r$, $i \neq t$, are parts of μ. Thus, η contributes $(r-1)(j+1)$ to B.

In total, we have

$$\begin{aligned} b_{j,r}(n) &= (r-1)\left(\sum_{\eta \in \mathcal{O}_{j,r}(n)} -m(\eta) + \sum_{\eta \in \mathcal{O}_{j,r}(n)} (m(\eta) - j) + \sum_{\eta \in \mathcal{O}_{j+1,r}(n)} (j+1)\right) \\ &= (r-1)(-j|\mathcal{O}_{j,r}(n)| + (j+1)|\mathcal{O}_{j+1,r}(n)|). \end{aligned}$$

3. A SECOND BECK-TYPE IDENTITY

Let $\mathcal{T}_{j,r}(n)$ denote the number of different parts with multiplicity between $r+1$ and $2r-1$ in all partitions in $\mathcal{D}_{j,r}(n)$. Let $b'_{j,r}(n)$ be the difference in the number of different parts in $\mathcal{D}_{j,r}(n)$ and the number of different parts in $\mathcal{O}_{j,r}(n)$. If we denote by $\overline{\ell}(\lambda)$ the number of different parts in λ, then

$$b'_{j,r}(n) = \sum_{\lambda \in \mathcal{D}_{j,r}(n)} \overline{\ell}(\lambda) - \sum_{\lambda \in \mathcal{O}_{j,r}(n)} \overline{\ell}(\lambda).$$

In [16], Yang showed that

$$b'_{0,r}(n) = \mathcal{T}_{1,r}(n). \tag{3.1}$$

This statement generalizes to a companion identity to Franklin's identity as follows. Denote by $b'_{\leq j,r}(n)$ the difference in the number of different parts in all partitions in $\mathcal{D}_{\leq j,r}(n)$ and the number of different parts in all partitions in $\mathcal{O}_{\leq j,r}(n)$.

Theorem 3.1. *Let n, j, r be non-negative integers with $r \geq 2$. Then,*

$$b'_{\leq j,r}(n) = \mathcal{T}_{j+1,r}(n).$$

In [7], Theorem 3.1 is proved by repeated application of the next theorem.

Theorem 3.2. *For all non-negative integers n, j, r with $r \geq 2$, we have*

$$b'_{j,r}(n) = \mathcal{T}_{j+1,r}(n) - \mathcal{T}_{j,r}(n).$$

Proof. For analytic and combinatorial proofs that are similar to the proofs of Theorem 2 we refer the reader to [7]. □

4. Further Generalizations

Let S_1 and S_2 be subsets of the positive integers. We define $\widetilde{\mathcal{O}}_{j,r}(n)$ to be the set of partitions of n with exactly j different parts from rS_1 and all other parts from S_2 and $\widetilde{\mathcal{D}}_{j,r}(n)$ to be the set of partitions of n with parts in S_1 and exactly j different parts repeated at least r times. Subbarao [14] proved the following theorem.

Theorem 4.1. *Let $r \geq 2$. Then, $|\widetilde{\mathcal{O}}_{0,r}(n)| = |\widetilde{\mathcal{D}}_{0,r}(n)|$ for all non-negative integers n if and only if $rS_1 \subseteq S_1$ and $S_2 = S_1 \setminus rS_1$.*

Andrews [2] first discovered this result for $r = 2$ and called a pair (S_1, S_2) such that $|\widetilde{\mathcal{O}}_{0,2}(n)| = |\widetilde{\mathcal{D}}_{0,2}(n)|$ an *Euler pair* since the pair $S_1 = \mathbb{N}$ and $S_2 = 2\mathbb{N} - 1$ gives Euler's identity. By analogy, Subbarao called a pair (S_1, S_2) such that $|\widetilde{\mathcal{O}}_{0,r}(n)| = |\widetilde{\mathcal{D}}_{0,r}(n)|$ an *Euler pair of order r*. In [6], we showed that, if (S_1, S_2) is an Euler pair, the identity of Theorem 4.1 has companion Beck-type identities analogous to (1.6) and (3.1).

It is straightforward to show that if (S_1, S_2) is an Euler pair of order r, then $|\widetilde{\mathcal{O}}_{j,r}(n)| = |\widetilde{\mathcal{D}}_{j,r}(n)|$. A similar argument to [6] establishes analogues of Theorems 1.2 and 3.2 and thus analogues of Theorems 1.1 and 3.1 for all Euler pairs of order r. Denote by $\widetilde{b}_{j,r}(n)$ the difference in the number of parts in $\widetilde{\mathcal{O}}_{j,r}(n)$ and the number of parts in $\widetilde{\mathcal{D}}_{j,r}(n)$. Denote by $\widetilde{b}'_{j,r}(n)$ the analogous difference of the number of different parts. Define $\widetilde{b}_{\leq j,r}(n)$ and $\widetilde{b}'_{\leq j,r}(n)$ in analogy to $b_{\leq j,r}(n)$ and $b'_{\leq j,r}(n)$. Let $\widetilde{\mathcal{T}}_{j,r}(n)$ denote the number of parts with multiplicity between $r+1$ and $2r-1$ in all partitions in $\widetilde{D}_{j,r}(n)$.

Theorem 4.2. *For all non-negative integers n, j and all integers $r \geq 2$, we have*

$$\begin{aligned}
\frac{1}{r-1}\widetilde{b}_{\leq j,r}(n) &= (j+1)|\widetilde{\mathcal{O}}_{j+1,r}(n)| \\
&= (j+1)|\widetilde{\mathcal{D}}_{j+1,r}(n)|. \\
\frac{1}{r-1}\widetilde{b}_{j,r}(n) &= (j+1)|\widetilde{\mathcal{O}}_{j+1,r}(n)| - j|\widetilde{\mathcal{O}}_{j,r}(n)| \\
&= (j+1)|\widetilde{\mathcal{D}}_{j+1,r}(n)| - j|\widetilde{\mathcal{D}}_{j,r}(n)|. \\
\widetilde{b}'_{\leq j,r}(n) &= \widetilde{\mathcal{T}}_{j+1,r}(n). \\
\widetilde{b}'_{j,r}(n) &= \widetilde{\mathcal{T}}_{j+1,r}(n) - \widetilde{\mathcal{T}}_{j,r}(n).
\end{aligned}$$

Since there are infinite families of Euler pairs of order r, we obtain infinite families of new Beck-type identities. Below we give some examples of pairs for which Theorem 4.2 gives new Beck-type identities.

The following pairs (S_1, S_2) are Euler pairs (of order 2).

(i) $S_1 = \{m \in \mathbb{N} : m \not\equiv 0 \,(\mathrm{mod}\ 3)\}$; $S_2 = \{m \in \mathbb{N} : m \equiv 1, 5 \,(\mathrm{mod}\ 6)\}$.
In this case, the identity $|\widetilde{\mathcal{O}}_{0,2}(n)| = |\widetilde{\mathcal{D}}_{0,2}(n)|$ is known as Schur's identity.

(ii) $S_1 = \{m \in \mathbb{N} : m \equiv 2, 4, 5 \,(\mathrm{mod}\ 6)\}$;
$S_2 = \{m \in \mathbb{N} : m \equiv 2, 5, 11 \,(\mathrm{mod}\ 12)\}$.
In this case, the identity $|\widetilde{\mathcal{O}}_{0,2}(n)| = |\widetilde{\mathcal{D}}_{0,2}(n)|$ is known as Göllnitz's identity.

(iii) $S_1 = \{m \in \mathbb{N} : m = x^2 + 2y^2 \text{ for some } x, y \in \mathbb{Z}\}$;
$S_2 = \{m \in \mathbb{N} : m \equiv 1 \,(\mathrm{mod}\ 2) \text{ and } m = x^2 + 2y^2 \text{ for some } x, y \in \mathbb{Z}\}$.

The following is an Euler pair of order 3.

(iv) $S_1 = \{m \in \mathbb{N} : m = x^2 + xy + y^2 \text{ for some } x, y \in \mathbb{Z}\}$;
$S_2 = \{m \in \mathbb{N} : \exists x, z \in \mathbb{Z}, \gcd(m, 3) = 1 \text{ and } m = x^2 + xy + y^2\}$.

The following pairs (S_1, S_2) are Euler pairs of order r.

(v) $S_1 = \{m \in \mathbb{N} : m \equiv \pm r \pmod{r(r+1)}\}$;
$S_2 = \{m \in \mathbb{N} : m \equiv \pm r \,(\mathrm{mod}\ r(r+1)) \text{ and } m \not\equiv \pm r^2 \left(\mathrm{mod}\ r^2(r+1)\right)\}$.

(vi) $S_1 = \{m \in \mathbb{N} : m \equiv \pm r, -1 \,(\mathrm{mod}\ r(r+1))\}$;
$S_2 = \{m \in \mathbb{N} : m \equiv \pm r, -1 \,(\mathrm{mod}\ r(r+1)) \text{ and } m \not\equiv \pm r^2, -r \left(\mathrm{mod}\ r^2(r+1)\right)\}$.
If $r = 2$, this Euler pair becomes Göllnitz's pair in (ii) above.

(vii) Let $r + 1$ be a prime.
$S_1 = \{m \in \mathbb{N} : m \not\equiv 0 \,(\mathrm{mod}\ r+1)\}$;
$S_2 = \{m \in \mathbb{N} : m \not\equiv tr, t(r+1) \left(\mathrm{mod}\ r^2 + r\right) \text{ for } 1 \leq t \leq r\}$.
If $r = 2$, this Euler pair becomes Schur's pair in (i) above.

(viii) Let p be a prime and r a quadratic residue $\pmod{p}$.
$S_1 = \{m \in \mathbb{N} : m \text{ quadratic residue } \pmod{p}\}$;
$S_2 = \{m \in \mathbb{N} : m \not\equiv 0 \,(\mathrm{mod}\ r) \text{ and } m \text{ quadratic residue} \,(\mathrm{mod}\ p)\}$.

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