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MOMENTS OF q-JACOBI POLYNOMIALS AND q-ZETA VALUES

FRÉDÉRIC CHAPOTON, CHRISTIAN KRATTENTHALER, AND JIANG ZENG

Abstract. We explore some connections between moments of rescaled little q-Jacobi polynomials, q-analogues of values at negative integers for some Dirichlet series, and the q-Eulerian polynomials of wreath products of symmetric groups.

Introduction

This article explores connections among the following three kinds of objects:

(A) q-analogues of Dirichlet series and their values at negative integers,
(B) basic hypergeometric polynomials and their sequences of moments,
(C) weighted enumeration of elements in coloured symmetric groups.

Let us give more details on these three points in order. Precise definitions are postponed to section 1 and some examples can be found at the end of section 2.

The point (A) is about a q-analogue of the Dirichlet series

$$L(s,c,r) = \sum_{\substack{m \geq 1 \\ m \equiv c \pmod r}} \frac{1}{m^s}, \tag{0.1}$$

where c, r are fixed integers. This is the Riemann zeta function when $(c, r) = (1, 1)$. For general c and r, the summands do not form a multiplicative sequence, so there is no Euler product. One defines as in [3] a q-analogue of

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this Dirichlet series as an operator

$$L_q(s,c,r) = \sum_{\substack{m\geq 1 \\ m\equiv c \pmod r}} \frac{1}{[m]_q^s} F_m, \tag{0.2}$$

where $[m]_q = (q^m-1)/(q-1)$ is the usual q-integer and F_m is the formal Frobenius operator, acting on formal power series in z with no constant term and coefficients in $\mathbb{Q}(q)$, defined by

$$F_m(f)(q,z) = f(q^m, z^m).$$

Whenever the Dirichlet series $L(s,c,r)$ factorizes as an Euler product, then so does the operator $L_q(s,c,r)$ as a product of commuting operators.

One then introduces some q-analogues of the values of $L(s,c,r)$ at non-positive integers, namely $L_q(-n,c,r)(z)$ for $n\geq 0$. As images of the formal power series $f(z)=z$, these are formal power series in the variable z with coefficients in $\mathbb{Q}(q)$. As we will see, these are in fact rational functions in q and z.

The point (B) is about the little q-Jacobi polynomials, a system of orthogonal polynomials in one variable. This is one of the families in the Askey–Wilson scheme of basic hypergeometric orthogonal polynomials (cf. [8]). The little q-Jacobi polynomials, orthogonal with respect to the variable x, depend on the variable q and two further parameters. For each choice of integers (c,r), by an appropriate choice of these parameters and some affine change in the variable x, one obtains a system of orthogonal polynomials involving the variables q and z. Their sequence of moments, which are evaluations of the associated linear functional at the monomials x^n, are therefore rational functions in q and z.

The point (C) is about the complex reflection groups $G(r,n)$ defined as the wreath product of the symmetric group S_n by the cyclic group $\mathbb{Z}_r$. The elements of these groups can be seen as coloured permutation matrices, where non-zero entries contain a root of unity of order dividing r. By using two combinatorial statistics on these elements, one can refine the number $r^n n!$ of elements of $G(r,n)$ into a polynomial in two variables q and z, with positive integer coefficients. In this context, the parameter c is absent.

The aim of this article is to show that (A), (B) and (C) all give essentially the same rational function in q and z. More precisely, the rational functions from (A) and (B) are essentially the quotients of the polynomial from (C) by simple denominators. The part (C) is involved only when the parameter c equals 1.

The relationship between (C) and (A) is merely a reformulation of the results by Biagioli and the third author in [1]. The relationship between (A) and (B) is a (q,z)-analogue of well-known results about Bernoulli numbers and Euler numbers. We refer the reader to [6] for a recent paper on some closely related topics, including Hankel determinants.

1. Preliminaries

1.1. **Orthogonal polynomials.** In this subsection we recall some fundamental results of the theory of orthogonal polynomials [4, 11]. Let $\mathbb{K}$ be a field.

Let $\delta_{n,n'}$ denote the Kronecker delta function.

Definition. *Let $\varphi : \mathbb{K}[x] \to \mathbb{K}$ be a linear functional. A sequence of polynomials $\{p_n(x)\}_{n\geq 0}$ in $\mathbb{K}[x]$ is said to be* orthogonal *with respect to the linear functional φ if:*

(i) *$p_n(x)$ is of degree n, for $n = 0, 1, \dots$;*
(ii) *$\varphi(p_n(x)\, p_{n'}(x)) = K_n\, \delta_{n,n'}$, $K_n \neq 0$, for $n = 0, 1, \dots$.*

The sequence $\{\mu_n\}_{n\geq 0}$ with $\mu_n = \varphi(x^n)$ for $n \geq 0$ is called the moment sequence associated *with φ.*

Sometimes the polynomials $\{p_n(x)\}$ are also said to be orthogonal with respect to the sequence of moments $\{\mu_n\}_{n\geq 0}$.

Let us write OPS as a shorthand for *orthogonal polynomial system.*

Theorem 1.1. [1] *A sequence of monic polynomials $\{p_n(x)\}_{n\geq 0}$ in $\mathbb{K}[x]$ is an OPS if and only if there is a sequence $\{b_n\}_{n\geq 0}$ and a non-zero sequence $\{\lambda_n\}_{n\geq 0}$ such that $p_0(x) = 1$, $p_1(x) = x - b_0$ and*

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x) \quad \text{for} \quad n \geq 1. \tag{1.1}$$

Theorem 1.2. *Let the polynomials $(p_n(x))_{n\geq 0}$ satisfy (1.1). Then, if we fix $\mu_0 := \lambda_0 \neq 0$, the functional φ with respect to which this OPS is orthogonal is unique. Furthermore, for $Q_n(x) = \alpha^{-n} p_n(\alpha x + \beta)$, $\alpha \neq 0$, we have*

$$Q_{n+1}(x) = \left(x - \frac{b_n - \beta}{\alpha}\right) Q_n(x) - \frac{\lambda_n}{\alpha^2} Q_{n-1}(x), \quad \text{for} \quad n \geq 1, \tag{1.2}$$

and, if $(p_n(x))_{n\geq 0}$ is the OPS with respect to the moments (μ_n), then $(Q_n(x))$ is the OPS with respect to the moments ν_n given by

$$\nu_n = \varphi\left(\left(\frac{x-\beta}{\alpha}\right)^n\right) = \alpha^{-n} \sum_{j=0}^{n} \binom{n}{j} (-\beta)^{n-j} \mu_j, \quad \text{for} \quad n \geq 0. \tag{1.3}$$

Theorem 1.3. *The generating function of the moments $\{\varphi(x^n)\}$ has the continued fraction expansion*

$$\sum_{n\geq 0} \varphi(x^n)\, t^n = \cfrac{\lambda_0}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\ddots}}}. \tag{1.4}$$

There is also an associated formula for Hankel determinants of the sequence of moments, see [11, 9, 10].

[1] This statement is usually called Favard's theorem, although it was certainly known and used before, notably in works of Thomas Jan Stieltjes, Marshal Stone and Aurel Wintner. For a precise historical account, see Section 2.5 in the book [7].

1.2. **Wreath product of a symmetric group by a cyclic group.** Let $r \geq 1$ and $n \geq 1$ be integers. Let S_n be the symmetric group on $\{1, \ldots, n\}$. A permutation $\sigma \in S_n$ will be denoted by $\sigma = \sigma(1) \cdots \sigma(n)$. The *wreath product* $\mathbb{Z}_r \wr S_n$ of $\mathbb{Z}_r$ by S_n is the set

$$G(r, n) := \{(c_1, \ldots, c_n; \sigma) \mid c_i \in \{0, \ldots, r-1\}, \sigma \in S_n\}. \tag{1.5}$$

Using a fixed primitive r-th root of unity ξ, one can see the elements in this set as square matrices, starting from the permutation matrix for σ and replacing the non-zero entry in column i by ξ^{c_i}.

This group is therefore also called the *group of r-coloured permutations.* We will represent its elements as

$$\gamma = [\gamma(1), \ldots, \gamma(n)] = [\sigma(1)^{c_1}, \ldots, \sigma(n)^{c_n}].$$

We denote by

$$\mathrm{col}(\gamma) := \sum_{i=1}^{n} c_i,$$

the *colour weight* of any $\gamma \in G(r, n)$. For example, if $\gamma = [4^1, 3^0, 2^4, 1^2] \in G(5, 4)$ then $\mathrm{col}(\gamma) = 7$.

We endow the set of possible values for the $\gamma(i)$ with the following total order:

$$n^{r-1} < \cdots < n^1 < \cdots < 1^{r-1} < \cdots < 1^1 < 0 < 1^0 < \cdots < n^0.$$

The 0 is inserted here to separate the "positive" values i^0 from the "negative" values i^c with $c \geq 1$. It will also be used in the statistics that we are going to define now.

The *descent set* of $\gamma \in G(r, n)$ is defined by

$$\mathrm{Des}_G(\gamma) \quad := \quad \{i \in \{0, \ldots, n-1\} \mid \gamma(i) > \gamma(i+1)\}, \tag{1.6}$$

where $\gamma(0) := 0$, and its cardinality is denoted by $\mathrm{des}_G(\gamma)$.

The *major index* is defined to be the sum of descent positions:

$$\mathrm{maj}(\gamma) = \sum_{i \in \mathrm{Des}_G(\gamma)} i,$$

and the *flag-major index* is defined by

$$\mathrm{fmaj}(\gamma) := r \cdot \mathrm{maj}(\gamma) + \mathrm{col}(\gamma).$$

For example, for $\gamma = [4^1, 3^0, 2^4, 1^2] \in G(5, 4)$ we have $\mathrm{Des}_G(\gamma) = \{0, 2\}$, $\mathrm{des}_G(\gamma) = 2$, $\mathrm{maj}(\gamma) = 2$, and $\mathrm{fmaj}(\gamma) = 17$.

Biagioli and the third author [1] defined the generating polynomials for $G(r, n)$ with respect to the bi-statistic $(\mathrm{des}, \mathrm{fmaj})$:

$$G_{r,n}(Z, q) = \sum_{\gamma \in G(r,n)} Z^{\mathrm{des}_G(\gamma)} q^{\mathrm{fmaj}(\gamma)}, \tag{1.7}$$

and they proved the following identity. We refer the reader to Subsection 1.3 for the meaning of the q-notations.

Theorem (Carlitz–MacMahon identity for $G(r,n)$). *Let r and n be positive integers. Then*

$$\frac{G_{r,n}(Z,q)}{(Z;q^r)_{n+1}} = \sum_{k\ge 0} Z^k [rk+1]_q^n. \tag{1.8}$$

The above formula gives a nice generalization of identities of Carlitz [2] for the symmetric group (corresponding to the case where $r=1$), and of Chow and Gessel [5] for the hyperoctahedral group (corresponding to the case where $r=2$).

1.3. **Little q-Jacobi polynomials.** We use the standard q-notations from [8], among which

$$[x]_q = \frac{1-q^x}{1-q},$$

the q-Pochhammer symbol

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

and the convenient shorthand

$$(a,b;q)_n = (a;q)_n\,(b;q)_n.$$

We furthermore need the q-binomial theorem [8, p. 16]

$$ {}_1\Phi_0\left(\begin{matrix} a \\ - \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty} \tag{1.9}$$

and the q-Chu–Vandermonde formula [8, p. 17]

$$ {}_2\Phi_1\left(\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, q\right) = \sum_{k=0}^{\infty} \frac{(q^{-n};q)_k\,(b;q)_k}{(c;q)_k\,(q;q)_k} q^k = \frac{(c/b;q)_n}{(c;q)_n} b^n. \tag{1.10}$$

The little q-Jacobi polynomials [8, p. 482] have the explicit representation
(1.11)

$$p_n(x;a,b\mid q) = {}_2\Phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx\right) = \sum_{k=0}^{\infty} \frac{(q^{-n};q)_k\,(abq^{n+1};q)_k}{(aq;q)_k\,(q;q)_k}(qx)^k,$$

and are orthogonal with respect to the inner product defined by

$$\int_0^1 f(x)g(x)d_q w(x) = \sum_{k=0}^{\infty} f(q^k)g(q^k)w(q^k),$$

where

$$w(x) = \frac{(aq,bq;q)_\infty}{(abq^2,q;q)_\infty}\cdot\frac{(qx;q)_\infty}{(bqx;q)_\infty} x^{\alpha+1}$$

with $a=q^\alpha$.

Let $p_n(x)$ be the monic little q-Jacobi polynomials, *i.e.*,

$$p_n(x) = \frac{(-1)^n q^{\binom{n}{2}}(aq;q)_n}{(abq^{n+1};q)_n} p_n(x;a,b\mid q).$$

Then the normalized recurrence relation [8, p. 483] reads

$$xp_n(x) = p_{n+1}(x) + (A_n + C_n)p_n(x) + A_{n-1}C_np_{n-1}(x), \tag{1.12}$$

where

$$A_n = q^n \frac{(1-aq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})},$$
$$C_n = aq^n \frac{(1-q^n)(1-bq^n)}{(1-abq^{2n})(1-abq^{2n+1})}.$$

By the q-binomial theorem (1.9), the nth moment is

$$\mu_n = \int_0^1 x^n d_qw(x) = \frac{(aq;q)_n}{(abq^2;q)_n}, \quad \text{for } n = 0, 1, 2, \dots. \tag{1.13}$$

We can also verify (1.13) by using the explicit formula (1.11) and the q-Chu-Vandermonde formula (1.10): namely, for $n \geq 1$, we have

$$\int_0^1 p_n(x; a, b \mid q) d_qw(x) = 0. \tag{1.14}$$

We can now prove the connection between (B) and (A).

Theorem 1.4. *For integers $r \geq 1$, the nth moment μ_n of the shifted little q-Jacobi polynomials $p_n(q^{-c}(1+(q-1)x); Zq^{-r}, 1 \mid q^r)$ is*

$$\mu_n = (1-Z)\sum_{k\geq 0}([rk+c]_q)^n Z^k. \tag{1.15}$$

For $c = 1$, we have

$$\mu_n = \frac{G_{r,n}(Z,q)}{(Zq^r;q^r)_n}. \tag{1.16}$$

Proof. By (1.3), the nth moment of $p_n(q^{-c}(1+(q-1)x); a, b \mid q^r)$ is

$$\nu_n = q^{nc}(q-1)^{-n}\sum_{j=0}^{n}\binom{n}{j}(-q^c)^{j-n}\frac{(aq^r;q^r)_j}{(abq^{2r};q^r)_j}.$$

Replacing a by Zq^{-r} and b by 1, we get

$$\begin{aligned}\nu_n &= q^{nc}(q-1)^{-n}\sum_{j=0}^{n}\binom{n}{j}(-q^c)^{j-n}\frac{1-Z}{1-Zq^{rj}}\\ &= (1-Z)(q-1)^{-n}\sum_{k\geq 0}Z^k\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}q^{(rk+c)j}\\ &= (1-Z)\sum_{k\geq 0}([rk+c]_q)^n Z^k.\end{aligned}$$

The last statement follows from (1.8). □

Theorem 1.5. *The generating function for the moments* μ_n *in* (1.15) *has the continued fraction expansion*

$$\sum_{n\geq 0}\mu_n t^n = \cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{1-b_1t-\cfrac{\lambda_2t^2}{\ddots}}}. \tag{1.17}$$

where the coefficients b_n *and* λ_n *are given by*

$$\lambda_n = \frac{Zq^{2r(n-1)+2c}\,[rn]_q^2\,(1-Zq^{r(n-1)})^2}{(1-Zq^{2rn})(1-Zq^{r(2n-1)})^2(1-Zq^{r(2n-2)})} \tag{1.18}$$

and

$$\begin{aligned} b_n = &\frac{q^c}{q-1}\\ &\cdot\left(\frac{q^{rn}(1-Zq^{rn})^2}{(1-Zq^{2rn})(1-Zq^{r(2n+1)})}+\frac{Zq^{r(n-1)}(1-q^{rn})^2}{(1-Zq^{r(2n-1)})(1-Zq^{2rn})}-q^{-c}\right). \end{aligned} \tag{1.19}$$

Proof. This follows by combining (1.12) and Theorems 1.4, 1.2 and 1.3 with $\alpha=(q-1)/q^c$ and $\beta=q^{-c}$. □

2. Zeta operators at negative integers

We define the q-difference operator on the formal power series f in z by

$$\Delta_z(f)=\frac{f(qz)-f(z)}{q-1}. \tag{2.1}$$

Note that $\Delta_z(z^m) = [m]_q z^m$. This implies that repeated application of the operator Δ_z creates the sequence of values at negative integers for the q-analogues of Dirichlet series. Indeed, for $n\geq 0$, we have

$$L_q(-n,c,r)(z)=\sum_{\substack{m\geq 1\\ m\equiv\, c (\mathrm{mod}\ r)}}[m]_q^n z^m. \tag{2.2}$$

and therefore

$$\Delta_z\left(L_q(-n,c,r)(z)\right)=L_q(-n-1,c,r)(z). \tag{2.3}$$

Computing the initial value for $n=0$, one finds

$$L_q(0,c,r)(z)=\frac{z^c}{1-z^r}. \tag{2.4}$$

By induction using (2.3), the expression $L_q(-n,c,r)(z)$ is a rational function in q and z with denominator $(z^r,q^r)_{n+1}$.

The general relation between (A) and (B) is therefore, by comparison of (2.2) with (1.15), using (2.4), that

$$L_q(-n,c,r)(z)/L_q(0,c,r)(z)=\mu_n\big|_{Z=z^r}. \tag{2.5}$$

For example, with $(c,r)=(1,2)$, the first few values of $L_q(-n,1,2)(z)$ are

$$\frac{z}{1-z^2},\frac{z(qz^2+1)}{(1-z^2)(1-q^2z^2)},\frac{z(q^4z^4+2q^3z^2+2q^2z^2+2qz^2+1)}{(1-z^2)(1-q^2z^2)(1-q^4z^2)}.$$

For $c=1$, comparison with (1.8) reveals the combinatorial expression

$$L_q(-n,c,r)(z)=z\frac{G_{r,n}(z^r,q)}{(z^r,q^r)_{n+1}}, \tag{2.6}$$

which makes the precise connection between (A) and (C).

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(Frédéric Chapoton) Institut de Recherche Mathématique Avancée Université de Strasbourg 7 rue René Descartes F-67084 STRASBOURG Cedex FRANCE
E-mail address: chapoton@math.unistra.fr

(Christian Krattenthaler) Fakultät für Mathematik Universität Wien Oskar-Morgenstern-Platz 1 A-1090 Vienna AUSTRIA
E-mail address: christian.krattenthaler@univie.ac.at

(Jiang Zeng) Institut Camille Jordan, CNRS UMR 5208, Université Claude Bernard Lyon 1, Université de Lyon, F-69622 Villeurbanne cedex, France
E-mail address: zeng@math.univ-lyon1.fr