THE SIMPLICITY INDEX OF TOURNAMENTS

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Abstract. An n-tournament T with vertex set V is simple if there is no subset M of V such that 2 ≤ |M| ≤ n − 1 and for every x ∈ V \ M, either M → x or x → M. The simplicity index of an n-tournament T is the minimum number s(T) of arcs whose reversal yields a nonsimple tournament. Müller and Pelant (1974) proved that s(T) ≤ (n − 1)/2, and that equality holds if and only if T is doubly regular. As doubly regular tournaments exist only if n ≡ 3 (mod 4), s(T) < (n − 1)/2 for n ̸≡ 3 (mod 4). In this paper, we study the class of n-tournaments with maximal simplicity index for n ̸≡ 3 (mod 4).

1. Introduction

A tournament T consists of a finite set V of vertices together with a set A of ordered pairs of distinct vertices, called arcs, such that for all x ̸= y ∈ V, (x, y) ∈ A if and only if (y, x) /∈ A. Such a tournament is denoted by T = (V, A). Given x ̸= y ∈ V, we say that x dominates y and we write x → y when (x, y) ∈ A. Similarly, given two disjoint subsets X and Y of V, we write X → Y if x → y holds for every (x, y) ∈ X × Y. Throughout this paper, we mean by an n-tournament a tournament with n vertices.

A tournament is regular if there is an integer k ≥ 1 such that each vertex dominates exactly k vertices. It is doubly regular if there is an integer k ≥ 1 such that every unordered pair of vertices dominates exactly k vertices.

A tournament is transitive, if for any vertices x, y and z, x → y and y → z implies that x → z. A tournament T = (V, A) is reducible if V admits a bipartition {X, Y} such that X → Y. The notion of simple tournament was introduced by Fried and Lakser [8], it was motivated by questions in algebra. It is closely related to modular decomposition [9] which involves the notion of module. Recall that a module of a tournament T = (V, A) is a subset M of V such that for every x ∈ V \ M either M → {x} or {x} → M. For example, ∅, {x}, where x ∈ V, and V are modules of T called trivial.
modules. An \( n \)-tournament is simple \([6, 15]\) (or prime \([4]\) or primitive \([5]\) or indecomposable \([10, 17]\)) if \( n \geq 3 \) and all its modules are trivial. The simple tournaments with at most 5 vertices are shown in Figure 1. A tournament is decomposable if it admits a nontrivial module.

Given an \( n \)-tournament \( T \), the Slater index \( i(T) \) of \( T \) is the minimum number of arcs that must be reversed to make \( T \) transitive \([18]\). It is not difficult to see that \( i(T) \leq n(n - 1)/4 \). However, we do not know an exact determination of the upper bound of \( i(T) \). Erdős and Moon \([7]\) proved that this bound is asymptotically equal to \( n^2/4 \). Recently, Satake \([16]\) proved that the Slater index of doubly regular \( n \)-tournaments is at least

\[
\frac{n(n-1)}{4} - n^2 \log_2(2n).
\]

Kirkland \([11]\) defined the reversal index \( i_R(T) \) of a tournament \( T \) as the minimum number of arcs whose reversal makes \( T \) reducible. Clearly, \( i_R(T) \leq i(T) \). Kirkland \([11]\) proved that \( i_R(T) \leq \lfloor (n - 1)/2 \rfloor \) and characterized all the tournaments for which equality holds.

The indices above can be interpreted in terms of distance between tournaments. The distance \( d(T_1, T_2) \) between two tournaments \( T_1 \) and \( T_2 \) with the same vertex set is the number of pairs \( \{x, y\} \) of vertices for which the arc between \( x \) and \( y \) has not the same direction in \( T_1 \) and \( T_2 \). Let \( F \) be a family of tournaments with vertex set \( V \). The distance from a tournament \( T \) to the family \( F \) is \( d(T, F) = \min \{d(T, T') : T' \in F\} \). If \( F \) is the family

\[\text{Figure 1. The simple tournaments with at most 5 vertices}\]
of transitive tournaments on $V$, then $i(T) = d(T, \mathcal{F})$. If $\mathcal{F}$ is the family of reducible tournaments on $V$, then $i_R(T) = d(T, \mathcal{F})$. 

By considering the family of decomposable tournaments, we obtain the simplicity index introduced by Müller and Pelant [15]. Precisely, consider an $n$-tournament $T$, where $n \geq 3$. The simplicity index $s(T)$ of $T$ (also called the arrow-simplicity of $T$ in [15]) is the minimum number of arcs that must be reversed to make $T$ nonsimple. For example, the tournaments shown in Figure 1 have simplicity index 1. Obviously, $s(T) \leq i_R(T)$ and $s(T) \leq (n - 1)/2$. Müller and Pelant proved that $s(T) = (n - 1)/2$ if and only if $T$ is doubly regular.

A dual notion of the simplicity index is the decomposability index [2], which is obtained by considering the family of simple tournaments.

In this paper, we provide an upper bound for $s(T)$, where $T$ is an $n$-tournament for $n \not\equiv 3 \pmod{4}$. More precisely, we obtain the following result.

**Theorem 1.1.** Given an $n$-tournament $T$, the following statements hold

1. if $n = 4k + 2$, then $s(T) \leq 2k$;
2. if $n = 4k + 1$, then $s(T) \leq 2k - 1$;
3. if $n = 4k$, then $s(T) \leq 2k - 2$.

To prove that the bounds in this theorem are the best possible, we use the double regularity as follows.

**Theorem 1.2.** Let $l \in \{1, 2, 3\}$. Consider a doubly regular tournament $T$ of order $4k + 3$, where $k \geq l$. The simplicity index of a tournament obtained from $T$ by removing $l$ vertices is $(2k + 1) - l$.

As shown by the next result, the opposite direction in Theorem 1.2 holds when $l = 1$.

**Theorem 1.3.** Given a tournament $T$ with $4k + 2$ vertices, where $k \geq 1$, if $s(T) = 2k$, then $T$ is obtained from a doubly regular tournament by removing one vertex.

The existence of doubly regular tournaments is equivalent to the existence of skew-Hadamard matrices [3]. Wallis [20] conjectured that $n \times n$ skew-Hadamard matrices exist if and only if $n = 2$ or $n$ is divisible by 4. Infinite families of skew-Hadamard matrices can be found in [12].

The most known examples of a doubly regular tournament are obtained from Paley construction. For a prime power $q \equiv 3 \pmod{4}$, the Paley tournament of order $q$ is the tournament whose vertex set is the finite field $\mathbb{F}_q$, such that $x$ dominates $y$ if and only if $x - y$ is a nonzero quadratic residue in $\mathbb{F}_q$. 

2. Preliminaries

Let $T = (V, A)$ be an $n$-tournament and let $x \in V$. The out-neighborhood of $x$ is

$$N^+_T(x) := \{ y \in V : x \rightarrow y \},$$

and the in-neighborhood of $x$ is

$$N^-_T(x) := \{ y \in V : y \rightarrow x \}.$$

The out-degree of $x$ (resp. the in-degree of $x$) is

$$\delta^+_T(x) := |N^+_T(x)| \quad \text{ (resp. } \delta^-_T(x) := |N^-_T(x)|).$$

The out-degree of $x$ is also called the score of $x$ in $T$. Recall that

$$\sum_{z \in V} \delta^+_T(z) = \sum_{z \in V} \delta^-_T(z) = \frac{n(n-1)}{2}.$$  

A tournament is near-regular if there exists an integer $k > 0$ such that the out-degree of every vertex equals $k$ or $k - 1$.

**Remark:** Let $T$ be an $n$-tournament. It follows from (2.1) that

1. $T$ is regular if and only if $n$ is odd and every vertex has out-degree $(n-1)/2$;
2. $T$ is near-regular if and only if $n$ is even and $T$ has $n/2$ vertices of out-degree $n/2$ and $n/2$ vertices of out-degree $(n-2)/2$.

**Notation.** Let $T = (V, A)$ be a near-regular tournament of order $4k + 2$. We can partition $V$ into two $(2k+1)$-subsets,

$$V_{\text{even}} := \{ z \in V, \delta^+_T(z) = 2k \} \text{ and } V_{\text{odd}} := \{ z \in V, \delta^+_T(z) = 2k + 1 \}.$$

Let $x, y$ be distinct vertices of an $n$-tournament $T = (V, A)$. The set $V \setminus \{ x, y \}$ can be partitioned into four subsets:

$$N^+_T(x) \cap N^+_T(y), \quad N^-_T(x) \cap N^-_T(y),$$

$$N^+_T(x) \cap N^-_T(y), \quad N^-_T(x) \cap N^+_T(y).$$

The out-degree (resp. the in-degree) of $(x,y)$ is

$$\delta^+_T(x,y) := |N^+_T(x) \cap N^+_T(y)| \quad \text{ (resp. } \delta^-_T(x,y) := |N^-_T(x) \cap N^-_T(y)|).$$

The elements of $(N^+_T(x) \cap N^-_T(y)) \cup (N^-_T(x) \cap N^+_T(y))$ are called separators of $x, y$ and their number is denoted by $\sigma_T(x,y)$.

**Lemma 2.3.** Let $T$ be an $n$-tournament with vertex set $V$. For any $x \neq y \in V$, we have

- $\sigma_T(x,y) + \delta^-_T(x,y) + \delta^+_T(x,y) = n - 2$;
- $\delta^-_T(x,y) - \delta^+_T(x,y) = \delta^-_T(x) - \delta^+_T(y)$.

In particular, if $T$ is regular, then for any $x \neq y \in V$, $\delta^-_T(x,y) = \delta^+_T(x,y)$. 
Proof. The first assertion is obvious. For the second assertion, we have
\[ |N_T^{-}(x)| = |N_T^{-}(x) \cap N_T^{-}(y)| + |N_T^{+}(x) \cap N_T^{+}(y)| + |N_T^{-}(x) \cap \{y\}| \]
and
\[ |N_T^{+}(y)| = |N_T^{+}(y) \cap N_T^{+}(x)| + |N_T^{+}(y) \cap N_T^{-}(x)| + |N_T^{+}(y) \cap \{x\}|. \]
Moreover, \( y \in N_T^{-}(x) \) if and only if \( x \in N_T^{+}(y) \). Then
\[ |N_T^{-}(x) \cap \{y\}| = |N_T^{+}(y) \cap \{x\}| \]
and hence
\[ |N_T^{-}(x) \cap N_T^{-}(y)| - |N_T^{+}(x) \cap N_T^{+}(y)| = |N_T^{-}(x)| - |N_T^{+}(y)|. \]

Let \( T = (V, A) \) be a tournament. For each vertex \( z \in V \), we have
\[ \delta_T^{-}(z) \delta_T^{+}(z) = \left| \{ \{x, y\} \in \binom{V}{2} : z \in (N_T^{-}(x) \cap N_T^{+}(y)) \cup (N_T^{+}(x) \cap N_T^{-}(y)) \} \right|. \]
By double-counting, we obtain
\[ \sum_{z \in V} \delta_T^{+}(z) \delta_T^{-}(z) = \sum_{\{x, y\} \in \binom{V}{2}} \sigma_T(x, y). \]

(2.2)

In the next proposition, we give some basic properties of doubly regular tournaments. For the proof, see [15].

Proposition 2.4. Let \( T = (V, A) \) be a doubly regular \( n \)-tournament. There exists \( k \geq 0 \) such that \( n = 4k + 3 \), \( T \) is regular, and for all \( x, y \in V \) such that \( x \to y \), we have
\[ |N_T^{-}(x) \cap N_T^{-}(y)| = |N_T^{+}(x) \cap N_T^{+}(y)| = |N_T^{-}(x) \cap N_T^{+}(y)| = k \]
and
\[ |N_T^{-}(x) \cap N_T^{+}(y)| = k + 1. \]

3. Proof of Theorem 1.1

Let \( T = (V, A) \) be a tournament. Given a subset \( B \) of \( A \), we denote by \( \text{Inv}(T, B) \) the tournament obtained from \( T \) by reversing all the arcs of \( B \). We also use the following notation:
\[ \delta_T^{+} = \min \{ \delta_T^{+}(x) : x \in V \}, \quad \delta_T^{-} = \min \{ \delta_T^{-}(x) : x \in V \}, \]
\[ \delta_T = \min(\delta_T^{+}, \delta_T^{-}), \quad \sigma_T = \min \{ \sigma_T(x, y) : x \neq y \in V \}. \]
The next proposition provides an upper bound of the simplicity index of a tournament.

Proposition 3.1. For a tournament \( T = (V, A) \) with at least 3 vertices, we have \( s(T) \leq \min(\delta_T, \sigma_T) \).
Proof. Let $x \in V$. Clearly, the subset $V \setminus \{x\}$ is a nontrivial module of $\text{Inv}(T, \{x\} \times N^+_T(x))$ and $\text{Inv}(T, N^-_T(x) \times \{x\})$. It follows that

$$s(T) \leq \min_{x \in V} (\delta^+_T(x), \delta^-_T(x)) = \delta_T.$$

Now, consider an unordered pair $\{x, y\}$ of vertices of $T$ and let

$$B := (\{x\} \times (N^+_T(x) \cap N^-_T(y)) \cup (N^+_T(y) \cap N^-_T(x)) \times \{x\}).$$

Clearly, $\{x, y\}$ is a module of $\text{Inv}(T, B)$. It follows that

$$s(T) \leq |B| = |N^+_T(x) \cap N^-_T(y)| + |N^+_T(y) \cap N^-_T(x)| = \sigma_T(x, y).$$

Hence, $s(T) \leq \sigma_T$. □

In addition to the previous proposition, the proof of Theorem 1.1 requires the following lemma.

Lemma 3.2. Given an $n$-tournament $T = (V, A)$ with $n \geq 2$, we have

$$\delta_T \leq \left\lfloor \frac{n-1}{2} \right\rfloor \text{ and } \sigma_T \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof. For every $x \in V$, we have

$$\min (\delta^+_T(x), \delta^-_T(x)) \leq (n-1)/2.$$ Thus,

$$\delta_T \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Now, to verify that $\sigma_T \leq \lfloor (n-1)/2 \rfloor$, observe that

$$\sigma_T \leq \frac{1}{\binom{|V|}{2}} \sum_{\{x, y\} \in \binom{V}{2}} \sigma_T(x, y).$$

It follows from (2.2) that

$$\sigma_T \leq \frac{2}{n(n-1)} \sum_{z \in V} \delta^+_T(z)\delta^-_T(z) \leq \frac{2}{n(n-1)} \left( \frac{\delta^+_T(z) + \delta^-_T(z)}{2} \right)^2 \leq \frac{(n-1)}{2}.$$

Proof of Theorem 1.1. For the first statement, suppose that $n = 4k + 2$. By Proposition 3.1 and Lemma 3.2, we have

$$s(T) \leq \delta_T \leq \left\lfloor \frac{n-1}{2} \right\rfloor = 2k.$$

For the second statement, suppose that $n = 4k + 1$. By Proposition 3.1, $s(T) \leq \delta_T$. If $T$ is not regular, then $\delta_T < (n-1)/2$ and hence $s(T) \leq 2k - 1$. Suppose that $T$ is regular and let $x \neq y \in V$. By Lemma 2.3,

$$\sigma_T(x, y) = n - 2 - \delta^-_T(x, y) - \delta^+_T(x, y) \text{ and } \delta^-_T(x, y) = \delta^+_T(x, y).$$
Therefore, $\sigma_T(x, y)$ is odd, and hence $\sigma_T$ is odd as well. By Lemma 3.2, $\sigma_T \leq \lceil (n-1)/2 \rceil = 2k$. Since $\sigma_T$ is odd, we obtain $\sigma_T \leq 2k - 1$. It follows from Proposition 3.1 that $\sigma(T) \leq 2k - 1$.

For the third statement, suppose that $n = 4k$. If $T$ is not near-regular, then $\delta_T < 2k - 1$, and hence $s(T) \leq 2k - 2$ by Proposition 3.1. Suppose that $T$ is near-regular. By Remark 2.1, for every $z \in V$, $\delta_T^+(z) \in \{2k, 2k - 1\}$. It follows from (2.2) that

$$\sum_{\{x,y\}\in (V^2)} \sigma_T(x, y) = \sum_{z\in V} \delta_T^+(z)\delta_T^-(z) = 8k^2(2k - 1).$$

Thus, we obtain

$$\sigma_T \leq \frac{1}{\binom{|V|}{2}} \sum_{\{x,y\}\in (V^2)} \sigma_T(x, y)$$

$$\leq \frac{2}{4k(4k - 1)} 8k^2(2k - 1)$$

$$\leq (2k - 1) + \frac{2k - 1}{4k - 1}$$

$$\leq 2k - 1.$$

Since $s(T) \leq \sigma_T$ by Proposition 3.1, we obtain $s(T) \leq 2k - 1$. Seeking a contradiction, suppose that $s(T) = 2k - 1$. We obtain $\sigma_T = 2k - 1$. Let $x \in V_{\text{even}}$ and $y \in V_{\text{odd}}$ (see Notation 2.2). It follows from Lemma 2.3 that $\sigma_T(x, y)$ is even and hence $\sigma_T(x, y) \geq 2k$. Thus, there are at least $(2k)^2$ unordered pairs $\{x, y\}$ satisfying $\sigma_T(x, y) \geq 2k$. For the other $2\binom{2k}{2}$ unordered pairs, we have $\sigma_T(x, y) \geq \sigma_T = 2k - 1$. It follows that

$$\sum_{\{x,y\}\in (V^2)} \sigma_T(x, y) \geq 2\binom{2k}{2}(2k - 1) + (2k)^2(2k) > 8k^2(2k - 1),$$

which contradicts (3.1). Consequently, $s(T) \leq 2k - 2$. \quad \square

4. Proof of Theorem 1.2

To begin, recall that a graph is defined by a vertex set $V$ and an edge set $E$. Two distinct vertices $x$ and $y$ of $G$ are adjacent if $\{x, y\} \in E$. For a vertex $x$ in $G$, the set

$$N_G(x) := \{y \in V : \{x, y\} \in E\}$$

is called the neighborhood of $x$ in $G$. The degree of $x$ is $\delta_G(x) := |N_G(x)|$.

Let $T = (V, A)$ be a tournament. To each subset $C$ of $V$, we associate a graph in the following way. Denote by $s_C(T)$ the minimum number of arcs that must be reversed to make $C$ a module of $T$. Clearly,

$$s(T) = \min \{s_C(T) : 2 \leq |C| \leq n - 1\}.$$
A graph $G = (V, E)$ is called a decomposability graph for $C$ if $|E| = s_C(T)$ and $C$ is a module of the tournament

$$\text{Inv}(T, \{(x, y) \in A : \{x, y\} \in E\})$$

obtained from $T$ by reversing the arc between $x$ and $y$ for each edge $\{x, y\}$ of $G$. In the next lemma, we provide some of the properties of decomposability graphs.

**Lemma 4.1.** Let $T = (V, A)$ be a $n$-tournament and let $C$ be a subset of $V$ such that $2 \leq |C| \leq n - 1$. Given a decomposability graph $G = (V, E)$ for $C$, the following assertions hold

- $G$ is bipartite with bipartition $\{C, V \setminus C\}$;
- for each $x \in V \setminus C$, $N_G(x) = N_T^+(x) \cap C$ or $N_G(x) = N_T^-(x) \cap C$, and $\delta_G(x) = \min(|N_T^-(x) \cap C|, |N_T^+(x) \cap C|)$.

**Proof.** The first assertion follows from the minimality of $|E| = s_C(T)$. For the second assertion, consider $x \in V \setminus C$. Since $C$ is a module of the tournament $\text{Inv}(T, \{(x, y) \in A : \{x, y\} \in E\})$, we have

$$N_G(x) = N_T^+(x) \cap C \text{ or } N_G(x) = N_T^-(x) \cap C.$$

Furthermore, it follows from the minimality of $|E| = s_C(T)$ that

$$\delta_G(x) = \min(|N_T^-(x) \cap C|, |N_T^+(x) \cap C|).$$

$\square$

The next proposition is useful to prove Theorems 1.2 and 1.3.

**Proposition 4.2.** Let $T = (V, A)$ be an $n$-tournament and let $C$ be a subset of $V$ such that $2 \leq |C| \leq n - 1$. Given a decomposability graph $G = (V, E)$ for $C$, the following statements hold

- if $n - \delta_T \leq |C|$, then $s_C(T) \geq \delta_T$;
- if $|C| \leq \sigma_T$, then $s_C(T) \geq \sigma_T$.

**Proof.** Before showing the first assertion, we establish

\begin{equation}
|E| \geq (n - |C|)(|C| - (n - 1 - \delta_T)).
\end{equation}

Let $x \in V \setminus C$. By the second assertion of Lemma 4.1

$$\delta_G(x) = \min(|N_T^-(x) \cap C|, |N_T^+(x) \cap C|)$$

$$= |C| - \max(|N_T^-(x) \cap C|, |N_T^+(x) \cap C|).$$

Therefore, we obtain

$$\delta_G(x) \geq |C| - \max(|N_T^-(x)|, |N_T^+(x)|)$$

\begin{equation}
\geq (|C| - (n - 1 - \delta_T)).
\end{equation}

Since $G$ is bipartite with bipartition $\{C, V \setminus C\}$, we have

$$|E| = \sum_{x \in V \setminus C} \delta_G(x).$$
It follows from (4.3) that
\[ |E| \geq |V \setminus C|(|C| - (n - 1 - \delta_T)) \]
\[ \geq (n - |C|)(|C| - (n - 1 - \delta_T)). \]
Thus, (4.2) holds. Moreover, we have
\[ (n - |C|)(|C| - (n - 1 - \delta_T)) - \delta_T = (n - 1 - |C|)(|C| - (n - \delta_T)). \]
Now, to prove the first assertion, suppose that \( n - \delta_T \leq |C| \). We obtain
\[ (n - |C|)(|C| - (n - 1 - \delta_T)) \geq 0, \]
and hence
\[ (n - |C|)(|C| - (n - 1 - \delta_T)) \geq \delta_T. \]
It follows that \( s_C(T) = |E| \geq \delta_T. \)
Before showing the second assertion, we establish
\[ (4.4) \quad |E| \geq \frac{|C|}{2}(2 - |C| + \sigma_T). \]
Consider two vertices \( x \neq y \in C \). For convenience, denote by \( S_T(x, y) \) the set of separators of \( \{x, y\} \). Clearly, we have \( (S_T(x, y) \setminus C) \subseteq N_G(x) \cup N_G(y) \).
It follows that
\[ \delta_G(x) + \delta_G(y) \geq |S_T(x, y) \setminus C| \geq \sigma_T(x, y) - (|C| - 2). \]
Consequently, we obtain
\[ (4.5) \quad \delta_G(x) + \delta_G(y) \geq \sigma_T - |C| + 2. \]
Furthermore, observe that
\[ \sum_{\{x, y\} \in \binom{C}{2}} (\delta_G(x) + \delta_G(y)) = (|C| - 1) \sum_{x \in C} \delta_G(x). \]
It follows from (4.5) that
\[ (|C| - 1) \sum_{x \in C} \delta_G(x) \geq \left(\frac{|C|}{2}\right)(2 - |C| + \sigma_T). \]
Therefore, we have
\[ \sum_{x \in C} \delta_G(x) \geq \frac{|C|}{2}(2 - |C| + \sigma_T). \]
Since \( G \) is bipartite with bipartition \( \{C, V \setminus C\} \), we have
\[ |E| = \sum_{x \in C} \delta_G(x). \]
We obtain
\[ |E| \geq \frac{|C|}{2}(2 - |C| + \sigma_T), \]
so (4.4) holds.
Finally, to prove the second assertion, suppose that $|C| \leq \sigma_T$. We obtain
\[
\frac{|C|}{2} (2 - |C| + \sigma_T) \geq \sigma_T.
\]
Since $s_C(T) = |E|$, it follows from (4.4) that $s_C(T) \geq \sigma_T$. \hfill \Box

Proof of Theorem 1.2. Let $l \in \{1, 2, 3\}$. Consider a tournament $R$ from $T$ by removing $l$ vertices $v_1, \ldots, v_l$. Set $V' := V \setminus \{v_1, \ldots, v_l\}$. It follows from Theorem 1.1 that $s(R) \leq (2k + 1) - l$. It remains to show that $s(R) \geq (2k + 1) - l$. By (4.1), it suffices to verify that $s_C(R) \geq (2k + 1) - l$ for every subset $C$ of $V'$ such that
\[
2 \leq |C| \leq (4k + 2) - l.
\]
Let $C \subseteq V'$ such that
\[
2 \leq |C| \leq (4k + 2) - l.
\]
We distinguish the following three cases.

Case 1: Suppose that $2 \leq |C| \leq (2k + 1) - l$.
Since $T$ is doubly regular, it follows from Proposition 2.4 that $\sigma_T = 2k + 1$. Therefore, $\sigma_R \geq (2k + 1) - l$. Since
\[
2 \leq |C| \leq (2k + 1) - l, \quad \sigma_R \geq |C|.
\]
It follows from Proposition 4.2 that $s_C(R) \geq \sigma_R$, and hence $s_C(R) \geq (2k + 1) - l$.

Case 2: Suppose that $2k + 2 \leq |C| \leq (4k + 2) - l$.
Since $T$ is doubly regular, it follows from Proposition 2.4 that $T$ is regular. Thus, $\delta_T = 2k + 1$. It follows that $\delta_R \geq (2k + 1) - l$. Since
\[
2k + 2 \leq |C| \leq (4k + 2) - l,
\]
we obtain $|C| + \delta_R \geq (4k + 3) - l$. It follows from Proposition 4.2 that $s_C(R) \geq \delta_R$, and hence $s_C(R) \geq (2k + 1) - l$.

Case 3: $(2k + 2) - l \leq |C| \leq 2k + 1$.
Let $G = (E', V')$ be a decomposability graph for $C$. We verify that
\[
|\{x \in V' \setminus C : \delta_G(x) \neq 0\}| \geq |V' \setminus C| - 1. \tag{4.6}
\]
Otherwise, there exist $x \neq y \in V' \setminus C$ such that $\delta_G(x) = \delta_G(y) = 0$. It follows from the second assertion of Lemma 4.1 applied to $R$ that $C$ is contained in one of the following intersections:
\[
(N_R^-(x) \cap N_R^+(y)), \quad (N_R^-(x) \cap N_R^-(y)),
\]
\[
(N_R^+(x) \cap N_R^+(y)), \quad \text{or} \quad (N_R^+(x) \cap N_R^-(y)).
\]
Thus, $C$ is contained in
\[
(N_T^-(x) \cap N_T^+(y)), \quad (N_T^-(x) \cap N_T^-(y)),
\]
\[
(N_T^+(x) \cap N_T^+(y)), \quad \text{or} \quad (N_T^+(x) \cap N_T^-(y)).
\]
It follows from Proposition 2.4 that $|C| \leq k + 1$, which contradicts $|C| \geq (2k + 2) - l$ because $k \geq l$. Consequently, (4.6) holds. Since $G$ is bipartite with bipartition $\{C, V' \setminus C\}$, we have

$$|E'| = \sum_{x \in V' \setminus C} \delta_G(x).$$

Since $|E'| = s_C(R)$, we obtain

$$s_C(R) = \sum_{x \in V' \setminus C} \delta_G(x)$$

$$\geq |V' \setminus C| - 1 \quad \text{(by (4.6))}$$

$$\geq (2k + 1) - l \quad \text{(because $|C| \leq 2k + 1$)}.$$

□

5. PROOF OF THEOREM 1.3

If a tournament $T$ is obtained from a doubly regular $(4k + 3)$-tournament by deleting one vertex, then $T$ is near-regular and it follows from Proposition 2.4 that

(C1) if $x, y \in V_{even}$ (see Notation 2.2) or $x, y \in V_{odd}$, then $\sigma_T(x, y) = 2k + 1$.

(C2) if $x \in V_{even}$ and $y \in V_{odd}$, then $\sigma_T(x, y) = 2k$.

Conversely, we have the following proposition.

Proposition 5.1. Let $T = (V, A)$ be a near-regular tournament of order $4k + 2$. If $T$ satisfies (C1) and (C2), then the tournament $U$ obtained from $T$ by adding a vertex $\omega$ which dominates $V_{odd}$ and is dominated by $V_{even}$ is doubly regular.

The proof of this proposition uses the following lemma.

Lemma 5.2. Under the notation and conditions of Proposition 5.1, for every $x, y \in V$ such that $x \rightarrow y$, we have

- if $x, y \in V_{odd}$, then
  $|N_T^-(x) \cap N_T^+(y)| = k + 1$ and $|N_T^+(x) \cap N_T^-(y)| = k$;

- if $x, y \in V_{even}$, then
  $|N_T^-(x) \cap N_T^+(y)| = k + 1$ and $|N_T^+(x) \cap N_T^-(y)| = k$;

- if $x \in V_{odd}$ and $y \in V_{even}$, then
  $|N_T^-(x) \cap N_T^+(y)| = k$ and $|N_T^+(x) \cap N_T^-(y)| = k$;

- if $x \in V_{even}$ and $y \in V_{odd}$, then
  $|N_T^-(x) \cap N_T^+(y)| = k + 1$ and $|N_T^+(x) \cap N_T^-(y)| = k - 1$. 

Then, \( |N_T^-(x) \cap N_T^-(y)| + |N_T^+(x) \cap N_T^+(y)| = |N_T^-(x)| \).

By using Lemma 2.3, we obtain

\[
|N_T^+(x) \cap N_T^+(y)| + |N_T^-(x) \cap N_T^-(y)| = |N_T^+(y)|. 
\]

Using Assertions (C1) and (C2), we obtain the desired values of

\[
|N_T^-(x) \cap N_T^+(y)| = \sigma(x, y) - |N_T^-(x) \cap N_T^+(y)|. 
\]

Proof of Proposition 5.1. Clearly, \( U \) is regular. Furthermore, by Lemma 2.3,

\[
\delta_U^+(x, y) = \frac{4k - \sigma_U(x, y) + 1}{2}
\]

for distinct \( x, y \in V \cup \{\omega\} \). Therefore, \( U \) is doubly regular if and only if
\( \sigma_U(x, y) = 2k + 1 \) for every \( x, y \in V \cup \{\omega\} \). This equality follows directly from (C1) and (C2) when \( x, y \in V \). Hence, it remains to prove that

\[
|N_T^-(x) \cap N_T^-(y)| = 2k + 1 \text{ for every } z \in V. 
\]

Consider \( z \in V \). It is not difficult to see that

\[
\sigma_U(\omega, z) = |N_T^+(z) \cap V_{even}| + |N_T^-(z) \cap V_{odd}| \quad \text{(see Notation 2.2).}
\]

Let

\[
A_{odd} := (N_T^+(z) \cap V_{odd}), \quad A_{even} := (N_T^+(z) \cap V_{even}), \\
B_{odd} := (N_T^-(z) \cap V_{odd}), \quad B_{even} := (N_T^-(z) \cap V_{even}).
\]

We determine \( |A_{odd}|, |A_{even}|, |B_{odd}|, \) and \( |B_{even}| \) as follows.

To begin, suppose that \( z \in V_{odd} \). By counting the number of arcs from \( N_T^+(z) \) to \( N_T^-(z) \) in two ways, we get

\[
\sum_{t \in A_{odd}} |N_T^-(z) \cap N_T^+(t)| + \sum_{t \in A_{even}} |N_T^-(z) \cap N_T^+(t)| \\
= \sum_{t \in B_{odd}} |N_T^-(z) \cap N_T^+(z)| + \sum_{t \in B_{even}} |N_T^-(z) \cap N_T^+(t)|. 
\]

It follows from Lemma 5.2 that

\[
(k + 1) |A_{odd}| + k |A_{even}| = (k + 1) (|B_{odd}| + |B_{even}|).
\]

Since \( z \in V_{odd} \), we have

\[
|A_{odd}| + |A_{even}| = 2k + 1, \quad |B_{odd}| + |B_{even}| = 2k,
\]

\[
|A_{odd}| + |B_{odd}| = 2k, \quad \text{and} \quad |A_{even}| + |B_{even}| = 2k + 1.
\]
It follows that \(|A_{\text{odd}}| = k, |B_{\text{odd}}| = k, |B_{\text{even}}| = k,\) and \(|A_{\text{even}}| = k + 1.\)

Similarly, if \(z \in V_{\text{even}},\) then \(|A_{\text{odd}}| = k, |B_{\text{odd}}| = k + 1, |B_{\text{even}}| = k,\) and \(|A_{\text{even}}| = k.\)

Consequently, (5.3) holds whatever the parity of \(\delta^+_T(z).\)

\[\square\]

Proof of Theorem 1.3. Given \(k \geq 1,\) consider a tournament \(T,\) with \(4k + 2\) vertices, such that \(s(T) = 2k.\) By Proposition 3.1, \(\delta_T \geq 2k.\) Thus, \(T\) is near-regular. We conclude by applying Proposition 5.1. Therefore, it suffices to verify that (C1) and (C2) are satisfied.

By Proposition 3.1, \(\sigma_T(x, y) \geq 2k\) for distinct \(x, y \in V.\) Moreover, it follows from Lemma 2.3 that if \(x, y \in V_{\text{even}}\) or \(x, y \in V_{\text{odd}}\) (see Notation 2.2), then \(\sigma_T(x, y)\) is odd and hence \(\sigma_T(x, y) \geq 2k + 1.\)

Lastly, seeking a contradiction, suppose that (C1) or (C2) are not satisfied. One of the following situations occurs

- there are distinct \(x, y \in V_{\text{even}}\) such that \(\sigma_T(x, y) > 2k + 1,\)
- there are distinct \(x, y \in V_{\text{odd}}\) such that \(\sigma_T(x, y) > 2k + 1,\)
- there are \(x \in V_{\text{even}}\) and \(y \in V_{\text{odd}}\) such that \(\sigma_T(x, y) > 2k.\)

We obtain

\[
\sum_{\{x,y\} \in \binom{V}{2}} \sigma_T(x, y) > (2k + 1) \left(\frac{|V_{\text{even}}|}{2}\right) + (2k + 1) \left(\frac{|V_{\text{odd}}|}{2}\right) + 2k |V_{\text{even}}||V_{\text{odd}}|
\]

\[= 4k(2k + 1)^2,\]

which contradicts (2.2). Consequently, (C1) and (C2) are satisfied. \(\square\)

6. Concluding Remarks

1. An \(n\)-tournament with \(n = 4k + 1\) is called near-homogeneous \([19]\) if every unordered pair of its vertices belongs to \(k\) or \((k + 1)\) 3-cycles. The existence of near-homogeneous tournaments is discussed in \([19, 1, 14].\) For \(n \equiv 1 \pmod{4}\) or \(n \equiv 0 \pmod{4},\) the \(n\)-tournaments given in Theorem 1.2 are not the only ones with a maximal simplicity index. Indeed, let \(T\) be a near-homogeneous tournament \(T\) with \(4k + 1\) vertices. By adapting the proof of Theorem 1.2, we can verify that \(s(T) = 2k - 1.\) Moreover, by removing one vertex from \(T,\) we obtain a \((4k)\)-tournament whose simplicity index is \(2k - 2.\) Consequently, an analogue of Theorem 1.3 does not exist when \(l = 2\) or 3.

2. The score vector of a \(n\)-tournament \(T\) is the ordered sequence of the scores of \(T\) listed in a nondecreasing order. Kirkland \([11]\) proved that the reversal index of an \(n\)-tournament \(T\) is equal to

\[
\min \left\{ \sum_{i=1}^{j} s_i - \left(\frac{j}{2}\right) : 1 \leq j \leq n \right\},
\]

where \((s_1, s_2, \ldots, s_n)\) is the score vector of \(T.\)

An equivalent form of this result was obtained earlier by Li and Huang \([13].\) As a consequence, two tournaments with the same score vector have the
same reversal index. This fact is not true for the simplicity index. Indeed, for an odd number \( n \), consider the \( n \)-tournament \( R_n \) whose vertex set is the additive group \( \mathbb{Z}_n = \{0, 1, \ldots, n-1\} \) of integers modulo \( n \), such that \( i \) dominates \( j \) if and only if \( i - j \in \{1, \ldots, (n-1)/2\} \). It is not difficult to verify that the tournament \( R_n \) is regular and simple. Moreover, by reversing the arc \((0, (n-1)/2)\), we obtain a nonsimple tournament. Hence, the simplicity index of \( R_n \) is 1. If \( n \) is prime and \( n \equiv 3 \pmod{4} \), the Paley tournament \( P_n \) is also regular but its simplicity index is \((n-1)/2\).

Let \( T \) be an \( n \)-tournament with vertex set \( \{v_1, \ldots, v_n\} \). The sequences \( L_1 = (\delta^+_T(v_i))_{1 \leq i \leq n} \) and \( L_2 = (\delta^+_T(v_i, v_j))_{1 \leq i < j \leq n} \) are frequently used in our study of the simplicity index. It is natural to ask whether the simplicity index of \( T \) can be expressed in terms of \( L_1 \) and \( L_2 \).

**References**


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