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MULTICOLOR COMPOSITIONS AND CONJUGATION OF N-COLOR COMPOSITIONS

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Abstract. We introduce n-multicolor compositions which generalize the n-color compositions that Agarwal first defined twenty-two years ago. A summand may bear a set of many integer-valued colors which it bounds from above. We enumerate such multicolored compositions using generating functions. Then we isolate the conjugable class of multicolored compositions known as cracked compositions. A concise exposition of the conjugation of n-color compositions is presented in the classical tradition of Percy Alexander MacMahon (1854 - 1929). Our set of conjugable n-color compositions turn out to be considerably larger than previously known ones.

1. Introduction

Colored compositions are generalized integer compositions in which a part size may come in a prescribed number of types or colors which are denoted with subscripts, $1_1, 1_2, \ldots, 2_1, 2_2, \ldots$.

The number of standard compositions of a positive integer n, that is, representations of n as sequences of positive integers that sum to n, is known to be 2^{n-1} [4, 7].

$$1 + \sum_{n=1}^{\infty} 2^{n-1} x^n = \frac{1-x}{1-2x}. \tag{1.1}$$

An n-color composition is a colored composition in which a part of size m may come in m different colors: $m_1, m_2, \ldots, m_m$, $m = 1, 2, 3, \ldots$.

The 'n-color' notion first appeared in a paper of Agarwal and Andrews [3] that investigated combinatorial identities for "partitions with n copies of n", that is, n-color partitions. Subsequently Agarwal [2] defined n-color compositions and explored many of their enumeration and combinatorial properties. For instance, he proved that the number of n-color compositions of a positive integer ν is the Fibonacci number $F_{2\nu}$, where $F_1 = F_2 = 1$ and $F_\nu = F_{\nu-1} + F_{\nu-2}$, $\nu > 2$.

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Many authors have since discovered further properties of n-color compositions, including their connections with other combinatorial objects (see Agarwal's survey [1], and [2, 6, 14, 15]).

In this paper we generalize n-color compositions to have parts that may bear more than one color simultaneously.

Recall that an n-color composition has the form

$$((g_1)_{c_1}, (g_2)_{c_1}, \dots, (g_k)_{c_k}),$$

where $g = (g_1, g_2, \dots, g_k)$ is the support (or underlying) standard composition with a sequence of integer valued colors, $(c_1, \dots, c_k)$, $1 \le c_i \le g_i \ \forall\, i$.

We consider the situation where a part g_j is assigned r distinct colors from the set $\{c_i \mid 1 \le c_i \le g_j\}$ for any $r \in [g_j] := \{1, 2, \dots g_j\}$. With this color assignment we say that g_j is *n-multicolored.* The resulting colored composition of ν, that is, a colored composition of ν consisting of n-multicolored parts, will be called an *n-multicolor composition* of ν.

The notation for an n-multicolor composition with support composition g and n-multicolor sequence $L = (L_1, \dots, L_k)$ is $C(g, L) = ((g_1)_{L_1}, \dots, (g_k)_{L_k})$ with $L_i \subseteq [g_i]$. We call L_i the *color set* of g_i. For example, an admissible n-multicolor sequence for $g = (5, 5, 9, 1, 3, 8)$ is

$$L = (\{1, 3\}, \{2, 4, 5\}, \{1\}, \{1\}, \{1, 2, 3\}, \{1, 4, 7\}).$$

Thus $C(g, L) = (5_{1,3}, 5_{2,4,5}, 9_1, 1_1, 3_{1,2,3}, 8_{1,4,7})$.

In particular an n-color composition is an n-multicolor composition in which every part bears a singleton color set.

In Section 2 we give the generating function for the number of n-multicolor compositions and provide a bijection with a class of single colored objects. In Section 3 we study the enumeration of n-multicolor compositions under certain restrictions by means of generating functions. In Section 4 we define cracked compositions and show that they form the subclass of conjugable n-multicolor compositions. Lastly, in Section 5 we discuss the conjugation of n-color compositions in the classical MacMahon tradition, via the techniques of tilings and manipulation of ternary sequences.

2. Preliminaries

For brevity we will denote n-multicolor compositions by n-mc compositions. Let $mc(n, k)$ be the number of unrestricted n-mc compositions of ν into k parts, and let $mc(\nu) = \sum_{k \ge 0} mc(\nu, k)$.

Theorem 2.1. *The generating function for the number of n-mc compositions of ν is*

$$\sum_{\nu=0}^{\infty} mc(\nu) x^{\nu} = 1 + \frac{x}{1 - 4x + 2x^2} = \frac{1 - 3x + 2x^2}{1 - 4x + 2x^2}, \tag{2.1}$$

and hence the exact formula is

$$mc(\nu) := \frac{\left(2+\sqrt{2}\right)^{\nu} - \left(2-\sqrt{2}\right)^{\nu}}{2\sqrt{2}}, \ \nu > 0.$$

Proof. Since a part of size i may bear any non-empty color set $A \subseteq [i]$, the number of distinct color assignments for i is $2^i - 1$. If x marks the size of the composition, its contribution to the generating function is x^i. Since i is arbitrary the generating function for the sequence $mc(\nu, k)$ is

$$\sum_{\nu \geq 0} mc(\nu, k) x^n = \left(\sum_{i \geq 1} (2^i - 1) x^i \right)^k = \left(\frac{x}{(1-2x)(1-x)} \right)^k. \tag{2.2}$$

Summing on k gives

$$\sum_{\nu \geq 0} mc(\nu) x^{\nu} := \sum_{k \geq 0} \left(\frac{x}{(1-2x)(1-x)} \right)^k = \frac{1-3x+2x^2}{1-4x+2x^2}.$$

The exact formula for $mc(\nu)$ is obtained by extracting coefficients of x^{ν}. ∎

For example, $mc(3) = 14$, which gives the following n-mc compositions:

(3_1)	(3_2)	(3_3)	$(3_{1,2})$	$(3_{1,3})$	$(3_{2,3})$	$(3_{1,2,3})$
$(2_1, 1_1)$	$(2_2, 1_1)$	$(2_{1,2}, 1_1)$				
$(1_1, 2_1)$	$(1_1, 2_2)$	$(1_1, 2_{1,2})$				
$(1_1, 1_1, 1_1)$						

The first few values of the sequence $mc(\nu)$, $\nu \geq 1$ are as follows

$$1, 4, 14, 48, 164, 560, 1912, 6528, 22288, 76096, 259808, 887040, \ldots.$$

We obtain a hit in the Online Encyclopedia of Integer Sequences [17, seq A007070]. A comment also describes $mc(\nu)$ as "the number of generalized compositions of ν when there are $2^i - 1$ different types of the part i, $i = 1, 2, \ldots$". We may distinguish the 'types' with subscripts and obtain (non-n-) color compositions in which any part u bears a single (globally restricted) color $\in \{1, 2, \ldots, 2^u - 1\}$.

Theorem 2.2. *There is a bijection between the set of n-mc compositions of ν and the set of (single-) colored compositions of ν in which a part m bears one color r, where $1 \leq r \leq 2^m - 1$.*

Proof. We describe a bijection between the two classes of compositions. Consider an n-mc composition $C(g, L) = ((g_1)_{L_1}, (g_2)_{L_2}, \ldots, (g_k)_{L_k})$. Note that $L_j \subseteq [g_j]$ gives $2^{g_j} - 1$ possible subsets $L_j \neq \emptyset$ for all possible j, where L_j is assumed to be totally ordered so as to be identified with the increasing sequence of its elements. The latter notation will be adopted in this proof.

For each part g_j sort the family of possible subscript sequences L_j in lexicographic order, within clusters of increasing lengths. Thus the sorted subscript sequences take the form

$$(1, 2, \ldots, g_j, 12, 13, \ldots, (g_j - 1)g_j, 123, 124, \ldots, \ldots, 12 \cdots g_j). \tag{2.3}$$

Then associate the rth member of (2.3) with $r \equiv r_{ji_j}$ for $j = 1, \ldots, k$, $i_j = 1, \ldots, 2^{g_j} - 1$. This produces a unique sequence of single colors

$$(r_{1i_1}, r_{2i_2}, \ldots, r_{ki_k})$$

for $C(g, L)$.

Thus a bijective map may be defined by associating $C(g, L)$ with the color composition obtained by assigning the subscripts $r_{1i_1}, r_{2i_2}, \ldots, r_{ki_k}$ to $g_1, g_2, \ldots, g_k$, in order.

This transformation is illustrated for parts of sizes 1, 3 and 4 below.

$$\begin{aligned}
g_j &= 1 : (1) \longmapsto (1) \\
g_j &= 3 : (1, 2, 3, 12, 13, 23, 123) \longmapsto (1, 2, 3, 4, 5, 6, 7) \\
g_j &= 4 : (1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 134, 234, 1234) \\
&\qquad \longmapsto (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15)
\end{aligned}$$

Thus for instance,

$$\begin{gathered}
(4_{1,2}, 3_2, 1_1) \mapsto (4_5, 3_2, 1_1), \\
(3_{1,2}, 1_1, 4_{2,3,4}) \mapsto (3_4, 1_1, 4_{14}),
\end{gathered}$$

and so forth. This map is clearly reversible. ∎

Even though Theorem 2.2 gives a single-color-per-part enumerative equivalence to n-mc compositions, we elect to study the latter because of the rich structure of the associated color sets which often yields interesting connections with other combinatorial objects. (See for example, Sections 4 and 5).

3. Some Restricted n-mc compositions

We will denote by $mc(\nu \mid P)$ the number of n-multicolor compositions of ν satisfying property P, and by $mc(\nu, k \mid P)$ the number of such compositions with k parts.

3.1. **Restricting part sizes.** In order to enumerate n-mc compositions with all parts of size at most m we adjust the generating function in the proof of Theorem 2.1 to have an upper limit of m on the i sum (see the second sum in (2.2)). Thus we obtain

Theorem 3.1. *The generating function for the number of n-mc compositions with part sizes at most m is*

$$\begin{aligned}
&\sum_{\nu \geq 0} mc(\nu \mid \text{parts} \leq m) x^\nu \\
&= \frac{1 - 3x + 2x^2}{1 - 4x + 2x^2 + (2^{m+1} - 1)\, x^{m+1} - (2^{m+1} - 2)\, x^{m+2}}. \qquad (3.1)
\end{aligned}$$

In particular, the coefficients of x^ν when $m=1$, and $m=2$ follow:

$$mc(\nu \mid \text{parts} \le 1) = 1;$$

$$mc(\nu \mid \text{parts} \le 2) = \frac{2^{-\nu-1}\left(\left(1+\sqrt{13}\right)^{\nu+1} - \left(1-\sqrt{13}\right)^{\nu+1}\right)}{\sqrt{13}}.$$

As an illustration of how this sequence grows for small m, see Table 1.

$m\backslash\nu$	1	2	3	4	5	6	7	8	9	20
1	1	1	1	1	1	1	1	1	1	1
2	1	4	7	19	40	97	217	508	1159	2683
3	1	4	14	33	103	300	840	2461	7081	20344

Table 1. Numbers of n-mc compositions with parts $\le m$

Theorem 3.2. *The generating function for the number of n-mc compositions with part sizes at least m is*

$$\sum_{\nu\ge 0} mc(\nu \mid \text{parts} \ge m)x^\nu = \frac{(x-1)(2x-1)}{(2^m-2)\,x^{m+1} - (2^m-1)\,x^m + 2x^2 - 3x + 1}$$

Note that for $m=1$ this is the same sequence as $mc(\nu)$, while $m=2$ accounts for n-mc compositions without 1's:

$$\sum_{\nu\ge 0} mc(\nu \mid \text{parts} \ge 2)x^\nu = \frac{(x-1)(2x-1)}{2x^3 - x^2 - 3x + 1}.$$

3.2. **Restricting color sizes.** Here we consider the number of n-mc compositions C in which a part may bear colors of size at most t.

Since t may be larger than some parts of C the color set of a part g_j is $\{1, 2, \ldots, \min(t, g_j)\}$.

A part of size $i \le t$ bears an unrestricted number of color assignments, i.e., $2^i - 1$, and a part of size $i > t$ bears a color set $A \subseteq [t]$, i.e., $2^t - 1$ possible assignments. Therefore,

$$\begin{aligned}\sum_{\nu\ge 0} mc(\nu, k|\text{color sets} \subseteq [t])x^\nu &= \left(\sum_{i=1}^{t}(2^i-1)x^i + \sum_{i\ge t+1}(2^t-1)x^i\right)^k \\ &= \left(\frac{x(1-2^t x^t)}{(1-2x)(1-x)}\right)^k\end{aligned}$$

$$\Longrightarrow \sum_{\nu\ge 0} mc(\nu|\text{color sets} \subseteq [t])x^\nu = \frac{1-3x+2x^2}{1-4x+2x^2+2^t x^{t+1}}.$$

It may be verified that $\lim\limits_{t\to\infty} mc(\nu|\text{color sets} \subseteq [t]) = mc(\nu)$, and the case $t=1$ enumerates all standard compositions, that is, gives (1.1).

3.3. **Restricting the number of colors.** We compute the number of n-mc compositions in which a part may bear at most t colors.

Note that this class contains the foregoing class. For example, $mc(30 \mid \text{color sets} \subseteq [3])$ enumerates

$$(5_2, 5_{1,2,3}, 9_{1,2}, 2_{1,2}, 1_1, 8_3)$$

but not

$$(5_2, 5_{1,2,4}, 9_{1,5}, 2_{1,2}, 1_1, 8_8)$$

because, for instance, in $9_{1,5}$, we have $5 > 3 = t$. On the other hand $mc(30 \mid \text{color sets have} \leq 3 \text{ colors})$ enumerates both objects.

A part of size $i \leq t$ bears an unrestricted number of color assignments, and a part of size $i > t$ bears a color set $A \subseteq [i]$ such that $|A| \leq t$, i.e., $\binom{i}{1} + \binom{i}{2} + \cdots + \binom{i}{t}$ possible assignments.

$$\begin{aligned}\sum_{\nu \geq 0} mc(\nu \mid \text{color sets have} \leq t \text{ colors}) x^\nu &= \sum_{k \geq 0} \left(\sum_{i \geq 1} \sum_{j=1}^{t} \binom{i}{j} x^i \right)^k \\ &= \sum_{k \geq 0} \left(\sum_{j=1}^{t} \frac{x^j}{(1-x)^{j+1}} \right)^k \\ &= \frac{(1-2x)(1-x)^{t+1}}{x^{t+1} + (2x^2 - 4x + 1)(1-x)^t}.\end{aligned}$$

3.3.1. *Exactly t colors.* To enumerate n-mc compositions where a part must bear exactly t colors (and necessarily be a part of size at least t), we compute

$$\sum_{\nu \geq 0} mc(\nu \mid \text{exactly } t \text{ colors}) x^\nu := \sum_{k \geq 0} \left(\sum_{i \geq t} \binom{i}{t} x^i \right)^k = \frac{(1-x)^{t+1}}{(1-x)^{t+1} - x^t}.$$

Note that, contrary to expectation, $mc(\nu \mid \text{exactly } t \text{ colors})$ is not equal to the difference

$$mc(\nu \mid \text{color sets have} \leq t \text{ colors}) - mc(\nu \mid \text{color sets have} \leq t-1 \text{ colors}).$$

For instance when $t = 2$, the composition $(2_{1,2}, 1_1)$ is not counted by $mc(\nu \mid \text{exactly } 2 \text{ colors})$ but is still counted by the difference.

4. Conjugation of n-mc Compositions

In this section we isolate a class of n-multicolor compositions that possess the conjugation involution property, in the classical tradition of Percy Alexander MacMahon (1854 - 1929) (see for example, [11, 10, 12]). Given a standard composition C there are several methods of obtaining the conjugate composition of C, denoted by C', including the zig-zag graph method and the line graph method.

A bar graph representation of an ordinary composition $g = (g_1, g_2, \ldots, g_k)$ is a sequence of columns composed of unit squares, where column j has g_j

squares. For example, see Figure 1 for the bar graphs corresponding to the compositions of $n = 3$.

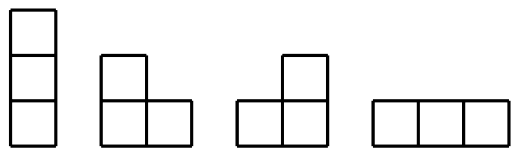


FIGURE 1. Bar graphs of $(3), (2, 1), (1, 2), (1, 1, 1)$.

Definition 4.1. *A* cracked composition *is an n-mc composition in which a part cannot bear the color 1 simultaneously with another color label.*

The name 'cracked composition' was originally used to describe a composition with (potential) "cracks" between the squares in the bar graph representation (see Heubach-Mansour [7, p. 77]). Thus cracks do not occur at the top or bottom of a column. Figure 2 shows the bar graphs of the cracked compositions of 3, where '×' denotes a crack.

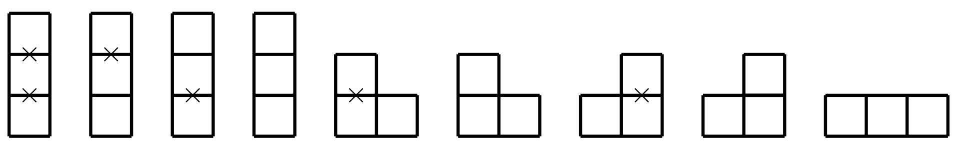


FIGURE 2. Bar graphs of cracked compositions of $n = 3$.

It is easy to see that Definition 4.1 of a cracked composition $C(g, L) = ((g_1)_{L_1}, \dots, (g_k)_{L_k})$ agrees with the graphical description. Indeed since each g_j corresponds to a column of g_j unit squares containing $g_j + 1$ horizontal bars/levels, an occurrence of $(g_j)_1 \in C(g, L)$ indicates that g_j has no crack at level 1, the bottom level, and elsewhere. An occurrence of $(g_j)_{v_1,\dots,v_s}, 2 \le v_1 \le \cdots \le v_s \le g_j$ indicates that g_j has cracks at levels $v_1, \dots, v_s$, $s > 0$.

Thus each part $(d)_L$ of a cracked composition is either 'cracked' (when $\emptyset \neq L \subseteq \{2, \dots, d\}$) or 'uncracked' (when the color is precisely 1, $L = \{1\}$), and not both.

For example, the bar graphs in Figure 2 correspond to the following respective n-mc compositions.

$$(3_{2,3}),\ (3_3),\ (3_2),\ (3_1),\ (2_2, 1_1),\ (2_1, 1_1),\ (1_1, 2_2),\ (1_1, 2_1),\ (1_1, 1_1, 1_1). \tag{4.1}$$

So cracked compositions share the essential property of standard compositions of being representable both algebraically (as in (4.1)) and graphically (as in Figure 2).

It will be demonstrated that the set of cracked compositions is closed under the operation of conjugation, to be described below.

4.1. **Enumeration of Cracked Compositions.** We exploit our previous experience with general n-mc compositions to obtain the following essential results.

Theorem 4.2. *The number $rc(n)$ of cracked compositions of n is given by*

$$\sum_{n=0}^{\infty} rc(n)x^n = 1 + \sum_{n=1}^{\infty} 3^{n-1}x^n = 1 + \frac{x}{1-3x} = \frac{1-2x}{1-3x}.$$

Proof. The generating function for the number of cracked compositions with k parts where each part can have no cracks (only color 1) or any subset (including the empty set) of $\{2, 3, \ldots, i\}$ cracks (any $i-1$ colors) is

$$\begin{aligned}\sum_{n\geq 0} rc(n)x^n &= \sum_{k\geq 0}\left(\sum_{i\geq 1}\left(x^i + (2^{i-1}-1)x^i\right)\right)^k \\ &= \sum_{k\geq 0}\left(\frac{1}{2}\frac{2x}{1-2x}\right)^k \\ &= \frac{1-2x}{1-3x}.\end{aligned}$$

■

4.1.1. *Exactly m Cracks Per Part.* Consider the number of cracked compositions of n when each part has exactly m cracks, where $m \geq 0$ is fixed. Then each part necessarily has size at least $m+1$.

$$\begin{aligned}\sum_{n\geq 0} rc(n \mid m \text{ cracks per part})x^n &= \sum_{k\geq 1}\left(\sum_{i\geq m+1}\binom{i-1}{m}x^i\right)^k \\ &= \frac{x^{m+1}}{(1-x)^{m+1}-x^{m+1}}.\end{aligned}$$

We extend this idea by considering the number of cracked compositions of n when each part has at most m cracks, where $m \geq 1$ is fixed. The generating function is

$$\begin{aligned}&\sum_{n\geq 0} rc(n \mid \text{at most } m \text{ cracks per part})x^n \\ &= \sum_{k\geq 1}\left(\sum_{i=1}^{\infty}\left(x^i + \sum_{j=1}^{m}\binom{i-1}{j}x^i\right)\right)^k \\ &= \sum_{k\geq 1}\left(\frac{x}{1-x} + \sum_{j=1}^{m} x\frac{x^j}{(1-x)^{j+1}}\right)^k \\ &= \frac{x(1-x)^{m+1}-x^{m+2}}{x^{m+2}+(1-3x)(1-x)^{m+1}}. \qquad (4.2)\end{aligned}$$

4.1.2. *Exactly m cracks overall.*

Theorem 4.3.

$$\sum_{n=0}^{\infty} rc(n \mid \text{exactly } m \text{ cracks})x^n = \frac{x^{m+1}}{(1-2x)^{m+1}}, \ m \geq 0.$$

Proof. We introduce a variable q to track the cracks in a composition. The generating function for a part of size i with j cracks is then

$$x^i \sum_{j=0}^{i-1} q^j \binom{i-1}{j} = \frac{((q+1)x)^i}{q+1}.$$

Consequently, (by considering the coefficient of q^m),

$$\begin{aligned}\sum_{n\geq 0} rc(n \mid \text{exactly } m \text{ cracks})x^n &= [q^m]\sum_{k=0}^{\infty}\left(\sum_{i=1}^{\infty}\frac{((q+1)x)^i}{q+1}\right)^k \\ &= [q^m]\frac{qx+x-1}{qx+2x-1} \\ &= \left(\frac{x}{1-2x}\right)^{m+1}.\end{aligned}$$

∎

Our main result for this section is the following three-way identity.

Theorem 4.4. *The following sets are equinumerous:*

I. *Cracked compositions of N with exactly m cracks.*
II. *Integer sequences obtained by concatenating $m+1$ non-empty compositions of sizes determined by compositions of N into $m+1$ parts.*
III. *Cracked compositions of N into $m+1$ parts.*

Proof. We assume that the cracked compositions $C(g, L) = R$ in the theorem are non-empty, and give bijections between the sets in the order I $\iff$ III, I $\iff$ II.

Proof of I $\iff$ III. Start with the bottom square of the first column in the bar graph $B(R)$ of R, and label it as follows: move from one square to the next (first up, then right) and mark it with a 0 if you go up with no crack, with c if you go up and there is a crack, and with p if you have a new part (and hence cannot have a crack). Then the resulting sequence of 0's, c's and p's will have as many c's as there are cracks in $B(R)$, and the number of parts of R is 1 greater than the number of p's (since the transition into the first square is not included in the sequence). The proof consists of interchanging the c's and p's.

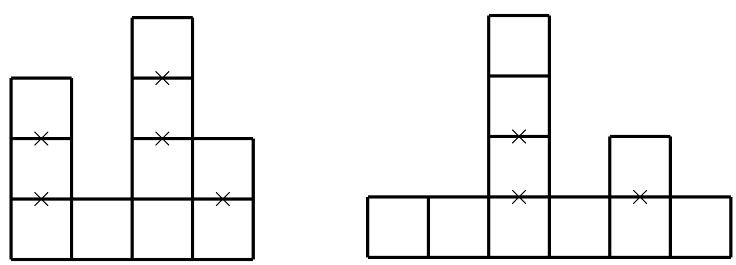
Assume that R has exactly m cracks and obtain the labels for $B(R)$ in terms of the code $\{0, c, p\}$. This will give a sequence with exactly m occurrences of c; the rest will be 0's or p's. Now replace every c by a p and every p by a c. This will give a unique cracked composition whose sequence has exactly m occurrences of p, which implies $m+1$ parts. The reverse map, III $\Rightarrow$ I, follows naturally.

Figure 3 illustrates this bijection for two cracked compositions of $n = 10$ that map to/from each other, namely

$$(3_{2,3}, 1_1, 4_{3,4}, 2_2)$$

and

$$(1_1, 1_1, 4_{2,3}, 1_1, 2_2, 1_1).$$



sequence $= c, c, p, p, 0, c, c, p, c$; $m = 5$ cracks.

sequence $= p, p, c, c, 0, p, p, c, p$; $m + 1 = 6$ parts.

FIGURE 3. Illustration of the bijection I $\Leftrightarrow$ III

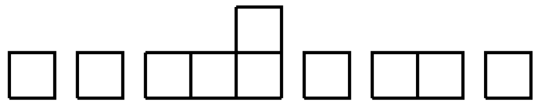
Proof of I $\iff$ II. We want a bijection between the cracked compositions of n with exactly m cracks and sequences that are created by concatenating $m+1$ non-empty compositions of weights $k_1, k_2 \ldots k_{m+1}$ where $\sum_{i=1}^{m+1} k_i = n$. Our proof generalizes the case $m = 1$ given in [7, p. 78].

First label the sequence of non-empty compositions by $C_1, C_2, \ldots, C_{m+1}$. Then place the first part of C_{i+1} on top of the last part of C_i and insert a crack between the two squares. In this way a unique cracked composition is formed by putting all of the $m + 1$ non-empty compositions next to each other, except that the first part of each successive non-empty composition is not a new part but added to the top of the last part of the previous non-empty composition (inducing a crack). See Figure 4 for an example with six ordinary compositions and $(3_{2,3}, 1_1, 4_{3,4}, 2_2)$.

This completes the proof of Theorem 4.4. ∎

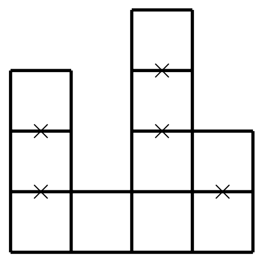
4.2. **Conjugation.** We note that the proof of I $\Leftrightarrow$ III constitutes a description of *conjugation of cracked compositions*, exemplified by $(3_{2,3}, 1_1, 4_{3,4}, 2_2)$ and $(1_1, 1_1, 4_{2,3}, 1_1, 2_2, 1_1)$. These are conjugates, one of the other, as depicted by the graphs in Figure 3. Thus if $R = (3_{2,3}, 1_1, 4_{3,4}, 2_2)$, then $R' = (1_1, 1_1, 4_{2,3}, 1_1, 2_2, 1_1)$. Using the more handy sequence transformation from the figure we have:

$$R \equiv (c, c, p, p, 0, c, c, p, c) \longmapsto (p, p, c, c, 0, p, p, c, p) \equiv R'. \tag{4.3}$$



marked integer sequence = $1|1|112|1|11|1$;

$m+1=6$ non-empty compositions.



sequence = $c, c, p, p, 0, c, c, p, c$;

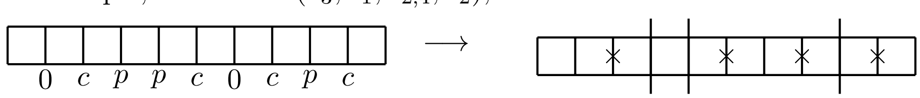
$m = 5$ cracks.

FIGURE 4. Illustration of bijection I $\Leftrightarrow$ II

We may use the same technique to arrange the cracked compositions of $n = 3$ (see (4.1)) in conjugate pairs as follows:

$$\begin{array}{ccccc} (3_1) & (3_2) & (3_3) & (3_{2,3}) & (2_2, 1_1) \\ (1_1, 2_1) & (2_1, 1_1) & (1_1, 1_1, 1_1) & (1_1, 2_2) & \end{array}$$

4.2.1. *The Tiling Representation.* The tiling representation of a cracked composition C of n consists of a tiling of a $1 \times n$ board with n unit squares in which the dividing lines have any of the three characteristics, $c, 0$ or p. The latter are depicted as $\times$, nothing, or a longer vertical line, respectively. For example, when $C = (3_3, 1_1, 4_{2,4}, 2_2)$, we have:



Conjugation now involves interchanging the $\times$'s and longer vertical lines. Thus the conjugate of C is given by $C' = (2_1, 3_{2,3}, 2_1, 2_2, 1_1)$. As a further illustration, note that Figure 5 gives an alternative depiction of the mutually conjugate cracked compositions shown in Figure 3.

In the next section we provide a more detailed discussion of the conjugation of n-color compositions which form the simplest subclass of cracked compositions.

5. Conjugation of n-Color Compositions

The n-color compositions are cracked compositions with at most one crack per part.

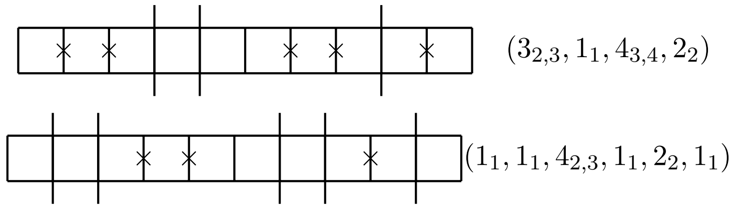


FIGURE 5. Tiling representation of conjugate cracked compositions.

In his seminal paper [2] Agarwal posed a problem regarding the graphical representation of n-color compositions analogous to the MacMahon's zig-zag representation of ordinary compositions. Subsequently Narang-Agarwal [13] solved the problem by depicting n-color compositions using weighted lattice paths. Hopkins [9] introduced a zig-zag graph representation using spotted tilings, and tackled the additional problem of conjugation of n-color compositions.

In this paper we have introduced the tiling representation of n-color compositions as the simplest subclass of cracked compositions (see Subsection 4.2.1). Our diagrams are clearly analogous to the MacMahon's line graph representation of standard compositions [10, 11].

Let $\ell(\nu)$ be the number of n-color compositions of ν.

When $m = 1$ Eqn (4.2) gives the familiar generating function (see [2])

$$(5.1)\quad \sum_{\nu=1}^{\infty} \ell(\nu)x^{\nu} = \frac{x}{1-3x+x^2} = x+3x^2+8x^3+21x^4+55x^5+144x^6+\cdots .$$

We show that the conjugation of n-color compositions is fully realizable as the conjugation of cracked compositions with at most one crack per part. However, this class of cracked compositions is not conjugate-closed. In other words, the conjugate of an n-color composition may contain a part with more than one crack (i.e., a part size with multiple subscripts) thus violating the definition of an n-color composition.

We will provide a complete description of the conjugate-closed subset of n-color compositions, and show that they are considerably more numerous than the only previously defined set of conjugable n-color compositions to our knowledge.

Brian Hopkins [9] devised a conjugation involution based on zig-zag graphs. However, the graphical conjugation operation he proposed yielded conjugates for a very small number of n-color compositions. His findings are summarized in the following theorem.

Theorem 5.1 (Hopkins). *(i) Conjugable n-color compositions have the form:*

$$(2_1^a, 1_1, 2_2^b, (3_2, 2_1^a, 1_1, 2_2^b)^c),$$

where exponents denote repetition, $a, b, c \geq 0$, and a, b may vary in each occurrence.
(ii) The number of conjugable n-color compositions of ν is $2^{(\nu-1)/2}$ if ν is odd, and 0 if ν is even.

Let $CC(\nu)$ denote the set of conjugate-closed n-color compositions of ν, and let $cc(\nu) = |CC(\nu)|$. In our discussion here $CC(\nu)$ consists entirely of conjugable n-color compositions.

Example 5.2. *When $n = 4$, we have $cc(4) = 15$, where the enumerated objects are*

$$\begin{array}{cccccccc} (4_1) & (4_2) & (4_3) & (4_4) & (1_1, 3_2) & (1_1, 3_3) & (2_1, 2_2) & (2_2, 2_2) \\ & (1_1, 3_1) & (2_1, 2_1) & (3_1, 1_1) & (2_2, 2_1) & (3_2, 1_1) & (3_3, 1_1) & (1_1, 2_2, 1_1) \end{array}$$

We will prove the following enumeration result.

Theorem 5.3. *The number $cc(\nu)$ of conjugable n-color compositions of ν is given by*

$$cc(\nu) = 2^\nu - 1.$$

The generating function is given by

$$\sum_{\nu=0}^{\infty} cc(\nu)x^\nu = \sum_{\nu=0}^{\infty}(2^\nu - 1)x^\nu = \frac{x}{1 - 3x + 2x^2}.$$

5.1. **Background to the Conjugation Theory.** In [10, Sec.IV, Ch.III, p.181] MacMahon initiated a generalization of the concept of a composition with the statement:

"If we think of the number n as a succession of n units arranged in a row, we have $n-1$ spaces between them, which we may fill with a + sign or leave blank. The 2^{n-1} different expressions so obtained are the compositions of n."

MacMahon then indicated that the $n-1$ spaces could, in general, be filled with any desired number of symbols to give generalized compositions of any prescribed order.

We first consider the classical case of two symbols which we denote with 0 and 1, with u's denoting units. The correspondence is illustrated with $n = 4$ in the first two columns of Table 2. The third column of the table is obtained by deleting the units from the entries in the first column. The correspondence of the objects in the last two columns then implies a generic proof of the following assertion.

Proposition 5.4. *There is a one-to-one correspondence between compositions of n and all binary strings $\beta \in \{0, 1\}^{n-1}$.*

We remark that this proposition may also be established by first encoding a composition of n as a set $\emptyset \neq A \subseteq \{1, \ldots, n-1\}$ via partial sums (see

[12, 18]). The corresponding binary sequence $(b_1, \dots, b_{n-1})$ is then obtained using the characteristic function

$$b_i = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{if } i \notin A. \end{cases}$$

MacMahon	Standard	Binary
$u1u1u1u$	$(1, 1, 1, 1)$	(1,1,1)
$u0u1u1u$	$(2, 1, 1)$	(0,1,1)
$u1u0u1u$	$(1, 2, 1)$	(1,0,1)
$u0u0u1u$	$(1, 1, 2)$	(0,0,1)
$u1u0u0u$	$(1, 3)$	(1,0,0)
$u0u0u1u$	$(3, 1)$	(0,0,1)
$u0u1u0u$	$(2, 2)$	(0,1,0)
$u0u0u0u$	(4)	(0,0,0)

TABLE 2. Representations of Compositions of 4

Observe that the conjugate C' of a composition C is obtained by swapping 0's and 1's in the binary representation of C, and converting back to a composition. For example,

$$C = (2, 1, 1) \equiv (0, 1, 1) \mapsto (1, 0, 0) \equiv (1, 3) = C'.$$

If we extend the classical MacMahon model by using three symbols to fill the $n - 1$ spaces, we would analogously obtain ternary sequences. Since ternary sequences $\tau \in \{0, 1, 2\}^{n-1}$ are enumerated by 3^{n-1}, we discover from Theorem 4.2, that the corresponding type of compositions are cracked compositions.

Theorem 5.5. *There is a one-to-one correspondence between cracked compositions of n and all ternary sequences $\tau \in \{0, 1, 2\}^{n-1}$.*

Remark: If we replace p by 1 and c by 2 in the sequence encodings in (4.3), we notice that the conjugate R' of a cracked composition R is obtained by swapping 1's and 2's in the ternary representation of R, and converting back to a cracked composition.

$$R \equiv (2, 2, 1, 1, 0, 2, 2, 1, 2) \mapsto (1, 1, 2, 2, 0, 1, 1, 2, 1) \equiv R'.$$

We now show that n-color compositions may be identified with certain restricted ternary sequences. Let $t(n)$ be the number of all ternary sequences $T = (v_1, v_2, \dots, v_{n-1}, v_n)$ with the property that 1 occurs between any pair of consecutive 2's.

When $v_n = 0$ or $v_n = 1$, delete v_n to obtain $2t(n - 1)$ objects T.

When $v_n = 2$, delete v_n to obtain $t(n - 1)$ objects. But this number includes sequences $E = (v_1, v_2, \dots, v_{n-1})$ with last terms of the form $2, 0^r$, $r \geq 0$, even though the string $2, 0^r, 2$ is forbidden in T. So we subtract the number of such sequences. If we delete the last term of E, which is 0 or 2, we obtain a sequence enumerated by $t(n-2)$. Conversely, given a sequence enumerated by $t(n-2)$, we append 0 if the last terms have the form $2, 0^r$, $r \geq 0$, and append 2 if the last terms have the form $1, 0^r$, to obtain E. So there are exactly $t(n-2)$ forbidden sequences. Hence the number of sequences T with $v_n = 2$ is given by $t(n-1) - t(n-2)$.

Combining the two cases, with obvious initial values, we obtain

$$t(0) = 1,$$
$$t(1) = 3,$$
$$t(n) = 3t(n-1) - t(n-2), n \geq 2.$$

It may be verified that $t(n)$ satisfies the same recurrence relation, with the same initial conditions, as the number $\ell(\nu + 1)$ of n-color compositions of $\nu + 1$, using Eqn (5.1). So we have proved the next result.

Theorem 5.7. *The number of n-color compositions of ν is equal to the number of ternary sequences $\in \{0, 1, 2\}^{\nu-1}$ in which 1 occurs between every pair of consecutive 2's.*

This theorem is immediately justified graphically because 2 represents a crack (see Remark 5.6) and 1 represents a demarcation between parts. So the appearance of 1's between every pair of consecutive 2's means that each cracked part has exactly one crack.

Theorem 5.7 may be compared with a different result in [5] in which n-color compositions of ν are identified with ternary sequences of length $\nu - 1$ avoiding 12.

5.2. **Good Compositions and Proof of Theorem 5.3.** Define a *good composition* of n as any standard support composition of a member of $CC(n)$. For example $(1_1, 3_2)$ has the support composition $(1, 3)$, so $(1, 3)$ is good. Denote the set of good compositions of n by $GC(n)$ with $gc(n) = |GC(n)|$. Thus from Example 5.2 we see that $gc(4) = 5$, where

$$GC(4) : (4), (1, 3), (2, 2), (3, 1), (1, 2, 1). \tag{5.2}$$

We found the following rather unexpected result.

Proposition 5.8. *We have*

$$gc(n) = F_{n+1}.$$

Proof. Let C be the good composition of n corresponding to a conjugable n-color composition CC, and let $t(CC)$ be the ternary sequence representation of CC. Then C has the following properties.

(i) C does not contain a pair of adjacent 1's for all $n > 2$. Indeed two adjacent 1's connote a pair of adjacent 1's in $t(CC)$, "$\ldots, 1, 1, \ldots$", which conjugates into "…,2,2,…" in $t(CC')$, and this entails a part ≥ 3 with at least two subscripts. The only exception occurs when $n = 2$ since $C = (1, 1) \to (1) \equiv (1_1, 1_1) = CC$ and $CC' = (2_2) \in CC(2)$.

(ii) C does not contain 1 between two big parts (i.e., parts > 1). A violation of this would also entail the presence of a part ≥ 3 with at least two subscripts as in case (i).

Hence C contains at most two 1's which may appear only as a first and/or last part. Such compositions of n are equinumerous with standard compositions of $n+2$ without 1's. An easy bijection is obtained by adding 1 to the first and last parts of C. (If C has only one part, add 2 to that part).

But it is well-known that compositions of $n+2$ without 1's are enumerated by F_{n+1} (see for example, [8, 12]). Hence the result. ∎

From the properties (i) and (ii) in the proof of Proposition 5.8 we deduce that the ternary encoding $t(CC)$ of a conjugable n-color composition CC satisfies the following property.

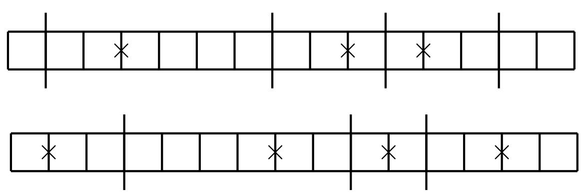
"There is a 1 between any pair of consecutive 2's in $t(CC)$, and a 2 between any pair of consecutive 1's, where the 0's are ignored".

Theorem 5.9. *Every conjugable n-color composition of ν may be identified with a ternary sequence $\tau \in \{0,1,2\}^{\nu-1}$ in which 1 occurs between every pair of consecutive 2's, and 2 occurs between every pair of consecutive 1's.*

For example, let $CC = (1_1, 6_3, 3_3, 3_2, 2_1) \in CC(15)$. Then

$$t(CC) = (1,0,2,0,0,0,1,0,2,1,2,0,1,0).$$

Interchange 1's with 2's to obtain $t(CC') = (2,0,1,0,0,0,2,0,1,2,1,0,2,0)$ which gives $CC' = (3_2, 6_5, 2_2, 4_3) \in CC(15)$. Alternatively, one may use the tiling representation of CC to obtain CC' as follows.



Note, for example, that the n-color composition $CC = (3_2, 1_1, 8_6, 2_2, 1_1) \notin CC(15)$, that is, CC is not conjugable because $CC' = (2_1, 7_{2,3}, 4_4, 2_2) \notin CC(15)$, as one may verify.

Remark: Since (n_1) corresponds to the sequence $(0,\ldots,0)$ which has no 1's and no 2's, it follows that (n_1) is the only self-conjugate n-color composition. See Example 5.2 for a pairing of members of $CC(4)$ according to conjugacy.

Proof of Theorem 5.3. Every ternary sequence $t(CC)$ satisfying Theorem 5.9 may be reduced to an alternating sequence of 1's and 2's by deleting 0's. The length of the latter is then j, where $0 \leq j \leq \nu - 1$. The case $j = 0$ corresponds to the unique sequence $(0,\ldots,0)$, and the case $j > 0$ corresponds to a sequence of the form $(1,2,\ldots)$ or $(2,1,\ldots)$. When $j > 0$ there are exactly $2\binom{\nu-1}{j}$ ways to fix the 1's and 2's in j positions, followed by one way of filling the remaining positions with 0's, to give $t(CC)$. Hence with the case $j = 0$ we obtain

$$cc(\nu) = 1 + 2\sum_{j\geq 1}\binom{\nu-1}{j} = 1 + 2\cdot(2^{\nu-1}-1) = 2^{\nu}-1.$$

∎

The following result is obtained by refining the proof of Theorem 5.3.

Theorem 5.11. *Let $cc(\nu, k)$ be the number of conjugable n-color compositions of ν into k parts. Then $cc(\nu, 1) = \nu$ and*

$$cc(\nu, k) = \binom{\nu+1}{2k-1}, \quad \nu \geq 2(k-1),\ k > 1.$$

Proof. If CC has k parts, $k > 1$, then $t(CC)$ contains exactly $k-1$ copies of 1 with $k-2$ copies of 2. Thus suppressing 0's (which may be restored in one way later) from $t(CC)$ we first fix the terms of the sequence $1, 2, \ldots, 2, 1$ among $\nu - 1$ available positions in $\binom{\nu-1}{2k-3}$ ways. Alternatively, we fix the terms of $2, 1, 2, \ldots, 2, 1$ or $1, 2, \ldots, 2, 1, 2$ in $2\binom{\nu-1}{2k-2}$ ways. A final possibility is to fix the terms of $2, 1, \ldots, 1, 2$ in $\binom{\nu-1}{2k-1}$ ways. Therefore,

$$cc(\nu, k) = \binom{\nu-1}{2k-3} + 2\binom{\nu-1}{2k-2} + \binom{\nu-1}{2k-1} = \binom{\nu+1}{2k-1}.$$

Note that $cc(\nu, 1) = \nu$, where the enumerated objects are $(\nu_1), \ldots, (\nu_\nu)$. Hence the proof. ∎

Remark: It may be verified that $\nu + \sum_{k\geq 2} cc(\nu, k) = 2^{\nu} - 1$.

From Example 5.2 and (5.2) we observe that (4) contributes 4 objects in $CC(4)$, and $(1, 3)$ contributes 3 objects, and so forth.

The following result is reminiscent of the known fact that the number of n-color compositions contributed by an ordinary composition C is equal to the product of all the parts of C (see for example, [1, 16]).

Corollary 5.13. *The number nC of conjugable n-color compositions contributed by a good composition $C = (a_1, \ldots, a_k)$ is obtained as follows:*

(i) if $k = 1$, then $nC = a_1$;

(ii) if $k = 2$, then $nC = a_1 a_2$;

(iii) if $k \geq 3$, then $nC = a_1(a_2 - 1)\cdots(a_{k-1} - 1)a_k$.

Proof. Let $CC = ((a_1)_{u_1}, \ldots, (a_k)_{u_k})$, $k > 2$. Then from the proof of Proposition 5.8 we have $a_2, \ldots, a_{k-1} > 1$. A key observation is that the color sequence $(u_1, \ldots, u_k)$ also satisfies the properties of a good composition. Hence $(u_1, \ldots, u_k)$ contains at most two 1's which may appear only as a first and/or last part. The rest of the proof follows straightforwardly. ∎

Lastly, the foregoing results imply the following easy test of conjugability of an n-color composition.

Corollary 5.14. *Let* $CC = ((g_1)_{c_1}, (g_2)_{c_2}, \ldots, (g_k)_{c_k})$ *be an n-color composition. Then* CC *is conjugable if and only if* $k = 1$*, or* $k > 1$ *and each of* $(g_1, g_2, \ldots, g_k)$ *and* $(c_1, c_2, \ldots, c_k)$ *contains at most two 1's which may appear only as a first and/or last part.*

References

1. A.K. Agarwal, *Combinatorial properties of n-color compositions*, in: *The Legacy of Srinivasa Ramanujan*, in: Ramanujan Math. Soc. Lect. Notes Ser., vol. 20, Ramanujan Math. Soc., Mysore, 2013, pp. 13–28.
2. A.K. Agarwal, *n-color compositions*, Indian J. Pure Appl. Math. 31 (2000), 1421–1427.
3. A.K. Agarwal and G.E. Andrews, *Rogers-Ramanujan identities for partitions with "N copies of N"*, J. Combin. Theory Ser. A 45, No. (1987), 40–49
4. G.E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading 1976; reprinted, Cambridge University Press, Cambridge, 1984, 1998.
5. A. Collins, C. Dedrickson, and H. Wang, *Binary words, n-color compositions and bisection of the Fibonacci numbers*, Fibonacci Quart. 51(2) (2013), 130–136.
6. Yu-Hong Guo, n-color even compositions, Ars Combin., 109 (2013), 425–432.
7. S. Heubach, T. Mansour, *Combinatorics of Compositions and Words*, Discrete Mathematics and its Applications, CRC Press, 2010.
8. S. Heubach and T. Mansour, *Compositions of n with parts in a set*, Congressus Numerantium 168 (2004), 127–143.
9. B. Hopkins, *Spotted tilings and n-color compositions*, Integers 12B (2012), A6.
10. P.A. MacMahon, *Combinatory Analysis*, Volume 1, Cambridge University Press, 1915.
11. P.A. MacMahon, *Second memoir on the theory of the compositions of numbers*, Philos. Trans. Roy. Soc. London Ser. A, 207 (1908), 65–134.
12. A.O. Munagi, *Euler-type Identities for integer compositions via zig-zag graphs*, Integers 12 (2012), A60, 10pp.
13. G. Narang and A.K. Agarwal, *Lattice paths and n-color compositions*, Discrete Math., 308 (2008), 1732–1740
14. R. Sachdeva, A.K. Agarwal, *Combinatorics of certain restricted n-color composition functions*, *Discrete Mathematics*, 340 (2017), 361-372.
15. C. Shapcott, *C-color compositions and palindromes*, Fibonacci Quart., 50 (2012), 297–303.
16. C. Shapcott, *New bijections from n-color compositions*, J. Comb., 4 (2013), 373–385.
17. N. J. A. Sloane, (2018), The On-Line Encyclopedia of Integer Sequences, published electronically at *http://www.research.att.com/ njas/sequences/*.
18. R. P. Stanley, Enumerative Combinatorics, Vol. 1., Cambridge Univ. Press, 1997.

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