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HOMOTHETIC COVERING OF CONVEX HULLS OF COMPACT CONVEX SETS

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Abstract. Let K be a compact convex set and m be a positive integer. The covering functional of K with respect to m is the smallest $\lambda \in [0,1]$ such that K can be covered by m translates of λK. Estimations of the covering functionals of convex hulls of two or more compact convex sets are presented. It is proved that, if a three-dimensional convex body K is the convex hull of two compact convex sets having no interior points, then the least number $c(K)$ of smaller homothetic copies of K needed to cover K is not greater than 8 and $c(K) = 8$ if and only if K is a parallelepiped.

1. Introduction

Let K be a compact convex subset of $\mathbb{R}^n$ that contains distinct points. We denote by $\operatorname{relint} K$, $\operatorname{relbd} K$, $\operatorname{int} K$, $\operatorname{bd} K$, and $\operatorname{ext} K$ the *relative interior*, *relative boundary*, *interior*, *boundary*, and the set of *extreme points* of K, respectively. For each $x \in \mathbb{R}^n$ and $\lambda \in (0,1)$, the set

$$x + \lambda K := \{x + \lambda y \mid y \in K\}$$

is called a *smaller homothetic copy* of K. We denote by $\mathcal{K}^n$ the set of *convex bodies* in $\mathbb{R}^n$, i.e., the set of compact convex sets in $\mathbb{R}^n$ having interior points.

For each compact convex set K, we denote by $c(K)$ the least number of translates of $\operatorname{relint} K$ needed to cover K. Concerning the least upper bound of $c(K)$ in $\mathcal{K}^n$, there is a long-standing conjecture (see [6], [3], and [2] for more information about this conjecture):

Conjecture 1 (Hadwiger's covering conjecture). *For each $K \in \mathcal{K}^n$, $c(K)$ is bounded from the above by 2^n, and this upper bound is attained only by parallelotopes.*

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The assertion "$c(K) \leq 2^n$, $\forall K \in \mathcal{K}^n$" will be referred to as the "inequality part" of Conjecture 1. This conjecture has been completely verified for several classes of convex bodies including all planar convex bodies (cf. [10]), zonotopes, zonoids, belt bodies (cf. [3, §34]), and convex hulls of a pair of compact convex sets contained in two parallel hyperplanes in $\mathbb{R}^3$ (cf. [15]). And the inequality part of Conjecture 1 has been verified for centrally symmetric convex bodies in $\mathbb{R}^3$ (cf. [7]), convex polyhedron in $\mathbb{R}^3$ having an affine symmetry (cf. [1]), convex bodies in $\mathbb{R}^3$ symmetric about a plane (cf. [5]).

For each $m \in \mathbb{Z}^+$, we use the short-hand notation

$$[m] = \{t \in \mathbb{Z}^+ \mid 1 \leq t \leq m\}.$$

Note that, for each compact convex set K, $c(K)$ equals the least number of smaller homothetic copies of K needed to cover K (see, e.g., [3, p. 262, Theorem 34.3]). Therefore, $c(K) \leq m$ for some $m \in \mathbb{Z}^+$ if and only if $\Gamma_m(K) < 1$, where $\Gamma_m(K)$ is defined by

$$\Gamma_m(K) := \min\left\{\gamma > 0 \;\middle|\; \exists\{x_i | i \in [m]\} \subseteq \mathbb{R}^n \text{ s.t. } K \subseteq \bigcup_{i=1}^{m}(x_i + \gamma K)\right\},$$

and is called *the covering functional of K with respect to m* (cf. [8], where $\Gamma_m(K)$ is called *the m-covering number of K*, and [16]).

In this paper, we extend the results in [15] by studying the homothetic covering problem for compact convex sets that can be expressed as convex hulls of two or more compact convex sets. In section 2, we provide an estimation of covering functionals for this class of convex bodies in $\mathcal{K}^n$. In section 3, we solve Hadwiger's conjecture with respect to convex bodies in $\mathcal{K}^3$ that are convex hulls of two compact convex sets having empty interiors.

2. Covering functional of convex hulls of compact convex sets

The following estimation of the covering functionals of convex hulls of compact convex sets $K_1, \dots, K_p$ uses only information about the covering functionals of each K_i.

Theorem 1. *Suppose that $K \in \mathcal{K}^n$ is the convex hull of convex compact sets $K_1, \dots, K_p$ and $m_1, \dots, m_p \in \mathbb{Z}^+$.*

(1) *If $p \leq n+1$, then*

$$\Gamma_{m_1+\dots+m_p}(K) \leq \max\left\{\frac{p-1+\Gamma_{m_i}(K_i)}{p} \;\middle|\; i \in [p]\right\}.$$

(2) *If $p > n+1$, then*

$$\Gamma_{m_1+\dots+m_p}(K) \leq \max\left\{\frac{n+\Gamma_{m_i}(K_i)}{n+1} \;\middle|\; i \in [p]\right\}.$$

Proof. Without loss of generality, we may assume that $o \in \operatorname{relint} K$. For each $i \in [p]$, put $\gamma_i = \Gamma_{m_i}(K_i)$. Then, for each $i \in [p]$, there exists a set $\{y_j^i | j \in [m_i]\}$ of m_i points such that

$$K_i \subseteq \bigcup_{j=1}^{m_i}(y_j^i + \gamma_i K_i) \subseteq \bigcup_{j=1}^{m_i}(y_j^i + \gamma_i K).$$

Let x be an arbitrary point in K.

Case 1: $p \leq n+1$.

By Theorem 3.13 in [13], there exist p points $x_1, \dots, x_p$, p numbers $\lambda_1, \dots, \lambda_p \in [0,1]$ such that

$$x_i \in K_i, \ \forall i \in [p], \quad \sum_{i\in[p]} \lambda_i = 1, \quad \text{and} \quad x = \sum_{i\in[p]} \lambda_i x_i.$$

We may assume, without loss of generality, that

$$\lambda_1 \geq \frac{1}{p} \quad \text{and} \quad x_1 \in y_1^1 + \gamma_1 K_1 \subseteq y_1^1 + \gamma_1 K.$$

Then

$$\begin{aligned} x = \lambda_1 x_1 + \sum_{i=2}^{p} \lambda_i x_i &= \frac{1}{p}x_1 + \left(\lambda_1 - \frac{1}{p}\right)x_1 + \sum_{i=2}^{p} \lambda_i x_i \\ &\in \frac{1}{p}y_1^1 + \frac{p-1+\gamma_1}{p}K \\ &\subseteq \frac{1}{p}y_1^1 + \max\left\{\frac{p-1+\gamma_i}{p} \;\middle|\; i \in [p]\right\} K. \end{aligned}$$

It follows that

$$K \subseteq \bigcup_{i\in[p]} \left\{\frac{1}{p}y_j^i \;\middle|\; j \in [m_i]\right\} + \max\left\{\frac{p-1+\gamma_i}{p} \;\middle|\; i \in [p]\right\} K.$$

Case 2: $p > n+1$.

By Carathéodory's theorem, there exist $n+1$ points $x_1, \dots, x_{n+1} \in \bigcup_{i=1}^{p} K_i$ and $n+1$ numbers $\lambda_1, \dots, \lambda_{n+1} \in [0,1]$ such that

$$x = \sum_{i=1}^{n+1} \lambda_i x_i \quad \text{and} \quad \sum_{i=1}^{n+1} \lambda_i = 1.$$

We may assume, without loss of generality, that

$$\lambda_1 \geq \frac{1}{n+1} \quad \text{and} \quad x_1 \in y_1^1 + \gamma_1 K_1 \subseteq y_1^1 + \gamma_1 K.$$

Then, in a similar way as above, we can show that

$$x \in \frac{1}{n+1}y_1^1 + \max\left\{\frac{n+\gamma_i}{n+1} \;\middle|\; i \in [p]\right\} K.$$

It follows that

$$K \subseteq \bigcup_{i\in[p]} \left\{ \frac{1}{n+1} y_j^i \;\middle|\; j \in [m_i] \right\} + \max \left\{ \frac{n+\gamma_i}{n+1} \;\middle|\; i \in [p] \right\} K. \qquad \square$$

In particular, we have the following:

Corollary 2. *Suppose that K is the convex hull of two nonempty compact convex sets L and M, and that $m_1, m_2 \in \mathbb{Z}^+$. Then*

$$\Gamma_{m_1+m_2}(K) \leq \max \left\{ \frac{1+\Gamma_{m_1}(L)}{2}, \frac{1+\Gamma_{m_2}(M)}{2} \right\}.$$

Corollary 3. *If K is the convex hull of segments $K_1, \ldots, K_p$, then*

$$\Gamma_{2p}(K) \leq \frac{2p-1}{2p}.$$

Remark 4. When applying Theorem 1 to get a good estimation of $\Gamma_m(K)$, a suitable representation of K as the convex hull of compact convex sets is necessary. For example, let K be a three-dimensional simplex with a, b, c, d as vertices. If we use the representation $K = \operatorname{conv}(\{a\} \cup \operatorname{conv}\{b, c, d\})$ then, by Theorem 1, we have $\Gamma_4(K) \leq 5/6$. But, if we use $K = \operatorname{conv}(\operatorname{conv}\{a, b\} \cup \operatorname{conv}\{c, d\})$, we will have the estimation $\Gamma_4(K) \leq 3/4$, which is much better.

When n is odd and K is an n-dimensional simplex, it is not difficult to verify that $\Gamma_{n+1}(K) = n/(n+1)$. By Corollary 3, we have $\Gamma_{n+1}(K) \leq n/(n+1)$. This shows that the estimation in Theorem 1 is tight in general. However, it can be improved in many other cases by taking the extremal structure of K into consideration.

3. The three-dimensional case

Let $K \subset \mathbb{R}^n$ be a compact convex set, $x \in \operatorname{relbd} K$, and $u \in \mathbb{R}^n$ be a nonzero vector. If there exists a scalar $\lambda > 0$ such that $x + \lambda u \in \operatorname{relint} K$, then we say that u *illuminates* x. It is not difficult to see that, a set D of directions illuminates $\operatorname{relbd} K$ if and only if D illuminates all extreme points of K. Moreover (cf. Theorem 34.3 in [3]), $c(K)$ equals to the minimal cardinality of a set of directions that can illuminate $\operatorname{relbd} K$.

A pair of points a, b in a set $X \subseteq \mathbb{R}^n$ is called *antipodal* provided there are distinct parallel hyperplanes H_a and H_b through a and b, respectively, such that X lies in the slab between H_a and H_b.

Lemma 5. *Let $K \in \mathcal{K}^n$ and x, $y \in \operatorname{bd} K$. If x and y are not antipodal, then there is a direction that illuminates both x and y.*

Proof. We only need to consider the case when $x \neq y$. Since x and y are not antipodal, the segment $[x, y]$ is not an affine diameter (cf. [12] for the definition and basic properties of affine diameters) of K. Let $[u, v]$ be an affine diameter of K parallel to $[x, y]$ and c be an interior point of K. Without loss of generality, we may assume that

$$\frac{x-y}{\|x-y\|} = \frac{u-v}{\|u-v\|}.$$

Then there exists a number $\lambda \in (0,1)$ such that $s - t = x - y$, where

$$s = \lambda c + (1-\lambda)u, \quad t = \lambda c + (1-\lambda)v.$$

Clearly, both s and t are interior points of K. Let $d = \frac{s+t}{2} - \frac{x+y}{2}$. Then

$$x + d = \frac{x-y}{2} + \frac{s+t}{2} = s, \quad \text{and} \quad y + d = \frac{y-x}{2} + \frac{s+t}{2} = t,$$

i.e., x and y are both illuminated by $d \neq o$. □

Theorem 6. *Let $K \in \mathcal{K}^3$ be a convex body. If there exist two compact convex sets L and M with empty interior such that $K = \operatorname{conv}(L \cup M)$, then $c(K) \leq 8$ and the equality holds if and only if K is a parallelepiped.*

Proof. We denote by $\operatorname{aff} L$ and $\operatorname{aff} M$ the affine dimensions of L and M, respectively. We distinguish four cases.

Case 1: $0 \in \{\operatorname{aff} L, \operatorname{aff} M\}$.

Assume without loss of generality that $\operatorname{aff} L = 0$. Then $\operatorname{aff} M = 2$. By Theorem 4 in [14] and the fact that $\Gamma_7(M) \leq 1/2$ holds for each planar convex body (cf. [9]), we have

$$\Gamma_8(K) \leq \frac{1}{2 - \Gamma_7(M)} \leq \frac{2}{3}.$$

By Corollary 2 and the fact that $\Gamma_4(M) \leq \sqrt{2}/2$ holds for each planar convex body M (cf. [8]), we have

$$\Gamma_5(K) \leq \frac{1 + \frac{\sqrt{2}}{2}}{2} \approx 0.854.$$

Case 2: $\operatorname{aff} L = \operatorname{aff} M = 1$.

In this situation, K is a three-dimensional simplex. We have (cf. [16])

$$\Gamma_8(K) \leq \Gamma_5(K) = \frac{9}{13}.$$

Case 3: $\{\operatorname{aff} L, \operatorname{aff} M\} = \{1, 2\}$.

Assume without loss of generality that $\operatorname{aff} L = 1$ and $\operatorname{aff} M = 2$. Then $\Gamma_2(L) = 1/2$ and $\Gamma_6(M) \leq \sin^2(3\pi/10)$ (cf. [9]). Then, Corollary 2 shows that

$$\Gamma_8(K) \leq \frac{1 + \Gamma_6(M)}{2} \leq \frac{1 + \sin^2 \frac{3\pi}{10}}{2} \approx 0.827.$$

In a similar way as in case 1, we have

$$\Gamma_6(K) \leq \frac{1 + \frac{\sqrt{2}}{2}}{2} \approx 0.854.$$

Case 4: $\operatorname{aff} L = \operatorname{aff} M = 2$.

In this case we have

$$\Gamma_8(K) \leq \frac{1 + \frac{\sqrt{2}}{2}}{2} \approx 0.854.$$

In the rest, we characterize the case when $c(K) = 8$. The foregoing statements imply that $\operatorname{aff} L = \operatorname{aff} M = 2$. If one of L and M, say L, is not a parallelogram, then $\Gamma_3(L) < 1$ and $\Gamma_4(M) < 1$. It follows from Corollary 2 that

$$\Gamma_7(K) \leq \max\left\{\frac{1 + \Gamma_3(L)}{2}, \frac{1 + \Gamma_4(M)}{2}\right\} < 1.$$

Thus, $c(K) \leq 7$, a contradiction. In the following we assume that both L and M are parallelograms. Since $c(K) = 8$, $\operatorname{ext} K = \operatorname{ext} L \cup \operatorname{ext} M$ consisting of 8 points. Lemma 5 shows that the points in $\operatorname{ext} L \cup \operatorname{ext} M$ are pairwise antipodal. By the main result in [4] (see also p. 225 in [11]), $\operatorname{ext} K$ is the set of vertices of a parallelepiped. When K is a parallelepiped, it is clear that $c(K) = 8$. □

References

1. K. Bezdek, *The problem of illumination of the boundary of a convex body by affine subspaces*, Mathematika **38** (1991), no. 2, 362–375 (1992). MR 1147835
2. K. Bezdek and Muhammad A. Khan, *The geometry of homothetic covering and illumination*, Discrete Geometry and Symmetry, Springer Proc. Math. Stat., vol. 234, Springer, Cham, 2018, pp. 1–30. MR 3816868
3. V. Boltyanski, H. Martini, and P. S. Soltan, *Excursions into Combinatorial Geometry*, Universitext, Springer-Verlag, Berlin, 1997. MR 1439963
4. L. Danzer and B. Grünbaum, *Über zwei Probleme bezüglich konvexer Körper von P. Erdös und von V. L. Klee*, Math. Z. **79** (1962), 95–99. MR 138040
5. B. V. Dekster, *Each convex body in E^3 symmetric about a plane can be illuminated by 8 directions*, J. Geom. **69** (2000), no. 1-2, 37–50. MR 1800455 (2001m:52003)
6. H. Hadwiger, *Ungelöste Probleme, No. 20*, Elem. Math. **12** (1957), 121.
7. M. Lassak, *Solution of Hadwiger's covering problem for centrally symmetric convex bodies in E^3*, J. London Math. Soc. (2) **30** (1984), no. 3, 501–511. MR 810959
8. ———, *Covering a plane convex body by four homothetical copies with the smallest positive ratio*, Geom. Dedicata **21** (1986), no. 2, 157–167. MR 861193 (88c:52013)
9. ———, *Covering plane convex bodies with smaller homothetical copies*, Intuitive geometry (Siófok, 1985), Colloq. Math. Soc. János Bolyai, vol. 48, North-Holland, Amsterdam, 1987, pp. 331–337. MR 910721
10. F. W. Levi, *Überdeckung eines Eibereiches durch Parallelverschiebungen seines offenen Kerns*, Arch. Math. (Basel) **6** (1955), 369–370. MR 0076368 (17,888b)
11. H. Martini and V. Soltan, *Antipodality properties of finite sets in Euclidean space*, Discrete Math. **290** (2005), no. 2-3, 221–228. MR 2123391
12. V. Soltan, *Affine diameters of convex bodies–a survey*, Expo. Math. **23** (2005), no. 1, 47–63. MR 2133336
13. ———, *Lectures on Convex Sets*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015. MR 3308530
14. S. Wu and K. Xu, *Covering functionals of cones and double cones*, J. Inequal. Appl. (2018), 2018:186. MR 3832153
15. S. Wu and Y. Zhou, *On the illumination of a class of convex bodies*, Contributions to Discrete Mathematics **14** (2019), no. 1, 190–202. MR 4050315

16. C. Zong, *A quantitative program for Hadwiger's covering conjecture*, Sci. China Math. **53** (2010), no. 9, 2551–2560. MR 2718847 (2012c:52040)

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