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HOMOGENEOUS STRUCTURES - A LIST OF OPEN PROBLEMS

Abstract. The workshop "Homogeneous Structures, A Workshop in Honour of Norbert Sauer's 70th Birthday" took place at the Banff International Research Station from November 8th to November 13th, 2015. The purpose of the following note is to gather the list of open problems that were posed during the problem sessions. Contributions appear in alphabetical order, according to the name of their author.

1. Itaï Ben Yaacov, Université Claude Bernard

Problem. Let G be a Polish Roelcke precompact group with compatible left invariant metric d_L. Define a new metric as:

$$d_u(g,h) = \sup_{k \in G} d_L(gk, hk)$$

Is there any relationship between discreteness of d_u and the fact that G cannot act transitively by isometries on a complete metric space?

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2. Gabriel Conant, University of Notre-Dame

Problem 1. Suppose that S is an infinite set of positive reals with the 4-values condition, as defined in [2]. Fix $A \subset S$ finite. Is there a finite S_0 with the 4-values condition such that $A \subseteq S_0 \subset S$?

Remark. In [5], Sauer shows that Problem 1 is true in the case that S is closed under the binary operation $u +_S v := \sup\{x \in S : x \leq u + v\}$. (Sauer also shows that, for such S, the 4-values condition is equivalent to associativity of $+_S$.)

Given a finite set S of positive reals, we write $S = \{s_1, \ldots, s_n\}_<$ to mean $S = \{s_1, \ldots, s_n\}$ and $s_1 < \cdots < s_n$.

Definition ([4]). Suppose $S = \{s_1, \ldots, s_n\}_<$ and $T = \{t_1, \ldots, t_n\}_<$ are finite sets of positive reals. Then $S \sim T$ if, for all $i, j, k \leq n$, $s_i \leq s_j + s_k$ if

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and only if $t_i \leq t_j + t_k$.

If $S = \{s_1, \ldots, s_n\}_<$ and $T = \{t_1, \ldots, t_n\}_<$, then $S \sim T$ if and only if the map $s_k \mapsto t_k$ is an isomorphism from $(S, +_S, \leq)$ to $(T, +_T, \leq)$. In particular, the 4-values condition is invariant under $\sim$-equivalence, and so it would be interesting to find somewhat canonical $\sim$-representatives for finite sets of reals satisfying the 4-values condition. An initial observation is that, for any finite set S of positive reals, there is a finite set T of positive *integers* such that $S \sim T$. (For instance, this follows easily from the fact that $(\mathbb{Q}, +, <, 0)$ is an elementary substructure of $(\mathbb{R}, +, <, 0)$.) Therefore, it suffices to look for isomorphism representatives for finite sets of positive integers.

Problem 2. Suppose $S = \{s_1, \ldots, s_n\}_<$ is a finite set of positive integers. Is there a set $T = \{t_1, \ldots, t_n\}_<$ of positive integers such that $S \sim T$ and, for all $k \leq n$, $2^k - 1 \leq t_k \leq 2^n - 1$?

Remarks.

(1) Problem 2 has been verified for $n \leq 6$ (see [1], [4]).
(2) The notion of $\sim$-equivalence has some similarity to *Freiman isomorphism* [3]. Conjecture 1 of [3] asks a similar question concerning representatives of Freiman isomorphism.

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3. Cameron Freer, Massachusetts Institute of Technology

Call a relational structure *highly homogeneous* when for every k, its automorphism group acts transitively on the k-element sets. Peter Cameron classified the countable highly homogeneous relational structures in 1976, and all of them turn out to satisfy the strong amalgamation property.

Problem: Is there an elementary proof of this latter fact, without referring to the classification?

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4. Robert Gray, University of East Anglia

A relational structure M is *set-homogeneous* if whenever two finite substructures U and V are isomorphic, there is an automorphism $g \in \mathrm{Aut}(M)$ such that $g(U) = V$. As an example, consider the countably infinite graph $R(3)$ defined as follows. Its vertex set is a countable dense subset of the unit circle with no two points make an angle of $2\pi/3$ at the centre of the circle. Its edge set is given by $x \sim y$ if and only if $0 < \arg(x/y) < 2\pi/3$.

Problem 1 ([DGMS94]). Are $R(3)$ and its complement $\overline{R(3)}$ the only countable set-homogeneous graphs which are not homogeneous?

Problem 2 ([GMPR12]). Is there a countable set-homogeneous tournament that is not homogeneous?

Related to this, there is also the problem:

Problem 3. Classify the countably infinite set-homogeneous digraphs.

A structure (with an age which has finitely many n-element structures up to isomorphism) is *homogenizable* if it can be made homogeneous by adding finitely many relations to the language (without changing the automorphism group).

Problem 4 ([DGMS94]).
Is there a set-homogeneous structure which is not homogenizable?

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5. Jordi Lopez-Abad, Instituto de Ciencias Matemáticas

Problem 1 ([1]). Does the dual Ramsey theorem hold for equidistributed partitions?

Problem 2. Let $\mathbb{G}$ be the Gurarij space, and $\mathrm{iso}(\mathbb{G})$ be its linear isometry group. Is $\mathrm{iso}(\mathbb{G})$ compactly approximable, i.e. does it admit an increasing chain of compact subgroups whose union is dense? Is $\mathrm{iso}(\mathbb{G})$ Lévy?

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6. Martino Lupini, California Institute of Technology

Problem 1 ([4])**.** Let $\mathbb{G}$ be the Gurarij space, and $\mathrm{iso}(\mathbb{G})$ be its linear isometry group. Does it have the automatic continuity property, i.e. is every algebraic homomorphism from $\mathrm{iso}(\mathbb{G})$ to a separable group continuous?

Problem 2. Given a Fraïssé class (or a metric Fraïssé class) $\mathcal{C}$ with limit M, is there a natural assumption on $\mathcal{C}$ that guarantees that for every countable substructure X of M, $\mathrm{Aut}(X)$ embeds in $\mathrm{Aut}(M)$ via some embedding j of X in M so that every automorphism of $j(X)$ extends to an automorphism of M?

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7. Lionel Nguyen Van Thé, Aix-Marseille University

Call a topological group G *extremely amenable* when every continous action of G on a compact Hausdorff space admits a fixed-point. One of the first natural groups for which this property was proved is the group of all linear isometries of the separable real Hilbert space ℓ_2, equipped with the topology of pointwise convergence [2]. According to the paper [3], this suggests that some Ramsey-type property could hold at the level of finite Euclidean metric spaces. For example:

Problem ([3])**.** Call a metric subspace X of ℓ_2 *affinely independent* when no $x \in X$ belongs to the affine span generated by $X \setminus \{x\}$. Let $\mathcal{E}^<$ denote the class of all finite, ordered, affinely independent subspaces of ℓ_2 with rational distances. Is this a Ramsey class? Equivalently: Given $A, B \in \mathcal{E}^<$, $k \in \mathbb{N}$, is there $C \in \mathcal{E}^<$ such that for every k-coloring of the (isometric, ordered) copies of A in C, there is a copy $\tilde{B}$ of B in C such that all copies of A in $\tilde{B}$ have the same color?

At the moment, the only positive result in that direction is essentially due to Frankl–Rödl in [1]. It corresponds to the case where A is reduced to a single point, and no order is taken into account.

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8. Micheal Pawliuk, University of Toronto

Problem ([1])**.** Let $\mathbb{S}$ denote the semigeneric digraph. Is $\mathrm{Aut}(\mathbb{S})$ uniquely ergodic, i.e. is there a unique invariant Borel probability measure whenever $\mathrm{Aut}(\mathbb{S})$ acts continuously and minimally on a compact Hausdorff space? Equivalently, is there a unique invariant Borel probability measure on the universal minimal flow of $\mathrm{Aut}(\mathbb{S})$?

Remark. A positive answer has been announced in [2].

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9. Michael Pinsker, Charles University

Problem 1. Let $F \subset \omega^{\omega^3}$ be topologically closed, closed under composition, and containing the set $\{\pi_1, \pi_2, \pi_3\}$, where π_i denotes the ith projection. Assume that there exists a map $\xi : F \to \{\pi_1, \pi_2, \pi_3\}$ which preserves composition. Is there a continuous such map?

Problem 2. Same question as above, assuming that the set of functions $\{x \mapsto f(x,x,x) : f \in F\} \subset \omega^\omega$ contains an oligomorphic group.

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10. Maurice Pouzet, Université Claude-Bernard / University of Calgary

First, here are two old problems about the notions of indivisibility and minimality introduced by Roland Fraïssé.

A relational structure R is *indivisible* (resp. *age-indivisible*) if for every partition of its domain into two parts, say A, B, one of the induced structures $R_{\restriction A}$ and $R_{\restriction B}$ embeds R (resp. has the same age as R). See [3]. An indivisible structure is obviously age-indivisible.

Problem 1. If R is age-indivisible, is there an indivisible relational structure with the same age as R? Note that the answer is yes if the maximum of the arities is 3. See [6].

A relational structure R is *minimal for its age* if every induced substructure with the same age as R embeds R (see [3]). Not every age has a structure which is minimal for this age (see [5]), but, trivially, if R is age-indivisible and minimal for its age, then R is indivisible.

Problem 2. If R is age-indivisible, is there R' with the same age that is also minimal for its age?

Next, here are some problems on the collection of orbits of an oligomorphic group.

The *profile* of a relational structure M is the function φ_M which counts for every nonnegative integer n the number $\varphi_M(n)$ of n-element substructures of M counted up to isomorphy, see [7]. When M is homogeneous, this counts the orbits of the action of $\mathrm{Aut}(M)$ on the n-element subsets of the domain of M. The *age of a group G of permutations on a set V* is the set $\mathrm{Age}(G)$ of orbits of finite subsets of V. This set can be ordered as follows: for two orbits O', O'' we set $O' \leq O''$ if there are subsets F' and F'' of V such that $F' \subseteq F''$, $O' = \mathrm{Orb}(F')$ and $O'' = \mathrm{Orb}(F'')$. As an ordered set, $\mathrm{Age}(G)$ is ranked; elements of rank n being the orbits of n-element subsets of V. The function φ_G which counts for each integer n the (cardinal) number $\varphi_G(n)$ of orbits of n-element subsets is the *orbital profile* of G. If the number of these orbits is finite for each integer n then G is *oligomorphic* (see [1]).

Problem 3. If φ_G is not bounded above by some exponential function of n, is it true that $\mathrm{Age}(G)$ contains an infinite antichain?

The following result yields groups whose ages have no infinite antichain (use the test given in [8]).

Theorem. Let M be a relational structure. If for every nonnegative integer n the class $\mathrm{Age}_{n^-}(M)$ of finite substructures N of M with n-labelled elements, say $(N, \{a_1, \ldots a_n\})$, has no infinite antichain, then there is some M' with the same age as M whose theory is $\aleph_0$-categorical and inductive. Moreover, if the set of initial segment of $\mathrm{Age}_{\omega^-}(M) := \bigcup_{n<\omega} \mathrm{Age}_{n^-}(M)$ has no infinite antichain then $\mathrm{Age}(G')$, where $G' := \mathrm{Aut}(M')$, has no infinite antichain.

Finally, I will present some problems about polynomially bounded profiles.

Cameron conjectured that if G is a permutation group, φ_G is bounded above by some polynomial function of n then $\varphi_G(n) \simeq a.n^k$ for some $a > 0$ and $k \in \mathbb{N}$ (see [1]). Macpherson [4] asked if the fact that φ_G is bounded above by some polynomial function implies that the Cameron algebra of G is finitely generated (this algebra is made of linear combinations of members of $\mathrm{Age}(G)$ (see [1]). A positive answer to this question has been announced by Falque and Thiery in [2]. It implies that the generating series associated to the profile is a rational fraction; Cameron's conjecture follows.

We may note that there are relational structures whose profile is bounded above by some polynomial and in fact the generating series is a rational fraction but for which the age algebra of Cameron is not finitely generated [9].

A description of those groups whose orbital profile is bounded above by some polynomial has yet to come. I propose the following approach: According to Schmerl [10] a relational structure M is *cellular* if its domain V is the disjoint union of a finite set F and a set which can be identified to the cardinal product $K \times L$ such that (1) for every permutation f of L the map $(1_K, f) \cup 1_F$ is an automorphism of M. I propose a variation of this notion, replacing (1) by the following condition:

(2) The substructures induced on two finite sets A and A' with the same cardinality are isomorphic provided that $(2a)$ $A \cap F = A' \cap F$ and $(2b)$ the frequency vectors $\chi_{A \setminus F}$ and $\chi_{A' \setminus F}$ are equal (the frequency vector $\chi_{A \setminus F}$ associates to every non-empty subset K' of K the number $\chi_{A \setminus F}(K') := |\{\ell \in L : A \cap (K \times \{\ell\}) = K' \times \{\ell\}\}|$).

I will say that M is *set-cellular* if it has such a decomposition.

Problem 4.

(1) Is it true that if $\varphi(M)$ is bounded above by some polynomial, then M is set-cellular (the converse holds trivially)?
(2) Is it true that if M is set-cellular, then the generating series of the profile is a rational fraction?

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11. Norbert Sauer, University of Calgary

Problem. Let R be a relational structure on $\mathbb{N}$. Consider the poset $\mathbb{P}$ of all copies of R in itself, ordered by inclusion. What does it look like? For example, does it have an infinite antichain? Note that it is known that the cardinality of $\mathbb{P}$ is either 1 or continuum. Furthermore, if R is ω-categorical, the second alternative always holds and $\mathcal{P}(\mathbb{N})$ embeds into $\mathbb{P}$.

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12. John Truss, University of Leeds

In [2] it is shown that if p is a partial endomorphism of a finite (simple) graph Δ, then there is a finite graph $\Delta' \supseteq \Delta$ and a (totally defined) endomorphism of Δ' extending p.

Problem. Prove that if Δ is a finite graph, then there is a finite graph $\Delta' \supseteq \Delta$, such that every partial endomorphism of Δ extends to an endomorphism of Δ'.

This would be the analogue of Hrushovski's Lemma for partial automorphisms in the endomorphism context. Here, by "endomorphism" is understood a map such that for any two points of its domain, if they are joined by an edge in the graph, then so are their images (but two points which are not joined are allowed to be mapped to an edge, a nonedge, or collapsed to a point).

Note that there are many examples where the corresponding property is known to hold for partial automorphisms. In fact there is a large literature on it, see for example [1]. For partial endomorphisms, no instances are known (apart from the trivial structure).

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13. Andrew Zucker, Université Paris-Diderot

Problem. Let G be a compactly approximable Polish group. Is G uniquely ergodic?

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