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A DIFFERENT APPROACH TO GAUSS FIBONACCI POLYNOMIALS

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Abstract. In this paper with the help of higher order Fibonacci polynomials, we introduce higher order Gauss Fibonacci polynomials that generalize the Gauss Fibonacci polynomials studied by Özkan and Taştan [14]. We give a recurrence relation, Binet-like formula, generating and exponential generating functions, summation formula for the higher order Gauss Fibonacci polynomials. Moreover, we give two special matrices that we call $Q^{(s)}(x)$ and $P^{(s)}(x)$, respectively. From these matrices, we obtain a matrix representation and derive the Cassini's identity of higher order Gauss Fibonacci polynomials.

1. Introduction

Gaussian numbers were examined by C. F. Gauss in 1832. The Gaussian integers are the set

$$\mathbb{Z}[i] = \{a + ib : \ a, b \in \mathbb{Z}\},$$

where $i^2 = -1$. Since the Gaussian integers are closed under addition and multiplication, they form a commutative ring. Therefore, these numbers have an important place in algebra [6]. Based on Gauss' definition, Horadam [8] defined and studied complex Fibonacci numbers (or Gauss Fibonacci numbers) in 1963. There are several papers related to Gauss Fibonacci numbers such as the works in [5, 7, 9, 17]. The Gaussian Fibonacci numbers $\{GF_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$$GF_{n+1} = GF_n + GF_{n-1} \qquad n \geq 1,$$

with the initial values $GF_0 = 0$ and $GF_1 = 1$. One can see that

$$GF_n = F_n + iF_{n-1},$$

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where F_n is the n-th Fibonacci number is defined by

$$F_{n+2} = F_{n+1} + F_n \qquad n \geq 0,$$

with $F_0 = 0$, $F_1 = 1$. Similarly, Lucas sequence is defined by the following recurrence relation: for $n \geq 0$,

$$L_{n+2} = L_{n+1} + L_n,$$

where $L_0 = 2$ and $L_1 = 1$. The Binet formulas for the Fibonacci numbers and Lucas numbers have the following forms

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.1}$$

and

$$L_n = \alpha^n + \beta^n, \tag{1.2}$$

respectively, where α and β are the roots of the characteristic equation $x^2 - x - 1 = 0$.

Fibonacci and Lucas numbers, their generalizations, and matrix representations have been extensively studied by many mathematicians from the past up to the present [4, 10, 11, 12, 19]. One of these generalizations is the Fibonacci polynomials studied by Belgian mathematician Catalan and the German mathematician E. Jacobsthal. For $n \geq 3$, the Fibonacci polynomials $F_n(x)$ are defined by

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x),$$

where $F_1(x) = 1$ and $F_2(x) = x$. Closed form expression of Fibonacci polynomial, namely Binet-like formula is

$$F_n(x) = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x) - \beta(x)}, \tag{1.3}$$

where $\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$, $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$. For $n \geq 2$, the Lucas polynomials $L_n(x)$, are defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x),$$

where $L_0(x) = 2$ and $L_1(x) = x$. Also, Binet-like formula for the Lucas polynomial is

$$L_n(x) = \alpha(x)^n + \beta(x)^n.$$

Higher order Fibonacci numbers (or Fibonacci divisor, conjugate to F_s) were defined by Pashaev and Özvatan [15, 16]. These numbers are defined for $s \geq 1$ integer, as follows:

$$F_n^{(s)} = \frac{F_{ns}}{F_s} = \frac{(\alpha^s)^n - (\beta^s)^n}{\alpha^s - \beta^s}. \tag{1.4}$$

Since F_{ns} is divisible by F_s, the ratio $\frac{F_{ns}}{F_s}$ is an integer. So, all higher order Fibonacci numbers are integers. Let us state here that for $s = 1$, the higher order Fibonacci numbers $F_n^{(1)}$, turn into the ordinary Fibonacci numbers.

The Fibonacci divisor numbers $F_n^{(s)}$ give factorization of Fibonacci numbers with factorized index ns :

$$F_{ns} = F_s \cdot F_n^{(s)}. \tag{1.5}$$

For $s = 1, 2, 3, 4, 5$ and $n = 1, 2, 3, 4, \ldots$, the first few numbers of the Fibonacci divisor $F_n^{(s)}$; as follows:

$$\begin{aligned}
s &= 1;\ F_n^{(1)} = F_n = 1, 1, 2, 3, \ldots \\
s &= 2;\ F_n^{(2)} = F_{2n} = 1, 3, 8, 21, \ldots \\
s &= 3;\ F_n^{(3)} = \frac{1}{2} F_{3n} = 1, 4, 17, 72, \ldots \\
s &= 4;\ F_n^{(4)} = \frac{1}{3} F_{4n} = 1, 7, 48, 329, \ldots \\
s &= 5;\ F_n^{(5)} = \frac{1}{5} F_{5n} = 1, 11, 122, 1353, \ldots.
\end{aligned}$$

In [16], Pashaev gave some important properties of Fibonacci divisor numbers and also applied to some physical examples for these numbers. Similar to equation (1.4), higher order Fibonacci polynomials can be written as follows:

$$F_n^{(s)}(x) = \frac{F_{ns}(x)}{F_s(x)} = \frac{(\alpha(x)^s)^n - (\beta(x)^s)^n}{\alpha(x)^s - \beta(x)^s}. \tag{1.6}$$

From $F_{ns}(x)$ is divisible by $F_s(x)$, $F_n^{(s)}(x)$ is a polynomial with respect to x variable.

We now return to a recent investigation by Özkan and Taştan [14], who defined the special polynomials which are related to Gauss Fibonacci polynomials. The authors obtained the recurrence relation, the Binet formula and the Cassini's identity with a matrix similar to the Fibonacci Q-matrix. Moreover, they obtained some determinant equations for this matrix.

Other articles on the generalizations of Gauss Fibonacci or Gauss Fibonacci type polynomials and numbers are available in references (see Aşçı and Gürel [1, 2, 3], Morales [13], Taştan and Özkan [18]).

Motivated by some of the above-cited recent papers, in the present paper, we introduce new polynomials whose components are higher order Fibonacci polynomials. We define the higher order Gauss Fibonacci polynomials and derive some fundamental properties. We obtain recurrence relation, Binet formula, summation formula, generating function and exponential generating function of higher order Fibonacci polynomials. Moreover we define two special matrices that we call $Q^{(s)}(x)$ and $P^{(s)}(x)$, respectively. From these matrices, we obtain a matrix whose entries are higher order Gauss Fibonacci polynomials and derive the Cassini's identity.

2. Higher Order Gauss Fibonacci polynomials

In this section, we define the higher order Gauss Fibonacci polynomials and derive some new identities concerning higher order Gauss Fibonacci polynomials.

Definition 2.1. *For $n \geq 1$, the higher order Gauss Fibonacci polynomials, $GF_n^{(s)}(x)$, is defined by*

$$GF_n^{(s)}(x) = F_n^{(s)}(x) + iF_{n-1}^{(s)}(x), \tag{2.1}$$

where $s \geq 1$ is an integer and $F_n^{(s)}(x)$ is defined by (1.6).

If we take $s = 1$ in (2.1), we get the Gauss Fibonacci polynomials defined and studied by Özkan and Taştan [14]. If we take $s = x = 1$ in (2.1), we obtain the Gauss Fibonacci numbers defined and studied by Horadam and Jordan [8, 9].

Now we give the Binet-like formula for the Gauss Fibonacci polynomials.

Theorem 2.2. *Binet-like formula for higher order Gauss Fibonacci polynomials is*

$$GF_n^{(s)}(x) = \frac{(\alpha^s(x))^{n-1}(i + \alpha^s(x)) - (\beta^s(x))^{n-1}(i + \beta^s(x))}{\alpha^s(x) - \beta^s(x)}. \tag{2.2}$$

Proof. By using (2.1) and (1.4), we have

$$\begin{aligned} GF_n^{(s)}(x) &= F_n^{(s)}(x) + iF_{n-1}^{(s)}(x) \\ &= \frac{(\alpha^s(x))^n - (\beta^s(x))^n}{\alpha^s(x) - \beta^s(x)} + i\frac{(\alpha^s(x))^{n-1} - (\beta^s(x))^{n-1}}{\alpha^s(x) - \beta^s(x)} \\ &= \frac{(\alpha^s(x))^{n-1}(i + \alpha^s(x)) - (\beta^s(x))^{n-1}(i + \beta^s(x))}{\alpha^s(x) - \beta^s(x)}. \end{aligned}$$

So the proof is completed. □

Corollary 2.3. *([14, Page 3]) The Binet-like formula for the Gauss Fibonacci polynomials is*

$$GF_n(x) = \frac{\alpha^{n-1}(x)(i + \alpha(x)) - \beta^{n-1}(x)(i + \beta(x))}{\alpha(x) - \beta(x)}.$$

Proof. This follows from substituting $s = 1$ in the Equation (2.2). □

Theorem 2.4. *For $n \geq 1$, the recurrence relation for higher order Gauss Fibonacci polynomials is given by*

$$GF_{n+1}^{(s)}(x) = L_s(x)GF_n^{(s)}(x) + (-1)^{s+1}GF_{n-1}^{(s)}(x), \tag{2.3}$$

where $L_s(x)$ is Lucas polynomial.

Proof. By using (2.2) and (1.2), we have

$$\begin{aligned}
GF_{n+1}^{(s)}(x) &= \frac{(\alpha^s(x))^n\,(i+\alpha^s(x)) - (\beta^s(x))^n\,(i+\beta^s(x))}{\alpha^s(x)-\beta^s(x)} \\
&= \frac{1}{\alpha^s(x)-\beta^s(x)} \\
&\quad \times \Big((\alpha^s(x))^{n-1}\,(i+\alpha^s(x))\,\alpha^s(x) - (\beta^s(x))^{n-1}\,(i+\beta^s(x))\,\beta^s(x) \Big) \\
&= \frac{1}{\alpha^s(x)-\beta^s(x)} \\
&\quad \times \left(\begin{array}{c} (\alpha^s(x))^{n-1}\,(i+\alpha^s(x))\,\alpha^s(x) - (\beta^s(x))^{n-1}\,(i+\beta^s(x))\,\alpha^s(x) \\ +(\beta^s(x))^{n-1}\,(i+\beta^s(x))\,\alpha^s(x) - (\beta^s(x))^{n-1}\,(i+\beta^s(x))\,\beta^s(x) \end{array} \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
GF_{n+1}^{(s)}(x) &= GF_n^{(s)}(x)\,(\alpha^s(x)+\beta^s(x)) - GF_n^{(s)}(x)\beta^s(x) \\
&\quad + \frac{1}{\alpha^s(x)-\beta^s(x)} \\
&\quad \times \Big((\beta^s(x))^{n-1}\,(i+\beta^s(x))\,\alpha^s(x) - (\beta^s(x))^{n-1}\,(i+\beta^s(x))\,\beta^s(x) \Big) \\
&= L_s(x)GF_n^{(s)}(x) \\
&\quad + \frac{\left(\begin{array}{c} -(\alpha^s(x))^{n-1}\,(i+\alpha^s(x))\,\beta^s(x) + (\beta^s(x))^n\,(i+\beta^s(x)) \\ +(\beta^s(x))^{n-1}\,(i+\beta^s(x))\,\alpha^s(x) - (\beta^s(x))^{n-1}\,(i+\beta^s(x))\,\beta^s(x) \end{array} \right)}{\alpha^s(x)-\beta^s(x)} \\
&= L_s(x)GF_n^{(s)}(x) \\
&\quad -\alpha^s(x)\beta^s(x) \left(\frac{(\alpha^s(x))^{n-2}\,(i+\alpha^s(x)) - (\beta^s(x))^{n-2}\,(i+\beta^s(x))}{\alpha^s(x)-\beta^s(x)} \right) \\
&= L_s(x)GF_n^{(s)}(x) + (-1)^{s+1}\,GF_{n-1}^{(s)}(x).
\end{aligned}$$

and that gives us the desired recursion formula (2.3). □

Corollary 2.5. *(*[14, Definition 1]*) Recurrence relation of the Gauss Fibonacci polynomials is*

$$GF_{n+1}(x) = xGF_n(x) + GF_{n-1}(x) \qquad n \geq 2.$$

Proof. This follows from substituting $s = 1$ in the Equation (2.3) and $L_1(x) = x$. □

We shall give the generating function and exponential generating function for the higher order Gauss Fibonacci polynomials.

Theorem 2.6. *The generating function of the higher order Gauss Fibonacci polynomials is given by*

$$W^{(s)}(x,t) = \frac{(-1)^s\,(-i + (iL_s(x) + (-1)^s)\,t)}{1 - L_s(x)\,t + (-1)^s\,t^2}. \tag{2.4}$$

Proof. Let

$$W^{(s)}(x,t)=\sum_{n=0}^{\infty}GF_n^{(s)}(x)t^n$$

be the generating function of $GF_n^{(s)}$, then using (2.1) and (1.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty}GF_n^{(s)}(x)t^n &= \sum_{n=0}^{\infty}\left(F_n^{(s)}(x)+iF_{n-1}^{(s)}(x)\right)\ t^n\\ &= \frac{1}{\alpha^s(x)-\beta^s(x)}\\ &\quad\times\left(\frac{(i+\alpha^s(x))}{\alpha^s(x)}\sum_{n=0}^{\infty}(\alpha^s(x)t)^n-\frac{(i+\beta^s(x))}{\beta^s(x)}\sum_{n=0}^{\infty}(\beta^s(x)t)^n\right)\\ &= \frac{1}{\alpha^s(x)-\beta^s(x)}\\ &\quad\times\left(\frac{(i+\alpha^s(x))}{\alpha^s(x)\left(1-\alpha^s(x)t\right)}-\frac{(i+\beta^s(x))}{\beta^s(x)\left(1-\beta^s(x)t\right)}\right).\end{aligned}$$

After some calculations, we have

$$W^{(s)}(x,t)=\frac{(-1)^s\left(-i+\left(iL_s(x)+(-1)^s\right)t\right)}{1-L_s(x)t+(-1)^s t^2}.$$

□

Corollary 2.7. *The generating function of the Gauss Fibonacci polynomial is*

$$\sum_{n=0}^{\infty}GF_n(x)t^n=\frac{i-(ix-1)t}{1-xt-t^2}.$$

Proof. This follows from substituting $s=1$ in the Equation (2.4) □

Theorem 2.8. *Exponential generating function of the higher order Gauss Fibonacci polynomials is*

$$\begin{aligned}&\sum_{n=0}^{\infty}GF_n^{(s)}(x)\frac{t^n}{n!}\\ &= \frac{e^{\alpha^s(x)t}-e^{\beta^s(x)t}+(-1)^{s+1}i\left(\alpha^s(x)e^{\beta^s(x)t}-\beta^s(x)e^{\alpha^s(x)t}\right)}{\alpha^s(x)-\beta^s(x)}.\end{aligned}\tag{2.5}$$

Proof. Let

$$U^{(s)}(x,t)=\sum_{n=0}^{\infty}GF_n^{(s)}(x)\frac{t^n}{n!},$$

be the exponential generating function of $GF_n^{(s)}(x)$, then using (2.2), we have

$$\begin{aligned}
U^{(s)}(x,t) &= \sum_{n=0}^{\infty} GF_n^{(s)}(x)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{(\alpha^s(x))^{n-1}(i+\alpha^s(x)) - (\beta^s(x))^{n-1}(i+\beta^s(x))}{\alpha^s(x)-\beta^s(x)}\right)\frac{t^n}{n!} \\
&= \frac{1}{\alpha^s(x)-\beta^s(x)} \\
&\quad \times \left(\frac{(i+\alpha^s(x))}{\alpha^s(x)}\sum_{n=0}^{\infty}\frac{(\alpha^s(x)t)^n}{n!} - \frac{(i+\beta^s(x))}{\beta^s(x)}\sum_{n=0}^{\infty}\frac{(\beta^s(x)t)^n}{n!}\right) \\
&= \frac{1}{\alpha^s(x)-\beta^s(x)}\left(\frac{(i+\alpha^s(x))}{\alpha^s(x)}e^{\alpha^s(x)t} - \frac{(i+\beta^s(x))}{\beta^s(x)}e^{\beta^s(x)t}\right) \\
&= \frac{e^{\alpha^s(x)t} - e^{\beta^s(x)t} + (-1)^{s+1} i\left(\alpha^s(x)e^{\beta^s(x)t} - \beta^s(x)e^{\alpha^s(x)t}\right)}{\alpha^s(x)-\beta^s(x)}.
\end{aligned}$$

□

Corollary 2.9. *Exponential generating function of the Gauss Fibonacci polynomials $GF_n(x)$ is*

$$\sum_{n=0}^{\infty} GF_n(x)\frac{t^n}{n!} = \frac{e^{\alpha(x)t} - e^{\beta(x)t} + i\left(\alpha(x)e^{\beta(x)t} - \beta(x)e^{\alpha(x)t}\right)}{\alpha(x)-\beta(x)}.$$

Proof. This follows from substituting $s=1$ in the Equation (2.5). □

Theorem 2.10. *For $n \geq 0$, the following equality holds:*

$$\sum_{k=0}^{n} GF_k^{(s)}(x) = \frac{1+(-1)^s F_n^{(s)}(x) - F_{n+1}^{(s)}(x) + (-1)^{s+1} i\gamma_n(x)}{1 - L_s(x) + (-1)^s}, \tag{2.6}$$

where $\gamma_n(x) = 1 + (-1)^s F_n^{(s)}(x) - F_2^{(s)}(x) - F_{n-1}^{(s)}(x)$.

Proof. By virtue of (2.2), we get

$$
\begin{aligned}
&\sum_{i=0}^{n}\left(\frac{(\alpha^{s}(x))^{k-1}(i+\alpha^{s}(x))-(\beta^{s}(x))^{k-1}(i+\beta^{s}(x))}{\alpha^{s}(x)-\beta^{s}(x)}\right)\\
=\;& \frac{1}{\alpha^{s}(x)-\beta^{s}(x)}\\
&\times\left(\frac{(i+\alpha^{s}(x))}{\alpha^{s}(x)}\sum_{k=0}^{n}(\alpha^{s}(x))^{k}-\frac{(i+\beta^{s}(x))}{\beta^{s}(x)}\sum_{k=0}^{n}(\beta^{s}(x))^{k}\right)\\
=\;& \frac{1}{\alpha^{s}(x)-\beta^{s}(x)}\\
&\times\left(\frac{(i+\alpha^{s}(x))\left(1-(\alpha^{s}(x))^{n+1}\right)}{\alpha^{s}(x)(1-\alpha^{s}(x))}-\frac{(i+\beta^{s}(x))\left(1-(\beta^{s}(x))^{n+1}\right)}{\beta^{s}(x)(1-\beta^{s}(x))}\right).
\end{aligned}
$$

After some calculations, we arrive at the desired result (2.6). □

Corollary 2.11. *For $n \geq 0$, the following equality holds:*

$$
\sum_{k=0}^{n} GF_{k}(x)=\frac{GF_{n+1}(x)+GF_{n}(x)-1+ix-i}{x}. \tag{2.7}
$$

Proof. This follows from substituting $s=1$ in the Equation (2.6). □

If we take $x=1$ in Equation (2.7), we obtain following result, which is well known for Gaussian Fibonacci numbers.

$$
\sum_{k=0}^{n} GF_{k}=GF_{n+2}-1.
$$

3. A matrix representation for higher order Gauss Fibonacci polynomials

In this part of the our paper, we derive the matrix representation of the higher order Gauss Fibonacci polynomials. We first introduce two matrices $Q^{(s)}(x)$ and $P^{(s)}(x)$ as follows:

$$
Q^{(s)}(x)=\begin{pmatrix} L_{s}(x) & (-1)^{s+1} \\ 1 & 0 \end{pmatrix},
$$

and

$$
P^{(s)}(x)=\begin{pmatrix} L_{s}(x)+i & 1 \\ 1 & (-1)^{s-1}\, i \end{pmatrix},
$$

where $L_{s}(x)$ are the Lucas polynomials. Now we give the following theorem regarding our result.

Theorem 3.1. *For $n \geq 0$, the following equality holds:*

$$
\left(Q^{(s)}(x)\right)^{n} P^{(s)}(x)=\begin{pmatrix} GF_{n+2}^{(s)}(x) & GF_{n+1}^{(s)}(x) \\ GF_{n+1}^{(s)}(x) & GF_{n}^{(s)}(x) \end{pmatrix}. \tag{3.1}
$$

Proof. For the proof, we use induction method on n. The equality holds for $n = 0$. Assume that our assertion holds for $n = k$. Namely,

$$\left(Q^{(s)}(x)\right)^k P^{(s)}(x) = \begin{pmatrix} GF_{k+2}^{(s)}(x) & GF_{k+1}^{(s)}(x) \\ GF_{k+1}^{(s)}(x) & GF_k^{(s)}(x) \end{pmatrix}.$$

Then for $n = k + 1$, we have

$$\begin{aligned} \left(Q^{(s)}(x)\right)^{k+1} P^{(s)}(x) &= \begin{pmatrix} L_s(x) & (-1)^{s+1} \\ 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} L_s(x) + i & 1 \\ 1 & (-1)^{s-1}\, i \end{pmatrix} \\ &= \begin{pmatrix} L_s(x) & (-1)^{s+1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} GF_{k+2}^{(s)}(x) & GF_{k+1}^{(s)}(x) \\ GF_{k+1}^{(s)}(x) & GF_k^{(s)}(x) \end{pmatrix} \\ &= \begin{pmatrix} GF_{k+3}^{(s)}(x) & GF_{k+2}^{(s)}(x) \\ GF_{k+2}^{(s)}(x) & GF_{k+1}^{(s)}(x) \end{pmatrix}. \end{aligned}$$

So the proof is completed. □

We give the Cassini identity for higher order Gauss Fibonacci polynomials in the following theorem.

Theorem 3.2. *For $n \geq 1$, we have*

(3.2)
$$GF_{n+1}^{(s)}(x) GF_{n-1}^{(s)}(x) - \left(GF_n^{(s)}(x)\right)^2 = (-1)^{sn-s} \left((-1)^{s-1} i \left(L_s(x) + i\right) - 1\right).$$

Proof. If we take the determinant of both sides of Equation (3.1), we find that

$$\begin{aligned} \begin{vmatrix} GF_{n+1}^{(s)}(x) & GF_n^{(s)}(x) \\ GF_n^{(s)}(x) & GF_{n-1}^{(s)}(x) \end{vmatrix} &= \det\left(\left(Q^{(s)}(x)\right)\right)^{n-1} \det\left(P^{(s)}(x)\right) \\ &= \begin{vmatrix} L_s(x) & (-1)^{s+1} \\ 1 & 0 \end{vmatrix}^{n-1} \begin{vmatrix} L_s(x) + i & 1 \\ 1 & (-1)^{s-1}\, i \end{vmatrix} \\ &= (-1)^{sn-s} \left((-1)^{s-1} i \left(L_s(x) + i\right) - 1\right). \end{aligned}$$

□

Corollary 3.3. *(*[14, Proposition 1]*) For $n \geq 1$, we have*

$$GF_{n+1}(x) GF_{n-1}(x) - \left(GF_n(x)\right)^2 = (-1)^n (2 - xi).$$

Proof. This follows from substituting $s = 1$ in the Equation (3.2). □

4. CONCLUSION

In our present investigation, we have introduced and studied higher order Gauss Fibonacci polynomials which are defined by means of the higher order Fibonacci polynomials. Then we have derived several fundamental properties of these higher order Gauss Fibonacci polynomials. All results obtained in the article can be varied according to the different integer values of s. For example, in the main results we have obtained, the special case of $s = 1$ gives various results about Gaussian Fibonacci polynomials defined and studied by Özkan and Taştan [14]. With this method, various Fibonacci type algebraic structures can be generalized.

References

[1] M. Asci and E. Gürel, *Gaussian Jacobsthal and Gaussian Jacobsthal Lucas Polynomials*, Notes Number Theory Discrete Math. **19(1)** (2013), 25-36.

[2] M. Asci and E. Gürel, *Bivariate Gaussian Fibonacci and Lucas polynomials*, Ars Combin. **109** (2013), 485-496.

[3] M. Asci and E. Gurel, *Gaussian Jacobsthal and Gaussian Jacobsthal Lucas Numbers*, Ars Combin. **111** (2013), 53-63.

[4] M. Asci, D. Tasci, and M. El-Mikkawy, *On Determinants and Permanents of k-Tridiagonal Toeplitz Matrices*, Util. Math. **89** (2012), 97-106.

[5] G. Berzsenyi, *Gaussian Fibonacci numbers*, Fibonacci Q. **15** (1977), 223-236.

[6] J. B. Fraleigh, *A First Course In Abstract Algebra*, 7th edn. University of Rhode Island, Pearson, 1976.

[7] C. J. Harman, Complex Fibonacci numbers. Fibonacci Q., 19:1, 82–86, (1981).

[8] A. F. Horadam, Complex Fibonacci numbers and Fibonacci quaternions, Amer. Math. Monthly, 70, 289–291, (1963).

[9] J. H. Jordan, *Gaussian Fibonacci and Lucas Numbers*, Fibonacci Q. **3** (1965), 315-318.

[10] C. Kızılateş and T. Kone, *On higher order Fibonacci hyper complex numbers*, Chaos Solitons Fractals **148** (2021), 1-6.

[11] C. Kızılateş and T. Kone, *On higher order Fibonacci quaternions*, J. Anal. **29** (2021), 1071-1082.

[12] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, Proc., New York-Toronto, 2001.

[13] G. C. Morales, *Gaussian third-order Jacobsthal and Gaussian third-order Jacobsthal–Lucas polynomials and their properties*, Asian-Eur. J. Math. **14(5)** (2020).

[14] E. Özkan and M. Taştan, *On Gauss Fibonacci Polynomials, Gauss Lucas Polynomials and Their Applications*, Commun. Algebra **48(3)** (2020), 952-960.

[15] M. Özvatan, *Generalized golden–Fibonacci calculus and applications*, Master of Science thesis, Izmir Institute of Technology, 2018.

[16] O. K. Pashaev, *Quantum calculus of Fibonacci divisors and infinite hierarchy of bosonic-fermionic golden quantum oscillators*, Internat. J. Geom. Methods Modern Phys. **18(5)** (2021), 32 pp.

[17] S. Pethe and A.F. Horadam, *Generalised Gaussian Fibonacci numbers*, Bull. Austral. Math. Soc. **33** (1986), 37-48.

[18] M. Taştan and E. Özkan, *On the Gauss Fibonacci Polynomials*, Electron. J. Math. Analysis Appl. **9(1)** (2021), 124-130.

[19] S. Vajda, *Fibonacci and Lucas Numbers the Golden Section*, Ellis Horrowood Limited Publ. England, 1989.

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