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CONFINING THE ROBBER ON COGRAPHS

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ABSTRACT. In a game of Cops and Robbers on graphs, usually the cops' objective is to capture the robber—a situation which the robber wants to avoid invariably. In this paper, we begin with introducing the notions of *trapping* and *confining* the robber and discussing their relations with capturing the robber. Our goal is to study the confinement of the robber on graphs that are free of a fixed path as an induced subgraph. We present some necessary conditions for graphs G not containing the path on k vertices (referred to as P_k-free graphs) for some $k \geq 4$, so that $k-3$ cops do not have a strategy to capture or confine the robber on G (Propositions 2.1, 2.3). We then show that for planar cographs and planar P_5-free graphs the confining cop number is at most one and two, respectively (Corollary 2.4). We also show that the number of vertices of a connected cograph on which one cop does not have a strategy to confine the robber has a tight lower bound of eight. Moreover, we explore the effects of twin operations—which are well known to provide a characterization of cographs—on the number of cops required to capture or confine the robber on cographs. Finally, we pose two conjectures on confining the robber on P_5-free graphs and the smallest planar graph of confining cop number of three.

1. INTRODUCTION

A *game of Cops and Robbers* is a pursuit game on graphs, or a class of graphs, in which a set of agents, called the *cops*, try to get to the same position as another agent, called the *robber*. Among several variants of such a game, we solely consider the one introduced in [1], which is played on finite undirected graphs. Hence, we will simply refer to this variant as "the" game of Cops and Robbers. Let G be a simple undirected graph. Consider a finite set of cops and a robber. The game on G goes as follows. At the beginning of the game (step 1), each cop will be positioned in a vertex of G and then the robber will be positioned in some vertex of G. In each of the subsequent steps, each agent either moves to a vertex adjacent to its current

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vertex or stays still, with the robber taking turns after all of the cops. The cops win if one of the cops *captures* the robber, i.e., gets to the vertex where the robber is located. The minimum number of cops that are guaranteed, irrespective of how the robber plays, to capture the robber on G in a finite number of steps is called the *cop number* of G and denoted $C(G)$. Graph G is said to be k-*copwin* ($k \in \mathbb{N}$) if $C(G) \leq k$. A 1-copwin graph is simply referred to as a *copwin* graph. Since the cop number of a graph is equal to the sum of the cop numbers of its components, whenever the cop number of a graph G is concerned G is considered to be connected, unless otherwise stated. A class $\mathcal{G}$ of graphs is called *cop-bounded* if there is $k \in \mathbb{N}$ such that $C(G) \leq k$ for every $G \in \mathcal{G}$. Among the cop-bounded classes of graphs, we can mention the class of trees, which is cop-bounded by one, and the class of planar graphs, which is cop-bounded by three [1]. For more background on the game of Cops and Robbers and the cop number, see [3].

Notation.

- *If a, b are integers with $a \leq b$, we denote the set of integers between a and b, both inclusive, by $[a, b]$.*
- *When v and v are vertices in a graph G, we denote their graph distance in G by $d_G(u, v)$.*
- *When G is a graph and U is a subset of its vertex set, $G[U]$ denotes the subgraph of G induced by U.*
- *Let U and W be disjoint subsets of the vertex set of a graph G. Then we write $U \Leftrightarrow_G W$ (or simply $U \Leftrightarrow W$ if the graph G is understood from the context) to mean that every vertex in U is adjacent to every vertex in W.*

Definition 1.1. *Let $G = (V, E)$ be a graph. For each $v \in V$ we define the* open neighborhood *$N_G(v)$ of v to be $\{w \in V : vw \in E\}$ and the* closed neighborhood *$N_G[v]$ to be $N_G(v) \cup \{v\}$.*

Definition 1.2. *We say a vertex x of G is a* dominated vertex *or a* corner *if there is another vertex y of G such that $N_G[x] \subseteq N_G[y]$, in which case we also say that y* dominates *x (in G). An* elimination ordering *of a graph G is an ordering, say, $v_1, \ldots, v_n$ of the vertices of G where each v_i ($i \in [1, n-1]$) is a corner of $G[\{v_j : j \in [i, n]\}]$. Graphs that admit an elimination ordering are called* dismantlable.

Theorem 1.3 ([9, 10, 11])**.** *A graph is copwin if and only if it is dismantlable.*

Definition 1.4. *Let $\mathcal{H}$ be a set of graphs. A graph G is called $\mathcal{H}$-*free *if no graph in $\mathcal{H}$ is an induced subgraph of G. If $\mathcal{H}$ is a singleton, say $\{H\}$, we will use $\{H\}$-free and H-free interchangeably.*

The game of Cops and Robbers on graphs with one forbidden induced subgraph was studied in [6]. The main results in [6] are summarized as follows:

Theorem 1.5 ([6])**.**

(1) *For a graph H, the class of H-free graphs is cop-bounded if and only if every component of H is a path.*
(2) *The class of P_k-free graphs ($k \geq 3$) is $(k-2)$-copwin.*

The results in [6] were extended [7, 8], mainly through the introduction of the *Train-chasing Lemma* (Lemma 1.7), to the game of Cops and Robbers on graphs with a set of forbidden induced subgraphs.

Definition 1.6 ([8])**.** *Let G be a graph and U be the set of all triples (u, v, H) where H is a connected subgraph of G, and $u, v \in V(H)$ with $d_H(u, v) \geq 2$. A* chasing function for G *is a function θ mapping every triple $(u, v, H) \in U$ onto the neighbor of u along a (u, v)-shortest path in H.*

Lemma 1.7 (Train-chasing Lemma [8])**.** *Consider an instance of the game of Cops and Robbers on a graph G. Let θ be a chasing function for G. Let $k \in \mathbb{N}$ and suppose on the cops' turn in step one there are k cops $C_1, \ldots, C_k$ in a vertex v_1 of the graph while the robber is located in a vertex w_1. Further, suppose the robber can and will play in such a way to survive the next k steps of the game, regardless of how the cops $C_1, \ldots, C_k$ play. Denote the following (generally not predetermined) robber's positions with $w_2, \ldots, w_k$. Then, let H_i ($i \in [1, k]$) and v_i ($i \in [2, k]$) be defined recursively by the following relations:*

- *$H_1 = G$;*
- *$v_{i+1} = \theta(v_i, w_i, H_i)$ for $i \in [1, k]$;*
- *$X_1 = N_{H_1}(v_1) \setminus \{v_2\}$;*
- *$X_i = N_{H_i}(v_i) \setminus \{v_{i-1}, v_{i+1}\}$ for $i \in [2, k]$;*
- *H_{i+1} : the component of v_1 in $H_i - X_i$ for $i \in [1, k]$.*

Then the following holds:

(1) *Every H_i is an induced subgraph of G.*
(2) *If $uv \in E(G) \setminus E(H_{k+1})$ such that $u \in V(H_{k+1})$, then $v \in \bigcup_{i=1}^{k} X_i$.*
(3) *Vertices $v_1, \ldots, v_{k+1}$, in that order, induce a path in H_k.*
(4) *The cops can play such that on the cops' turn in step k every C_i, $i \in [1, k]$, is located in vertex v_i.*
(5) *Keeping every C_i in v_i for the rest of the game forces the robber to stay in H_{k+1}.*

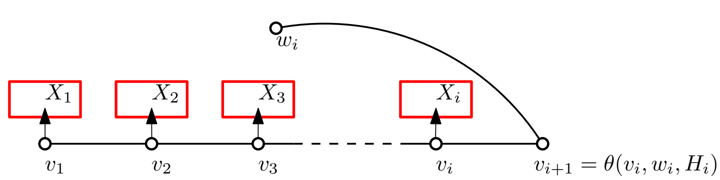

FIGURE 1. Train-chasing the robber according to Lemma 1.7 [8].

In [8], the Train-chasing Lemma was, in particular, used to characterize classes $\mathcal{F}$ of graphs such that $\mathcal{F}$-free graphs are cop-bounded, under the condition that there is a constant bounding the diameter of the components of elements of $\mathcal{F}$. The resultant characterization generalizes Theorem 1.5(a). It is worth mentioning that the following extension of Theorem 1.5(b) is also an immediate corollary of the Train-chasing Lemma. (See [12] for the definition of the *one-active-cop* version of the game of Cops and Robbers.)

Theorem 1.8 ([8]). *For $k \geq 3$, $k-2$ cops require no more than $k-1$ steps of the game to capture the robber on a P_k-free graph in the one-active-cop version of the game of Cops and Robbers.*

In this paper, we consider P_k-free graphs from the viewpoint of some new notions relevant to the cop number of graphs, described below.

Definition 1.9. *The* trapping cop number *of a graph G, denoted* $\mathrm{tcn}(G)$, *is the minimum number of cops that can force an arrangement of the cops and the robber on vertices of G in which the robber has to invariably stay in the closed neighborhood $N_G[v]$ of a vertex v in order to avoid immediate capture, in which case we say that the cops have* trapped *the robber.*

Definition 1.10. *The* confining cop number *of a graph G, denoted* $\mathrm{ccn}(G)$, *is the minimum number of cops that can force an arrangement of the cops and the robber on vertices of G in which the robber has to stay in its position in order to avoid capture in the next move of the cops, in which case we say that the cops have* confined *the robber.*

Definition 1.11. *Let G be a graph with $|G| \geq 3$. We call a vertex v of G a* confined corner *of G if there exists a vertex w such that $d_G(v,w) = 2$ and $N_G(v) \subseteq N_G(w)$, in which case w is said to* confine *v in G.*

On a graph, the of cop number, trapping cop number, and confining cop number are related through the following inequalities.

Proposition 1.12. *For every graph G we have*

$$\mathrm{tcn}(G) \leq \mathrm{ccn}(G) \leq C(G) \leq \mathrm{tcn}(G) + 1.$$

Proof. The first two inequalities are obvious. As for the last one, note that with $\mathrm{tcn}(G) + 1$ cops available, $\mathrm{tcn}(G)$ of them eventually force the robber to stay in $N_G[v]$ for some vertex v. By keeping those cops stationary and placing the remaining cop in v, the capture of the robber by the following step of the game will be guaranteed. □

It is known that the cop number of any graph having girth ≥ 5 is at least as large as its minimum degree:

Proposition 1.13 ([1]). *For a graph G with minimum degree δ we have $C(G) \geq \delta$ provided the girth of G is at least 5.*

The proof of Proposition 1.13 indeed establishes the following stronger result, which is in terms of the confining cop number of graphs.

Proposition 1.14. *For a graph G with minimum degree δ we have* $\mathrm{ccn}(G) \geq \delta$ *provided the girth of G is at least 5.*

Corollary 1.15. *For every graph G of order ≤ 9 we have* $\mathrm{ccn}(G) \leq 2$. *Moreover, the Petersen graph is the only graph on 10 vertices whose confining cop number is equal to 3.*

Proof. As shown in [2], the cop number of the Petersen graph is three, whereas every graph G on at most 10 vertices which is not the Petersen graph has $C(G) \leq 2$. Moreover, by Proposition 1.14, the confining cop number of the Petersen graph is at least three. Hence, in light of Proposition 1.12 the desired claims follow. □

In light of Proposition 1.12, the following result can be presented as an extension of Theorem 1.8.

Theorem 1.16. *If G is a P_k-free graph for some $k \geq 3$, then* $\mathrm{tcn}(G) \leq k-3$. *Furthermore, $k-3$ cops need no more than $k-3$ steps of the game to trap the robber in the one-active-cop version of the game of Cops and Robbers.*

Sketch of proof. The proof is just an adaptation of the proof of Theorem 1.8 with $k-3$ cops in play. See [8] for details. □

Remark: The case $k = 3$ is a triviality. Also, note that by Propositions 1.12 and Theorem 1.16, for a P_k-free graph G we have $\mathrm{tcn}(G) > k-3$ if and only if $\mathrm{tcn}(G) = C(G) = k-2$.

Notation. *Given $k \geq 4$, we will denote the class of all connected P_k-free graphs G satisfying* $\mathrm{ccn}(G) = k-2$ *(resp. $C(G) = k-2$) by $\mathcal{G}_{k,c}$ (resp. $\mathcal{G}_k$).*

In section 2 we will establish some necessary conditions for elements of $\mathcal{G}_k$ and $\mathcal{G}_{k,c}$. In light of such conditions, in section 3 we will consider the game of Cops and Robbers on P_4-free graphs, also known as *cographs*.

Definition 1.18. *Distinct vertices u, v in a graph G are said to be* twins *(or to form a* twin pair*) if every other vertex in G is adjacent to both u and v, or nonadjacent to both u and v. A pair u, v of twin vertices in G is called* true *(resp.* false*) whenever $N_G[u] = N_G[v]$ (resp. $N_G(u) = N_G(v)$). Given $G = (V, E)$, a* twin operation *on G is an operation of adding a new vertex w' to G so that $N_{G'}(w') = N_G(w)$ (*false-twin operation*) or $N_{G'}(w') = N_G[w]$ (*true-twin operation*) for some $w \in V(G)$.*

Several characterizations of cographs were established in [5], one of which states that a graph G is a cograph if and only if every nontrivial induced subgraph of G has a pair of twins. It can be easily seen that the latter implies the following characterization, which is of our special interest in section 3:

Theorem 1.19. *A connected nontrivial graph G is a cograph if and only if it can be obtained from K_2 by a sequence of twin operations.*

Remark: For general graph theoretic definitions see [4].

2. Some properties of $\mathcal{G}_k$ and $\mathcal{G}_{k,c}$

It is easy to see that $\mathcal{G}_{k,c} \subseteq \mathcal{G}_k$. In that regard, first, in Proposition 2.1, we present some properties of $\mathcal{G}_k$, and then, in Proposition 2.3, refine those properties for the subclass $\mathcal{G}_{k,c}$ of $\mathcal{G}_k$. We point out that both of these technical propositions are similar to the Train-chasing Lemma and are established in a more-or-less similar fashion.

Proposition 2.1. *Let $G \in \mathcal{G}_k$ and $v_1 \in V(G)$. With $k-3$ cops available, suppose the robber uses any winning strategy against the cops. In addition, suppose the cops start at v_1 and play according to any chasing function θ for G in the first $k-3$ steps of the game. Denote the position at the end of step $k-3$ of the robber by w. Let H_i and v_i be as in Lemma 1.7. Furthermore, for $j \in [1,(k-3)]$ let*

$$M_j := N_G(v_j) \setminus \bigcup \{N_G[v_i] : 1 \leq i \leq k-2,\ i \neq j\},$$

and for $j > k-3$ let M_j be the jth neighborhood of v_1 in H_{k-2}.

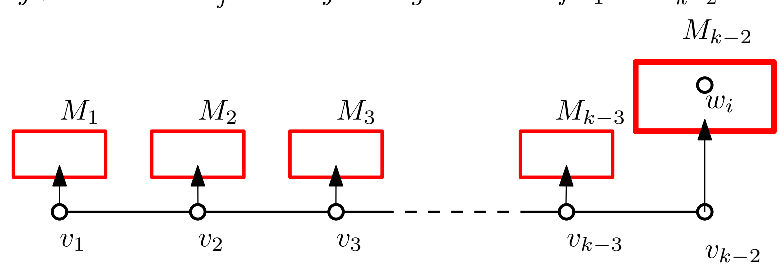

Figure 2. An illustration of M_js defined in Proposition 2.1.

Then:

1. *$M_j = \varnothing$ for $j \geq k-1$;*
2. *$M_j \neq \varnothing$ for each $j \in [1,(k-2)]$;*
3. *$M_1 \Leftrightarrow M_{k-2}$;*
4. *for each $u \in M_1$ and $z \in M_{k-2}$, $G[\{u, z, v_1, \ldots, v_{k-2}\}]$ is a k-cycle; in particular, every vertex of G belongs to an induced k-cycle; and*
5. *we have*

$$w \in \bigcap \{N_G(M_j) : j \in [1,(k-3)]\}. \tag{2.1}$$

In particular,

$$M_{k-2} \cap \left(\bigcap \{N_G(M_j) : j \in [1,(k-3)]\} \right) \neq \varnothing, \tag{2.2}$$

and G contains a vertex that belongs to an induced j-cycle in G for each $j \in [4,k]$.

Proof. Since the cops play according to θ, at the end of step $k-3$ of the game we have the cops along the induced path $P:\ v_1 = v, v_2, \ldots, v_{k-3}$ in H_{k-2} and the robber at $w \in M_{k-2}$. Hence, in particular, $M_{k-2} \neq \varnothing$ and at the end of step $k-3$, the game is restricted to H_{k-2} with the properties set forth in Lemma 1.7. Therefore, if $M_j \neq \varnothing$ for some $j \geq (k-1)$ then H_{k-2} and, hence, G would contain an induced k-path from v_1 to M_j; a contradiction. This establishes (a). Then, observe that since v_{k-2} dominates M_{k-2}, the robber has to stay in M_{k-2} as long as the cops cover the vertices of P. Hence, if $M_j = \varnothing$ for some $j \in [1,(k-3)]$ then keeping cops in all v_i with $i \in [1,(k-3)] \setminus \{j\}$ would suffice to keep the robber in M_{k-2} and allow the cops to cover all vertices in $\{v_i : i \in [1,(k-2)] \setminus \{j\}\}$ in the next step of the game. But then the cops will be able to capture the robber by the following step of the game. The latter contradicts the assumption that $G \in \mathcal{G}_k$. Therefore, (b) also holds. Next, note that if there exist $x \in M_1$ and $y \in M_{k-2}$ such that $xy \notin E(G)$, then $G[\{x, v_1, \ldots, v_{k-2}, y\}]$ would be a k-path; a contradiction. Hence, (c) must also hold. Note that (d) is immediate from (c) alongside the fact that any vertex $v \in V(G)$ can be set as the initial position v_1 of the cops. Finally, if, given the position w of the robber at the end of step $k-3$ of the game, there exists $j_0 \in [1,(k-3)]$ so that $w \notin N_G(M_{j_0})$ then, as argued for (a), covering all vertices in $\{v_i : i \in [1,(k-2)] \setminus \{j\}\}$ by the cops forces the robber to stay within the neighborhood of at least one cop; thereby, the robber will be captured by the very next step of the game; a contradiction. Hence, we have

$$w \in \bigcap \{N_G(M_j) : j \in [1,(k-3)\},$$

from which the other claims in (e) follow. □

Corollary 2.2. *Every $G \in \mathcal{G}_k$ is 2-connected.*

Proof. In light of Proposition 2.1(d), it suffices to show that no induced k-cycle in G contains a cut-vertex of G. To this end, consider an induced k-cycle C of G and assume, toward a contradiction, that C contains a cut-vertex x of G. Let B be the block of G that contains C, and B' be another block of G that contains x. Pick a neighbor y of x in C, and any neighbor z of x in B'. Then, the graph

$$G[(V(C) \setminus \{y\}) \cup \{z\}]$$

will be a P_k; a contradiction. □

Proposition 2.3. *Let $G \in \mathcal{G}_{k,c}$ and $v_1 \in V(G)$. We consider the assumptions and notations of Proposition 2.1 with the exception that we assume the robber uses any winning strategy against confinement by the cops. Then:*

1. *$|M_j| \geq 2$ for $j \in \{1, k-2\}$.*
2. *$E(G[M_j])$ is nonempty for $j \in \{1, k-2\}$.*
3. *$|V(G)| \geq 2k-2$.*

Proof. **(a) and (b)** Suppose the cops stay still after step $k-3$ of the game so that the robber has to stay in M_{k-2} for the rest of the game. Since the robber's strategy avoids confinement, the robber at w must have a neighbor $w' \in M_{k-2}$. Hence, $E(G[M_{k-2}]) \neq \varnothing$ and $|M_{k-2}| \geq 2$. Likewise, by having the cops occupy vertices $v_2, \ldots, v_{k-2}$ in step $k-2$, the robber has to leave

w to a vertex $u \in M_1$ to avoid capture. In that situation, since $G \in \mathcal{G}_{k,c}$, keeping the cops stationary in the next step of the game leads to the existence of a vertex u' satisfying $u' \in N_G(u) \setminus N_G(\{v_i : i \in [2,(k-2)\})$ so that the robber can move to u' in step $k-1$ of the game. As such, considering the graph

$$G[\{u,u'\} \cup \{v_i : i \in [1,(k-2)\}]$$

shows that u' must be in $N_G(v_1)$; thereby, we have $u' \in M_1$. As a result, we also have $E(G[M_1]) \neq \varnothing$ and $|M_1| \geq 2$. (See Figure 3.)

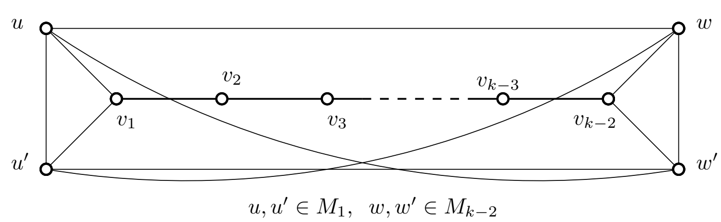

Figure 3. An illustration for an induced subgraph of G in Proposition 2.3.

(c) Since the $k-1$ sets $M_1, \ldots, M_{k-2}$ and $\{v_i : i \in [1,(k-2)\}$ are mutually disjoint subsets of $V(G)$, according to (a) and Proposition 2.1(b) we have

$$|V(G)| \geq 2(k-4) + 2 \times 3 = 2k-2,$$

as desired. □

Corollary 2.4. *Let G be a connected planar graph that is P_k-free for some $k \geq 4$. Then, $\mathrm{ccn}(G) \leq k-3$. In other words, every element of $\mathcal{G}_{k,c}$ (with $k \geq 4$) is nonplanar.*

Proof. If $G \in \mathcal{G}_{k,c}$ then, in terms of the notations of Proposition 2.3 and its proof, we have $|M_1|, |M_{k-3}| \geq 2$ with $M_1 \Leftrightarrow M_{k-3}$. Then, for all pairs of 2-sets $\{u,u'\} \subseteq M_1$ and $\{w,w'\} \subseteq M_{k-3}$ the graph

$$G[\{u,u',w,w',v_1,\ldots,v_{k-2}\}]$$

contains a subdivision of $K_{3,3}$ with partite sets $\{u,u',v_{k-2}\}$ and $\{w,w',v_1\}$. (See Figure 3.) Hence, G is nonplanar according to the Kuratowski Theorem. □

The following is also immediate in light of Proposition 2.3.

Corollary 2.5. *For every $G \in \mathcal{G}_{k,c}$ we have $\delta(G) \geq 3$ and $\Delta(G) \geq k$.*

Proof. We implement the notations of Proposition 2.3 and its proof. In that regard, for any typical vertex v_1 of G we have $N_G(v_1) \supseteq \{u,u',v_2\}$. Hence, $\delta(G) \geq 3$. Furthermore, since $\{w',u,u',v_{k-2}\} \subseteq N_G(w)$ and $N_G(w) \cap M_j$ is nonempty for each $j \in [2,(k-3)]$, we also have $|N_G(w)| \geq 4 + (k-4) = k$. Thus, $\Delta(G) \geq k$, as desired. □

For $k \geq 5$ we can strengthen the first part of Corollary 2.5.

Proposition 2.6. *For every $G \in \mathcal{G}_{k,c}$ with $k \geq 5$ we have $\delta(G) \geq 4$.*

Proof. Toward a contradiction, let $G \in \mathcal{G}_{k,c}$ with $\delta(G) \leq 3$. Then, by Corollary 2.5, it follows that $\delta(G) = 3$. Pick any vertex $v_1 \in V(G)$ with $\deg_G(v_1) = 3$. Let there be $k-3$ cops available. Then, with the assumptions and notations of Propositions 2.1 and 2.3, we have $N_G(v_1) = \{u, u', v_2\}$ and $M_1 = \{u, u'\}$. (See Figure 3.) By having cops at vertices $v_2, \ldots, v_{k-2}$ in step $k-2$ of the game, the robber will be forced to move to one of the vertices in M_1, say u. Then, for the following step, moving the cop at v_{k-2} to w and keeping the other cops stationary force the robber to move to a neighbor, say, z of u so that the robber will avoid being captured in the very next cop moves—recall that $G \in \mathcal{G}_{k,c}$. In that regard, we must have $z \in V(G) \setminus N(v_j)$ for each $j \in [2, (k-3)]$. Moreover, by Proposition 2.1(c), we must have $z \notin M_1$; thereby, $z \notin N(v_1)$. Therefore, z, which is apparently a nonneighbor of w, must belong to M_{k-2}, for otherwise $G[\{v_j : j \in [1, (k-2)]\} \cup \{u, z\}]$ would be a P_k; a contradiction.

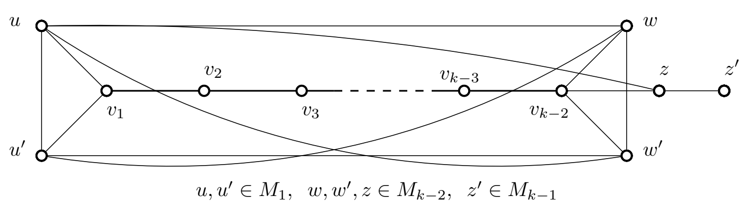

Figure 4. Proof of Proposition 2.6 by contradiction.

After the robber's move to z, in the following step of the game the cop presently at w can move to u while the rest of the cops stay put. This arrangement of the cops forces the robber to move to a neighbor z' of z where the robber can avoid capture. As with z, now we must have $z' \in V(G) \setminus N(v_j)$ for each $j \in [2, (k-3)]$, and $z' \notin M_j$ for $j \in \{1, k-2\}$. Consequently, we also have $z' \in V(G) \setminus N(v_j)$ for $j \in [1, (k-2)]$; i.e. $z' \in M_{k-1}$. The latter contradicts Proposition 2.1(a). (See Figure 4 for an illustration.) □

3. Cops and Robbers on Cographs

By Theorem 1.16, for every cograph G we have $\mathrm{tcn}(G) = 1$. In this section, we consider the effects of twin operations on the cop number and confining cop number of cographs.

Proposition 3.1. *Let G_1 be a cograph and $x \in V(G_1)$.*

1. *If G_2 is obtained from G_1 by adding a true twin y of x, then $C(G_1) = C(G_2)$.*
2. *If G_3 is obtained from G_1 by adding a false twin z of x, then $C(G_1) \leq C(G_3)$.*

Proof. We will use Theorem 1.3 and the fact that the cop number of a cograph is either one or two. **(a)** First, we show that $C(G_1) \leq C(G_2)$. To this end, note that if there is a winning strategy for, say, k cops on G_2, then k cops on G_1 can follow the same strategy on G_1 except that a cop's move to or from y is replaced with that cop's move to or from x. It is easy to see that using this simple shadow strategy, ultimately the cops capture the robber on G_1. Hence, $C(G_1)$ does not exceed $C(G_2)$. Also note that if $C(G_1) = 1$, then pasting y in front of any elimination ordering of G_1 gives an elimination ordering of G_2; therefore, in light of Theorem 1.3, we will have $C(G_2) = 1$. We also have $C(G_1) = C(G_2)$ whenever $C(G_1) = 2$, since $C(G_1) \leq C(G_2)$ and cographs are cop-bounded by two. **(b)** By the fact that cographs are cop-bounded by two, one only needs to consider the case where $C(G_1) = 2$. In this case, the robber has a winning strategy $\mathcal{S}$ against one cop on G_1. Then on G_3 and against one cop, the robber can react to any move of the cop to or from y as if the cop has moved to or from x and, as such, simply move among $V(G_1)$ according to $\mathcal{S}$. It can be easily seen that the latter is a winning strategy for the robber on G_2; therefore, $C(G_3) = 2$ whenever $C(G_1) = 2$. □

Remark: Note that the false twin operation can indeed increase the cop number of a cograph, as is the case with C_4 (with $c(C_4) = 2$) which is obtained by the false twin operation on the degree-two vertex of the copwin graph P_3.

Theorem 3.3. *Let G_1 be a cograph and $x \in V(G_1)$.*

(1) *If G_2 is obtained from G_1 by adding a true twin y of x, then we have $\operatorname{ccn}(G_1) \leq \operatorname{ccn}(G_2)$.*
(2) *If G_3 is obtained from G_1 by adding a false twin z of x, then we have $\operatorname{ccn}(G_1) = \operatorname{ccn}(G_3)$.*

Proof. **(a)** It suffices to consider the case where $\operatorname{ccn}(G_1) = 2$ so that the robber has a strategy $\mathcal{S}$ against one cop on G_1 to avoid confinement. Then, the robber can mimic $\mathcal{S}$ on G_2, as shown in the proof of Proposition 3.1(b), to avoid confinement by one cop on G_2. Therefore, $\operatorname{ccn}(G_2) = 2$ when $\operatorname{ccn}(G_1) = 2$. **(b)** Likewise the proof of (a), we can easily see that $\operatorname{ccn}(G_3) = 2$ whenever $\operatorname{ccn}(G_1) = 2$. Hence, in any case we have $\operatorname{ccn}(G_1) \leq \operatorname{ccn}(G_3)$. Therefore, to complete the proof, we assume $\operatorname{ccn}(G_1) = 1$ and $\operatorname{ccn}(G_3) = 2$, and show that these assumptions together give rise to a contradiction. To this end, consider a fixed strategy $\mathcal{S}'$ for one cop leading to confining or capturing the robber on G_1. Then, in the game of Cops and Robbers on G_3 with one cop, move the cop within $V(G_1)$ by using the following strategy shadowing $\mathcal{S}'$: If the robber moves to or from z, follow $\mathcal{S}'$ pretending that the robber has moved to or from x. Eventually, the game will reach a situation corresponding to confining or capturing the robber on G_1. The latter case, in turn, corresponds to the capture of the robber on G_3 unless the cop and the robber on G_3 are located at x and z, respectively, in which case the robber is confined by the cop. Hence, we may assume the game on G_3 has reached a situation corresponding to the following situation on G_1: The robber and the cop are at vertices, say, x' and y' of G_1 and the robber is confined on G_1. Note that in the latter situation we will have $d_{G_1}(x', y') = 2$. Then, we pick a vertex $z' \in N_{G_1}(x') \cap N_{G_1}(y')$—keep in mind that $z \notin \{x', y', z'\}$ since $z \notin V(G_1)$.

For the present position of the robber in the actual game (i.e., the game on G_3), we consider the possible cases, as follows:

(i) The robber is not at x'.
(ii) The robber is at x'.

First, assume (i). Due to the shadow cop-strategy $\mathcal{S}'$ on G_3, the present robber's position in G_3 must be z and, additionally, we must have $x' = x$ and $N_{G_3}(y') = N_{G_1}(y')$. But from the latter, it follows that

$$N_{G_3}(y') \supseteq N_{G_1}(x') = N_{G_3}(x') = N_{G_3}(z); \tag{3.1}$$

consequently, in the game on G_3 the cop (at y') has also confined the robber (at z). (See Figure 3.1 for an illustration.)

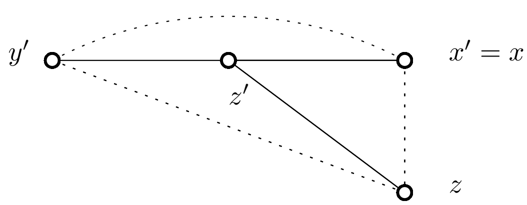

FIGURE 5. An illustration of the situation leading to (3.1).

We now assume (ii). Note that if we additionally have $z \notin N_{G_3}(x')$ or $z \in N_{G_3}(x') \cap N_{G_3}(y')$, then $N_{G_3}(x') \subseteq N_{G_3}(y')$, implying that the robber has been confined on G_3; contradiction the assumption that $\operatorname{ccn}(G_3) = 2$. Hence, we must have

$$z \in N_{G_3}(x') \setminus N_{G_3}(y').$$

Thus, since x, z are twins in G_3, we have $xx' \in V(G)$. Moreover, since y' dominates x in G_1, we have $xy' \in E(G_1) \subset E(G_3)$. Therefore, y is adjacent in G_3 to only one of the twin vertices x and z; a contradiction.

□

Corollary 3.4. *If G is a cograph with $\operatorname{ccn}(G) = 2$, then for every graph H obtained from G by a sequence of twin operations we have $\operatorname{ccn}(H) = 2$.*

Adding a true twin vertex to a cograph can indeed increase the confining cop number. This claim, according to Theorem 1.19 and Theorem 3.3, is equivalent to the statement that there exists a cograph G with $\operatorname{ccn}(G) = 2$. We shall show that the smallest order of such a graph is eight:

Theorem 3.5. *The confining cop number of every connected cograph on fewer than 8 vertices is equal to one. Moreover, for every $n \geq 8$ there is a connected cograph G on n vertices such that $ccn(G) = 2$.*

Proof. Let G be a graph in $\mathcal{G}_{4,c}$ with the minimum number of vertices. By Proposition 2.3(c), we have $|V(G)| \geq 6$. Indeed, by Proposition 2.3 and in accordance with its notations, G must have the graph G_1 of Figure 6 as an induced subgraph.

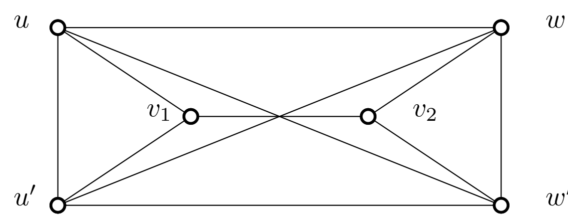

FIGURE 6. The induced subgraph G_1 of G in the Proof of Theorem 3.5.

As such, if $|V(G)| = 6$, we have $G = G_1$, in which case placing a cop at w in the first step of the game forces the robber to choose v_1 as its first position, at which vertex the robber is confined; a contradiction. Hence, we have

$$|V(G)| \geq 7.$$

Next, we will show that $|V(G)| \geq 8$. To this end, we show that each of the following three cases gives rise to a contradiction:

Case 1: $|V(G)| = 7$ and $|M_1| = 3$.
Case 2: $|V(G)| = 7$, $|M_1| = 2$, and $\deg_G(v_1) = 4$.
Case 3: $|V(G)| = 7$ and $|M_2| = 3$.

CASE 1: $|V(G)| = 7$ and $|M_1| = 3$.
In this case, we can easily examine that placing a cop at w_1 leads to the confinement or capture of the robber, hence, $\mathrm{ccn}(G) = 1$; a contradiction.

CASE 2: $|V(G)| = 7$, $|M_1| = 2$, and $\deg_G(v_1) = 4$.
Let $\{x\} = N_G(v_1) \setminus (M_1 \cup \{v_2\})$. Since $x \notin M_1$, we have $x \in N_G(v_2)$. If x is adjacent to a vertex in M_1 (resp. M_2), placing a cop at that vertex leads to either the confinement of the robber at v_2 (resp. v_1) in step 1 or the capture of the robber in step 2; a contradiction. Hence, $N_G(x) = \{v_1, v_2\}$. But then the the graphs $G[\{u, w, v_2, x\}]$ will be a P_4; contradicting the assumption that G is P_4-free. (See Figure 7.)

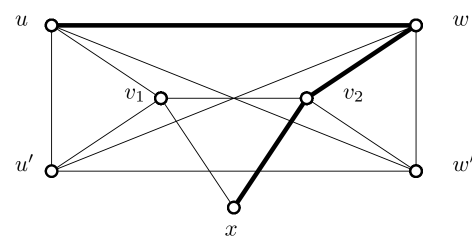

FIGURE 7. Proof of Theorem 3.5: An induced P_4 under Case II.

CASE 3: $|V(G)| = 7$ and $|M_2| = 3$.

This case also leads to a contradiction; likewise Case I.

Hence, $|V(G)| \geq 8$. Therefore, in light of Corollary 3.4 , to complete the proof it suffices to present a cograph of H of order eight so that $\operatorname{ccn}(H) = 2$. It can be easily checked that the graph of Figure 8 satisfies these conditions; indeed, it is the only cograph of order eight with the confining cop number of two.

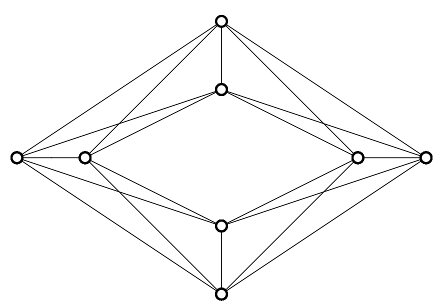

FIGURE 8. The smallest cograph with the confining cop number of two.

□

4. Concluding remarks

Since the cop number of the cycle on four vertices is two, the upper bound of two for the cop number of cographs is tight. However, it is an open question whether there exists a P_5-free graph which requires three cops to capture the robber. As shown in Corollary 2.4, though, the planarity of a connected P_5-free graph G implies $\operatorname{ccn}(G) \leq 2$. We conjecture that this planarity condition can be relaxed:

Conjecture. *For every connected P_5-free graph G we have* $\operatorname{ccn}(G) \leq 2$.

We conclude with another conjecture about the planar graphs. In light of Propositions 1.12 and 1.14 and the fact that planar graphs are 3-copwin, it can be easily seen that the dodecahedral graph has its cop-number and confining cop-number both equal to three. It has been conjectured that the dodecahedral graph (which has 20 vertices) is the smallest planar graph with cop-number three. Here, we pose the counterpart of this conjecture in terms of the confining cop number:

Conjecture. *For every connected planar graph G on at most 19 vertices we have* $\mathrm{ccn}(G) \leq 2$.

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