CANONICAL FUNCTIONS:
A PROOF VIA TOPOLOGICAL DYNAMICS

MANUEL BODIRSKY AND MICHAEL PINSKER

ABSTRACT. Canonical functions are a powerful concept with numerous applications in the study of groups, monoids, and clones on countable structures with Ramsey-type properties. In this short note, we present a proof of the existence of canonical functions in certain sets using topological dynamics, providing a shorter alternative to the original combinatorial argument. We moreover present equivalent algebraic characterisations of canonicity.

1. INTRODUCTION

When $f \colon (\mathbb{Q}; <) \to (\mathbb{Q}; <)$ is any function from the order of the rational numbers to itself, then there are arbitrarily large finite subsets of $\mathbb{Q}$ on which f "behaves regularly"; that is, it is either strictly increasing, strictly decreasing, or constant. A direct (although arguably unnecessarily elaborate) way to see this is by applying Ramsey's theorem: two-element subsets of $\mathbb{Q}$ are coloured with three colours according to the local behaviour of f on them (this yields, by the infinite version of Ramsey's theorem, even an infinite set on which f behaves regularly, but this is beside the point for us). In particular, it follows that the closure of the set $\{\beta f \alpha \mid \alpha, \beta \in \operatorname{Aut}(\mathbb{Q}; <)\}$ in $\mathbb{Q}^{\mathbb{Q}}$, equipped with the pointwise convergence topology, contains a function which behaves regularly everywhere. This function of regular behaviour is called *canonical*.

More generally, a function $f \colon \Delta \to \Lambda$ between two structures Δ, Λ is called canonical if it behaves regularly in an analogous way, that is, if it sends tuples in Δ of the same type (in the sense of model theory, as in [23]) to tuples the same type in Λ [14, 15, 20]. Similarly as in the example above, canonical functions can be obtained from f, in the fashion stated above, if Δ has sufficient Ramsey-theoretic properties (for example, if it is a countable

Ramsey structure in the sense of [24]) and if Λ is sufficiently small (for example countable and ω-categorical) [14, 15, 20].

The concept of canonical functions has turned out useful in numerous applications: for classifying first-order reducts they are used in [1, 2, 9, 15, 18, 29, 31, 32], for computing the model companion and the model-complete core of such reducts [30], for complexity classification for constraint satisfaction problems (CSPs) in [10, 12, 16, 21, 28], for decidability of meta-problems in the context of the CSPs in [20], for lifting algorithmic results from finite-domain CSPs to CSPs over infinite domains in [13], for lifting algorithmic results from finite-domain CSPs to homomorphism problems from definable infinite structures to finite structures [26], and for decidability questions in computations with atoms in [27]. Most of these applications are covered by a survey article published shortly after their invention [14].

As indicated above, the technique is available for a function $f \colon \Delta \to \Lambda$, in particular, whenever Δ is a countable Ramsey structure and Λ is countable and ω-categorical, and the existence of canonical functions in the set $\overline{\{\beta f \alpha \mid \alpha \in \mathrm{Aut}(\Delta), \beta \in \mathrm{Aut}(\Lambda)\}} \subseteq \Lambda^{\Delta}$ was originally shown under these conditions by a combinatorial argument [14, 15, 20]. By the Kechris–Pestov–Todorčević correspondence [24], a countable structure Δ is Ramsey (with respect to colourings of embeddings) if and only if its automorphism group $\mathrm{Aut}(\Delta)$ is *extremely amenable*, meaning that every continuous action of it on a compact Hausdorff space has a fixed point. Moreover, by the theorem of Ryll-Nardzewski, Engeler, and Svenonius, two tuples in a countable ω-categorical structure have the same type if and only if they lie in the same orbit with respect to the componentwise action of its automorphism group on tuples, and a countable structure is ω-categorical if and only if its automorphism group is oligomorphic. Therefore both the definition of canonicity as well as the above-mentioned conditions implying their existence in sets of the form $\overline{\{\beta f \alpha \mid \alpha \in \mathrm{Aut}(\Delta), \beta \in \mathrm{Aut}(\Lambda)\}}$ can be formulated in the language of permutation groups.

It is therefore natural to ask for a perhaps more elegant proof of the existence of canonical functions via topological dynamics, reminiscent of the numerous proofs of combinatorial statements obtained in a similar fashion (cf. the survey [8] for Ergodic Ramsey theory; [25] mentions some applications of extreme amenability). In this short note, we present such a proof. The proof was discovered by the authors at the Workshop on Algebra and CSPs at the Fields Institute in Toronto in 2011, where it was also presented (by the second author), but has so far not appeared in print. We use the occasion of this note to present various equivalent characterisations of canonicity of functions that facilitate their use and better explain their significance.

2. Canonicity

We use the notation $\mathbf{G} \curvearrowright X$ to denote a permutation group $\mathbf{G}$ acting on a set X. We make the convention that if $f \colon X \to Y$ is a function and $t = (t_1, \ldots, t_k) \in X^k$, where $k \geq 1$, then $f(t) := (f(t_1), \ldots, f(t_k)) \in Y^k$ denotes the k-tuple obtained by applying f to t componentwise.

The following is an algebraic formulation of Definition 6 in [20].

Definition 1. Let $\mathbf{G} \curvearrowright X$ and $\mathbf{H} \curvearrowright Y$ be permutation groups. A function $f \colon X \to Y$ is called *canonical with respect to* $\mathbf{G}$ *and* $\mathbf{H}$ if for every finite tuple $t \in X^{<\omega}$ and every $\alpha \in \mathbf{G}$ there exists $\beta \in \mathbf{H}$ such that $f\,\alpha(t) = \beta\,f(t)$.

Hence, functions that are canonical with respect to $\mathbf{G}$ and $\mathbf{H}$ induce for each integer $k \geq 1$ a function from the orbits of the componentwise action of $\mathbf{G}$ of X^k to the orbits of the componentwise action of $\mathbf{H}$ on Y^k.

In order to formulate properties equivalent to canonicity we require some topological notions. We consider the set Y^X of all functions from X to Y as a topological space equipped with the topology of pointwise convergence, i.e., the product topology where Y is taken to be discrete. When $S \subseteq Y^X$, then we write $\overline{S}$ for the closure of S in this space. In particular, when $\mathbf{G} \curvearrowright X$ is a permutation group, then $\overline{\mathbf{G}}$ is the closure of $\mathbf{G}$ in X^X. Note that $\overline{\mathbf{G}}$ might no longer be a group, but it is still a monoid with respect to the composition of functions. For example, in the case of the full symmetric group $\mathbf{G} = \mathrm{Sym}(X)$ consisting of all permutations of X, $\overline{\mathbf{G}}$ is the transformation monoid of all injections in X^X.

A permutation group $\mathbf{G} \curvearrowright X$ is called *oligomorphic* if for each $k \geq 1$ the componentwise action of $\mathbf{G}$ on X^k has finitely many orbits. For oligomorphic permutation groups we have the following equivalent characterisations of canonicity.

Proposition 1. *Let $\mathbf{G} \curvearrowright X$ and $\mathbf{H} \curvearrowright Y$ be permutation groups, where X, Y are countable and $\mathbf{H} \curvearrowright Y$ is oligomorphic. Then for any function $f \colon X \to Y$ the following are equivalent.*

   (1) *f is canonical with respect to $\mathbf{G}$ and $\mathbf{H}$;*
   (2) *for all $\alpha \in \mathbf{G}$ we have $f\alpha \in \overline{\mathbf{H}f} := \overline{\{\beta f \mid \beta \in \mathbf{H}\}}$;*
   (3) *for all $\alpha \in \mathbf{G}$ there are $e_1, e_2 \in \overline{\mathbf{H}}$ such that $e_1 f \alpha = e_2 f$.*

A stronger condition would be to require that for all $\alpha \in \mathbf{G}$ there is an $e \in \overline{\mathbf{H}}$ such that $f\alpha = ef$. To illustrate that this is strictly stronger, already when $\mathbf{G} = \mathbf{H}$, we give an explicit example.

Example 2 (Thanks to Trung Van Pham). Let $\mathbf{G} := \mathbf{H} := \mathrm{Aut}(\mathbb{Q}; <)$. Note that $(\mathbb{Q}; <)$ and $(\mathbb{Q} \setminus \{0\}; <)$ are isomorphic, and let f be such an isomorphism. Then f, viewed as a function from $\mathbb{Q} \to \mathbb{Q}$, is clearly canonical with respect to $\mathbf{G}$ and $\mathbf{H}$. But f does not satisfy the stronger condition above. To see this, choose $a \in \mathbb{Q}$ such that $f(a) < 0$, and pick $\alpha \in \mathbf{G}$ such that $f\alpha(a) > 0$. Since the image of $f\alpha$ equals the image of f, any $e \in \overline{\mathbf{H}}$

such that $f\alpha = ef$ must fix 0. Since e must also preserve $<$, it cannot map $f(a) < 0$ to $f\alpha(a) > 0$. Hence, there is no $e \in \overline{\mathbf{H}}$ such that $f\alpha = ef$. $\qquad\square$

In Proposition 1, the implications from (1) to (2) and from (3) to (1) follow straightforwardly from the definitions. For the implication from (2) to (3) we need a lift lemma, which is in essence from [19]. This lemma has been applied frequently lately [6, 7, 10, 13], in various slightly different forms. We need yet another formulation here; since the lemma is a consequence of a compactness argument which we need in any case for the canonisation theorem in Section 3, we present its proof.

Let $\mathbf{H} \curvearrowright Y$ be a permutation group, and let $f, g \in Y^X$, for some X. We say that $f = g$ *holds locally modulo* $\mathbf{H}$ if for all finite $F \subseteq X$ there exist $\beta_1, \beta_2 \in \mathbf{H}$ such that $\beta_1 f{\restriction}_F = \beta_2 g{\restriction}_F$. We say that $f = g$ *holds globally modulo* $\mathbf{H}$ *(modulo* $\overline{\mathbf{H}}$*)* if there exist $e_1, e_2 \in \mathbf{H}$ $(e_1, e_2 \in \overline{\mathbf{H}}$, respectively) such that $e_1 f = e_2 g$.

Of course, if $f = g$ holds globally modulo $\overline{\mathbf{H}}$, then it holds locally modulo $\mathbf{H}$. On the other hand, if $f = g$ holds locally modulo $\mathbf{H}$, then it need not hold globally modulo $\mathbf{H}$: an example are the functions f and $f\alpha$ in Example 2, for the reasons explained above. However, there exist $e_1, e_2 \in \overline{\mathbf{H}}$ such that $e_1 f = e_2 f \alpha$, so $f = f\alpha$ holds globally modulo $\overline{\mathbf{H}}$. This is true in general, as we see in the following lift lemma.

Lemma 3. *Let* $\mathbf{H} \curvearrowright Y$ *be an oligomorphic permutation group acting on a countable set* Y*, let* I *be a countable index set, and let* X_i *be a countable set for every* $i \in I$*. Let* f_i, g_i *be functions in* Y^{X_i} *such that* $f_i = g_i$ *holds locally modulo* $\mathbf{H}$ *for all* $i \in I$*. Then* $f_i = g_i$ *holds globally modulo* $\overline{\mathbf{H}}$ *for all* $i \in I$*, and in fact there exist* $e, e_i \in \overline{\mathbf{H}}$ *such that* $e f_i = e_i g_i$ *for all* $i \in I$*.*

To prove Lemma 3, it is convenient to work with a certain compact Hausdorff space that we also use for the canonisation theorem in Section 3. Let $\mathbf{H} \curvearrowright Y$ be a permutation group, and X be a set. On Y^X, define an equivalence relation $\sim$ by setting $f \sim g$ if $f \in \overline{\mathbf{H} g}$, i.e., if $f = g$ holds locally modulo $\mathbf{H}$; here, transitivity and symmetry follow from the fact that $\mathbf{H}$ is a group. The following has essentially been shown in [17] (though for the finer equivalence relation of global equality modulo $\mathbf{H}$), but we give an argument for the convenience of the reader since it is used so often (cf. for example [3–5, 11, 19]).

Lemma 4. *If* $\mathbf{H} \curvearrowright Y$ *is oligomorphic, and* X *is countable, then the space* $Y^X/_\sim$ *is a compact Hausdorff space.*

Proof. We represent the space in such a way that this becomes obvious. Extend the definition of the equivalence relation $\sim$ to all spaces Y^F, where $F \subseteq X$. When F is finite, then $Y^F/_\sim$ is finite and discrete, because $\mathbf{H}$ is oligomorphic. Hence, the space

$$\prod_{F \in [X]^{<\omega}} Y^F/_\sim$$

is compact. The mapping ξ from $Y^X/_\sim$ into this space defined by

$$[g]_\sim \mapsto \big([g{\restriction}_F]_\sim \mid F \in [X]^{<\omega}\big)$$

is well-defined. In fact, ξ is a homeomorphism onto a closed subspace thereof. To see this, note that injectivity follows from the definition of the equivalence relation $\sim$, and likewise continuity, since the topology on $Y^X/_\sim$ is precisely given by the behaviour of functions on finite sets, modulo the equivalence $\sim$. The fact that the image of ξ is closed follows from the fact that X is countable: when we have, in the range of ξ, tuples $([g_i{\restriction}_F]_\sim \mid F \in [X]^{<\omega})$ for each $i \in \omega$, and the sequence of these tuples converges in $\prod_{F \in [X]^{<\omega}} Y^F/_\sim$, then a function $g \in Y^X$ such that $([g{\restriction}_F]_\sim \mid F \in [X]^{<\omega})$ is the limit of the sequence can be constructed by a standard argument using Kőnig's tree lemma. The openness of the mapping ξ is then also obvious. It follows that $Y^X/_\sim$ is indeed a compact Hausdorff space. $\square$

We remark that when $\mathbf{H}$ is the automorphism group of an ω-categorical first-order structure on Y, then the space $Y^X/_\sim$ in Lemma 4 is nothing but the type space for the theory of that structure with variables indexed by the set X. Let us also mention that the condition of X being countable is necessary; cf. Examples 4.5 and 4.7 in [33].

Proof of Lemma 3. First assume that I is finite, and write $I = \{0, \ldots, n-1\}$. For each $0 \leq i \leq n-1$, we have $f_i \in \overline{\mathbf{H}\,g_i}$; since X_i is countable, there is a sequence $(\beta_i^j\,g_i)_{j \in \omega}$ converging to f_i. Now consider the set

$$S := \{(\mathrm{id},\ \beta_0^j, \ldots, \beta_{n-1}^j) \mid j \in \omega\}\,,$$

viewed as a subset of the space $\overline{\mathbf{H}}^{n+1}$, with id denoting the identity function in $\mathbf{H}$. The space $\overline{\mathbf{H}}^{n+1}$ can be viewed naturally as a closed subspace of $(Y^{n+1})^{(Y^{n+1})}$, and the equivalence relation $\sim$ induced on the latter by the componentwise, oligomorphic action of $\mathbf{H}$ on Y^{n+1} restricts to $\overline{\mathbf{H}}^{n+1}$ since this space is invariant under that action. Factoring $\overline{\mathbf{H}}^{n+1}$ by $\sim$, we obtain a compact space by Lemma 4. The equivalence classes of the elements of S have an accumulation point in $(\overline{\mathbf{H}}^{n+1})_\sim$, which we write as $[(e, e_0, \ldots, e_{n-1})]_\sim$, for some $e, e_0, \ldots, e_{n-1} \in \overline{\mathbf{H}}$. Hence, there exist $\delta^j \in \mathbf{H}$, for $j \in \omega$, such that $(\delta^j, \delta^j\beta_0^j, \ldots, \delta^j\beta_{n-1}^j)$ converges to $(e, e_0, \ldots, e_{n-1})$. Since for every $0 \leq i \leq n-1$ we have that $(\beta_i^j\,g_i)_{j \in \omega}$ converges to f_i, we obtain that $(\delta^j\beta_i^j\,g_i)_{j \in \omega}$ converges to ef_i; on the other hand, it converges to $e_i g_i$, proving $ef_i = e_i g_i$.

Now assume that I is countably infinite, and assume $I = \omega$. By the above, we obtain for every $n \geq 1$ elements $e^n, e_0^n, \ldots, e_{n-1}^n \in \mathbf{H}$ such that $e^n f_i = e_i^n g_i$ for all $0 \leq i \leq n-1$. We can embed the sequences $(e^n, e_0^n, \ldots, e_{n-1}^n) \in \overline{\mathbf{H}}^{n+1}$ into the product space

$$\prod_{n \geq 1} \overline{\mathbf{H}}^{n+1}$$

by first expanding them to a sequence in $\mathbf{H}^\omega$ by adding, an infinite number of times, the identity function $\mathrm{id} \in \mathbf{H}$, and then via the identification of $\mathbf{H}^\omega$ with a closed subspace of above product space, as in Lemma 4. Factoring every component $\overline{\mathbf{H}}^{n+1}$ of the latter by the equivalence relation $\sim$ induced by the action of $\mathbf{H}$ on the left, we obtain a compact space. There the equivalence classes of the sequences $(e^n, e_0^n, \ldots, e_{n-1}^n)$ have an accumulation point, namely the equivalence class induced by a sequence $(e, e_0, \ldots) \in \mathbf{H}^\omega$. Similarly as in the case where I was finite, we conclude $e f_i = e_i g_i$ for all $i \in \omega$. □

The implication from (2) to (3) in Proposition 1 now is a direct consequence of Lemma 3.

3. Canonisation

The following is the *canonisation theorem*, first proved combinatorially in [20] in a slightly more specialized context.

Theorem 5. *Let* $\mathbf{G} \curvearrowright X$, $\mathbf{H} \curvearrowright Y$ *be permutation groups, where* X *is countable,* $\mathbf{G}$ *is extremely amenable, and* $\mathbf{H}$ *is oligomorphic. Let* $f \colon X \to Y$. *Then*

$$\overline{\mathbf{H} f \mathbf{G}} := \overline{\{\beta f \alpha \mid \alpha \in \mathbf{G}, \beta \in \mathbf{H}\}}$$

contains a canonical function with respect to $\mathbf{G}$ *and* $\mathbf{H}$.

Proof. The space $\overline{\mathbf{H} f \mathbf{G}}/_\sim$ is a closed subspace of the compact Hausdorff space $Y^X/_\sim$ from Lemma 4, and hence is a compact Hausdorff space as well. We define a continuous action of $\mathbf{G}$ on this space by

$$(\alpha, [g]_\sim) \mapsto [g \, \alpha^{-1}]_\sim \ .$$

Clearly, this assignment is a function, it is a group action, and it is continuous. Since $\mathbf{G}$ is extremely amenable, the action has a fixed point $[g]_\sim$. Any member g of this fixed point is canonical: whenever $\alpha \in \mathbf{G}$, then $[g \, \alpha]_\sim = [g]_\sim$, which is the definition of canonicity. □

In applications of Theorem 5 (e.g., in [1, 2, 9, 10, 12–16, 18, 20, 21, 27–29, 31, 32]), one usually needs the following special case of the above situation. It states, roughly, that whenever we have a finite arity function f on a countable set, and an oligomorphic extremely amenable permutation group $\mathbf{G}$ on the same set, then we can obtain from f and $\mathbf{G}$, using composition and topological closure, a canonical function whilst retaining finite information about f.

In the following statement, for $m \geq 1$ we write $\mathbf{G}^m$ for the natural action of $\mathbf{G}^m$ on X^m given by $((\alpha_1, \ldots, \alpha_m), (x_1, \ldots, x_m)) \mapsto (\alpha_1(x_1), \ldots, \alpha_m(x_m))$. Moreover, we denote the pointwise stabilizer of $c^1, \ldots, c^n \in X^m$ in $\mathbf{G}^m$ by $(\mathbf{G}^m, c^1, \ldots, c^n)$.

Corollary 6. *Let* $\mathbf{G} \curvearrowright X$ *be an oligomorphic extremely amenable permutation group acting on a countable set* X. *Let* $f \colon X^m \to X$ *for some* $m \geq 1$, *and let* $c^1, \ldots, c^n \in X^m$ *for some* $n \geq 1$. *Then there exists*

$$g \in \overline{\mathbf{G}\, f\, \mathbf{G}^m}$$

such that

- *g agrees with f on $\{c^1, \ldots, c^n\}$, and*
- *g is canonical with respect to the groups $(\mathbf{G}^m, c^1, \ldots, c^n)$ and $\mathbf{G}$.*

Proof. The group $\mathbf{G}^m$ is obviously extremely amenable. Moreover, it is known that so is any stabilizer of it (in fact, every open subgroup; cf. [20]). The statement therefore follows from Theorem 5. $\square$

4. An Open Problem

Is there a converse of Theorem 5 in the sense that extreme amenability of $\mathbf{G}$ is equivalent to some form of the statement of the canonisation theorem? More precisely, we ask the following question.

Question 7. *Let* $\mathbf{G} \curvearrowright X$ *be a closed permutation group on a countable domain* X. *Is it true that* $\mathbf{G}$ *is extremely amenable if and only if it has the* canonisation property *of Theorem 5, i.e., for every oligomorphic permutation group* $\mathbf{H} \curvearrowright Y$ *and every* $f \colon X \to Y$ *the set* $\overline{\mathbf{H}\, f\, \mathbf{G}}$ *contains a function that is canonical with respect to* $\mathbf{G}$ *and* $\mathbf{H}$?

We remark that the canonisation property above implies, for example, that $\mathbf{G}$ preserves a linear order, as is the case when $\mathbf{G}$ is extremely amenable. For when $\mathbf{H} \curvearrowright X$ is any oligomorphic extremely amenable permutation group, and $g \in \overline{\mathbf{H}\,\mathrm{id}\,\mathbf{G}}$ is canonical, then it is easy to see that the preimage under g of any linear order preserved by $\mathbf{H}$ must be preserved by $\mathbf{G}$.

After publication of a draft of the present article, Trung Van Pham provided a positive answer to the above question for the case that $\mathbf{G}$ has an extremely amenable oligomorphic subgroup $\mathbf{H}$. This is an important case, since the first example of an oligomorphic group $\mathbf{G}$ not satisfying this condition was discovered only recently [22]. Pham's argument is combinatorial, using the Ramsey property; the following proof in the language of groups is the result of discussions with Antoine Mottet and Jakub Opršal.

Assuming that $\mathbf{G}$ is not extremely amenable, we show that $\overline{\mathbf{H}\,\mathrm{id}\,\mathbf{G}} = \overline{\mathbf{G}}$ does not contain any canonical function with respect to $\mathbf{G}$ and $\mathbf{H}$. To this end, let S be a compact Hausdorff space such that $\mathbf{G}$ acts continuously on S without a fixed point. Since $\mathbf{H}$ is extremely amenable, the restriction of this action $\mathbf{G} \curvearrowright S$ to $\mathbf{H}$ does have a fixed point $s \in S$. By restricting $\mathbf{G} \curvearrowright S$ to the closure of the orbit of s in S, we may assume that the orbit of s is dense in S.

As in the proof of Theorem 5, let $\mathbf{G}$ now act on $\overline{\mathbf{G}}/_\sim$ by $(\alpha, [g]_\sim) \mapsto [g\, \alpha^{-1}]_\sim$. Then the action $\mathbf{G} \curvearrowright S$ is a factor of the action $\mathbf{G} \curvearrowright \overline{\mathbf{G}}/_\sim$ via the mapping $\phi \colon \overline{\mathbf{G}}/_\sim \to S$ which sends every $[g]_\sim$ to the limit of $(\alpha_n^{-1}(s))_{n \in \omega}$,

for any sequence $(\alpha_n)_{n\in\omega}$ converging to g: it is well-defined since $\mathbf{H}$ fixes s, and if $(\beta_n)_{n\in\omega}$ is another such sequence, then $(\beta_n^{-1}\alpha_n)_{n\in\omega}$ converges to the identity, which fixes s, and so $(\alpha_n^{-1}(s))_{n\in\omega}$ converges to the limit of $(\beta_n^{-1}(s))_{n\in\omega}$ by continuity. Moreover, by definition ϕ is compatible with the two actions, i.e., $\phi([g\,\alpha^{-1}]_\sim) = \alpha(\phi([g]_\sim)$ for all $g \in \overline{\mathbf{G}}$ and all $\alpha \in \mathbf{G}$.

Since $\mathbf{G} \curvearrowright S$ does not have a fixed point, and since it is a factor of $\mathbf{G} \curvearrowright \overline{\mathbf{G}}/_\sim$, the latter cannot have a fixed point either. As in the proof of Theorem 5, fixed points of $\mathbf{G} \curvearrowright \overline{\mathbf{G}}/_\sim$ correspond precisely to canonical functions with respect to $\mathbf{G}$ and $\mathbf{H}$ in $\overline{\mathbf{G}}$, and we conclude that $\overline{\mathbf{G}}$ does not contain any canonical function.

References

1. L. Agarwal, *The reducts of the generic digraph*, Annals of Pure and Applied Logic **167** (2016), 370–391.
2. L. Agarwal and M. Kompatscher, $2^{\aleph_0}$ *pairwise nonisomorphic maximal-closed subgroups of Sym(ℕ) via the classification of the reducts of the Henson digraphs*, Journal of Symbolic Logic **83** (2018), no. 2, 395–415.
3. L. Barto, M. Kompatscher, M. Olšák, T. V. Pham, and M. Pinsker, *The equivalence of two dichotomy conjectures for infinite domain constraint satisfaction problems*, Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science – LICS'17, 2017, Preprint arXiv:1612.07551.
4. ———, *Equations in oligomorphic clones and the constraint satisfaction problem for ω-categorical structures*, Journal of Mathematical Logic **19** (2019), no. 2, #1950010.
5. L. Barto, J. Opršal, and M. Pinsker, *The wonderland of reflections*, Israel Journal of Mathematics **223** (2018), no. 1, 363–398.
6. L. Barto and M. Pinsker, *The algebraic dichotomy conjecture for infinite domain constraint satisfaction problems*, Proceedings of the 31th Annual IEEE Symposium on Logic in Computer Science – LICS'16, 2016, Preprint arXiv:1602.04353, pp. 615–622.
7. ———, *Topology is irrelevant*, SIAM Journal on Computing **49** (2020), no. 2, 365–393.
8. V. Bergelson, *Ergodic Ramsey theory: a dynamical approach to static theorems*, Proceedings of the International Congress of Mathematicians (Zürich), vol. II, European Mathematical Society, 2006, pp. 1655–1678.
9. M. Bodirsky, P. Jonsson, and T. V. Pham, *The reducts of the homogeneous binary branching C-relation*, Journal of Symbolic Logic **81** (2016), no. 4, 1255–1297, Preprint arXiv:1408.2554.
10. ———, *The Complexity of Phylogeny Constraint Satisfaction Problems*, ACM Transactions on Computational Logic (TOCL) **18** (2017), no. 3, An extended abstract appeared in the conference STACS 2016.
11. M. Bodirsky and M. Junker, $\aleph_0$-*categorical structures: interpretations and endomorphisms*, Algebra Universalis **64** (2011), no. 3-4, 403–417.
12. M. Bodirsky, B. Martin, M. Pinsker, and A. Pongrácz, *Constraint satisfaction problems for reducts of homogeneous graphs*, SIAM Journal on Computing **48** (2019), no. 4, 1224–1264, A conference version appeared in the Proceedings of the 43rd International Colloquium on Automata, Languages, and Programming, ICALP 2016, pages 119:1-119:14.
13. M. Bodirsky and A. Mottet, *Reducts of finitely bounded homogeneous structures, and lifting tractability from finite-domain constraint satisfaction*, Proceedings of the 31th Annual IEEE Symposium on Logic in Computer Science (LICS), 2016, Preprint available at ArXiv:1601.04520, pp. 623–632.

14. M. Bodirsky and M. Pinsker, *Reducts of Ramsey structures*, AMS Contemporary Mathematics, vol. 558 (Model Theoretic Methods in Finite Combinatorics) (2011), 489–519.

15. _____, *Minimal functions on the random graph*, Israel Journal of Mathematics **200** (2014), no. 1, 251–296.

16. _____, *Schaefer's theorem for graphs*, Journal of the ACM **62** (2015), no. 3, 52 pages (article number 19), A conference version appeared in the Proceedings of STOC 2011, pages 655–664.

17. _____, *Topological Birkhoff*, Transactions of the American Mathematical Society **367** (2015), 2527–2549.

18. M. Bodirsky, M. Pinsker, and A. Pongrácz, *The 42 reducts of the random ordered graph*, Proceedings of the LMS **111** (2015), no. 3, 591–632, Preprint available from arXiv:1309.2165.

19. _____, *Projective clone homomorphisms*, Journal of Symbolic Logic (2019), 1–13, doi:10.1017/jsl.2019.23.

20. M. Bodirsky, M. Pinsker, and T. Tsankov, *Decidability of definability*, Journal of Symbolic Logic **78** (2013), no. 4, 1036–1054, A conference version appeared in the Proceedings of the Twenty-Sixth Annual IEEE Symposium on. Logic in Computer Science (LICS 2011), pages 321–328.

21. M. Bodirsky and M. Wrona, *Equivalence constraint satisfaction problems*, Proceedings of Computer Science Logic, LIPICS, vol. 16, Dagstuhl Publishing, September 2012, pp. 122–136.

22. D. M. Evans, J. Hubička, and J. Nešetřil, *Automorphism groups and Ramsey properties of sparse graphs*, Proceedings of the London Mathematical Society **119** (2019), 515–546.

23. W. Hodges, *A shorter model theory*, Cambridge University Press, Cambridge, 1997.

24. A. Kechris, V. Pestov, and S. Todorčević, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geometric and Functional Analysis **15** (2005), no. 1, 106–189.

25. A. S. Kechris, *Dynamics of non-archimedean Polish groups*, Proceedings of the European Congress of Mathematics, Krakow, European Math. Society, 2014, pp. 375–397.

26. B. Klin, E. Kopczynski, J. Ochremiak, and S. Toruńczyk, *Locally finite constraint satisfaction problems*, 30th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2015, Kyoto, Japan, 2015, pp. 475–486.

27. B. Klin, S. Lasota, J. Ochremiak, and S. Torunczyk, *Homomorphism Problems for First-Order Definable Structures*, 36th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2016) (Dagstuhl, Germany) (Akash Lal, S. Akshay, Saket Saurabh, and Sandeep Sen, eds.), Leibniz International Proceedings in Informatics (LIPIcs), vol. 65, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2016, pp. 14:1–14:15.

28. M. Kompatscher and T. V. Pham, *A complexity dichotomy for poset constraint satisfaction*, IfCoLog Journal of Logics and their Applications (FLAP) **5** (2018), no. 8, 1663–1696, An extended abstractappeared at the 34th Symposium on Theoretical Aspects of Computer Science (STACS) 2017.

29. J. Linman and M. Pinsker, *Permutations on the random permutation*, Electronic Journal of Combinatorics **22** (2015), no. 2, 1–22.

30. A. Mottet and M. Pinsker, *Cores over Ramsey structures*, To appear in the Journal of Symbolic Logic (2021).

31. P. Pál Pach, M. Pinsker, G. Pluhár, A. Pongrácz, and C. Szabó, *Reducts of the random partial order*, Advances in Mathematics **267** (2014), 94–120.

32. A. Pongrácz, *Reducts of the Henson graphs with a constant*, Annals of Pure and Applied Logic **168** (2017), no. 7, 1472–1489.

33. F. M. Schneider, *A uniform Birkhoff theorem*, Algebra Universalis **78** (2017), no. 3, 337–354.

Institut für Algebra, TU Dresden, 01062 Dresden, Germany
E-mail address: `Manuel.Bodirsky@tu-dresden.de`
URL: `http://www.math.tu-dresden.de/~bodirsky/`

Institute of Discrete Mathematics and Geometry, Technische Universität Wien, Austria, and Department of Algebra, Charles University in Prague, Czech Republic
E-mail address: `marula@gmx.at`
URL: `http://dmg.tuwien.ac.at/pinsker/`