Contributions to Discrete Mathematics

Volume 16, Number 2, Pages 23–35
ISSN 1715-0868

FRAÏSSÉ THEORY AND THE POULSEN SIMPLEX

MARTINO LUPINI

ABSTRACT. We present a Fraïssé-theoretic perspective on the study of the Poulsen simplex and its properties.

1. INTRODUCTION

The *Poulsen simplex* is a canonical object in Choquet theory and convexity theory. Let us briefly recall the basic notions and definitions from the theory of compact convex sets. By a *compact convex set* we mean a compact subset K of a locally convex topological vector space. A compact convex set is endowed with a notion of convex combinations. A point of a compact convex set is *extreme* if it cannot be written in a nontrivial way as a convex combination of elements of K. The *extreme boundary* $\partial_e K$ of K is the collection of extreme points of K. Natural examples of compact convex sets are the *standard simplices* Δ_n for $n \in \mathbb{N}$. Concretely, one can describe Δ_n as the space of stochastic vectors in $\mathbb{R}^{n+1}$. A *metrizable Choquet simplex* is a compact convex set that can be realized as the limit of an inverse sequence of standard simplices with surjective continuous affine maps as connective maps. Metrizable Choquet simplices can be regarded as the well-behaved compact convex sets. Many equivalent characterizations of metrizable Choquet simplices are known, including ones due to Choquet–Meyer, Bishop–de Leeuw, and Namioka–Phelps [7, 26, 27]. Many compact convex sets that arise in the applications are metrizable Choquet simplex. For example, if A is a separable unital C*-algebra, then the space of linear functionals of norm 1 on A satisfying $\tau(1) = 1$ and $\tau(xy) = \tau(yx)$ for $x, y \in A$ is a metrizable Choquet simplex, called the *trace simplex* of A. Any metrizable Choquet simplex can be obtained in this way.

One can associate with any compact metrizable space X the metrizable Choquet simplex $P(X)$ of Borel probability measures on X. The metrizable Choquet simplices of this form are precisely those with *closed* extreme

Received by the editors July 8, 2016, and in revised form June 2, 2017.

2000 *Mathematics Subject Classification.* Primary 46L07, 46A55; Secondary 46L89, 03C30, 03C98.

Key words and phrases. Gurarij space, Poulsen simplex, Choquet simplex, Fraïssé limit operator space, operator system, universal operator.

boundary (*Bauer simplices*). An example from 1961 due to Poulsen shows that such a property can fail spectacularly. Poulsen constructed in [28] a (nontrivial) metrizable Choquet simplex whose extreme boundary is *dense* in the whole simplex. While Poulsen's original construction does not suggest any canonicity in such an object, Lindenstrauss, Olsen, and Sternfeld proved in 1978 that there exists a *unique* nontrivial metrizable Choquet simplex with dense extreme boundary, hence called the *Poulsen simplex* $\mathbb{P}$ [22].

It is furthermore proved in [22] that the Poulsen simplex has the following *universality* and *homogeneity* properties within the class of metrizable Choquet simplices: a compact convex set is a metrizable Choquet simplex if and only if it is affinely homeomorphic to a closed proper face of $\mathbb{P}$, and any affine homeomorphism between closed proper faces of $\mathbb{P}$ extends to an affine homeomorphism of $\mathbb{P}$. Additionally, one can assert that the Poulsen simplex is *generic* among metrizable Choquet simplices, in the sense that the collection of metrizable Choquet simplices affinely homeomorphic to $\mathbb{P}$ is a dense G_δ set in the space of metrizable Choquet simplices endowed with a canonical Polish topology. Since then, the Poulsen simplex has appeared in many places in the literature. For instance, the simplex of invariant probability measures for the Bernoulli action of countable discrete group on an alphabet with two letters is the Poulsen simplex whenever the group does *not* have Kazhdan's property (T), and it is a Bauer simplex otherwise . Furthermore, when G is a countable nonabelian free group, the space of weak equivalence classes of measure-preserving G-actions endowed with its canonical topology and convex structure is the Poulsen simplex [9].

The present survey contains an exposition of some recent work from [23], where the Poulsen simplex and its properties are studied from the perspective of Fraïssé theory for metric structures as recently developed in [4]. Some of these results had previously been obtained in an unpublished work of Conley and Törnquist. Another approach to the study of the Poulsen simplex from Fraïssé-theoretic methods is considered in [20]. While the main goal of [23] is to give a unified proof of various results about Fraïssé limits in functional analysis—including results about the Gurarij space from [12, 17, 18, 19, 21]—in this short survey we consider the special instance of the framework of [23] in the case of compact convex sets. This allows one to give Fraïssé-theoretic proofs of many known results about the Poulsen simplex from [22]. This framework has been used in [23] to define and construct the natural *noncommutative analog* of the Poulsen simplex. The model-theoretic properties of the Poulsen simplex and its noncommutative analog are investigated in [14]. Finally, the Fraïssé-theoretic description of the Poulsen simplex together with the Kechris–Pestov–Todorcevic correspondence between the Ramsey property and extreme amenability [15, 25, 29] is used in [2] to compute the universal minimal flow of the Polish group of affine homeomorphisms of $\mathbb{P}$. This turns out to be the canonical action on the Poulsen simplex itself.

2. Function systems and the Kadison representation theorem

Suppose that T is a compact Hausdorff space. Let $C(T)$ be the space of real-valued continuous functions on T. The space $C(T)$ has a distinguished element $\mathbf{1}$ (the *unit*) which is the function constantly equal to 1. Concretely, a *function system* is a closed subspace V of $C(T)$ that is *unital*, in the sense that it contains the unit. Function systems can be abstractly characterized as those real Banach spaces V with a distinguished element $\mathbf{1}$ with the property that, for any $x \in V$ such that $\|x\| = 1$, one has that $\|x + \mathbf{1}\| = 2$ or $\|x - \mathbf{1}\| = 2$ [8, Section 2]. A linear map between function systems is *unital* if it maps the unit to the unit.

A function system $V \subset C(T)$ is endowed with a canonical notion of *positivity*, defined by letting an element a of V be positive if it is a positive-valued function when regarded as an element of $C(T)$. Such a notion does not depend on the concrete unital isometric representation of V as a space of continuous functions on some compact Hausdorff space. A linear map between function systems is *positive* if it maps positive elements to positive elements. For a unital linear map being positive is equivalent to having norm 1.

Suppose that K is a compact set. Let $A(K)$ be the space of real-valued continuous *affine* functions on K. Then $A(K)$ is a function system, as witnessed by the inclusion $A(K) \subset C(K)$. Kadison's representation theorem [1, Theorem II.8.1] asserts that any function system arises in this way. Indeed, suppose that V is a function system. Let $S(V)$ be the space of unital positive linear functionals. Then $S(V)$ is a weak*-compact convex set, called the *state space* of V. The norm of an element a of V can be described as the supremum of $s(a)$ where s ranges in $S(V)$.

Any element a of V gives rise to a real-valued continuous function f_a on $S(V)$ by point-evaluation. The induced function $V \to A(S(V))$, $a \mapsto f_a$ is a surjective unital isometric linear map, which allows one to identify V with $A(S(V))$. Furthermore the assignment $V \mapsto S(V)$ is a *contravariant equivalence of categories* from the category of function systems and unital positive linear maps to the category of compact convex sets and continuous affine maps. Indeed a unital positive linear map $\phi : V \to W$ induces a continuous affine function $\phi^\dagger : S(W) \to S(V)$, $s \mapsto s \circ \phi$, and all continuous affine functions from $S(W)$ to $S(V)$ are of this form. Furthermore ϕ is an isometry if and only if $\phi^\dagger$ is surjective. Using such a correspondence between compact convex sets and function systems, one can equivalently formulate statements about compact convex sets as statements about function systems. For instance, a compact convex set K is *metrizable* if and only if $A(K)$ is *separable*. If V is a separable function system, we let $\mathrm{Aut}(V)$ be the Polish group of surjective unital isometric linear maps from V to V, endowed with the topology of pointwise convergence. Then $\mathrm{Aut}(V)$ admits a canonical continuous action on the state space $S(V)$, defined by $(\alpha, s) \mapsto s \circ \alpha^{-1}$. More generally if W is another separable function system, then the space of unital positive linear maps from V to W is a Polish $\mathrm{Aut}(V)$-space with respect to the topology of pointwise convergence and the action $(\alpha, t) \mapsto t \circ \alpha^{-1}$.

Consider now a *standard simplex* Δ_{n-1} for $n \geq 1$. This is the set of stochastic vectors $\boldsymbol{s} = (s_1, \ldots, s_n) \in \mathbb{R}^n$. The corresponding function system $A(\Delta_{n-1})$ is the space ℓ_n^∞ with norm $\|(x_1, \ldots, x_n)\| = \max\{|x_1|, \ldots, |x_n|\}$ and unit $(1, 1, \ldots, 1)$. Indeed any element of Δ_{n-1} defines a state on ℓ_n^∞ by $\boldsymbol{s}(x_1, \ldots, x_n) = s_1x_1 + \cdots + s_nx_n$, and any state on ℓ_n^∞ is of this form. The function systems ℓ_n^∞ for $n \in \mathbb{N}$ are precisely the finite-dimensional function systems that are *injective objects* in the category of function systems and unital positive linear maps. This can be seen as a consequence of the Hahn–Banach theorem.

Metrizable Choqet simplices are precisely those compact convex sets that can be realized as limits of an inverse sequence of standard simplices with surjective continuous affine maps as connective maps. Equivalently, a compact convex set K is a metrizable Choquet simplex if and only if the function system $A(K)$ can be realized as the limit of an inductive sequence of finite-dimensional injective function systems. In this case the function system $A(K)$ is called a separable *simplex space*. In the following we will assume all the function systems to be separable, and all the compact convex sets to be metrizable.

3. The Fraïssé limit of finite-dimensional function systems

The goal of this section is to prove that the class of finite-dimensional function systems is a Fraïssé class in the sense of Fraïssé theory for metric structures [4]. Suppose that X, Y are function systems. An *embedding* from X to Y is a unital isometric linear map. The *amalgamation property* for operator systems asserts that if $f_0 : X \to Y_0$ and $f_1 : X \to Y_1$ are embeddings, then there exists a function system Z and embeddings $\phi_0 : Y_0 \to Z$ and $\phi_1 : Y_1 \to Z$ such that $\phi_0 \circ f_0 = \phi_1 \circ f_1$. (Here and in the following, for an injective linear map $f : X \to Y$ we denote by f^{-1} the linear map $f^{-1} : f[X] \to Y$.) This can be verified as follows. Consider the *algebraic* direct sum $Y_0 \oplus Y_1$. Define a *function system structure* Z on $Y_0 \oplus Y_1$ by letting the norm of (y_0, y_1) be the supremum of $|s_0(y_0) + s_1(y_1)|$ where $(s_0, s_1) \in S(Y_0) \times S(Y_1)$ are such that $s_0 \circ f_0 = s_1 \circ \phi_1$. This gives the desired function system Z, where the embeddings $\phi_0 : Y_0 \to Z$ and $\phi_1 : Y_1 \to Z$ are the first and second coordinate inclusion map, respectively. In categorical terms, the function system Z is the *pushout* of the morphisms f_0 and f_1.

The same argument allows one to prove an approximate version of the result above. In this case one considers (not necessarily isometric) positive unital linear maps $f_0 : X \to Y_0$ and $f_1 : X \to Y_1$ that are invertible, and furthermore the inverses are *almost* isometric in the sense that $\|f_0^{-1}\| \leq 1 + \delta$ and $\|f_1^{-1}\| \leq 1 + \delta$ for some δ. One can now define the function system structure Z on $Y_0 \oplus Y_1$ similarly as above, by considering the pairs $(s_0, s_1) \in S(Y_0) \times S(Y_1)$ such that $\|s_0 \circ f_0 - s_1 \circ f_1\| \leq 2\delta$. We call the corresponding function system Z the *approximate pushout* of f_0 and f_1 with tolerance δ. Observe that when Y_0, Y_1 are finite-dimensional, then Z is also finite-dimensional. If furthermore Y_0, Y_1 are finite-dimensional and injective, i.e. $Y_0 = \ell^\infty_{d_0}$ and $Y_1 = \ell^\infty_{d_1}$ for some $d_0, d_1 \in \mathbb{N}$, then one can simply take $Z = Y_0 \oplus^\infty Y_1 = \ell^\infty_{d_0+d_1}$.

To verify that the coordinate inclusions maps $\phi_0 : Y_0 \to Z$ and $\phi_1 : Y_1 \to Z$ are embeddings one needs to observe that "approximately contractive" unital linear maps can be approximated by positive unital linear maps. Precisely, if W is a function system and $t : W \to \ell^\infty_n$ is a unital linear functional on W of norm at most $1+\delta$, then there exists a positive unital linear map $s : W \to \ell^\infty_n$ such that $\|s - t\| \leq 2\delta$. This is a consequence of the fact that an element w of W can be written as $w = w_0 - w_1$ where w_0, w_1 are positive and $\|w\| = \|w_0\| + \|w_1\|$; see [1, Proposition II.1.14].

Using the *approximate amalgamation property* for function systems one can verify that the class of finite-dimensional function systems is a (metric) Fraïssé class in the sense of [4]. One can also construct explicitly the Fraïssé limit as follows. Fix, for every $n \in \mathbb{N}$, a countable dense subset D_n of ℓ^∞_n. Let $(\bar{a}_{n,k})$ be an enumeration of the finite injective tuples of elements of D_n, and set $E_{n,k} := \mathrm{span}\{1, \bar{a}_n\} \subset \ell^\infty_n$. For every $n, k, m, d \in \mathbb{N}$ let $\mathcal{E}_{n,k,d,m}$ be a 2^{-m}-dense set of embeddings from $E_{n,k}$ to ℓ^∞_d, where the distance in the space of embeddings is given by $d(\phi, \psi) = \|\phi - \psi\|$.

Using the approximate pushout construction from above, one can recursively define sequences (d_j), (η_j), (F) such that

(1) $d_j \in \mathbb{N}$ and $\eta_k : \ell^\infty_{d_j} \to \ell^\infty_{d_{j+1}}$ are unital isometric linear maps,
(2) $\mathcal{F}_{n,k,j}$ is a $2^{-(j+1)}$-dense collection of injective positive unital linear maps from $E_{n,k}$ to $\ell^\infty_{d_j}$,
(3) for every $d, n, k \leq j$, $f \in \mathcal{F}_{n,k,j}$, and $\phi \in \mathcal{E}_{n,k,d,j}$ there exists $\hat{f} : \ell^\infty_d \to \ell^\infty_{d_{j+1}}$ such that $\|\hat{f} \circ \phi - \eta_j \circ ft\| \leq 2^{-j}$.

Let now $\mathbb{V}$ be the limit of the sequence $(\ell^\infty_{d_j})$ with connective maps $\eta_j : \ell^\infty_{d_j} \to \ell^\infty_{d_{j+1}}$. We claim that $\mathbb{V}$ has the following property, which we call the *stable extension property*: whenever E, F are finite-dimensional function systems, and $\phi : E \to F$ and $f : E \to \mathbb{V}$ are injective unital positive linear maps such that $\|\phi^{-1}\| < 1 + \delta$ and $\|f^{-1}\| < 1 + \delta$, then there exists a unital isometric linear map $\hat{f} : F \to \mathbb{V}$ such that $\|\hat{f} \circ \phi - f\| < 2\delta$. This is a consequence of the construction above together with the following observations. The class of finite-dimensional injective function systems is *approximately cofinal*, in the sense that any finite-dimensional function system approximately embeds into ℓ^∞_d for some (large enough) $d \in \mathbb{N}$. Furthermore the small perturbation lemma in Banach space theory guarantees that the topology in the space of linear maps from a finite-dimensional function system X to a function system Y induced by the distance $d(\varphi, \psi) = \|\varphi - \psi\|$ coincides with the topology of pointwise convergence.

A back-and-forth argument, similar to the one in the proof of [21, Theorem 1.1] allows one to deduce from the stable extension property that $\mathbb{V}$ has the following *stable homogeneity property*: if E is a finite-dimensional function system, and $\phi : E \to \mathbb{V}$ and $f : E \to \mathbb{V}$ are injective unital positive linear maps such that $\|\phi^{-1}\| < 1+\delta$ and $\|f^{-1}\| < 1+\delta$, then there exists an automorphism α of $\mathbb{V}$ such that $\|\alpha \circ \phi - f\| < 2\delta$. The same back-and-forth argument shows that there exists a unique separable function system that satisfies the stable extension property. Furthermore, the stable extension property as stated above is equivalent to the same assertion where E, F are injective finite-dimensional operator systems and ϕ, f are unital isometric linear maps. Furthermore, a one-sided version of the same argument shows that any separable function system embeds into $\mathbb{V}$.

By construction, $\mathbb{V}$ is a separable *simplex space*, being realized as the limit of a sequence of finite-dimensional injective function systems with unital isometric linear connective maps. The state space $S(\mathbb{V})$ is therefore a metrizable *Choquet simplex*. We will explain in the next section how to see that $S(\mathbb{V})$ has dense extreme boundary.

From a model-theoretic perspective, one can regard function systems as a metric structure in the sense of [5]. The corresponding language contains function symbols for the vector space operations, a constant symbol for the unit, and a predicate symbol for the norm. One can see that the stable homogeneity property of $\mathbb{V}$ is elementary, and therefore $\mathbb{V}$ is the unique separable model of its first-order theory. As a consequence the automorphism group $\mathrm{Aut}(\mathbb{V})$ of $\mathbb{V}$ is a Roelcke precompact Polish group; see [6, Definition 2.2]. Furthermore using [5, Proposition 13.6] one can conclude that the first-order theory of $\mathbb{V}$ admits elimination of quantifiers, and that the theory of $\mathbb{V}$ is the model completion of the theory of function systems. Finally, one can give the following model-theoretic characterization of simplex spaces: a function system W is a simplex space if and only if it is *existentially closed*. This means that whenever a quantifier-free type is approximately realized in a function system $V \supset W$, then it is also approximately realized in W; see also [14].

4. Function systems with a distinguished state

In this section we still denote by $\mathbb{V}$ the Fraïssé limit of the class of finite-dimensional function systems as constructed in the previous section. We will prove that the state space $S(\mathbb{V})$ has dense extreme boundary. To this purpose, we consider the class of function systems *with a distinguished state*. A similar argument as the one in the previous section shows that such a class satisfies amalgamation. Indeed suppose that $f_0 : X \to Y_0$ and $f_1 : X \to Y_1$ are invertible unital positive linear maps such that $\|f_0^{-1}\| \leq 1+\delta$ and $\|f_1^{-1}\| \leq 1+\delta$, and that Z is the approximate pushout of f_0, f_1 with tolerance δ defined as in the previous section. Let $\phi_0 : Y_0 \to Z$ and $\phi_1 : Y_1 \to Z$ be the canonical embeddings. Then it follows from its definition that the approximate pushout has the following universal property: if W is a function system, and $g_0 : Y_0 \to W$ and $g_1 : Y_1 \to W$ are unital positive linear maps such that $\|g_0 \circ f_0 - g_1 \circ f_1\| \leq 2\delta$, then there exists a unique unital positive linear map $t : Z \to W$ such that $t \circ \phi_0 = g_0$ and $t \circ \phi_1 = g_1$. In particular, if $s_0 \in S(Y_0)$ and $s_1 \in S(Y_1)$ are states such that $\|s_0 \circ f_0 - s_1 \circ f\| \leq 2\delta$, then there exists a unique state s on Z such that $s \circ \phi_0 = s_0$ and $s \circ \phi_1 = s_1$.

It follows from the remarks above that the class of finite-dimensional function systems with a distinguished state is a Fraïssé class. By uniqueness of the Fraïssé limit, one can regard the corresponding Fraïssé limit as a distinguished state $s_{\mathbb{V}}$ on $\mathbb{V}$. Such a state is uniquely characterized, up to an automorphism of $\mathbb{V}$, by the following property: if E_0, E_1 are finite-dimensional function systems, $s \in S(E_1)$, and $\phi : E_0 \to F$ and $f : E_0 \to \mathbb{V}$ are injective unital positive linear maps such that $\|\phi^{-1}\| < 1 + \delta$, $\|f^{-1}\| < 1 + \delta$, and $\|s_{\mathbb{V}} \circ f - s \circ \phi\| < 2\delta$, then there exist a unital linear isometry $\hat{f} : E_1 \to \mathbb{V}$ such that $\|\hat{f} \circ \phi - f\| < 2\delta$ and $s_{\mathbb{V}} \circ \hat{f} = s$. This is in turn equivalent to the same assertion where E_0, E_1 are finite-dimensional injective operator systems, and f, ϕ are unital isometric linear maps. Such a characterization implies that the $\mathrm{Aut}(\mathbb{V})$-orbit of $s_{\mathbb{V}}$ is a dense G_δ subspace of the state space $S(\mathbb{V})$. In order to conclude that $S(\mathbb{V})$ has dense extreme boundary, it remains to observe that $s_{\mathbb{V}}$ is an extreme point.

Suppose that $t_0, t_1 \in S(\mathbb{V})$ and $\lambda \in (0,1)$ are such that $\lambda t_0 + (1-\lambda)\, t_1 = s$. Fix a unital isometric linear map $f : \ell^\infty_d \to \mathbb{V}$ and $\varepsilon > 0$. Consider the unital linear isometry $\phi : \ell^\infty_d \to \ell^\infty_{d+1}, (x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_d, (s_{\mathbb{V}} \circ f)(x_1, \ldots, x_d))$, and let $e \in S(\ell^\infty_{d+1})$ be the state $e(x_1, \ldots, x_{d+1}) = x_{d+1}$. Then there exists a unital linear isometry $\hat{f} : \ell^\infty_{d+1} \to \mathbb{V}$ such that $\|\hat{f} \circ \phi - f\| \leq \varepsilon$ and $s_{\mathbb{V}} \circ \hat{f} = e$. Since e is an extreme point of $S(\ell^\infty_{d+1})$ we have that $t_0 \circ \hat{f} = t_1 \circ \hat{f} = e$. Hence $\|t_0 \circ f - s_{\mathbb{V}} \circ f\| \leq \varepsilon$. Since this is true for any unital linear isometry $f : \ell^\infty_d \to \mathbb{V}$ and any $\varepsilon > 0$, we can conclude that $t_0 = s_{\mathbb{V}}$. This concludes the proof that $s_{\mathbb{V}}$ is an extreme point of the state space of $\mathbb{V}$.

The argument above shows that $S(\mathbb{V})$ is a metrizable Choquet simplex with dense extreme boundary. In the following section, we will observe that in fact the set of extreme points of $S(\mathbb{V})$ is equal to the $\mathrm{Aut}(\mathbb{V})$-orbit of $s_{\mathbb{V}}$, and that $S(\mathbb{V})$ is the *unique* nontrivial metrizable Choquet simplex with dense extreme boundary.

5. Existence and uniqueness of the Poulsen simplex

In this section we still denote by $\mathbb{V}$ the Fraïssé limit of the class of finite-dimensional function systems, and by $s_{\mathbb{V}}$ the state on $\mathbb{V}$ obtained as the Fraïssé limit of the class of finite-dimensional function systems with a distinguished state. We have proved before that $s_{\mathbb{V}}$ is an extreme point of the state space of $\mathbb{V}$. We now observe that, conversely, any extreme point belongs to the $\mathrm{Aut}(\mathbb{V})$-orbit of $s_{\mathbb{V}}$.

Indeed, suppose that W is a simplex space. Then one can characterize the extreme points of the state space of W as those states t that satisfy the following: for any finite-dimensional function systems $E_0 \subset E_1$, unital positive linear map $f : E_0 \to W$, state $s \in S(E_1)$ such that $\|t \circ f - s|_{E_0}\| < \varepsilon$, there exists a unital positive linear map $\hat{f} : E_1 \to W$ such that $t \circ \hat{f} = s$ and $\|\hat{f}|_{E_0} - f\| < 3\varepsilon$; see [23, Proposition 6.21]. From this and the homogeneity property characterizing the $\mathrm{Aut}(\mathbb{V})$-orbit of $s_{\mathbb{V}}$ mentioned in the previous section, it follows that any extreme point of the state space of $\mathbb{V}$ belongs to the $\mathrm{Aut}(\mathbb{V})$-orbit of $s_{\mathbb{V}}$. In other words, $\mathrm{Aut}(\mathbb{V})$ acts transitively on the extreme points of $S(\mathbb{V})$.

The characterization of extreme points of a Choquet simplex recalled above admits the following generalization. Suppose that $A(K), A(F)$ are simplex spaces, and $P : A(K) \to A(F)$ is a unital linear *quotient mapping*. This means that P maps the open unit ball of $A(K)$ onto the open unit ball of $A(F)$. Then the image $\{s \circ P : s \in F\}$ of F under the dual map $P^{\dagger}$ is a closed face of K if and only if for any finite-dimensional function systems $E_0 \subset E_1$, unital positive linear maps $\phi : E_0 \to E_1$ and $g : E_0 \to V$ such that $\|t \circ f - g|_{E_0}\| < \varepsilon$, there exists a unital positive linear map $\hat{f} : E_1 \to W$ such that $P \circ \hat{f} = g$ and $\left\|\hat{f}|_{E_0} - f\right\| < 3\varepsilon$; see [23, Proposition 6.21]. Furthermore in this case one can regard P as the function $A(K) \to A(F)$, $a \mapsto a|_F$ after one identifies F with its image under the dual map P. One recovers the characterization of extreme points mentioned above in the particular case when F is the trivial simplex.

We now observe that the state space of $\mathbb{V}$ is the unique nontrivial metrizable Choquet simplex with dense extreme boundary. Indeed, suppose that K is a nontrivial metrizable Choquet simplex and $A(K)$ be the corresponding simplex space. Our goal is to prove that $A(K)$ satisfies the characterizing property of the Fraïssé limit $\mathbb{V}$ of finite-dimensional function systems. To this purpose, fix $d \in \mathbb{N}$, $\varepsilon > 0$, and unital linear isometries $f : \ell^{\infty}_d \to A(K)$ and $\phi : \ell^{\infty}_d \to \ell^{\infty}_{d+1}$. Since f is a unital linear isometry, without loss of generality, we can assume that $\ell^{\infty}_d \subset A(K)$ and $f : \ell^{\infty}_d \to A(K)$ is the inclusion map. One can also choose a standard basis $e_1, \ldots, e_d$ of ℓ^{∞}_d and a standard basis $f_1, \ldots, f_{d+1}$ of ℓ^{∞}_{d+1} such that $\phi(e_i) = f_i + t(e_i) f_{n+1}$ for some state t on ℓ^{∞}_{d+1}. Pick extreme points $s_1, \ldots, s_n$ of K such that $s_i(e_j) = \delta_{ij}$. Since K by assumption is nontrivial and has dense extreme boundary, there exists an extreme point s_{n+1} of K different from $s_1, \ldots, s_n$ such that $|s_{n+1}(e_j) - t(e_j)| < \varepsilon$ for $j = 1, 2, \ldots, d$. Define now the unital positive quotient mapping $Q : A(K) \to \ell^{\infty}_{d+1}$ by $a \mapsto (s_1(a), \ldots, s_{n+1}(a))$. Observe that the image of the state space of ℓ^{∞}_{d+1} under the dual mapping $Q^{\dagger}$ is precisely the convex hull of $s_1, \ldots, s_{n+1}$. This is a closed face of K, since $s_1, \ldots, s_{n+1}$ are extreme points. Furthermore by the choice of $s_1, \ldots, s_{n+1}$ the map Q satisfies $\|Q|_E - \phi\| < \varepsilon$. Therefore there exists a unital positive (necessarily isometric) linear map $\hat{f} : \ell^{\infty}_{d+1} \to A(K)$ such that $Q \circ \hat{f}$ is the identity map of ℓ^∞_{d+1} and $\|\hat{f} \circ \phi - f\| < 3\varepsilon$. This concludes the proof that $A(K)$ is unitally isometrically isomorphic to $\mathbb{V}$, and hence K is affinely homeomorphic to the state space of $\mathbb{V}$.

The argument above—although presented in a slightly different language—is essentially the original argument of Lindenstrauss–Olsen–Sternfeld in their proof of the uniqueness of the Poulsen simplex [22]; see also [27]. From now on we will denote by $\mathbb{P}$ the unique nontrivial Choquet simplex with dense extreme boundary, and call it the *Poulsen simplex*. Consistently, we will identify the Fraïssé limit $\mathbb{V}$ of the class of finite-dimensional function systems with the space $A(\mathbb{P})$ of real-valued continuous affine functions on $\mathbb{P}$. We will also identify the group $\mathrm{Aut}(A(\mathbb{P}))$ of surjective unital linear isometries of $A(\mathbb{P})$ with the group of affine homeomorphisms of $\mathbb{P}$.

6. Homogeneity of the Poulsen simplex

Besides the uniqueness of the Poulsen simplex, the main result of the paper [22] of Lindenstrauss–Olsen–Sternfeld is that the Poulsen simplex $\mathbb{P}$ satisfies the following *universality* and *homogeneity* properties: any metrizable Choquet simplex is affinely homeomorphic to a closed proper face of $\mathbb{P}$, and any affine homeomorphism between closed proper faces of $\mathbb{P}$ extends to an affine homeomorphism of $\mathbb{P}$. In this section we will observe that these results can be proved by adapting the techniques of the previous section.

Indeed, suppose that K is a metrizable Choquet simplex, and $A(K)$ is the corresponding simplex space. Then one can show that finite-dimensional function systems E with a distinguished unital positive linear map $s : E \to A(K)$ form a Fraïssé class. When K is the trivial simplex, one recovers the class of finite-dimensional function systems with a distinguished state. The proof in the general case is entirely analogous, after observing that $A(K)$ satisfies the following property (which in fact characterizes simplex spaces among function systems): whenever $E_0 \subset E_1$ are finite-dimensional function systems, $f : E_0 \to A(K)$ is a unital positive linear map, and $\varepsilon > 0$, there exists a unital positive linear map $\hat{f} : E_1 \to A(K)$ such that $\|\hat{f}|_{E_0} - f\| < \varepsilon$. This follows from a small perturbation argument and the fact that $A(K)$ is the limit of an inductive sequence of finite-dimensional injective function systems.

The Fraïssé limit of the class of finite-dimensional function systems with a distinguished positive linear map to $A(K)$ can be seen as a distinguished unital positive linear map $\Omega^{A(K)}_{A(\mathbb{P})} : A(\mathbb{P}) \to A(K)$. Observe that the space $\mathrm{Aut}(A(\mathbb{P}))$ has a canonical continuous action on the space of unital positive linear maps from $A(\mathbb{P})$ to $A(K)$. As in the case when K is the trivial simplex, one can characterize the $\mathrm{Aut}(A(\mathbb{P}))$-orbit of $\Omega^{A(K)}_{A(\mathbb{P})}$ as those maps of the form $f \mapsto f|_F$, where F is a closed proper face of $\mathbb{P}$ affinely homeomorphic to K and $A(K)$ is identified with $A(F)$. From the existence of the Fraïssé limit $\Omega^{A(K)}_{A(\mathbb{P})}$ one can conclude that K is affinely homeomorphic to a closed proper face of $\mathbb{P}$. From the functoriality of the assignment $K \mapsto \Omega_{A(\mathbb{P})}^{A(K)}$ one can conclude that any affine homeomorphism between closed proper faces extends to an affine homeomorphism of $\mathbb{P}$.

It is proved in [2] that the class of finite-dimensional function systems satisfies the approximate Ramsey property for embeddings. This is the natural analog in the setting of metric structures of the Ramsey property for discrete structures, where colorings are replaced with $[0,1]$-valued Lipschitz maps (*continuous colorings*), and monochromatic sets are replaced with sets where the oscillation of the given continuous coloring is less than a given strictly positive ε. The Kechris–Pestov–Todorcevic correspondence between the Ramsey property of a Fraïssé classes of finitely-generated discrete structures and the extreme amenability of the automorphism group of the corresponding Fraïssé limit [15] admits a natural generalization to the metric setting [25]. From this one can deduce that, for any closed proper face F of $\mathbb{P}$, the group of affine homeomorphisms of $\mathbb{P}$ that fix F pointwise is an extremely amenable group. Particularly, the stabilizer $\mathrm{Aut}(\mathbb{P}, e)$ of an extreme point of $\mathbb{P}$ is extremely amenable. The latter fact allows one to compute the universal minimal flow of the group $\mathrm{Aut}(\mathbb{P})$ of affine homeomorphisms of $\mathbb{P}$. This is the canonical action $\mathrm{Aut}(\mathbb{P}) \curvearrowright \mathbb{P}$. Minimality of such an action is a result of Glasner [13], while universality follows by observing that $\mathbb{P}$ is $\mathrm{Aut}(\mathbb{P})$-equivariantly homeomorphic to the completion of the homogeneous space $\mathrm{Aut}(\mathbb{P})\,/\mathrm{Aut}(\mathbb{P}, e)$.

7. The noncommutative Poulsen simplex

Noncommutative mathematics studies, broadly speaking, the mathematical structures that arise when classical physics is replaced with quantum physics. In this setting, many classical notions admit a natural noncommutative generalization. The noncommutative analog of function systems is given by *operator systems*. Recall that, concretely, a function system is a unital subspace of $C(T)$ for some compact Hausdorff space T. The spaces $C(T)$ are precisely the *abelian* unital C*-algebra. In general, a (concrete) unital C*-algebra is a space A of operators on a Hilbert space H with the property that A is invariant under taking compositions and adjoints, it contains the identity operator (the *unit*), and it is closed in the topology induced by the operator norm. Unital C*-algebras can be regarded as the noncommutative analog of compact Hausdorff spaces.

An *operator system* is a closed subspace V of a unital C*-algebra A that contains the identity operator and it is invariant under taking adjoints. An operator system V is a function system when it can be represented inside an *abelian* unital C*-algebra A. Operator systems can be regarded as the noncommutative analog of compact convex sets.

Operator systems are in canonical correspondence with geometrical objects called *compact matrix convex sets* [31, 10, 11, 30]. Suppose that W is a locally convex topological vector space. We denote by $M_n(W)$ the space of $n \times n$ matrices with entries in W, endowed with its canonical topological vector space structure. We also denote by $M_n(\mathbb{C})$ the space of $n \times n$ complex matrices, and by $M_{n,k}(\mathbb{C})$ the space of $n \times k$ complex matrices. A compact matrix convex set is a sequence $\boldsymbol{K} = (K_n)$ of compact convex sets $K_n \subset M_n(W)$ that is invariant under *matrix convex combinations.* A matrix convex combination is an expression of the form $\alpha_1^* v_1 \alpha_1 + \cdots + \alpha_\ell^* v_\ell \alpha_\ell$ where $v_i \in K_{n_i}$, $\alpha_i \in M_{n_i,n}(\mathbb{C})$ is right invertible, and $\alpha_1^*\alpha_1 + \cdots + \alpha_\ell^*\alpha_\ell$ is the identity $n \times n$ matrix. Several notions from convexity theory admits natural matrix convex analogs, obtained by replacing convex combinations with matrix convex combinations. For instance, a *matrix affine function* between matrix convex sets $\boldsymbol{K}$ and $\boldsymbol{K'}$ is a sequence of functions $f_n : K_n \to K'_n$ that preserves matrix convex combinations. An element of a compact matrix convex set is a *matrix extreme point* if it cannot be written in a nontrivial way as a matrix convex combination.

One can assign in a natural way to a compact matrix convex set $\boldsymbol{K}$ an operator system $A(\boldsymbol{K})$ whose elements are the matrix affine functions from $\boldsymbol{K}$ to $(M_n(\mathbb{C}))_{n\in\mathbb{N}}$. It is proved in [30] that, conversely, any operator system V arises from a compact matrix convex set $\boldsymbol{K}$ in this way. Here K_n is the collection of all morphisms (in the category of operator systems) from V to $M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ is identified with the algebra of bounded linear operators on an n-dimensional Hilbert space. Furthermore such a correspondence between compact matrix convex sets and operator systems is functorial, and extends the correspondence between compact convex sets and function systems.

The notion of noncommutative Choquet simplex has been introduced and studied in [23]. A compact matrix convex set $\boldsymbol{K}$ is a (metrizable) noncommutative Choquet simplex if the corresponding operator system $A(K)$ is a (separable) *nuclear* operator system. Nuclearity is a regularity condition for operator systems that requires the identity map to be the pointwise limit of morphisms that factor through finite-dimensional injective operator systems. For function systems such a condition is equivalent to being a simplex space.

The natural noncommutative analog of the Poulsen simplex is defined and studied in [23]. The *noncommutative Poulsen simplex* $\mathbb{NP}$ is a (nontrivial) metrizable noncommutative Choquet simplex with a dense set of matrix extreme points. It is proved in [24] that the noncommutative Poulsen simplex is unique, and its corresponding operator system $A(\mathbb{NP})$ is the unique nontrivial separable nuclear operator systems that is universal in the sense of Kirchberg and Wassermann [16]. The operator system $A(\mathbb{NP})$ can be realized as the Fraïssé limit of the class of finite-dimensional operator systems. A unified approach to the study of the Poulsen simplex and its noncommutative analog, as well as other Fraïssé limits in functional analysis, is presented in [23]. It is also proved in [23] that any metrizable noncommutative Choquet simplex can be realized as a proper noncommutative face of $\mathbb{NP}$. Finally, the methods of [3] apply to show that the universal minimal flow of the group Aut($\mathbb{NP}$) of matrix affine homeomorphisms of $\mathbb{NP}$ is the canonical action of Aut($\mathbb{NP}$) on the space of positive unital linear functionals on $A(\mathbb{NP})$; see [3].

References

1. E. M. Alfsen, *Compact convex sets and boundary integrals*, Springer-Verlag, New York-Heidelberg, 1971.
2. D. Bartośova, J. López-Abad, M. Lupini, and B. Mbombo, *The Ramsey property for Banach spaces and Choquet simplices*, Journal of the European Mathematical Society, in press.
3. D. Bartośova, J. López-Abad, M. Lupini, and B. Mbombo, *The Ramsey property for operator spaces and noncommutative Choquet simplices*, arXiv:2006.04799 (2020).
4. I. Ben Yaacov, *Fraïssé limits of metric structures*, Journal of Symbolic Logic **80** (2015), no. 1, 100–115.
5. I. Ben Yaacov, A. Berenstein, C. W. Henson, and A. Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis. Vol. 2, London Mathematical Society Lecture Note Series, vol. 350, Cambridge University Press, 2008, pp. 315–427.
6. I. Ben Yaacov and T. Tsankov, *Weakly almost periodic functions, model-theoretic stability, and minimality of topological groups*, Transactions of the American Mathematical Society, to appear.
7. E. Bishop and K. De Leeuw, *The representations of linear functionals by measures on sets of extreme points*, Annales de l'Institut Fourier **9** (1959), 305–331.
8. D. P. Blecher and M. Neal, *Metric characterizations of isometries and of unital operator spaces and systems*, Proceedings of the American Mathematical Society **139** (2011), no. 3, 985–998.
9. P. Burton, *Topology and convexity in the space of actions modulo weak equivalence*, Ergodic Theory and Dynamical Systems, to appear.
10. E. G. Effros, *Aspects of noncommutative order*, C*-algebras and applications to physics (Proc. Second Japan-USA Sem., Los Angeles, Calif., 1977), Lecture Notes in Mathematics, vol. 650, Springer, Berlin, 1978, pp. 1–40.
11. E. G. Effros and S. Winkler, *Matrix convexity: operator analogues of the bipolar and Hahn-Banach theorems*, Journal of Functional Analysis **144** (1997), no. 1, 117–152.
12. J. Garbulińska-Węgrzyn and W. Kubiś, *A universal operator on the Gurariĭ space*, Journal of Operator Theory **73** (2015), no. 1, 143–158.
13. S. Glasner, *Distal and semisimple affine flows*, American Journal of Mathematics **109** (1987), no. 1, 115–131.
14. I. Goldbring and M. Lupini, *Model-theoretic aspects of the Gurarij operator system*, Israel J. Math. **226** (2018), no. 1 87–118.
15. A. S. Kechris, V. Pestov, and S. Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geometric & Functional Analysis **15** (2005), no. 1, 106–189.
16. E. Kirchberg and S. Wassermann, *C*-algebras generated by operator systems*, Journal of Functional Analysis **155** (1998), no. 2, 324–351.
17. W. Kubiś, *Metric-enriched categories and approximate Fraïssé limits*, arXiv:1210.6506 (2012).
18. ———, *Fraïssé sequences: category-theoretic approach to universal homogeneous structures*, Annals of Pure and Applied Logic **165** (2014), no. 11, 1755–1811.
19. ———, *Injective objects and retracts of Fraïssé limits*, Forum Mathematicum **27** (2015), no. 2, 807–842.
20. W. Kubiś and A. Kwiatkowska, *The Lelek fan and the Poulsen simplex as Fraïssé limits*, arXiv:1512.09252 (2015).

21. W. Kubiś and S. Solecki, *A proof of uniqueness of the Gurariĭ space*, Israel Journal of Mathematics **195** (2013), no. 1, 449–456.
22. J. Lindenstrauss, Gunnar Hans Olsen, and Yaki Sternfeld, *The Poulsen simplex*, Annales de l'Institut Fourier **28** (1978), no. 1, 91–114.
23. M. Lupini, *Fraïssé limits in functional analysis*, Advances in Mathematics **338** (2018), 93–174.
24. ———, *The Kirchberg–Wassermann operator system is unique*, Journal of Mathematical Analysis and Applications **459** (2018), 1251–1259.
25. J. Melleray and T. Tsankov, *Extremely amenable groups via continuous logic*, arXiv:1404.4590 (2014).
26. I. Namioka and R. R. Phelps, *Tensor products of compact convex sets*, Pacific Journal of Mathematics **31** (1969), 469–480.
27. G. H. Olsen, *On simplices and the Poulsen simplex*, Functional analysis: surveys and recent results, II (Proc. Second Conf. Functional Anal., Univ. Paderborn, Paderborn, 1979), Notas Mat., vol. 68, North-Holland, Amsterdam-New York, 1980, pp. 31–52.
28. E. T. Poulsen, *A simplex with dense extreme points*, Annales de l'Institut Fourier **11** (1961), 83–87.
29. L. Nguyen Van Thé, *A survey on structural Ramsey theory and topological dynamics with the Kechris-Pestov-Todorcevic correspondence in mind*, Selected Topics in Combinatorial Analysis, vol. 25, Zbornik Radova, no. 17, Beograd, 2015, pp. 189–207.
30. C. Webster and C. Winkler, *The Krein-Milman theorem in operator convexity*, Transactions of the American Mathematical Society **351** (1999), no. 1, 307–322.
31. G. Wittstock, *On matrix order and convexity*, Functional analysis: surveys and recent results, III (Paderborn, 1983), North-Holland Math. Stud., vol. 90, North-Holland, Amsterdam, 1984, pp. 175–188.

School of Mathematics and Statistics
Victoria University of Wellington
PO Box 600, 6140 Wellington, New Zealand
E-mail address: `martino.lupini@vuw.ac.nz`