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SIBLINGS OF AN $\aleph_0$-CATEGORICAL RELATIONAL STRUCTURE.

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Abstract. A *sibling* of a relational structure R is any structure S which can be embedded into R and, vice versa, such that R can be embedded into S. Let $\mathrm{sib}(R)$ be the number of siblings of R, these siblings being counted up to isomorphism. Thomassé conjectured that for countable relational structures made of at most countably many relations, $\mathrm{sib}(R)$ is either one, countably infinite, or the size of the continuum; but even showing the special case $\mathrm{sib}(R)1$ is one or infinite is unsettled when R is a countable tree.

We prove that if R is countable and $\aleph_0$-categorical, then indeed $\mathrm{sib}(R)$ is one or infinite. Furthermore, $\mathrm{sib}(R)$ is one if and only if R is finitely partitionable in the sense of Hodkinson and Macpherson [14]. The key tools in our proof are the notion of monomorphic decomposition of a relational structure introduced in [35] and studied further in [23], [24], and a result of Frasnay [11].

Dedicated to Roland Fraïssé and Claude Frasnay. In memoriam.

1. Introduction

A *sibling* of a given relational structure R is any structure S which can be embedded into R, and vice versa, such that R can be embedded into S. If R is finite, there is just one sibling but generally one cannot expect equimorphic structures to be necessarily isomorphic. However, the famous

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Cantor–Bernstein–Schroeder theorem states that this is the case for structures in a language with pure equality: if there is an injection from one set to another and vice versa, then there is a bijection between these two sets. The same situation occurs in other structures such as vector spaces, where embeddings are linear injective maps. But, as expected, the case that equimorphic structures are isomorphic does not hold in general.

Thus, let $\mathrm{sib}(R)$ be the number of siblings of R, these siblings being counted up to isomorphism. Thomassé conjectured that $\mathrm{sib}(R) = 1$, $\aleph_0$, or $2^{\aleph_0}$ for countable relational structures made of at most countably many relations (see [44, Conjecture 2]). We verified this conjecture for chains in [17]. The special case, $\mathrm{sib}(R)$ is one or infinite, is unsettled, even in the case of trees. It is connected to the Bonato–Tardif conjecture which asserts that for every tree T the number of trees which are siblings of T is either one or infinite, see [1, 2, 46]. The connection is through the following observation. Every sibling of a tree T is a tree if and only if $T \oplus 1$, the graph obtained by adding to T an isolated vertex, is not a sibling of T (more generally, note that every sibling of a connected graph is connected, just in case $G \oplus 1$ is not a sibling). Hence, for a tree T not equimorphic to $T \oplus 1$, the Bonato–Tardif conjecture and the special case of Thomassé's conjecture are equivalent. It turns out that for these trees, these conjectures are open (for an example, it is open for ternary trees decorated with pendant vertices). On the other hand, if a tree T is equimorphic to $T \oplus 1$, the number of siblings of T is infinite, hence the special case of the Thomassé conjecture holds, but we do not know if the Bonato–Tardif conjecture holds).

In this paper we prove the following:

Theorem 1.1. *The number of siblings of a countable $\aleph_0$-categorical relational structure R is either one or infinite. Furthermore, it is one if and only if R is finitely partitionable, that is, there is a partition of the domain E of R into finitely many sets such that every permutation of E which preserves each block of the partition is an automorphism of R.*

Our result extends a result of Hodkinson and Macpherson [14]. Indeed, they proved that a countable structure R in a finite language is such that every R' with the same age is isomorphic to R (in which case every R' with the same age is equimorphic to R), if and only if R is finitely partitionable. They indicate that their result holds if the language is infinite and, in addition $\mathrm{Aut}(R)$, the automorphism of R, is oligomorphic, that is, for each integer n, the number of orbits of n-element subsets of the base set is finite.

The fact that a countable relational structure R is $\aleph_0$-categorical is equivalent to the fact that $\mathrm{Aut}(R)$ is oligomorphic (Engeler, Ryll-Nardzewski, and Svenonius, see for example Cameron [5, p. 30]). In this context, our result applies to countable homogeneous structures with an oligomorphic automorphism group. Indeed, let G be a group acting on a set E. We recall that a partial map f with domain A and codomain A', subsets of E, is *adherent* to G with respect to the pointwise convergence topology if for every finite subset F of A there is some $g \in G$ such that f and g coincide on F. In our setting, we will instead say that such a map is a G-*local embedding*; if $A = E$ then we say that this is a G-*embedding*, and if furthermore $A' = E$ we say that this is a G-*automorphism*. We write $\overline{G}$ for the set of G-embeddings, and we write $\overline{G}^{\mathfrak{S}}$ for the set of G-automorphisms which are easily seen to form a group. If $G = \overline{G}^{\mathfrak{S}}$, we say that G is *closed* (this is the case if $G = \mathrm{Aut}(R)$ for some relational structure R). We say that two subsets of E are *equivalent*, resp. *weakly-equivalent*, if each is the image of the other by some G-local embedding, resp. each one *contains* the image of the other by some G-local embedding. A G-*copy* is the image of E under some G-embedding, that is, a member of the equivalence class of E. A G-*sibling* is a subset of E which contains a G-copy; equivalently, this is a subset weakly equivalent to E. We denote by $\mathrm{sib}(G)$ the number of equivalence classes of G-siblings, under isomorphism.

In this setting, Theorem 1.1 yields the following.

Theorem 1.2. *If G is a closed oligomorphic group on a countable set E, then* $\mathrm{sib}(G)$ *is one or infinite. That is either the weak-equivalence classes of E coincide with the equivalence classes of E (the set of copies), or each is the union of infinitely many equivalence classes. In the first case there is a partition of E into finitely many sets such that every permutation of E which preserves each block of the partition belongs to G.*

Proof. Since G is closed, there exist some homogeneous relational structure R such that $\mathrm{Aut}(R) = G$ (see for example Cameron [5, p. 26]). Since G and hence $\mathrm{Aut}(R)$ is oligomorphic, R is $\aleph_0$-categorical. This R is such that a partial map is a local embedding of R if and only if it is a G-local embedding. Hence, the number of equivalence classes of G-siblings is exactly the number of siblings of R. □

The number of siblings of a countable $\aleph_0$-categorical structure can be one or $\aleph_0$, but our proof does not show if $2^{\aleph_0}$ is the only other possibility.

1.1. **Ideas behind the proof. An outline.** A natural idea in the study of siblings of a structure R is to study extensions of R with the same age. When R is universal for its age, these extensions are automatically siblings.

To illustrate, let us consider countable homogeneous graphs. Thanks to the classification result of Lachlan–Woodrow [16] we have a precise description. Each such graph is (up to complement) the Rado graph (where the age is all finite graphs); the generic structure whose age is all K_n-free graphs ($n \geq 3$); mK_n (where $m + n$ is infinite, $m, n \geq 1$). Using the idea of non-isomorphic extensions, we can easily produce $2^{\aleph_0}$ siblings for G, the Rado graph, or G, the homogeneous K_n-free graph. Indeed, let $\{G_n : n \in \mathbb{N}\}$ be an antichain (for graph embedding) of finite connected graphs without triangles (e.g., take for G_n an $(n + 4)$-element cycle). For $S \subseteq \mathbb{N}$, form $G_S := G \cup \sum_{n \in S} G_n$, the disjoint union of G and some of the G_n. Since G is connected, these graphs are not isomorphic; since G is universal for its age, they are equimorphic to G. Hence $\mathrm{sib}(G) = 2^{\aleph_0}$. When $G = mK_n$, three cases need to be considered.

CASE 1: m, n are infinite.

For $S \subseteq \mathbb{N}$, form $G_S = G \cup \sum_{n \in S} K_n$. Clearly, G_S embeds in G and this produces $2^{\aleph_0}$ siblings.

CASE 2: m is finite.

In this case, $\mathrm{sib}(G) = 1$.

CASE 3: m is infinite; we may suppose $n \geq 2$.

In this case, by extending G to isolated vertices, $\mathrm{sib}(G) = \aleph_0$.

It is not difficult to use the same idea to show that for the countable ultrahomogeneous tournaments one has the same trichotomy. It is tempting to try to generalize the results to relational structures R that are universal for their own age. But this goes beyond the techniques we have. By restricting the classes to $\aleph_0$-categorical structures, and by using the idea of monomorphic decomposition, one can get some general results showing $\mathrm{sib}(R)$ is one or infinite. We sketch the outline of the proof.

We start with a countable structure R which is $\aleph_0$-categorical in its complete theory. As is well-known, there is a countable structure R' equimorphic to R which is $\aleph_0$-categorical, but for which the complete theory is axiomatizable by universal-existential sentences (see Saracino [37], see also Pouzet [27]). Since R' is equimorphic to R, then $\mathrm{sib}(R') = \mathrm{sib}(R)$, and hence we may replace R by R'.

Structures R for which the complete theory is axiomatizable by universal-existential sentences have a combinatorial definition that we recall in Section 2 (Theorem 2.2). They are uniformly prehomogeneous and their profile (the function which counts for each integer n the number of restrictions to the n-elements subsets, these restrictions being counted up to isomorphy) take only finite values.

Starting with such a structure R, we consider its monomorphic decomposition. This notion appears in full generality in [35], [23], and [24]). In our case it is given by an equivalence relation that is definable by a universal sentence.

We first study a special case, when the decomposition consists of one class, that is, in the terminology of Fraïssé, R is monomorphic. In this case, we prove that $\mathrm{sib}(R)$ is one, in which case $\mathrm{Aut}(R)$ is the full symmetric group or $2^{\aleph_0}$ (Theorem 3.1). To do this we use both Frasnay's result on chainable structures and Cameron's result on monomorphic groups. More generally, we show that if R has an infinite class which is not a strongly indiscernible subset of R (that is some permutation of that class does not extend to an automorphism of R by the identity on the remainder) then R has $2^{\aleph_0}$ siblings (Theorem 5.2). From this, it follows that if R has a finite monomorphic decomposition then R has one or $2^{\aleph_0}$ siblings (Theorem 5.1). Next, we consider the case where R has no finite monomorphic decomposition. Here, we prove that $\mathrm{sib}(R)$ is infinite (Theorem 6.1, (a)). Indeed, since R is universal for its age, every countable extension with the same age will be equimorphic to R. With Ramsey's theorem and the compactness theorem of first order logic, we can build an extension R' of R whose domain E' is an extension of the domain E of R, and where $E' \setminus E$ is an infinite monomorphic part of R'. Then for H a finite subset of $E' \setminus E$ and $R'_H = R' \restriction E \cup H$, we will obtain that R'_H is equimorphic to R. We aim to get R' such that for infinitely many integers k, the various R'_H's with $|H| = k$ are pairwise nonisomorphic, hence $\mathrm{sib}(R)$ will be infinite. Using the fact that R has infinitely many components, we get R' such that the trace over E of the component of R' containing $E' \setminus E$ is finite. This will suffices to realize our aim. Finally, using again the compactness theorem of first order logic, we prove that, if R has infinitely many infinite components, then $\mathrm{sib}(R) = 2^{\aleph_0}$ siblings (Theorem 6.1, (b)).

The value of $\mathrm{sib}(R)$ remains unsettled if R has infinitely many finite monomorphic classes and all infinite classes are strongly indiscernible. If R is the Rado graph or an infinite direct sum of copies of the complete graph K_m ($m \in \mathbb{N}$) all classes are finite (the classes of the Rado graph are singletons, while the class of the direct sum of copies of K_m are these copies). But, the number of siblings of the Rado graph is the continuum, while the number of siblings of this direct sum is countable. We conjecture that $\mathrm{sib}(R)$ is at most countable if and only if R is cellular (see Problem 8.18 in Section 8).

1.2. **Structure of the paper.** Basic definitions are introduced in Section 2. Five sections focus on the proof of the main theorem. In Section 3 we present the notion of monomorphy and prove that if a countable relational structure R is monomorphic, uniformly prehomogeneous and if $\mathrm{Aut}(R)$ is not the symmetric group then $\mathrm{sib}(R) = 2^{\aleph_0}$ (Theorem 3.1). We introduce in Section 4 the notion of the monomorphic decomposition of a relational structure. In Section 5 we prove that if a countable relational structure is uniformly prehomogeneous and has a finite monomorphic decomposition then it has one or $2^{\aleph_0}$ siblings (Theorem 5.1). In Section 6 we consider the case of structures without finite monomorphic decomposition. We reassemble our results in Theorem 7.1 of Section 7. Theorem 1.1 follows.

In Section 8, the last section, we present several problems around the notion of equimorphy.

2. Basic definitions

Our terminology follows that of Fraïssé [9]. A *relational structure* of *signature* $\mu = (n_i)_{i \in I}$ and *domain* E is a pair $R = (E, (\rho_i)_{i \in I})$ where each ρ_i is an n_i-ary relation on E. If I' is a subset of I, then $R' = (E, (\rho_i)_{i \in I'})$ is called a *reduct* of R, and called a *finite reduct* if I' is finite. A relational structure $R = (E, (\rho_i)_{i \in I})$ is a *binary relational structure*, *binary structure* for short, if it is made only of binary relations. It is *ordered* if one of the relations ρ_i is a linear order.

2.1. **Embeddability, age, profile.** The *substructure induced by R on a subset A of E*, simply called the *restriction of R to A*, is the relational structure $R_{\restriction A} = (A, (A^{n_i} \cap \rho_i)_{i \in I})$. For simplicity the restriction to $E \setminus \{x\}$ is denoted R_{-x}. The notion of *isomorphism* between relational structures is defined in the natural way. A map f from a subset F of the domain E onto a subset F' of a relational structure R' is a *local isomorphism of R into R'* if f is an isomorphism of $R_{\restriction F}$ onto $R'_{\restriction F'}$. If $R = R'$, we say that f is a *local isomorphism of R* (or a local embedding of R). A relational structure R is *embeddable* into a relational structure R' if R is isomorphic to some restriction of R'. Embeddability is a quasi-order on the class of structures having a given signature.

The *age* of a relational structure R is the set $\mathrm{age}(R)$ of restrictions of R to finite subsets of its domain, these restrictions being considered up to isomorphy. The *profile* of a relational structure R is the function φ_R which gives for every nonnegative integer n, the number of n-element restrictions counted up to isomorphy. This function depends only on the age of R.

2.2. **Homogeneity.** A relational structure R is *homogeneous* if every finite local isomorphism extends to an automorphism of the structure (the notion has been introduced independently by several authors, the current terminology comes from Fraïssé; the reader must be aware that it is called ultra-homogeneous in some of the early literature). We present below three generalizations of this notion. We focus on the notion of *uniform prehomogeneity* which we characterize in terms of the notion of local 1-embedding.

Let R and R' be two relational structures on E and E' respectively; we say that a map f defined on a subset F of E with values in a subset F' of E' is a *local* 1*-embedding of R into R'* if its restriction to every finite subset H of F extends to every finite set $\overline{H} \subseteq E$ containing H to a local isomorphism of R into R'. If f^{-1}, the set inverse of f, is also a local 1-embedding, we say that f is a *local* 1*-isomorphism*; if such f exists, we say that F and F' are 1*-isomorphic* or have the same 1*-isomorphism type*.

Let R be a relational structure with base E. An *extension* of R is any relational structure R' such that $R'_{\restriction E} = R$. An extension R' is a 1*-extension of R* if for every finite subset F of E, the identity map $\mathrm{Id}_{\restriction F}$ on F is a 1-local embedding *from R' to R*. This means that for every finite subset F' of $E' \setminus E$ there is a local isomorphism of R' to R which is the identity on F and maps F' into E. Then, we say that a relational structure R is *existentially closed* if every extension of R with the same age is a 1-extension. We say that R is *existentially universal* if for every extension R' with the same age, every finite F in the domain of R, every finite F' in the domain of R', the identity map on F extends to F' to a local 1-embedding of R' to R (its role is discussed in the last section).

We say that R is *prehomogeneous* if, for every finite set F of the domain E of R, there is a finite superset F' of F such that every local isomorphism of R with domain F extends to an automorphism of R provided that it extends to F'. We say that R is *uniformly prehomogeneous* if in addition the cardinality of F' is bounded by some function θ of the cardinality of F.

Slightly different notions of existentially closed and existentially universal structures were introduced by Robinson in syntactical terms by means of existential sentences and existential types [36]. Notions of prehomogeneity and uniform prehomogeneity were introduced by Pabion [25] for multirelations (relational structures with finitely many relations); a syntactical definition is in [27]. If the profile of R takes only integer values (particularly if the signature is finite), our definitions given here are equivalent to the syntactical definitions. In this case, (a) every structure extends to an existentially closed structure with the same age; (b) R is existentially closed if and only if every local 1-embedding of R with finite domain in an extension with the same age is a local 1-isomorphism.

A characterization of prehomogeneity was given by Pabion ([25, Proposition 1, p. 530]) for multirelations. It is given in terms of complete types. With our condition below, his proof extends to structures with infinitely many relations. For more about prehomogeneity, see [28, 41, 33].

Theorem 2.1. *A relational structure R on a countable set E is prehomogeneous if and only if for each finite subset F of E there exists $\overline{F}$ finite containing F such that every local isomorphism defined on F which extends to $\overline{F}$ is a local* 1*-isomorphism.*

The following result summarizes the main properties of uniform prehomogeneity. Equivalences from (ii) to (v) are in [25, Proposition 3, p. 531], (see also [27, Proposition 3.1, p. 696]); statement (i) is new.

Theorem 2.2. *Let G be a permutation group acting on a countable set E, and R be a relational structure on E. Then the following properties are equivalent:*

- *(i) G is oligomorphic,* $\mathrm{Aut}(R) = \overline{G}^{\mathfrak{S}}$ *and* $\mathrm{Emb}(R)$*, the monoid of embeddings of R, is equal to $\overline{G}$;*
- *(ii) R is uniformly prehomogeneous and its profile takes only finite values;*
- *(iii)* (a) *every local* 1*-embedding of R with finite domain is a local* 1*-isomorphism and* (b) *for each integer n, the number of* 1*-isomorphism types of n-element subsets of R is finite;*
- *(iv) R is prehomogeneous and* $\mathrm{Aut}(R)$ *is oligomorphic;*
- *(v) R is $\aleph_0$ categorical and* $\mathrm{Th}(R)$ *is axiomatizable by universal-existential sentences.*

Proof. Given Pabion's result [25, Proposition 3, p. 531] it is actually enough to show $(i) \Rightarrow (iii)$ and $(iv) \Rightarrow (i)$. However, we also show that $(iii) \Rightarrow (i)$.

$(i) \Rightarrow (iii)$. Since G is oligomorphic, $\mathrm{Aut}(R)$ is oligomorphic too, hence (b) holds. To prove that (a) holds, let f be a local 1-embedding of R mapping a finite subset F of E onto F'. The map f extends to an embedding $\overline{f}$ from R into some extension R', such that F' has the same 1-isomorphism type in R and in R' (indeed, if $|F| = n$, add n constants to the language of R interpreted as the n elements $a_1, \dots, a_n$ of F and $f(a_1), \dots, f(a_n)$ of F'; the universal theory T of $(R, a_1, \dots, a_n)$ contains the universal theory T' of $(R, f(a_1), \dots, f(a_n))$ hence there is some extension R' of $(R, f(a_1), \dots, f(a_n))$ whose universal theory is T' (a well-known consequence of the compactness theorem of first order logic). Since $\mathrm{Aut}(R)$ is oligomorphic, the profile of R takes only finite values, hence the identity map on F' is a local 1-isomorphism of R into R'. Now, since $\mathrm{Aut}(R)$ is oligomorphic, the complete theory of R is $\aleph_0$-categorical. It follows that R is universal in the universal theory of R and thus there is an embedding g of R' into R. The map $g \circ \overline{f}$ is an embedding of R, hence, according to our hypothesis, its restriction to F is the restriction of an automorphism. It follows that this restriction is a 1-embedding, hence f is a 1-isomorphism, as claimed.

$(iii) \Rightarrow (iv)$. The condition in Theorem 2.1 is satisfied, hence R is prehomogeneous. Due to $(iii)(b)$, $\mathrm{Aut}(R)$ is oligomorphic.

$(iv) \Rightarrow (i)$. It suffices to see that $\mathrm{Emb}(R) = \overline{\mathrm{Aut}(R)}$. Without any condition, $\overline{\mathrm{Aut}(R)} \subseteq \mathrm{Emb}(R)$. Let $g \in \mathrm{Emb}(R)$. We need to show that given F, a finite subset of E, there is an automorphism $\overline{g}$ which agrees with g on f. Since the restriction $g_{\restriction F}$ of g to F extends to every finite subset of E it extends to $\overline{F}$; since R is prehomogeneous, $g_{\restriction F}$ extends to an automorphism $\overline{g}$, hence $g \in \overline{\mathrm{Aut}(R)}$. $\square$

Since our main result is on $\aleph_0$-categorical structures which have finite profile, *we consider only relational structures with finite profile.* This allows us to code restrictions of such relational structures by open formulas.

3. The number of siblings of monomorphic structures

The purpose of this section is to prove a first result that allows us to count the number of siblings based on structural properties.

Theorem 3.1. *If a countable relational structure R is monomorphic, uniformly prehomogeneous and* $\mathrm{Aut}(R)$ *is not the symmetric group, then* $\mathrm{sib}(R) = 2^{\aleph_0}$.

3.1. **Free-interpretability, chainability and monomorphy.** Let R and S be two relational structures on the same domain E. We say that R is *freely interpretable by S* if every local isomorphism of S is a local isomorphism of R. If S is a chain, we say that S *chains* R, and thus we say that R is *chainable* if some chain S chains R.

Now let p be a nonnegative integer; a relational structure R is said to be *p-monomorphic* if its restrictions to finite sets of the same cardinality p are all isomorphic; the relational structure is *monomorphic* if it is p-monomorphic for every p. Since two finite chains with the same cardinality are isomorphic, chains are monomorphic structures and hence so are chainable relational structures. Conversely, Fraïssé [9] showed that every infinite monomorphic relational structure is chainable.

We now consider three well-known structures associated to a chain $C = (E \leq)$:

- The *betweenness relation* $B_C = (E, b_C)$ *associated to* C, where b_C is the set of triples (x_1, x_2, x_3) such that either $x_1 < x_2 < x_3$ or $x_3 < x_2 < x_1$.
- The *circular order* $T_C = (E, t_C)$ *associated to* C, where t_C is the set of triples (x_1, x_2, x_3) such that $x_{\sigma(1)} < x_{\sigma(2)} < x_{\sigma(3)}$ for some circular permutation σ of $\{1, 2, 3\}$.
- The *betweenness relation* $D_C = (E, d_C)$ *associated to the circular order*, where d_C is the set of quadruples (x_1, x_2, x_3, x_4) such that $x_{\sigma(1)} < x_{\sigma(2)} < x_{\sigma(3)} < x_{\sigma(4)}$ or $x_{\sigma(4)} < x_{\sigma(3)} < x_{\sigma(2)} < x_{\sigma(1)}$ for some circular permutation σ of $\{1, 2, 3, 4\}$.

By construction, these three structures are chainable by C. Furthermore, if C is isomorphic to the chain of rational numbers, these three structures are actually homogeneous.

Moving to group properties and following Cameron [4], a group of permutations on a set E is *monomorphic* if it has just one orbit for every n-element set (another terminology is *set-homogeneous*). Cameron proved that on a countable set there are essentially five monomorphic closed groups:

Theorem 3.2 ([4]). *A monomorphic closed group on a countable set is isomorphic, as a permutation group, to one of the following groups:*

(a) $\mathfrak{S}(\mathbb{Q})$*, the full symmetric group on the set of rationals;*
(b) $\mathrm{Aut}(\mathbb{Q})$*, the automorphism group of the chain of rational numbers;*
(c) $\mathrm{Aut}(B_{\mathbb{Q}})$ *the automorphism group of the betweenness relation associated to the chain of rational numbers;*
(d) $\mathrm{Aut}(T_{\mathbb{Q}})$ *the automorphism group of the circular order associated with the chain of rational numbers;*
(e) $\mathrm{Aut}(D_{\mathbb{Q}})$ *the automorphism group of the betweenness relation associated to the circular order on the rationals.*

We will need the following consequence of Theorem 3.2 in the proof of Lemma 4.18 (a).

Lemma 3.3. *A descending chain of monomorphic closed groups on a countable set has at most four terms.*

Proof. It suffices to show that if R and R' are two relational structures on the same set E such that $\mathrm{Aut}(R') \subseteq \mathrm{Aut}(R)$ and R isomorphic to R', then $\mathrm{Aut}(R) = \mathrm{Aut}(R')$. This conclusion in fact simply follows if $\mathrm{Aut}(R)$ and $\mathrm{Aut}(R')$ are oligomorphic. Indeed, for each integer n, the partition of E^n into orbits for the action of $\mathrm{Aut}(R')$ is included in the partition of E^n into orbits for the action of $\mathrm{Aut}(R)$. Since these groups are oligomorphic and isomorphic as permutation groups, these partitions have finitely many classes and have the same number of classes, hence are equal. The fact that these groups are equal follows. □

Cameron's theorem implies that every monomorphic closed group G on a countable set is the automorphism group of some relational structure R chainable by a chain isomorphic to the chain of rational numbers. In fact, it implies that every R such that $\mathrm{Aut}(R) = G$ has this property, and thus the following result.

Theorem 3.4. *Let R be a countable structure; then the following properties are equivalent:*

(i) R is chainable by a chain isomorphic to the chain of rational numbers;
(ii) R is monomorphic and uniformly prehomogeneous;
(iii) $\mathrm{Aut}(R)$ is monomorphic.

Theorem 3.4 was proved in [32] (see 2.6 and 2.7 and line 18 of p. 321) by direct arguments. It was a step in a proof of Cameron's theorem based on Frasnay's result. We outline a proof.

Proof. $(i) \Rightarrow (iii)$. If R is chainable by the chain of rational numbers then $\mathrm{Aut}(R)$ is an overgroup of $\mathrm{Aut}(\mathbb{Q})$ hence it is monomorphic.
$(iii) \Rightarrow (ii)$. If $\mathrm{Aut}(R)$ is monomorphic then trivially, R is not only monomorphic but any two finite subsets of the same size are 1-isomorphic. Thus to show that R is uniformly prehomogeneous, it suffices by Theorem 2.2 to show that every local 1-embedding f with finite domain F included in the domain E of R is invertible by a local 1-embedding. Indeed, let F' be the image of F. Since $\mathrm{Aut}(R)$ is monomorphic, there is an automorphism, say σ, which carries F' onto F. Evidently, σ is a 1-local embedding, hence $\sigma_{\restriction F'} \circ f$ is a 1-local embedding; furthermore all the iterates of that map are 1-local embeddings. Since F is finite, an nth iterate is the identity on F, hence $f^{-1} = (\sigma \circ f)^{n-1} \circ \sigma$ is a 1-local embedding as claimed.
$(ii) \Rightarrow (i)$. Part of the argument is based on the following fact about free interpretability. Let R and S have the same base and let μ and ν be the respective signatures of R and S. If the signature ν is finite, then R is freely interpretable by S if and only if there exists a map P associating to every relational structure S' of signature ν a relational structure R' of signature μ on the same domain in such a way that (a) $P(S) = R$ and (b) every local isomorphism f of S' into S'' is a local isomorphism of $P(S')$ into $P(S'')$ (see Fraïssé [9]). Now the proof of the implication goes as follows. Suppose that R is monomorphic, then R is chainable by some chain, say C. The free operator transforming C into R will transform $\mathbb{Q}$ into some structure R'. It turns out that R' is isomorphic to R. Indeed, according to the implication $(i) \Rightarrow (ii)$, already proven, R' is uniformly prehomogeneous; it has the same age as R which is uniformly prehomogeneous, hence it is isomorphic to R (this is essentially the argument in [32, 2.5 Lemme de préservation, p. 320]). Hence R is chainable by some chain D isomorphic to the chain of rationals. □

Remark: If R is only known to be chainable, then it does not follow that $\mathrm{Aut}(R)$ is monomorphic, even if $\mathrm{Aut}(R)$ is oligomorphic as the example $R = 1 + \mathbb{Q}$ shows.

3.2. **Group-sequences, bichains, and indicative sequences.**

3.2.1. *Group-sequences.* Let R be a chainable relational structure with domain E and C be a chain (with same domain E) chaining R. Let n be an integer, $n \leq |E|$, let A be a n-element subset of E and c_A be the unique isomorphism of the natural chain on $\underline{n} = \{1, \ldots, n\}$ onto $C_{\restriction A}$. The set of permutations σ of $\underline{n}$ of the form $c_A^{-1} \circ \tau \circ c_A$ for $\tau \in \mathrm{Aut}(R_{\restriction A})$ forms a group. Since C chains R, this group is independent of the n-element set A and we denote it by $\mathrm{Ind}_n(R, C)$. The sequence of these groups is called the *group-sequence* of the pair (R, C).

For each positive integer n we define the following permutation groups on $\underline{n}$:

- $\mathfrak{S}(n)$ consisting of all permutations;
- $\mathfrak{I}(n)$ consisting of only the identity;
- $\mathfrak{J}(n)$ consisting of the identity and the reversal r transforming each k into $n - k + 1$;
- $\mathfrak{T}(n)$ consisting of circular permutations;
- $\mathfrak{D}(n)$ consisting of the product of $\mathfrak{T}(n)$ and $\mathfrak{J}(n)$, i.e., $\mathfrak{D}(n)$ is the dihedral group.

Let $\mathfrak{S}$, $\mathfrak{I}$, $\mathfrak{J}$, $\mathfrak{T}$, $\mathfrak{D}$ each be the sequence of the above corresponding groups for $n \in \mathbb{N}$. Then clearly we have the following result connecting these sequences and our previous structures.

Lemma 3.6. *The sequences* $\mathfrak{S}$, $\mathfrak{I}$, $\mathfrak{J}$, $\mathfrak{T}$, $\mathfrak{D}$ *are the sequences* $(\mathrm{Ind}_n(R, \mathbb{Q}))_{n \in \mathbb{N}}$ *where* R *is successively* $(\mathbb{Q}, =)$, $(\mathbb{Q}, \leq)$, $B_{\mathbb{Q}}$, $T_{\mathbb{Q}}$, *and* $D_{\mathbb{Q}}$.

3.2.2. *Bichains and their indicative sequences.* As before, let R be a chainable relational structure with domain E and $C = (E, \leq)$ be a chain chaining R. We may observe that for every embedding φ of R into R, the inverse image of $\leq$ by φ again provides a chain chaining R. Frasnay studied the relationship between two chains chaining the same structure, and we briefly recall some elements of his theory (for more, see [10], [11], and Fraïssé [9]).

A *bichain* is a relational structure with two linear orders on the same set. To each bichain we associate a sequence of permutations groups, called the *indicative sequence* of the bichain. Consider a bichain $B = (E, \leq_0, \leq_1)$, and set each component as $B_i = (E, \leq_i)$ for $i = 0, 1$. Let n be a positive integer no larger than the cardinality of E and let A be an n-element subset of E. The chains $B_0 \restriction A$ and $B_1 \restriction A$ are isomorphic via a unique permutation h of A which transforms the first to the second; if we order A into the sequence $a_1 <_0 \cdots <_0 a_n$, there is a unique permutation σ of $\underline{n} = \{1, \ldots n\}$ which reorders it into $a_{\sigma(1)} <_1 \cdots <_1 a_{\sigma(n)}$, and satisfies $h(a_k) = a_{\sigma(k)}$ for $k \in \{1, \ldots, n\}$. The collection of these permutations σ for n fixed and A

belonging to all the n-element subsets of E, generates a subgroup $\mathrm{Ind}_n(B)$ of $\mathfrak{S}(n)$, called the *nth indicative group of* B. The sequence of these indicative groups is the *indicative sequence* of B. We can now recall the following result of Frasnay [11, Lemme, p. 263].

Theorem 3.7 ([11]). *Let $B = (E, \leq_0, \leq_1)$ be a bichain. If B_0 has no minimum and no maximum then the indicative sequence of B is one of the five sequences $\mathfrak{S}$, $\mathfrak{I}$, $\mathfrak{J}$, $\mathfrak{T}$, $\mathfrak{D}$ listed above.*

This together with Lemma 3.6 yields the following.

Corollary 3.8. *The indicative sequence of a bichain whose components have no extreme elements is the group-sequence of a homogeneous monomorphic countable structure.*

Recall that a chain is *scattered* if it does not embed the chain of the rationals.

Lemma 3.9. *Let $B = (E, \leq_0, \leq_1)$ be a bichain such that B_0 is nonscattered and B_1 is scattered. Then the indicative sequence of B is $\mathfrak{S}$.*

Proof. We will make use of the following.

Claim 3.10. *There is a subset A of E such that $B_0 \restriction A$ is isomorphic to the chain of rational numbers, and $B_1 \restriction A$ is isomorphic to either ω or ω^*.*

Proof of Claim 3.10. This readily follows from a famous unpublished result of Galvin, expressing that if the pairs of rational numbers are divided into finitely many classes, then there is a subset of the rationals which is isomorphic to the rationals and such that all pairs are contained in the union of at most two classes; for a proof see Todorcevic [45, Theorem 6.3, p. 44], or Vuksanovic [47].

Indeed, pick a subset E' of E such that $B_0 \restriction E'$ is isomorphic to the rationals and let $\leq_2$ be an ordering of E' in type ω. Distribute the pairs (x, y) of E' with $x <_0 y$ into four classes according to how x and y compare with $\leq_1$ and $\leq_2$. Galvin's theorem yields a subset A of E' such that $B_0 \restriction A$ is isomorphic to the rationals and $B_1 \restriction A$ either agrees with $\leq_2$ or its reverse, hence either of type ω or ω^*. $\square$

We can thus assume that $B = (E, \leq_0, \leq_1)$ is a bichain where B_0 is isomorphic to the rationals and B_1 is isomorphic to ω. Under this assumption, we have the following.

Claim 3.11. *For each integer n, the set of permutations σ of $\underline{n}$ which reorders an n-element ordered set $a_1 <_0 a_2 <_0 \cdots <_0 a_n$ of A into $a_{\sigma(1)} <_1 a_{\sigma(2)} <_1 \cdots <_1 a_{\sigma(n)}$ is the full symmetric group $\mathfrak{S}(n)$.*

With these claims, it follows that the indicative sequence of (the original) B is $\mathfrak{S}$ as required.

Proof of Claim 3.11. We proceed by induction on n. Let $\sigma \in \mathfrak{S}(n)$ and let $i = \sigma(n)$. It suffices to consider the case $1 < i < n$, and by induction we may find an $n-1$-element ordered set $a_1 <_0 a_2 <_0 \cdots <_0 a_{i-1} <_0 a_{i+1} <_0 \cdots <_0 a_n$ of A such that $a_{\sigma(1)} <_1 a_{\sigma(2)} <_1 \cdots <_1 a_{\sigma(n-1)}$. Since the interval (a_{i-1}, a_{i+1}) in B_0 is infinite and there exist only finitely many elements less than $a_{\sigma(n-1)}$ in B_1, we may find a_i such that $a_{i-1} <_0 a_i <_0 a_{i+1}$ and $a_{\sigma(n-1)} <_1 a_i$. Then σ reorders this n-element set as required for the claim. □

This completes the proof of Lemma 3.9. □

From this, we can deduce the following which is key to our structural result.

Theorem 3.12. *Let G be a monomorphic closed group on a countable set. Then all relational structures R such that $\mathrm{Aut}(R) = G$ have the same number of siblings: this number is 1 if G is the full symmetric group, and $2^{\aleph_0}$ otherwise.*

Proof. Suppose first that R is one of the the five previously listed homogeneous relational structures defined on $\mathbb{Q}$. If R is the equality relation, there is just one sibling. If R is one of the four others, we prove that there are $2^{\aleph_0}$ nonisomorphic siblings. For that, we define subsets C_s of $\mathbb{Q}$ for each $s \in \{0,1\}^{\mathbb{N}}$ such that the restrictions $R_{\restriction C_s}$ are equimorphic to R and pairwise nonisomorphic. The structure of the C_s's is such that an isomorphism of some $R_{\restriction C_s}$ onto some $R_{\restriction C_{s'}}$ will necessarily be an isomorphism from the chain C_s onto the chain C'_s and hence $s = s'$. Each C_s is the union of three sets A_0, A_1^s, A_2, where A_0 and A_2 are respectively a nonempty initial and final segment of $\mathbb{Q}$ without a largest element and a least element, and A_1^s is a scattered chain of the form $\sum_{n<\omega} C_n^{s(n)}$, where $C_n^{s(n)}$ is a chain of order type ω (resp. ω^*), if $s(n) = 0$ (resp. $s(n) = 1$). It can be easily verified that distinct sequences provide nonisomorphic chains. But now if R is any of the other four homogeneous structures, then each structure $R\restriction C_s$ is a sibling of R, and an isomorphism from $R\restriction C_s$ onto $R\restriction C_{s'}$ has to be an order isomorphism from A_1^s onto $A_1^{s'}$ or its reverse; the first case happens only if $s = s'$ while the second case never happens due to the form of A_1^s and $A_1^{s'}$. Hence $\mathrm{sib}(R) = 2^{\aleph_0}$ as required.

Next we deal with the general case. According to Theorem 3.4 we may suppose that R is chainable by the chain $C = (\mathbb{Q}, \leq)$ of rational numbers and that $G = \mathrm{Aut}(M)$ for some of the homogeneous relations occuring in Cameron's theorem; we show that $\mathrm{sib}(R) = \mathrm{sib}(M) = 2^{\aleph_0}$. If the restrictions of M to two subsets A and A' are isomorphic, then the restrictions $R_{\restriction A}$ and $R_{\restriction A'}$ are isomorphic. From this, and the fact that the embeddings of R coincide with the embeddings of M, it follows that $\mathrm{sib}(R) \leq \mathrm{sib}(M)$.

Conversely, suppose that $R_{\restriction A}$ and $R_{\restriction A'}$ are isomorphic. It suffices to prove the following.

Claim 3.13. *Every isomorphism f of $R_{\restriction A}$ onto $R_{\restriction A'}$ is a local isomorphism of M provided that A, as a subset of $\mathbb{Q}$, has no extreme elements.*

Indeed, to conclude the proof using the claim, if we take $2^{\aleph_0}$ subsets C_s of $\mathbb{Q}$ with no extreme elements such that their restrictions to M are pairwise nonisomorphic and equimorphic to M, as we did above, then by the claim the restrictions of R to these will also yield pairwise nonisomorphic and equimorphic structures to R, and hence $2^{\aleph_0} = \mathrm{sib}(M) \leq \mathrm{sib}(R)$.

Now toward proving the claim, consider the nth indicative group $\mathrm{Ind}_n(B)$ associated to the bichain $B = (A, \leq_A, \leq'_A)$, where $\leq'_A$ is the image of $\leq_{A'}$ by f^{-1}, and also consider the group-sequences $\mathrm{Ind}_n(M, \mathbb{Q})$ and $\mathrm{Ind}_n(R, \mathbb{Q})$. In order to prove the claim it suffices to prove the following:

Claim 3.13.a. $\mathrm{Ind}_n(B) \subseteq \mathrm{Ind}_n(M, \mathbb{Q})$ *for each integer n.*

Indeed, let A_n be an n-element subset of A, let σ be the permutation of $\{1, \dots n\}$ such that if $a_1 <_A \cdots <_A a_n$ is an enumeration of A_n, then the sequence $a'_1 <_{A'} \cdots <_{A'} a'_n$ with $a'_i = f(a_{\sigma(i)})$ provides an enumeration of $A'_n = f(A_n)$. Then by definition $\sigma \in \mathrm{Ind}_n(B)$, and thus if the subclaim holds we have $\sigma \in \mathrm{Ind}_n(M, \mathbb{Q})$, and hence $\sigma^{-1} \in \mathrm{Ind}_n(M, \mathbb{Q})$ as well. Now let t be the unique order-isomorphism from $f(A_n)$ onto A_n and define $g = t \circ f_{\restriction A_n}$; this map is represented on $\{1, \dots n\}$ by σ^{-1} hence it is an automorphism of $M_{\restriction A_n}$. It follows that f induces an isomorphism from $M_{\restriction A_n}$ onto $M_{\restriction A'_n}$ from which follows that f is a local isomorphism of M.

Proof of Claim 3.13.a. We have easily $\mathrm{Ind}_n(B) \subseteq \mathrm{Ind}_n(R, \mathbb{Q})$. According to Frasnay's Theorem 3.7 above, $(\mathrm{Ind}_n(B))_n$ is the group-sequence of some homogeneous structure belonging to the Cameron list, and thus let M' be such a structure with domain $\mathbb{Q}$. We have $\mathrm{Ind}_n(M', \mathbb{Q})) \subseteq \mathrm{Ind}_n(R,\ Q)$, which implies $\mathrm{Aut}(M') \subseteq \mathrm{Aut}(R)$. But now since $\mathrm{Aut}(R) = \mathrm{Aut}(M) = G$ the subclaim follows. □

This completes the proof of Theorem 3.12. □

Proof of Theorem 3.1. Let R be monomorphic and uniformly prehomogeneous such that $\mathrm{Aut}(R)$ is not the symmetric group. Then, according to Theorem 3.4, $\mathrm{Aut}(R)$ is monomorphic, and thus $\mathrm{sib}(R) = 2^{\aleph_0}$ by Theorem 3.12. □

4. Monomorphic decomposition of a relational structure

In this section, we extend some notions of the previous section bringing the concept of the monomorphic decomposition of a relational structure into play, and we specialize it to permutation groups. This notion was introduced in [35] and will form a main tool in this work. Our presentation follows [23], see [24, Chapter 7] for details.

Let R be a relational structure on a set E. A subset E' of E is a *monomorphic part* of R (or a monomorphic block) if for every integer k and every pair

A, A' of k-element subsets of E, the induced structures on A and A' are isomorphic whenever $A \setminus E' = A' \setminus E'$ (we do not require that an isomorphism of A onto A' sends $A \setminus E'$ onto $A' \setminus E'$). A *monomorphic decomposition* of R is a partition of E into monomorphic parts. A monomorphic part which is maximal for inclusion is called a *monomorphic component* of R, and taken together form a monomorphic decomposition of R of which every monomorphic decomposition of R is a refinement ([35, Proposition 2.12]).

This partition can also be defined in a direct way as follows, see [34]. For two elements x and y of E and F a finite subset of $E \setminus \{x, y\}$, we say that x and y are *F-equivalent*, written $x \simeq_{F,R} y$, if the restrictions of R to $\{x\} \cup F$ and $\{y\} \cup F$ are isomorphic (we do not require that an isomorphism of $\{x\} \cup F$ onto $\{y\} \cup F$ sends x to y). For k a nonnegative integer, we set $x \simeq_{k,R} y$ if $x \simeq_{F,R} y$ for every k-element subset F of $E \setminus \{x, y\}$. We set $x \simeq_{\leq k,R} y$ if $x \simeq_{k',R} y$ for every $k' \leq k$ and $x \simeq_R y$ if $x \simeq_{F,R} y$ for every finite set F. The following property holds ([23]; for a proof, see Lemma 7.48 and Lemma 7.49 in Section 7.2.5 of [24]).

Lemma 4.1. *The relations $\simeq_{k,R}$, $\simeq_{\leq k,R}$, and $\simeq_R$ are equivalence relations on E. Furthermore, the equivalence classes of $\simeq_R$ are the components of R.*

From the definition of these equivalence, we deduce:

Lemma 4.2. *Let R be a relational structure with base E and R' be the restriction of R to a subset E' of E. If $|E' \cap C| \geq \mathrm{Min}\{k+2, |C|\}$ for every equivalence class C of $\simeq_{\leq k,R}$ then $\simeq_{\leq k,R'}$ coincide with the restriction of $\simeq_{\leq k,R}$ to E'.*

Proof. Clearly, the restriction to E' of $\simeq_{\leq k,R}$ is included into $\simeq_{\leq k,R'}$. For the converse, let $x, y \in E'$ such that $x \simeq_{\leq k,R'} y$. This means that for every subset F' of $E' \setminus \{x, y\}$ with at most k elements, the restrictions of R to $\{x\} \cup F'$ and $\{y\} \cup F'$ are isomorphic. Let F be a subset of $E \setminus \{x, y\}$ with at most k elements. Since E' keeps at least $k + 2$ elements of each equivalence class of $\simeq_{\leq k,R}$, we may find a subset F' of $E' \setminus \{x, y\}$ such that $|F' \cap C| = |F \cap C|$ for every equivalence class C of $\simeq_{\leq k,R}$. As in the proof of [35, Lemma 2.10] we may transform $\{x\} \cup F$ into $\{x\} \cup F'$ by adding and removing one element at a time, from which follows that the restrictions of R to $\{x\} \cup F'$ and to $\{x\} \cup F$ are isomorphic. Similarly, the restriction of R to $\{y\} \cup F'$ and to $\{y\} \cup F$ are isomorphic. It follows that the restrictions of R to $\{x\} \cup F$ and to $\{y\} \cup F$ are isomorphic. Hence, $x \simeq_{\leq k,R} y$ as required. $\square$

Lemma 4.3. *Let R be a relational structure with base E, then there is an integer k such that the equivalence relations $\simeq_{\leq k,R}$ and $\simeq_R$ coincide whenever*

(1) $\simeq_R$ has finitely many classes or
(2) $\mathrm{Aut}(R)$ has finitely many orbits of pairs.

Proof. (1) Let ℓ be the number of equivalence classes of $\simeq_R$. Pick an element x_i in each class X_i, $i < \ell$. For $i \neq j$ there is a finite set $F_{i,j} \subseteq E \setminus \{x_i, x_j\}$ such that the restrictions of R to $\{x_i\} \cup F_{i,j}$ and $\{x_j\} \cup F_{i,j}$ are not isomorphic. Set $k := \mathrm{Max}\{|F_{i,j}| : i, j < \ell\} + 1$. (2) For each orbit C of a pair $\{x, y\}$ such that $x \not\simeq_R y$, witness this fact by selecting a finite subset $F_C \subseteq E \setminus \{x, y\}$. Let k be the maximality of $|F_C| + 1$ where C runs trough these orbits. $\square$

A consequence of item (1) of Lemma 4.3 is the following result ([35, Lemma 2.15]) obtained by a more complicated argument.

Lemma 4.4. *If a relational structure R on a set E has a finite monomorphic decomposition, then there is an integer d such that every finite subset F is contained in a finite subset F' with $|F' \setminus F| \leq d$ and such that the monomorphic decomposition of $R_{\restriction F'}$ into components is induced by the decomposition of R into components.*

Note that in the case of binary structures or of ordered structures there is a threshold phenomenon indicated below. But, using a result of [30] one can show that there is no threshold for ternary relations.

Lemma 4.5. *The equivalences relations $\simeq_{\leq 6,R}$ and $\simeq_R$ coincide on a binary structure. If R is a directed graph, resp. an ordered graph, we may replace 6 by 3, resp. by 2. If T is a tournament, the number of equivalences classes of $\simeq_{\leq 3,T}$ is finite provided that the number of equivalence classes of $\simeq_{\leq 2,T}$ is finite. There is an integer $i(m)$ such that on an ordered structure of arity at most m the equivalences relations $\simeq_{\leq i(m),R}$ and $\simeq_R$ coincide.*

The case of binary structures follows from a reconstruction result of Lopez [18, 19]. The case of directed graphs was obtained by Oudrar, Pouzet [23], and independently Boudabbous [3]. The case of ordered structures follows from a result of Ille [15].

This notion of equivalence is particularly well-adapted for permutation groups. Let G be a permutation group acting on a set E, and x and y be two elements of E. Set $x \simeq_G y$ if for every finite subset F of $E \setminus \{x, y\}$, the sets $\{x\} \cup F$ and $\{y\} \cup F$ are in the same G-orbit. Now if R is a homogeneous relational structure on E such that $\mathrm{Aut}(R) = \overline{G}^{\mathfrak{S}}$, then clearly $x \simeq_G y$ if and only if $x \simeq_R y$. From this simple observation follows that the relation $\simeq_G$ is an equivalence relation. We call the equivalence classes, the *G-monomorphic components*.

An immediate consequence of (2) of Lemma 4.3 is this:

Corollary 4.6. *If the automorphism group of a relational structure R is oligomorphic then for some nonnegative integer k the equivalence relations $\simeq_{\leq k,R}$ and $\simeq_R$ coincide, hence $\simeq_R$ is definable by a universal formula with at most k universal quantifiers.*

A crucial use of this notion of equivalence is illustrated by the following lemma

Lemma 4.7. *Let R be a relational structure on a set E. Suppose that there is some nonnegative integer k such that the equivalence relations $\simeq_{\leq k,R}$ and*

$\simeq_R$ coincide and furthermore that the size of finite equivalence classes is bounded by some integer ℓ. If D is an infinite equivalence class, resp., a countable union of infinite equivalence classes, then there are $\aleph_0$, resp., $2^{\aleph_0}$, pairwise nonisomorphic restrictions of R of the form $R_{\restriction E\setminus X}$ where X is a subset of D.

Proof. Suppose that D is an equivalence class. Pick in D infinitely many finite subsets B_n with different sizes larger than $\mathrm{Max}\{k,\ell\}+1$. According to Lemma 4.2, B_n is a monomorphic component of $R_n := R_{\restriction (E\setminus D)\cup B_n}$. Since the decompositions of R_n and R_m into monomorphic components do not yield the same sequence of cardinality classes, these structure are not isomorphic. Suppose that D is a countable union of classes, say $C_0,\dots,C_n,\dots$. In each C_n, pick a finite set B_n with size larger than $\mathrm{Max}\{k,\ell\}+1$. Let $s := \{|B_n| : n<\omega\}$. According to Lemma 4.2, the B_n's are monomorphic components of $R_s := R_{\restriction (E\setminus D)\cup\bigcup_{n<\omega} B_n}$. If s and s' are two different sequences (up to permutations) the monomorphic decompositions of R_s and $R_{s'}$ do not yield the same sequence of cardinality classes hence R_s and R'_s are not isomorphic. Since the number of sequences s as above is $2^{\aleph_0}$, the conclusion follows. □

We do not claim that the restrictions of R in Lemma 4.7 are siblings. We will show in Section 6 that if R is countable, uniformly prehomogeneous, with infinitely many infinite classes one may select a countable union of infinite equivalence classes C such that R is embeddable into $R_{\restriction E\setminus C}$. With this lemma, we get that R has $2^{\aleph_0}$ siblings.

A variant of these notions is of interest to us. A subset E' of E is a *strongly monomorphic part* of R if for every integer k and every pair A, A' of k-element subsets of E' there is an isomorphism of $R_{\restriction A}$ to $R_{\restriction A'}$ which can be extended by the identity on $E\setminus E'$ to a local isomorphism of R. A *strongly monomorphic component* is a strongly monomorphic part which is maximal with respect to inclusion (which may not be a monomorphic component). A *strongly monomorphic decomposition* of R is a partition of E into strongly monomorphic parts. Also, call E' a *chainable part* of R if there is a linear order $\le$ on E' such that every local isomorphism of $(E',\le)$ extended by the identity on $E\setminus E'$ is a local isomorphism of R.

A strengthening of the model theoretic notion of indiscernibility plays a natural role in our context. We say that a subset E' of E is a *strongly indiscernible subset* of R if for every integer k and every pair A, A' of k-element subsets of E' every bijective map from A to A' can be extended by the identity on $E\setminus E'$ to a local isomorphism of R. This amounts to saying that every permutation of E' can be extended by the identity on $E\setminus E'$ to an automorphism of R.

The following proposition assembles several properties relating these notions.

Proposition 4.8. *(a) Every strongly monomorphic part is a monomorphic part.*

(b) Every strongly monomorphic part is contained in a maximal one, which extends to a monomorphic component.

(c) There is a strongly monomorphic decomposition of R from which every other is finer; it is made of strongly monomorphic components.

(d) A chainable part is a strongly monomorphic part.

(e) The converse holds for infinite strongly monomorphic parts, furthermore:

(f) Every infinite monomorphic component is a strongly monomorphic component (and a chainable part).

The proofs of the first four items are immediate or easy. The proof of item (e) uses compactness and Ramsey's theorem via Fraïssé's theorem on chainability (Theorem 4.9) given below; the proof of item (f) is implication $(iii) \Rightarrow (i)$ of [35, Theorem 2.25, p.17] and uses properties of $\mathrm{Ker}(R)$, the *kernel* of R (the set of x of the base E of R such that $\mathrm{age}(R_{-x})$ is distinct from $\mathrm{age}(R)$).

Theorem 4.9 (Fraïssé). *Let $R = (E, (\rho_i)_{i\in I})$ be a relational structure on an infinite set E, F a finite subset of E and $\leq$ a linear order on $E \setminus F$. Then for each finite subset I' of I there is an infinite subset X of $E \setminus F$ such that X is a chainable part of the reduct $R^{I'}_{\restriction F\cup X}$ (and the linear order $\leq$).*

From item (c) of Proposition 4.8, the existence of a finite monomorphic decomposition is equivalent to the existence of a finite strongly monomorphic decomposition; this is also equivalent to the existence of a linear order on E and a partition of E into finitely many intervals such that every partial map which preserves the order on each interval is a local isomorphism of R.

We now come to a key tool we will use to estimate the number of siblings.

Lemma 4.10. *Let R be a relational structure with domain E and $n \in \mathbb{N}$. Then:*

1. *(1) The equivalence relations defining the monomorphic components of R are preserved by every member of* $\mathrm{Aut}(R)$.
2. *(2) If the number of orbits of singletons with respect to* $\mathrm{Aut}(R)$ *is finite then the set S of integers k such that some monomorphic component has cardinality k is finite.*
3. *(3) If R is* 1*-homogeneous (that is, two elements x, y such that $R_{\restriction\{x\}}$ and $R_{\restriction\{y\}}$ are isomorphic belong to the same orbit), then the orbit of any $x \in E$ is a union of monomorphic components of R, and all those components have the same cardinality.*
4. *(4) If R is prehomogeneous, then every infinite monomorphic component is contained in the orbit of some singleton.*
5. *(5) If R is homogeneous, then the equivalences $\simeq_R$ and $\simeq_{Aut(R)}$ coincide.*

Proof. (1) Being definable (by infinitary formulae), the monomorphic decomposition is preserved under automorphisms. (2) If two elements are in the same orbit of $\mathrm{Aut}(R)$, then the monomorphic component containing x and the monomorphic component containing y have the same size, hence (2) follows. (3) If two elements are in the same monomorphic component, the restriction of R to these elements is isomorphic; if R is 1-homogeneous, then there is some automorphism carrying one onto the other. (4) Let x, y be in the same monomorphic component X. Since R is prehomogeneous, there is some finite set F_x containing $\{x\}$ such that every local isomorphism defined on $\{x\}$ which can be extended to F_x can be extended to an automorphism. Now, since X is infinite, then by Proposition 4.8 X is strongly monomorphic and hence chainable over E by some chain $(X, \leq)$. Set $F'_x = F_x \cap X$. Now, every local isomorphism of $(X, \leq)$ defined on X and extendable by the identity on $E \setminus X$ will carry x onto some x' belonging to the same orbit, and the set S_x of elements of X which cannot be attained from x in this manner (if any) is by chainability the union of an initial interval and a final interval of X whose size is at most $|F'_x| - 1$. The same reasoning with y in place of x yields a set F'_y of size at most $|F'_y| - 1$. Since those sets are finite and X is infinite, there are elements which can be reached from x and y, and hence x can the transformed to y by some automorphism as required. (5) The fact that $\simeq_{\mathrm{Aut}(R)}$ is included in $\simeq_R$ holds with no condition on R; the homogeneity of R is used for the converse. □

Remark: Consider, as a comparative example, the direct sum of infinitely many copies of a 2-element chain. It is uniformly prehomogeneous, and the automorphism group has two orbits of singletons: the set of maximal elements and the set of minimal elements. The monomorphic components are the 2-element chains and none are contained in an orbit.

We now revisit the action of a group on a set. Let G be a permutation group acting on a set E. For A a subset of E, we denote by $G_{\underline{A}}$, resp. G_A, the pointwise, resp. setwise stabilizer of A. If G leaves A globally invariant (i.e., $G = G_A$), we set $G\restriction A = \{\sigma\restriction A : \sigma \in G\}$.

We first deal with prehomogeneous structures.

Proposition 4.12. *Let R be a prehomogeneous structure on a countable set E, $G = \mathrm{Aut}(R)$, and let A be an infinite monomorphic component of R with $B = E \setminus A$ its complement. Then*

(a) $G_B\restriction A$ is a monomorphic group;

(b) If G is oligomorphic, then $G_{\underline{B}}\restriction A$ is also oligomorphic, and there is a dense linear order on A such that for $C = (A, \leq)$, $\mathrm{Aut}(C) \subseteq G_{\underline{B}}\restriction A$ and hence $G_{\underline{B}}\restriction A$ is monomorphic.

Proof. (a) We must show that for every integer $n \geq 1$, if F_1 and F_2 are two n-element subsets of A, then there is some $\sigma \in G_B\restriction A$ which carries F_1 onto F_2. Let $\overline{F}_i$ be finite and containing F_i such that local embeddings defined on F_i which extend to $\overline{F}_i$ extend to automorphisms of R. Since A is an infinite monomorphic component, hence strongly monomorphic by Proposition 4.8, there is a linear order $\leq$ on A such that finite local isomorphisms of $(A, \leq)$

extend by the identity on B to local isomorphisms of R. Since A is infinite, we may find an n-element subset F in A such that, via order preserving mappings on $\overline{F}_i \cap A$, F_i is carried to F by an isomorphism which extends to a local isomorphism fixing B pointwise (and in particular $\overline{F}_i \cap B$). Since R is prehomogeneous, this provides an automorphism σ_i which carries F_i to F for $i = 1, 2$. Now, $\sigma_2^{-1} \circ \sigma_1$ carries F_1 to F_2 and is an automorphism. Since automorphisms must preserve the equivalence relation $\simeq_R$ and $F_i \subseteq A$, for $i = 1, 2$, this automorphism fixes A set wise. Hence it fixes B set wise, that is belongs to G_B. Without invoking a stronger condition, e.g., oligomorphic action as in (b), we have not been able to show that there is an automorphism carrying F_1 on F_2 and fixing B pointwise.

(b) Once we know that $G_{\underline{B}}\restriction A$ is oligomorphic, the existence of a dense order on A follows from Theorem 3.4. In fact, we prove directly the existence of a dense order as follows. On each infinite component A_i, we may put a linear order $\leq_i$ in such a way that the local isomorphisms of $(A_i, \leq_i)$ extended by the identity on the complement of A_i are local isomorphism of R (Proposition 4.8). Extend each infinite component A_i to a set A'_i and $\leq_i$ to a dense order $\leq'_i$ in such a way that for distinct i's the A'_i's are disjoint. We may extend R to a relation R' on $E' = E \cup \bigcup_i A'_i$ in such a way that any 1–1 map of finite domain $F \subseteq E'$ which sends each $A'_i \cap F$ into A_i and respect the order and fixes all other elements is a local isomorphism from R into R'. The extension R' has the same age as R. Since R is prehomogeneous, R' is a 1-extension of R. Furthermore, if R'' is an extension of R' with the same age, this is a 1-extension, that is R' is existentially closed. Since G is oligomorphic, R is the unique countable existentially closed structure for its age, hence R' is isomorphic to R. Clearly, $\mathrm{Aut}(A'_i, \leq'_i) \subseteq \mathrm{Aut}(R'_{\underline{B'_i}}\restriction A'_i)$. Any isomorphism will transform the A'_i's into the A'_i's, and hence the image of the dense orders will give dense orders on the A_i's with the required property. □

Clearly, the fact that $G_{\underline{B}}\restriction A$ is monomorphic implies that $G_B\restriction A$ is monomorphic. We do not know if the hypothesis of oligomorphy is really needed to prove the converse. In Proposition 4.18, we show that the fact that R is homogeneous suffices.

The following properties are folklore and straightforward.

Lemma 4.13. *Let G be a group acting on a set E. If G is closed in $\mathfrak{S}(E)$ then, for every subset A of E, the groups G_A and $G_{\underline{A}}$ are closed in $\mathfrak{S}(E)$ and the group $G_{\underline{A}}\restriction (E \setminus A)$ is closed in $\mathfrak{S}(E \setminus A)$. Provided that A or $E \setminus A$ is finite, the group $G_A\restriction (E \setminus A)$ is closed in $\mathfrak{S}(E \setminus A)$.*

Remark: Without some condition on A, $G_A\restriction (E \setminus A)$ is not necessarily closed. For an example, let $R = (\mathbb{Q}, \leq, U)$ where $\leq$ is the natural order on the rationals and U is a unary relation which divides the rationals into two dense sets. Let $G = \mathrm{Aut}(R)$ and $A = \{x \in \mathbb{Q} : U(x) = 1\}$. Then $G_{\restriction A}$ is different from $\overline{G_{\restriction A}}^{\mathfrak{S}}$, the closure of $G_{\restriction A}$ into $\mathfrak{S}$. Indeed, since R is homogeneous, the latter group is equal to $\mathrm{Aut}(R_{\restriction A})$. This group contains permutations which cannot extend to $\mathbb{Q}$; indeed if we choose q with $U(q) = 0$ and an irrational r we may find $\sigma \in \mathrm{Aut}(R_{\restriction A})$ whose extension carries q to r; this map σ cannot be extended to $\mathbb{Q}$. In this example, $G_{\restriction A}$ is monomorphic, hence oligomorphic, but not closed. The set A is invariant under the action of G, but it is not a monomorphic component of R; in fact we may separate every pair of distinct elements x and y by some subset F of $\mathbb{Q}$ with at most two elements.

However, there is a powerful duality for a permutation group acting on two globally invariant sets.

Lemma 4.15. *Let G be a permutation group acting on a set E which is the union of two disjoint sets A_0 and A_1, leaving each of these sets globally invariant. Then the subgroup H of G generated by $\bigcup_{i<2} G_{\underline{A}_i}$ is a normal subgroup of G; the group $G_{\underline{A}_{1-i}}\restriction A_i$ is a normal subgroup of $G\restriction A_i$ for every $i < 2$; and if H_i denotes the quotient of $G\restriction A_i$ by $G_{\underline{A}_{1-i}}\restriction A_i$, then H_0 and H_1 are isomorphic to the quotient of G by H.*

Proof. Let $\varphi_i : G \to G\restriction A_i$ defined by setting $\varphi_i(f) = f\restriction A_i$. Then $\mathrm{Ker}(\varphi) = G_{\underline{A}_i}$. Hence the quotient $G/G_{\underline{A}_i}$ is isomorphic to $G_{\restriction A_i}$. Now $G_{\underline{A}_0}$ and $G_{\underline{A}_1}$ commute, and in particular H is isomorphic to the product $G_{\underline{A}_0} \times G_{\underline{A}_1}$. It follows that H is a normal subgroup of G (for $f \in G$ and $h = h_0 \circ h_1 \in H$ with $h_i \in G_{\underline{A}_i}$ $(i < 2)$, let $f' = f^{-1}\circ h\circ f = f^{-1}\circ h_0\circ f\circ f^{-1}\circ h_1\circ f$; since $G_{\underline{A}_i}$ is normal in G, it contains $f^{-1}\circ h_i\circ f$, thus $f' \in H$). Thus the quotient G/H is unambiguously defined. Next, $G_{\underline{A}_{1-i}}\restriction A_i$ is a normal subgroup of $G\restriction A_i$. Indeed, we only need to check that $h^{-1} \circ G_{\underline{A}_{1-i}}\restriction A_i \circ h \subseteq G_{\underline{A}_{1-i}}$ for every h in $G\restriction A_i$. For that, let $g \in G_{\underline{A}_{1-i}}\restriction A_i$. Let $\underline{g} \in G_{\underline{A}_{1-i}}$ such that $\underline{g}\restriction A_i = g$ and let $h' \in G$ such that $h'_{\restriction A_i} = h$. We have readily $h'^{-1} \circ \underline{g} \circ h' \in G_{\underline{A}_{1-i}}$. Hence H_i is unambiguously defined. With the notation of Lemma 4.15, we have:

Claim 4.16. *For every $f \in G$, the following properties are equivalent:*

(i) *$f\restriction A_0$ extends to some $g_1 \in G_{\underline{A}_1}$;*
(ii) *$f\restriction A_1$ extends to some $g_0 \in G_{\underline{A}_0}$;*
(iii) *$f \in H$.*

Proof of Claim 4.16. $(i) \Rightarrow (ii)$ Set $g_0 = g_1^{-1} \circ f$. By the same token we have $(ii) \Rightarrow (i)$. $(i) \Rightarrow (iii)$ We have $f = g_1 \circ g_0$. $(iii) \Rightarrow (i)$ Immediate. □

Claim 4.17. *$\varphi_i^{-1}(G_{\underline{A}_{1-i}}\restriction A_i) = H$.*

Proof of 4.17. By symmetry it suffices to prove the case $i = 0$. Let $h \in H$; then $h = h_0 \circ h_1$ with $h_i \in G_{\underline{A}_i}$ $(i < 2)$. Hence, $\varphi_0(h) = \varphi_0(h_0 \circ h_1) = \varphi_0(h_0)\circ\varphi_0(h_1) = \varphi_0(h_1) \in G_{\underline{A}_1}\restriction A_0$. Thus, $H \subseteq \varphi_0^{-1}(G_{\underline{A}_1}\restriction A_0)$. Conversely, let $f \in \varphi_0^{-1}(G_{\underline{A}_1}\restriction A_0)$. Then, $f\restriction A_0 = \varphi_0(f) \in G_{\underline{A}_1}\restriction A_0$. Hence, $f\restriction A_0$ satisfies (i) of Claim 4.16, and hence satisfies (iii) as well and $f \in H$. Thus $\varphi_0^{-1}(G_{\underline{A}_1}\restriction A_0) \subseteq H$. Consequently, $\varphi_0^{-1}(G_{\underline{A}_1}\restriction A_0) = H$, as claimed. □

From Claim 4.17 it follows that the quotient G/H is isomorphic to the quotient H_i of $G{\restriction} A_i$ by $G_{\underline{A}_{1-i}}{\restriction} A_i$. This proves the lemma. $\Box$

We are now in a position to better describe closed permutation groups with an infinite monomorphic component.

Lemma 4.18. *Let G be a closed permutation group acting on a countable set. Suppose there is an infinite G-monomorphic component A, and let B be its complement. Then*

(a) $G_{\underline{B}}{\restriction} A$, $G_B{\restriction} A$ and its closure $\overline{(G_B{\restriction} A)}^{\mathfrak{S}}$ are monomorphic groups;
(b) The quotient of $G_B{\restriction} A$ by $G_{\underline{B}}{\restriction} A$ has at most two elements;
(c) When that quotient has size 2 and $C = (A, \leq)$ is a dense linear order such that $\mathrm{Aut}(C) \subseteq G_{\underline{B}}{\restriction} A$, then $G_{\underline{B}}{\restriction} A$ and $\overline{(G_B{\restriction} A)}^{\mathfrak{S}}$ are either respectively equal to $\mathrm{Aut}(C)$ and $\mathrm{Aut}(B_C)$, or else to $\mathrm{Aut}(T_C)$ and $\mathrm{Aut}(D_C)$.

Proof. (a) Let R be an homogeneous structure such that $\mathrm{Aut}(R) = G$. Then A is an infinite monomorphic component of R. According to Proposition 4.8, this is a strong monomorphic component of R. It follows that, for every finite subset B' of B and every integer n, any two n-elements subsets of A are in the same orbit of $G_{\underline{B'}} \restriction A$. Hence, each group $G_{\underline{B'}} \restriction A$ is monomorphic. Next, observe that $G_{\underline{B}} \restriction A = \bigcap\{\overline{G_{\underline{B'}} \restriction A}^{\mathfrak{S}} : B' \in [B]^{<\omega}\}$. The inclusion $G_{\underline{B'}} \restriction A \subseteq \bigcap\{\overline{G_{\underline{B'}} \restriction A}^{\mathfrak{S}} : B' \in [B]^{<\omega}\}$ follows immediately from the obvious inclusions $G_{\underline{B}} \restriction A \subseteq G_{\underline{B'}} \restriction A \subseteq \overline{G_{\underline{B'}} \restriction A}^{\mathfrak{S}}$. The reverse inclusion is immediate: let σ be in the above intersection, then σ extended by the identity on A belongs to G, that is $\sigma \in G_{\underline{B}} \restriction A$ (indeed, for every finite subset F of A and B' finite in B, some $\tau \in G_{B'} \restriction A$ coincides with σ on F). The groups $\overline{G_{\underline{B'}} \restriction A}^{\mathfrak{S}}$ are monomorphic and closed. Due to Cameron's theorem, there is no infinite descending sequence of such groups (cf. Lemma 3.3). Hence, $G_{\underline{B}} \restriction A = \overline{G_{\underline{B'}} \restriction A}^{\mathfrak{S}}$ for some $B' \in [B]^{<\omega}$. This proves that $G_{\underline{B}}{\restriction} A$ is a closed monomorphic group. Being overgroups of that group, the groups $G_B{\restriction} A$ and its closure are also monomorphic.

(b) and (c) As said, the group $G_{\underline{B}}{\restriction} A$ is closed (a fact which follows directly from Lemma 4.13). By Theorem 3.4 there is a dense linear order $C = (A, \leq)$ such that $\mathrm{Aut}(C) \subseteq G_{\underline{B}}{\restriction} A$.

For each integer $n \in \mathbb{N}$, the nth member of the group sequence of $G_{\underline{B}}{\restriction} A$, resp. $\overline{(G_B{\restriction} A)}^{\mathfrak{S}}$, is the group of permutations of an n element subset F of A induced by members of $G_{\underline{B}}{\restriction} A$, resp. $\overline{(G_B{\restriction} A)}^{\mathfrak{S}}$; we denote these groups by $G_{\underline{B}}{\restriction} F$, resp. $\overline{(G_B{\restriction} F)}^{\mathfrak{S}}$.

Now let F, F' be two finite subsets of A with $F \subseteq F'$. According to Lemma 4.15, the quotient $H_F = G_F{\restriction} B/G_{\underline{F}}{\restriction} B$ is isomorphic to the quotient $G_B{\restriction} F/G_{\underline{B}}{\restriction} F$ hence this quotient is finite. We claim that it can only decrease when the cardinality of F increases; from Frasnay's result (Theorem 3.7), the cardinalities of members of these two groups sequences are either 1, 2, or goes to infinity, and hence in our case this quotient can be only 1 or 2. This will prove (b). When this quotient is constantly 1, the groups $G_{\underline{B}}\restriction A$ and $\overline{(G_B\restriction A)}^{\mathfrak{S}}$ are identical. When this quotient is constantly 2, the only possibilities are those given in (c).

It remains to verify our claim, thus let us consider two finite sets $F \subseteq F' \subseteq A$.

Claim 4.19. $G_{F'}\restriction B \subseteq G_F\restriction B$.

Proof of Claim 4.19. Let $\sigma \in G_{F'}\restriction B$. Then consider $F_1 = \sigma(F)$ and let $\overline{\sigma} \in G_{F'}$ such that $\overline{\sigma} \restriction B = \sigma$. Since $\mathrm{Aut}(C) \subseteq G_{\underline{B}}\restriction A$, there is some $\theta \in G_{\underline{B}}$ such that $\theta(F_1) = F$. Let $\varphi' = \theta \circ \overline{\sigma}$. Then $\varphi' \in G_F$ and $\varphi'\restriction B = \overline{\sigma}\restriction B = \sigma$, hence $\sigma \in G_F\restriction B$. □

Claim 4.20. *$G_{\underline{F}}\restriction B = G_{\underline{F'}}\restriction B$, provided that $|F| \geq 4$. In fact, in this case $G_{\underline{F}}\restriction B = G_{\underline{A}}\restriction B$.*

Proof of Claim 4.20. Clearly, $G_{\underline{F'}}\restriction B \subseteq G_{\underline{F}}\restriction B$. Conversely, let $\sigma \in G_{\underline{F}}\restriction B$ and let $\overline{\sigma} \in G_{\underline{F}}$ such that $\overline{\sigma}\restriction B = \sigma$. Let θ be the extension of σ (and $\overline{\sigma}$) by the identity on A. We prove that θ is an automorphism of R from which follows that $\sigma \in G_{\underline{A}}\restriction B$, hence in $G_{\underline{F'}}\restriction B$.

Observe that the group $\overline{(G_B\restriction A)}^{\mathfrak{S}}$, being monomorphic and closed, the classification given in Cameron's Theorem asserts that any homogeneous structure R' on A with $\mathrm{Aut}(R') = \overline{(G_B\restriction A)}^{\mathfrak{S}}$ will have the same local isomorphisms as some relation which is at most 4-ary. Local isomorphisms of R' of finite domains are the finite restrictions of members of $G_B\restriction A$. Hence we may suppose that $R' = R_{\restriction A}$, and that if ρ_i is a relation occurring in R, then each n_i-tuple $a \in \rho_i$ has at most four components in A. Now, if these four components are in F, we will have $\theta(a) = \overline{\sigma}(a) \in \rho_i$ since $\overline{\sigma} \in G_{\underline{F}}$; if these four elements are not all in F, then since $G_{\underline{B}}$ is monomorphic, it contains some τ which sends these four components into F; but now the previous case shows that $\theta(a) = \tau^{-1} \circ \theta \circ \tau(a) \in \rho_i$. □

This completes the proof of Lemma 4.18.

□

By (c) of this lemma we get:

Corollary 4.21. *Under the hypothesis of Lemma 4.18, if $\overline{(G_B\restriction A)}^{\mathfrak{S}}$ is the full symmetric group on A, then $G_{\underline{B}}\restriction A$ is also the full symmetric group on A.*

Remark: Corollary 4.21 also follows from the Schreier–Ulam theorem [39] on permutation groups (see also Scott [40, p. 305]).

Indeed, $G_{\underline{B}}\restriction A$ is a normal subgroup of $\overline{(G_B\restriction A)}^{\mathfrak{S}}$. To see this let $g \in G_{\underline{B}}\restriction A$, and $h \in \overline{(G_B\restriction A)}^{\mathfrak{S}}$. Now $h = \lim_n h_n$ where $h_n \in G_B\restriction A$, and choose $\overline{h}_n \in G_B$ such that $\overline{h}_n \restriction A = h_n$. Further choose $\overline{g} \in G_{\underline{B}}$ such that $\overline{g} \restriction A = g$. But now we have $\overline{h}_n \circ \overline{g} \circ \overline{h}_n^{-1} = id_B \cup h_n \circ g \circ h_n^{-1} \in G_{\underline{B}}$, and since $\lim_n \ h_n^{-1} \circ g \circ h_n = h \circ g \circ h^{-1}$we conclude that $h \circ g \circ h^{-1} \in G_{\underline{B}}\restriction A$.

Now, the Schreier–Ulam theorem asserts that the only proper normal subgroups of the symmetric group on a countable set are the group of permutations with finite support and the alternating subgroup. Neither of these groups is closed. Thus $G_{\underline{B}}\restriction A$, being closed, must be the full symmetric group.

The following examples illustrate quotients having two elements in Lemma 4.18.

Example 4.23. *Consider as a first example the countable set $E = \mathbb{Q} \times \{0, 1\}$, naturally partitioned as $A = \mathbb{Q} \times \{0\}$, $B = \mathbb{Q} \times \{1\}$. Define a quaternary relation $\rho(x, y, z, w)$ if $x, y \in A$, $z, w \in B$, and (abusing notation) satisfies $x < y$ if and only if $z < w$. Then $G = \mathrm{Aut}(E, \rho) = \{(f, g) : f, g \in \mathrm{Aut}(\mathbb{Q}),$ or $f, g \in \mathrm{Aut}(B_{\mathbb{Q}}) \setminus \mathrm{Aut}(\mathbb{Q})\}$. One easily verifies that in this case $G_{\underline{B}}\restriction A = \mathrm{Aut}(\mathbb{Q})$, and $G_B\restriction A = \mathrm{Aut}(B_{\mathbb{Q}})$.*

Toward a second example, consider a chain $C = (E, \leq)$ and for an n-tuple $u = (u_1, u_2, \ldots, u_n)$ of distinct elements of E, let σ_u be the unique permutation of $\mathfrak{S}_n$ such that $u_{\sigma(1)} < u_{\sigma(2)} < \cdots < u_{\sigma(n)}$. Now for a subgroup $H \leq \mathfrak{S}_n$, form the n-ary relation $\rho_H = \{u \in E^n : \sigma_u \in H\}$. More generally consider two disjoint sets A and B, H a subgroup of $\mathfrak{S}_n \times \mathfrak{S}_n$, and define a relation ρ_H on $A \cup B$ by $\rho_H = \{(u, v) \in A^n \times B^n : (\sigma_u, \sigma_v) \in H\}$. Then $H = T_3^2 \cup (D_3 \setminus T_3)^2$ is a group, and taking A and B as two disjoint copies of the rationals and $G = \mathrm{Aut}(A \cup B, \rho_H)$ we obtain that $G_{\underline{B}}\restriction A = \mathrm{Aut}(T_{\mathbb{Q}})$, and $G_B\restriction A = Aut(D_{\mathbb{Q}})$.

In that setting the first example can be restated using $H = \mathfrak{I}(2)^2 \cup (\mathfrak{S}(2) \setminus \mathfrak{I}(2))^2$.

5. Finite monomorphic decomposition

We prove the following result.

Theorem 5.1. *If a countable relational structure R is prehomogeneous and has a finite monomorphic decomposition, then it has one or $2^{\aleph_0}$ siblings.*

Case 1: R has just one component.

If so the conclusion follows from Theorem 3.12: $\mathrm{sib}(R)$ is one if $\mathrm{Aut}(R)$ is the symmetric group and $2^{\aleph_0}$ otherwise.

Case 2: R has several monomorphic components.

In this case the result follows from Theorem 5.2 below. Indeed suppose no infinite component is as in Theorem 5.2. Then taking the partition whose classes are the infinite components and then singletons shows the structure is finitely partitioned. It is clear such structures have only one sibling.

Theorem 5.2. *Let R be a countable structure which is prehomogeneous and such that $G = \mathrm{Aut}(R)$ is oligomorphic. If R has an infinite monomorphic component A which is not a strongly indiscernible subset of R then R has $2^{\aleph_0}$ siblings.*

Proof. The fact that A is not an indiscernible subset of R means that $G_{\underline{B}}\restriction A$ (where B is the complement of A) is not the full symmetric group on A. According to Proposition 4.12, $G_{\underline{B}}\restriction A$ and hence $\overline{G_B\restriction A}^{\mathfrak{S}}$ are monomorphic groups; their structure is given by Lemma 4.18. According to Corollary 4.21, since $G_{\underline{B}}\restriction A$ is assumed not to be the full symmetric group, $\overline{G_B\restriction A}^{\mathfrak{S}}$ is not the full symmetric group either. According to Theorem 3.12, any structure S on A with $\mathrm{Aut}(S) = \overline{G_B\restriction A}^{\mathfrak{S}}$ has $2^{\aleph_0}$ siblings. That is there are $2^{\aleph_0}$ subsets $(A_\alpha)_{\alpha<2^{\aleph_0}}$ of A such that for each α, α' there is a $\overline{G_B\restriction A}^{\mathfrak{S}}$-embedding of A into A_α, and no $\overline{G_B\restriction A}^{\mathfrak{S}}$-embedding of A_α onto $A_{\alpha'}$.

Claim 5.3. *Each restriction $R_\alpha = R\restriction E_\alpha$, where $E_\alpha = B\cup A_\alpha$, is a sibling of R.*

Proof of Claim 5.3. Since $G = \mathrm{Aut}(R)$ is oligomorphic, there is a dense linear ordering $C = (A, \leq)$ on A such that $\mathrm{Aut}(C) \subseteq G_{\underline{B}}\restriction A$ (Proposition 4.12 (b)). Since there is some $\overline{G_B\restriction A}^{\mathfrak{S}}$-embedding σ of A into A_α, the members of the indicative sequence of the bichain $(C, C_{\sigma^{-1}})$ (where $C_{\sigma^{-1}} = (A, \leq_{\sigma^{-1}})$ and $x \leq_{\sigma^{-1}} y$ if and only if $\sigma(x) \leq \sigma(y)$) are termwise included into the group sequence of $\overline{G_B\restriction A}^{\mathfrak{S}}$. If $C_{\sigma^{-1}}$ is scattered, then by Lemma 3.9 this indicative sequence is $\mathfrak{S}$, and hence the group sequence of $\overline{G_B\restriction A}^{\mathfrak{S}}$ is $\mathfrak{S}$ which is excluded. Consequently $C_{\sigma^{-1}}$ is nonscattered, that is $C_{\sigma(A)} = (\sigma(A), \leq)$ is nonscattered, and there is an embedding of C into $C_{\sigma(A)}$, this embedding extended by the identity on B is an embedding of R into R_α. This proves the claim. $\square$

Now let $\Gamma = \{\{\alpha, \beta\} : R_\alpha \cong R_\beta\}$, and κ the number of monomorphic components of R.

Claim 5.3.a. $|[C]^2| \leq \kappa^2$ *for each isomorphism equivalence class C of siblings of R.*

Proof of Claim 5.3.a. The structures R, R_α, and R_β have the same induced monomorphic decomposition. Hence, if α and β are equivalent, an isomorphism of R_α onto R_β induces a permutation of the classes of the decomposition. Such a permutation sends A_α to some class and A_β to another one. If $|[C]^2| > \kappa^2$ then two pairs $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$ would be sent to the same pair of classes of the decomposition, but then A_α would be sent onto $A_{\alpha'}$. This map, being a $G_B\restriction A$-embedding from A_α to $A_{\alpha'}$, would be a $\overline{G_B\restriction A}^{\mathfrak{S}}$-embedding from A_α to $A_{\alpha'}$, but there are none. $\square$

Since κ is countable, there are $2^{\aleph_0}$ inequivalent elements, that is $2^{\aleph_0}$ siblings. This proves Theorem 5.2. $\square$

Theorem 5.1 appears rather weak to us, we make the following conjecture.

Conjecture. *Under the assumption that a relational structure R has a finite monomorphic decomposition, R has one or infinitely many siblings.*

We proved that it holds if the structure is a chain [17]. One would need to extend this conclusion to the case of an infinite monomorphic relational structure R; such a structure is chainable. Next, one must go from a monomorphic structure to one admitting a finite monomorphic decomposition.

6. Structures with no finite monomorphic decomposition

In this section, we prove the following result:

Theorem 6.1. *Let R be a countable prehomogeneous relational structure such that* $\mathrm{Aut}(R)$ *is oligomorphic.*

(a) If R has infinitely many monomorphic components then $\mathrm{sib}(R)$ *is infinite;*

(b) If R has infinitely many infinite monomorphic components then $\mathrm{sib}(R) = 2^{\aleph_0}$.

We prove (a) in Subsection 6.2. For that, we show in the next subsection that R has a 1-extension R' such that $E' \setminus E$, the difference of the two domains, is an infinite monomorphic part and for which the component of R' containing it meets E on a finite set (Lemma 6.8). Then, we show that the extensions of R to subsets of $E' \setminus E$ having finite distinct cardinalities provide distinct siblings (Proposition 6.6).

We prove (b) in Subsection 6.3. We prove that if R has infinitely many infinite classes, one may select a countable union D of infinite equivalence classes such that R is embeddable into $R_{\restriction E\setminus D}$ (Lemma 6.11). Then, we apply Lemma 4.7.

6.1. Adding an infinite monomorphic part.

Lemma 6.2. *Let R and R' be two relational structures with domains E and E' respectively. If R' is an extension, resp. a 1-extension, of R then the partition of E' into the monomorphic components of R' induces a partition of E into monomorphic parts, resp. monomorphic components of R.*

Proof. Let $(E'_j)_{j\in J}$ be the monomorphic decomposition of R' and let $(E'_j \cap E)_{j\in J}$ be the family of the induced blocks. Trivially, these sets are monomorphic parts of R, hence our first assertion holds. Furthermore, the partition into those parts is finer than the partition given by the monomorphic decomposition of R. To prove that these two partitions coincide whenever R' is a 1-extension of R, let x and y be two different blocks of the induced partition. They are inequivalent for R', hence, there is a finite subset F' of $E' \setminus \{x, y\}$ witnessing that fact. Since R' is a 1-extension of R, there is a local isomorphism from R' to R which fixes x, y, and $F' \cap E$. The image F of

F' witnesses that x and y are inequivalent modulo R, that they are into two parts of the monomorphic decomposition. This proves our assertion. $\square$

Lemma 6.3. *Let R be a relational structure with domain E and let R' be a 1-extension of R with domain E' such that $E' \setminus E$ is an infinite monomorphic part of R'. If the trace over E of the component C' of R' containing $E' \setminus E$ is infinite then R' is a 1-extension of $R_{-C'} := R_{\restriction E \setminus C'}$ and the monomorphic decomposition of $R_{-C'}$ is made of components of R.*

Proof. Let F be a finite subset of E'. We have to show that there is a local isomorphism h of R' that fixes $F \setminus C'$ and maps $F \cap C'$ into $E' \setminus C'$. Since $E' \setminus E$ is infinite, it contains a subset X with the same cardinality as $F \setminus C'$. Since C' is an infinite monomorphic component, it is strongly monomorphic (cf. (f) of Proposition 4.8), hence there is a local isomorphism f that fixes $F \setminus C'$ and maps $F \cap C'$ onto X. Since R' is 1-extension of R and $C' \cap E$ is finite, there is a local isomorphism of R' that fixes $(F \setminus C') \cup (C' \cap E)$ and maps X on a subset of $E' \setminus E$. Since this subset is disjoint from C', we may set $h := g \circ f$.

Since R' is a 1-extension of R and $R_{-C'}$, then according to Lemma 6.2 the monomorphic decomposition of R' induces the monomorphic decompositions of R and $R_{-C'}$. It follows that the monomorphic decomposition of R_{-C} is made of components of R. $\square$

Lemma 6.4. *Let $(E_i)_{i \in I}$ be the monomorphic decomposition into components of a relational structure R with base E. Let R' be a 1-extension of R with base E' such that $E' \setminus E$ is a monomorphic part of R'.*
Then either:

1. *There is some index i such that $E_i \cup (E' \setminus E)$ is a monomorphic component of R' and for every index $j \neq i$, E_j is a monomorphic component of R'; or*
2. *There is some nonnegative integer k, $k \leq |E' \setminus E|$, such that if $H' \subseteq E' \setminus E$, has at least k elements, the monomorphic decomposition of $R'_{\restriction E \cup H'}$ is made of the E_i's and of H'.*

Proof. We start with the following claim.

Claim 6.5. *Let $z \in E' \setminus E$ and $R'_z = R'_{\restriction E \cup \{z\}}$. Then, either there is a unique index i such that $E_i \cup \{z\}$ is a monomorphic component of R'_z, or $\{z\}$ is a monomorphic component of R'_z.*

Proof of Claim 6.5. If there is some index i and some $x_i \in E_i$ such that z and x_i are equivalent modulo R'_z, then we show that i is unique. Indeed suppose on the contrary that we have $j \neq i$ and $x_j \in E_j$ such that z and x_j are equivalent modulo R'_z. It follows that x_i and x_j are equivalent modulo R'_z. This implies that they are equivalent modulo R, a contradiction. This proves the uniqueness of i if it exists. Since R' is a 1-extension of R, R'_z is a 1-extension too, hence Lemma 6.2 applies. Thus, if there is no such i, then $\{z\}$ must be a monomorphic component of R'_z. $\square$

With this claim, the proof of the lemma goes as follows. According to Lemma 6.2, the decomposition of R is induced by the decomposition of R'. Hence, either $E' \setminus E$ union some E_i (in fact a unique one) forms a component, or not. In this latter case, we claim that there exists some integer k such that every finite subset H' with at least k elements of $E' \setminus E$ is a monomorphic component of $R'_{\restriction E \cup H'}$.

Indeed, let $\mathcal{H}$ be the set of finite $H' \subseteq E' \setminus E$ such that H' is not a monomorphic component of $R'_{\restriction E \cup H'}$. If there is some finite subset $H \notin \mathcal{H}$ then for $k = |H|$ we will have the conclusion of our claim. Suppose that every finite subset H' of $E' \setminus E$ belongs to $\mathcal{H}$. For each finite H' there is some index i' such that $E_{i'} \cup H'$ is a monomorphic component of $R'_{\restriction E \cup H'}$. If H' is nonempty it follows from Claim 6.5 that this i' is unique (pick any $z \in E' \setminus E$ and observe that $\{z\}$ cannot form a component of R'_z). If i' depends on H', i.e., there is some H'' and some $i'' \neq i'$ with the same property, then for $H = H' \cup H''$ there will be no i such that $E_i \cup H$ is a monomorphic component. Hence H will be a monomorphic component of $R'_{\restriction E \cup H}$, contradicting our hypothesis on $\mathcal{H}$. Hence i' is independent of H', meaning that there is a unique index i such that $E_i \cup H$ is a monomorphic component of $R'_{\restriction E \cup H}$ for every finite subset H of $E' \setminus E$. It follows that $E_i \cup (E' \setminus E)$ is a monomorphic part of R' and, in fact, a component of R'. Indeed, if not, we may pick $x_i \in E_i$, $z \in E' \setminus E$ and $F \subseteq E' \setminus \{x_i, z\}$ witnessing that they are not equivalent modulo R'; setting $H = F \cap (E' \setminus E) \cup \{z\}$ we will have that x_i and z are not equivalent modulo $R'_{\restriction E \cup H}$, which is impossible since $E_i \cup H$ is a component of $R'_{\restriction E \cup H}$. □

As a consequence, we get:

Proposition 6.6. *Let R be a relational structure with domain E and S be the set of nonnegative integers n such that R has* no *monomorphic component of size n, and suppose that S is infinite. If R has a* 1*-extension R' such that $E' \setminus E$ is an infinite monomorphic part of R' and the trace over E of the component C' of R' containing $E' \setminus E$ is finite then R has infinitely many* 1*-extensions which are pairwise nonisomorphic.*

Proof. Let $R'_{C'} = R'_{\restriction E' \setminus C'}$. We apply Lemma 6.4 to R'_{-C} and R'. Since C' is a monomorphic component of R' then for no monomorphic component E_i of $R_{-C'}$ can $E_i \cup C'$ be a monomorphic component of R', that is, Case 1 of Lemma 6.4 cannot happen. Thus the second case must hold. That is, there is some nonnegative integer k, $k \leq |C'|$ such that for every $H \subseteq C'$ with at least k elements, the monomorphic decomposition of $R'_H = R'_{\restriction (E' \setminus C') \cup H}$ is made of the E_i's and of H. For distinct values of $k = |H|$ with $k \in S$, the R_H's cannot be isomorphic, otherwise the decomposition of a R_H will be carried over the decomposition of an R'_H. Taking for H subsets of C' containing $C' \cap E$ yields the desired conclusion. □

We need the following result

Lemma 6.7. *Let $R = (E, (\rho_i)_{i\in I})$ be a relational structure of signature $\mu = (n_i)_{i\in I}$ on an infinite set E and A be an infinite subset of E. If the profile is finite, then, on any superset E' of E such that $E' \setminus E$ is infinite, there is some extension R' of R such that:*

(a) $E' \setminus E$ is a strong monomorphic part of R'; and
(b) for every finite subset F of E' there is some local isomorphism of R' which fixes $E \cap F$ and maps $F \setminus E$ into A.

In the case $A = E$, this is [30, Lemme III-2.2.3]. The proof uses Theorem 4.9 of Fraïssé and the compactness theorem of first order logic. In our case, the same proof applies.

We say that an extension R' of R as above is a *good extension above A*.

Lemma 6.8. *Let R be prehomogeneous on a countable set E, O be an infinite orbit of a singleton with respect to $\mathrm{Aut}(R)$ that meets infinitely many components of R, A a subset of O such that $|A \cap C| = 1$ for each component C of R meeting A, R' an extension of R to a superset E' that is good above A, and C' the component of R' containing $E' \setminus E$. Then $C' \cap E$ contains at most one element and this element belongs to A.*

Proof. Since $E' \setminus E$ is a monomorphic part of R', it is included in a component of R', say C'. Either $C' = E' \setminus E$, that is, $E' \setminus E$ is a component, or not. In the first case $C' \cap E = \emptyset$ is a subset of A. In the latter case, since R' is a 1-extension of R, the decomposition of R into components is induced by the decomposition of R', hence $C := C' \cap E$ is a component of R. We prove first that $C \subseteq O$. Suppose not. Let $b \in C \setminus O$. Since R is prehomogeneous, there is some finite set $F \subseteq E$ containing b such that every local isomorphism f of R defined on b that extends to F to a local isomorphism of R can be extended to an automorphism of R. Since $b \notin O$ and O is an orbit, no local isomorphism f can map b into O and extend to F. Since C' is an infinite component of R', it is a strongly monomorphic part, hence there is map h from $F \cap C'$ into $E' \setminus E$ that extends by the identity on $F \setminus C'$ to a local isomorphism of R'. Since R' is a good extension above A, there is a local isomorphism g of R' that fixes $F \setminus C'$ and sends $h(F \cap C')$ into A. But then $g \circ h$ maps b into A thus into O. A contradiction. Suppose that C contains at least two elements. Since C is a component, every automorphism of R sending some element of C into C sends all the others into C. Since R is prehomogeneous, if X is a two element subset of C there is some finite set $F \subseteq E$ containing X such that every local isomorphism f of R defined on X that extends to F to a local isomorphism of R can be extended to an automorphism of R. Furthermore, if X' is another 2-element subset of C, we may find F', the image of F by some automorphism of R, satisfying the same property. Fix a 2-element subset X of C. Since C' is an infinite component of R', it is a strongly monomorphic part, hence there is map h from $F \cap C$ onto $((F \cap C) \setminus \{a\}) \cup \{b\}$ for some $a \in X \cap C$, $b \in E' \setminus E$, that extends by the identity on $F \setminus C$ to a local isomorphism of R'. Since R' is a good extension above A, there is a local isomorphism g of R' that fixes $F \cup (A \cap C)$ and sends $h(F \cap C)$ into A. But, since $|A \cap C| = 1$ then $g \circ h$ maps a into $A \setminus C$ and the other elements of $F \cap C$ into C. A contradiction. □

Remark: From this lemma, it follows that $E' \setminus E$ is a component of R' whenever the components of R meeting O are nontrivial. But it is not true in general that $E' \setminus E$ is a component of R'. For an example, take for R the Rado graph, fix a vertex, say a, add an infinite independent set, say H, and for every x in R, if x is joined to a by an edge, join x to every vertex of H, otherwise x is joined to no vertex of H. Then the resulting graph G' is a 1-extension over E (as well as the nonneighbour of a) and $H \cup \{a\}$ is a component of R'. However, there are extensions of the Rado graph for which $E' \setminus E$ is a component.

Problem 6.10. *Is it true that a countable prehomogeneous structure R with infinitely many components and* $\mathrm{Aut}(R)$ *oligomorphic has a* 1*-extension R' with $E' \setminus E$ an infinite component?*

6.2. **Proof of Theorem 6.1** (a)**.** Since $\mathrm{Aut}(R)$ is oligomorphic, it has only finitely many orbits of singletons. One, say O, meets infinitely many classes. Let A be a subset A of O such that $|A \cap C| = 1$ for each component C of R meeting A. Since $\mathrm{Aut}(R)$ is oligomorphic, the profile of R is finite, hence Lemma 6.7 applies and there is some extension R' of R above A. According to Lemma 6.8, either $E' \setminus E$ is a component of R' or the component C' of R' containing $E' \setminus E$ is made of $E' \setminus E$ and a singleton belonging to O. Since $C' \cap E$ is a finite component of R we may then apply Proposition 6.6. □

6.3. Adding infinitely many monomorphic components. A proof of Theorem 6.1 (b).

Lemma 6.11. *If R is countable, uniformly prehomogeneous, with infinitely many infinite equivalence classes of $\simeq_R$, one may select a countable union D of infinite equivalence classes such that R is embeddable into $R_{\restriction E \setminus D}$.*

Proof. We prove a slightly different statement. Namely, under the conditions of the lemma, R has an extension R' which is isomorphic to R and such that $E' \setminus E$, the difference of their bases, contains infinitely many infinite components of R'. For that, we will use the diagram method due to Robinson and apply the compactness theorem of first order logic.

We enumerate the elements of E to form a sequence a_n, $n < \omega$. To the language of R we add these elements as constants and we also add a new infinite set of constants $c_{i,j}$, $i, j < \omega$. We add some sentences and we prove that they form a consistent set, and thus the compactness of first order logic ensures that there is some countable model. Due to our choice of sentences, this model will be a model of the universal theory of R, hence it will extend to a copy $R'$of R. The constants $c_{i,j}$ will satisfy the following three properties:

(1) $c_{i,j} \neq c_{i',j'}$ when $(i, j) \neq (i', j')$;

(2) $c_{i,j} \simeq_{R'} c_{i',j'}$ if and only if $i = i'$;
(3) $c_{i,j} \not\simeq_{R'} e$ for $e \in E$.

Hence, $E' \setminus E$, the difference of their bases, contains infinitely many infinite components of R' which avoid E. The sentences we add fall into three categories.

a) Those of the diagram of R. That is, we add the sentences of the form $\rho_i(a_{i,1}, \ldots, a_{i,m_i})$ for every $(a_{i,1}, \ldots, a_{i,m_i}) \in \rho_i$, the sentences of the form $\neg\rho_i(a_{i,1}, \ldots, a_{i,m_i})$ for every $(a_{i,1}, \ldots, a_{i,m_i}) \notin \rho_i$ and the sentences of the form $a_i \neq a_j$ for every $i \neq j$. Clearly, any model of the diagram will be an extension of R.

b) The sentences of the form $\forall x_1, \ldots, \forall x_p \neg F(x_1, \ldots, x_p)$ where F is a quantifier-free formula in the language of R describing a finite reduct which cannot be embedded in R. These sentences, added to the previous one, form a consistent set; indeed R is a model. Furthermore, any model R' is an extension, and in fact a 1-extension, hence a model of the universal theory of R.

c) Sentences expressing that (1), (2), and (3) hold. We note that there is an integer k such that $\simeq_{\leq k,R}$ and $\simeq_R$ coincide (Lemma 4.3). Since the profile of R is finite, it follows that there is an existential formula $F(x, y)$ (using at most k quantifiers) such that $a \not\simeq_R b$ if and only if $F(a, b)$ holds in R. We add to the diagram of R the sentences $F(c_{i,j}, c_{i'j'})$ for $i < i'$, $F(a_i, c_{i',j})$ for all i, i' and $\neg F(c_{i,j}, c_{i,j'})$ for all i, j, j'.

This set of sentences added to the previous ones is consistent. Indeed, taking finitely many, they will determine a finite subset A of E and a finite subset C of the (i, j)'s and will define an equivalence relation on C. Since R contains infinitely many infinite components, there are infinitely many that are disjoint from A, hence we may select in these components elements reproducing the structure of the equivalence relation over C to obtain the consistency of this finite set of sentences. As noted above, the compactness theorem of first order logic will give a copy R' of R extending R. In that copy, two elements a, b satisfy $a \simeq_{R'} b$ if and only if $\neg F(a, b)$. Since $F(x, y)$ is an existential formula, the $c_{i,j}$'s will not be $\simeq_{R'}$-equivalent and not $\simeq_{R'}$-equivalent to any element of E. In that copy, the union D of the equivalence classes of the $c_{i,j}$'s is disjoint from R. □

With Lemma 6.11 and Lemma 4.7 we get that R has $2^{\aleph_0}$ siblings. Hence, Theorem 6.1 (b) holds.

7. Proof of Theorem 1.1

Reassembling Theorem 5.1, Theorem 5.2, and Theorem 6.1, we get:

Theorem 7.1. *Let R be a countable prehomogeneous relational structure such that* $\mathrm{Aut}(R)$ *is oligomorphic. Then R has $2^{\aleph_0}$ siblings if R has some infinite monomorphic component which is not an indiscernible set of R, or* *infinitely many infinite components. If not then R has one sibling provided that R has finitely many components, and infinitely many siblings otherwise.*

Theorem 1.1 follows.

8. Conclusion

8.1. **A possible improvement of Theorem 5.2.** Let us recall that a relational structure R with base E is *cellular* [38] if there is a finite subset $F \subset E$ and an enumeration $(a_{(x,y)})(x, y) \in V \times L$ of the elements of $E \setminus F$ by a set $V \times L$, where V is finite such that for every bijective map f of L the map $(1_V, f)$ extended by the identity on F is a local isomorphism of R (the map $(1_V, f)$ is defined by $(1_V, f)(a_{(x,y)}) := a_{(x,f(y))}$). Note that a finitely partitionable structure is cellular, but the converse does not hold.

The age $\mathcal{A}$ of a cellular structure R is well-quasi-ordered (w.q.o., for short), that is every infinite sequence $(S_n)_{n\in\mathbb{N}}$ of members of A contains an increasing sequence with respect to embeddability. In fact, the set $\mathcal{A}_{[m]}$ of structures $S \in \mathcal{A}$ with m unary relations added, is also w.q.o. for every $m \in \mathbb{N}$. It follows from [27, Théorème 3.4, p. 697] that if for an age $\mathcal{A}$, the set $\mathcal{A}_{[m^-]}$ of structures $S \in \mathcal{A}$ with m constants added, is w.q.o. for every $m \in \mathbb{N}$ then there is a uniformly prehomogeneous structure with age $\mathcal{A}$ (we do not know if these two w.q.o. conditions are equivalent). In particular if R is cellular, some R' equimorphic to R is uniformly prehomogeneous (and cellular). If R is cellular then $\mathrm{sib}(R)$ is at most countable; this is a straightforward consequence of a result of [21] (see below).

Under this setting we propose the following problem.

Problem 8.1. *Let R be a countable and $\aleph_0$-categorical relational structure. Prove that either* $\mathrm{sib}(R) = 1$*,* $\aleph_0$*, or* $2^{\aleph_0}$*. Furthermore, show that* $\mathrm{sib}(R) \leq \aleph_0$ *if and only if R is cellular.*

Note that Theorem 5.2 does not give the value of $\mathrm{sib}(R)$ when R has infinitely many finite components and finitely many infinite components which are strongly indiscernible. We know that $\mathrm{sib}(R)$ is infinite, but there are examples such that the number of siblings is countable and some for which it is the continuum. Note that if our problem has a positive answer, then $\mathrm{sib}(R) = 2^{\aleph_0}$ whenever all the components are singletons.

8.2. **A possible extension to universal structures.** A countable structure R is *universal for its age* if every countable structure with the same age embeds into R.

Problem 8.2. *Get the same conclusion as in Problem 8.1 under the weaker requirement that R is universal for its age and the profile takes only integer values.*

Some condition, e.g. that the profile of R takes only integer values, is necessary. Indeed, let R_ω be the relational structure made of a countable set and infinitely many distinct constants, such that the set not covered by the constants is infinite. With our definition of age, R is unique for its age, but not finitely partitionable. On the other hand, the universal theory $T_{\forall}(R)$ of R has countably many countable models, namely $R_0, \dots, R_n, \dots, R_\omega$, where R_n is the restriction of R_ω to the constants plus n extra elements. Each of those structures have only one sibling.

If R is universal for its universal theory $T_{\forall}(R)$, then it is equimorphic to a countable existentially universal structure, [26]. This structure, unique up to isomorphism, could play the role that uniform prehomogeneity plays in the case of $\aleph_0$ categoricity.

If the conclusion of Problem 8.2 holds, a consequence is that if the profile of R is finite, R is universal, and $\mathrm{Ker}(R)$, the *kernel* of R, (the set of $x \in E$ such that $\mathrm{age}(R_{-x}) \neq \mathrm{age}(R)$) is infinite, then the number of siblings of R is $2^{\aleph_0}$. Indeed, if $\mathrm{Ker}(R)$ is infinite, R cannot be finitely partitionable nor cellular. So an obvious strategy is to prove the following directly.

Problem 8.3. *Let R be a countable relational structure with finite profile and an infinite kernel. Prove that if R is universal for its age,* $\mathrm{sib}(R) = 2^{\aleph_0}$.

As shown below, a positive answer to Problem 8.2 has some consequences on the number of countable models of a universal theory; one of which we know is true, the other conjectured.

8.3. **Problems on the number of countable models.** Thomassé's conjecture is a specific question about the number of models of universal theories. As it is well-known, there are complete theories with any $n, n \geq 3$, countable models, and Ehrenfeucht's families of examples provide such theories. Indeed set $R := (\mathbb{Q}, \leq, (c_n)_{n\in\mathbb{N}})$ where c_n is the constant n; then the theory of R contains, up to isomorphy, exactly three countable models: R, $R + \mathbb{Q} := (\mathbb{Q} + \mathbb{Q}', \leq, (c_n)_{n\in\mathbb{N}})$, $R + \{a\} + \mathbb{Q}' := (\mathbb{Q} + \{a\} + \mathbb{Q}, \leq, (c_n)_{n\in\mathbb{N}})$. The last two are equimorphic, but there are $2^{\aleph_0}$ equimorphic models.

It is possible that Thomassé's conjecture holds for any countable relational structure and that the solution comes from set theoretical or model theoretical techniques. Structures having 1 or $\aleph_0$ siblings must be exceptional and their description seems to be an interesting task. In that respect, we believe that the significant part of our result is the characterization of the structures R such that $\mathrm{sib}(R)$ is one.

Now let $\mathcal{C}$ be a hereditary class of finite relational structures in a finite language. Let $\overline{\mathcal{C}}_{\aleph_0}$ be the class of countable R (up to isomorphy) such that $\mathrm{age}(R) \subseteq \mathcal{C}$, and let $\overline{\mathcal{C}}_{\aleph_0}/\equiv$ be the set of equimorphism classes of members of $\overline{\mathcal{C}}_{\aleph_0}$. If further $\mathcal{A}$ is an age, let $I(\mathcal{A})$ be the number of countable R (up to isomorphy) such that $\mathrm{age}(R) = \mathcal{A}$ and let $I(\mathcal{A})/\equiv$ be the number of equimorphism classes of countable R such that $\mathrm{age}(R) = \mathcal{A}$. In this case the possible values of $I(\mathcal{A})$ are known. Indeed it was shown by Macpherson, Pouzet, and Woodrow in [21] that $I(\mathcal{A})$ is either 1, $\aleph_0$, or $2^{\aleph_0}$. They further proved that $I(\mathcal{A})$ is at most countable if and only if all R with age $\mathcal{A}$ are cellular. A positive answer to Problem 8.2 would be a generalization of the first part of their result; consider two cases:

a) The number of maximal existential types which appear in those R is uncountable. In this case the number of these types is $2^{\aleph_0}$, simply because these types form a G_δ set, and thus it follows that the number of isomorphic types of countable R is $2^{\aleph_0}$.

b) This number is at most countable. In this case there is some countable R' with age $\mathcal{A}$ which is universal (see [26]), and thus we apply the conclusion of Problem 8.2.

The following problem remains.

Problem 8.4. *Let $\mathcal{A}$ be an age, find the possible values of $I(\mathcal{A})/\equiv$.*

Concerning this problem, Pouzet, Sauer, and Thomassé had conjectured in 2006 that $I(\mathcal{A})/\equiv$ is either 1, $\aleph_0$, $\aleph_1$, or $2^{\aleph_0}$; note that these values do occur as the age of an infinite path yields $\aleph_0$, and the age of an infinite chain yields $\aleph_1$. In fact, as observed by Melleray in [22] it follows from a theorem of Burgess (see [42, Theorem 5.13.4, p. 230] that the number is $2^{\aleph_0}$ whenever it is larger than $\aleph_1$. Indeed, the set of relational structures R with age $\mathcal{A}$ and domain $\mathbb{N}$ is a G_δ set and the equivalence relation of equimorphy is analytic. One can say more. Let $\kappa := I(\mathcal{A})/\equiv < 2^{\aleph_0}$, then there is a countable universal structure, say U, with age $\mathcal{A}$. Indeed, as mentioned above, the number of maximal existential types which appear in the R with age $\mathcal{A}$ is at most κ (by taking a representative in each equimorphy class and the maximal existential types appearing in that representative). Since these types form a G_δ set, their number in this case must be countable. From the criteria given in [26], there is a countable structure U which is universal. The equivalence relation of equimorphy on subsets of U is analytic and we may now apply Burgess' result.

These same three authors had also conjectured that $I(\mathcal{A})/\equiv$ is 1 if and only if all countable R with $\mathrm{age}(R) = \mathcal{A}$ are cellular. This last conjecture is also somewhat related to Problem 8.2 since all countable R are universal. We do not know the answer to this simple question: Is $\mathrm{sib}(R) = 2^{\aleph_0}$ and $|I(\mathcal{A}(R))/\equiv$ equal to 1 impossible?

Now consider two ages $\mathcal{A} \subseteq \mathcal{A}'$. It follows, from a result of Hodkinson and Macpherson and the result of Macpherson, Pouzet, Woodrow [21] already mentioned, that $I(\mathcal{A}) \leq I(\mathcal{A}')$. Indeed $I(\mathcal{A})$ is either 1, $\aleph_0$, or $2^{\aleph_0}$. Moreover, $I(\mathcal{A})$ is 1 if and only if some (in fact every) countable structure with age $\mathcal{A}$ is finitely partitioned (Hodkinson–Macpherson), and $I(\mathcal{A})$ is at most countable if and only if some (in fact every) countable structure with age $\mathcal{A}$ is cellular. Thus, either $I(\mathcal{A}') = 2^{\aleph_0}$, in which case $I(\mathcal{A}) \leq I(\mathcal{A}')$, or $I(\mathcal{A}') = \aleph_0$, in which case $\mathcal{A}'$ is the age of a countable cellular structure, hence $\mathcal{A}$ is such an age as well, and thus $I(\mathcal{A}) \leq \aleph_0 = I(\mathcal{A}')$, or else $I(\mathcal{A}') = 1$, in which case $\mathcal{A}'$ is the age of a finitely partitionnable structure, hence $\mathcal{A}$ too and thus $I(\mathcal{A}) = 1 = I(\mathcal{A}')$.

Thus once again the following problem naturally follows.

Problem 8.5. *If $\mathcal{A} \subseteq \mathcal{A}'$ are both ages, is $I(\mathcal{A})/\equiv \leq\ I(\mathcal{A}')/\equiv$?*

The following particular case could shed some light.

Problem 8.6. *Let $\mathcal{A}$ be an age, and suppose that for every $n \in \mathbb{N}$ the collection of countable R with n unary relations added and such that* $\mathrm{age}(R) = \mathcal{A}$ *is w.q.o. Does it follow that $I(\mathcal{A})/\equiv$ is at most $\aleph_1$? Conversely, if $I(\mathcal{A})/\equiv$ is at most $\aleph_1$ (and $\aleph_1 < 2^{\aleph_0}$), does it follows that the collection of countable R such that* $\mathrm{age}(R) = \mathcal{A}$ *is w.q.o.?*

We mentioned earlier that it is known (Laver 1971 [20]) that $I(\mathcal{A})/\equiv$ is $\aleph_1$ if $\mathcal{A}$ is the age of an infinite chain. Is the same true for the age of finite cographs? This is relevant here as countable relations with this age and n unary relations added do form a w.q.o. (Thomassé [43]).

A related result is the following. Let R be relational structure and $\varphi_R(\kappa)$ be the number of restrictions of R to subsets A of size κ, these restrictions counted up to isomorphy. According to Gibson, Pouzet, and Woodrow [13] $\varphi_R(n) \leq \varphi_R(\aleph_0)$ for $n < \omega$, and $\varphi_R(\aleph_0)$ can be finite, $\aleph_0$ or $2^{\aleph_0}$.

Problem 8.7. *Let R' be a countable structure; it is well-known that, for every countable R with* $\mathrm{age}(R) \subseteq \mathcal{A}' = \mathrm{age}(R')$, *there is a countable extension of R and R' with age $\mathcal{A}'$. Consequently, the collection of countable R' with a given age, say $\mathcal{A}'$, is up-directed. Is it true that for every R' in this set, the number of R'' above R' is equal to the cardinality of this set?*

If this is true, then the answer to Problem 8.2 is positive. Indeed, if there is a universal R, the number of siblings will be the number of structures with the same age.

We conclude with the following general problem.

Problem 8.8. *If $\mathcal{C}$ is a hereditary class, find the possible values of $|\overline{\mathcal{C}}_{\aleph_0}|$ and $|\overline{\mathcal{C}}_{\aleph_0}/\equiv|$.*

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