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COMBINATORIAL PROOF OF THE GIRARD-WARING FORMULA

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ABSTRACT. The well-known and celebrated identity of Girard (1629) and Waring (1762) states that

$$x^n + y^n = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{n}{n-m} \binom{n-m}{m} (xy)^m (x+y)^{n-2m}$$

and can be easily proven algebraically (see H.W. Gould, *The Girard–Waring power sum formulas for symmetric functions and Fibonacci sequences,* Fibonacci Quart. **37** (1999), no. 2, 135–140). In this note, we provide a combinatorial proof of this identity.

Fix a natural number $n \geq 2$ and consider graphs whose set of vertices is $\mathbb{Z}_n := \{0, 1, \ldots, n-1\}$ and whose set of edges E satisfies

$$E \subseteq \{\{i, i+1\} \colon i \in \mathbb{Z}_n\}, \qquad \forall_{e_1, e_2 \in E}\ e_1 \neq e_2 \implies e_1 \cap e_2 = \varnothing,$$

where the addition is taken modulo n (so basically $n \equiv 0$, $-1 \equiv n-1$). Call such graph an *n-division*. This is just an arrangement of n objects in a cycle with some nonoverlapping pairs of adjacent elements connected. Clearly, $|E| \leq \lfloor n/2 \rfloor$.

Lemma 0.1. *For $m \leq \lfloor n/2 \rfloor$ the number of n-divisions with $|E| = m$ equals*

$$\frac{n}{n-m}\binom{n-m}{m}.$$

Proof. Let us count n-divisions according to whether $\{0, n-1\}$ is an edge. Suppose that $\{0, n-1\} \notin E$ in some n-division and consider a sequence $(a_i)_{i=0}^{n-1} \in \{0,1\}^n$ defined by

$$a_i = \begin{cases} 0, & \text{if } \exists_{e \in E}\ i \in e, \\ 1, & \text{otherwise.} \end{cases}$$

Then the sequence (a_n) consists of m blocks '00' and $n - 2m$ blocks '1'. On the other hand, every such sequence corresponds to exactly one n-division. Thus the number of n-divisions in this case equals $\binom{n-m}{m}$.

Similarly if $\{0, n-1\} \in E$ in some n-division then consider a sequence $(a_i)_{i=1}^{n-2} \in \{0,1\}^{n-2}$ defined as above. Now we obtain a sequence with $m-1$

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blocks '00' and $n-2m$ blocks '1' and again every such sequence corresponds to exactly one n-division. We get additional $\binom{n-m-1}{m-1}$ graphs, hence the final answer is

$$\binom{n-m}{m}+\binom{n-m-1}{m-1}=\binom{n-m}{m}+\frac{m}{n-m}\binom{n-m}{m}=\frac{n}{n-m}\binom{n-m}{m}$$

as desired. □

Let $c \geq 1$ be a natural number and let $G=(V,E)$ be an n-division with m edges. Call a *c-coloring* a function $f\colon V \to \mathbb{Z}_{c+1}$ such that

$$\forall_{\{i,j\}\in E}\ f(i)=f(j)\neq 0.$$

The number of c-colorings of G is clearly equal to $c^m(c+1)^{n-2m}$, as each edge is to be colored with one of c colors and each of the remaining vertices with one of $c+1$ colors. Call a pair (G,f) with G being an n-division and f being a c-coloring on G an *(n,c)-division.*

Theorem 0.2. *Fix $n \geq 2$, $c \geq 1$. Let $\mathcal{C}$ be the set of all (n,c)-divisions with $1 \leq |f^{-1}(\{0\})| \neq n$ or $|E| \geq 1$. Then $\mathcal{C}$ consists of equally many (n,c)-divisions with $2 \mid |E|$ and (n,c)-divisions with $2 \nmid |E|$.*

Proof. We will define a fixed-point-free involution $\Phi\colon \mathcal{C} \to \mathcal{C}$ such that in every pair $\{(G,f),\Phi(G,f)\}$ the numbers of edges differ by one (and thus have different parity). Let $(G,f)\in\mathcal{C}$ be an (n,c)-division and let

$$k=\min\{i\in\mathbb{Z}_n\colon \{i-1,i\}\in E \text{ or } (f(i-1)=0\neq f(i) \text{ and } \{i,i+1\}\notin E)\}.$$

Such a k exists as elements of $\mathcal{C}$ with $|E|=0$ have at least one vertex of color 0 and at least one vertex of a different color. Define $\Phi(G,f)=((V,E'),f')$:

$$E'=\begin{cases} E\setminus\{k-1,k\}, & \text{if } \{k-1,k\}\in E,\\ E\cup\{k-1,k\}, & \text{otherwise,}\end{cases}$$

$$f'(i)=\begin{cases} 0 & \text{if } i=k-1 \text{ and } \{k-1,k\}\in E,\\ f(k) & \text{if } i\in\{k-1,k\} \text{ and } \{k-1,k\}\notin E,\\ f(i), & \text{otherwise.}\end{cases}$$

So k is the least vertex which is either a non-0-colored vertex not belonging to an edge and preceded by a 0-colored vertex or the least edge vertex (being the greater among two ends); and the function Φ swaps the two options (keeping the color of k and all the remaining structure of an (n,c)-division). It is clear that a function defined in such a way induces a matching of elements of $\mathcal{C}$ with an even number of edges and elements of $\mathcal{C}$ with an odd number of edges. □

Corollary 0.3. *Fix $n \geq 2$, $c \geq 1$. Then*

$$c^n+1=\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\frac{n}{n-m}\binom{n-m}{m}c^m(c+1)^{n-2m}. \tag{0.1}$$

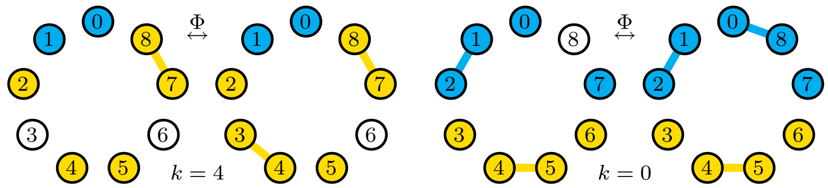

FIGURE 1. Examples of pairs induced by Φ with $(n, c) = (9, 2)$ (white, yellow, and blue correspond to colors 0, 1, and 2, respectively).

Proof. Note that (n, c)-divisions not belonging to $\mathcal{C}$ are exactly the ones with $|E| = 0$ and either none of the vertices are 0-colored or all of them are. Thus the number of such divisions is $c^n + 1$. Note that the number of (n, c)-divisions with m edges is equal to

$$\frac{n}{n-m}\binom{n-m}{m}c^m(c+1)^{n-2m},$$

so by the previous theorem, we get

$$\sum_{2\nmid m}\frac{n}{n-m}\binom{n-m}{m}c^m(c+1)^{n-2m} = -(c^n+1)+\sum_{2\mid m}\frac{n}{n-m}\binom{n-m}{m}c^m(c+1)^{n-2m},$$

which is equivalent to the desired identity. □

Corollary 0.4. *The equation in the abstract holds for all $x, y \in \mathbb{R} \setminus \{0\}$ and all $n \geq 2$.*

Proof. Fix $n \geq 2$. The identity (0.1) holds for all (i.e., more than n) natural $c \geq 1$. But it is an identity of polynomials of variable c and degree n. Thus the identity holds for all real $c \neq 0$. In particular, taking $c = x/y$ and multiplying both sides of identity (0.1) by y^n, we obtain the desired result. □

Remark. *All the identities remain valid for $n = 1$ or $xy = 0$.*

REFERENCES

1. H.W. Gould, *The Girard–Waring power sum formulas for symmetric functions and fibonacci sequences*, Fibonacci Quart. **37** (1999), no. 2, 135–140.

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