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BAILEY AND DAUM'S q-KUMMER THEOREM AND EXTENSIONS

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Abstract. By means of the linearization method, we establish four analytical formulae for the q-Kummer sum extended by two integer parameters. Ten closed formulae are presented as examples.

1. Introduction and Motivation

For the classical hypergeometric series, Kummer's summation theorem (cf. Bailey [3, §2.3]) is well–known

$$ {}_2F_1\left[\begin{matrix} a,\ c \\ 1+a-c \end{matrix}\,\middle|\, -1\right] = \frac{\Gamma(1+\frac{a}{2})\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-c)} \quad \text{for} \quad \Re(c) < 1. $$

When the well–poised condition is perturbed by an integer, Apagodu and Zeilberger [2], and Chu [7] found analytical formulae for the corresponding terminating ${}_2F_1$-series. The q-analogue of Kummer's formula was established independently by Bailey [4] and Daum [8] (see also Gasper–Rahman [9, II.9]):

$$ (1.1) \qquad {}_2\phi_1\left[\begin{matrix} a,\ c \\ qa/c \end{matrix}\,\middle|\, q; -q/c\right] = \frac{(qa;q^2)_\infty}{(qa/c^2;q^2)_\infty}\left[\begin{matrix} qa/c^2, -q \\ qa/c, -q/c \end{matrix}\,\middle|\, q\right]_\infty, $$

where the notation related to the q-series will be given on the next page. By applying the Heine transformation (Gasper–Rahman [9, III.2])

$$ (1.2) \qquad {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, q; z\right] = \left[\begin{matrix} c/b, bz \\ c, z \end{matrix}\,\middle|\, q\right]_\infty {}_2\phi_1\left[\begin{matrix} abz/c,\ b \\ bz \end{matrix}\,\middle|\, q; c/b\right] $$

and then the q-binomial series (Gasper–Rahman [9, II.3])

$$ {}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix}\,\middle|\, q; z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}. $$

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Andrews [1] gave an elementary proof of (1.1), that can be reproduced as follows:

$$\begin{aligned}{}_2\phi_1\left[\begin{matrix}a,\ c\\ qa/c\end{matrix}\,\middle|\, q;-q/c\right] &= \left[\begin{matrix}qa/c^2,-q\\ qa/c,-q/c\end{matrix}\,\middle|\, q\right]_\infty \times {}_2\phi_1\left[\begin{matrix}c,\ -c\\ -q\end{matrix}\,\middle|\, q;qa/c^2\right]\\ &= \left[\begin{matrix}qa/c^2,-q\\ qa/c,-q/c\end{matrix}\,\middle|\, q\right]_\infty \times {}_1\phi_0\left[\begin{matrix}c^2\\ -\end{matrix}\,\middle|\, q^2;qa/c^2\right]\\ &= \left[\begin{matrix}qa/c^2,-q\\ qa/c,-q/c\end{matrix}\,\middle|\, q\right]_\infty \times \frac{(qa;q^2)_\infty}{(qa/c^2;q^2)_\infty}.\end{aligned}$$

By making use of the q-integral representation, Kim et al. [10] derived two contiguous results of (1.1). The purpose of this short paper is to examine, for a given pair of integers λ and ρ, the following general series

$$\Omega_\lambda^\rho := \Omega_\lambda^\rho(a,c) = {}_2\phi_1\left[\begin{matrix}a,\quad c\\ q^{1+\rho}a/c\end{matrix}\,\middle|\, q;-q^{1+\lambda}/c\right], \tag{1.3}$$

whose very special case $\lambda=\rho=0$ becomes the q-Kummer series (1.1). By means of the linearization method employed in [5, 6, 7, 11], we shall prove that (see Theorem 4.1) the $\Omega_\lambda^\rho(a,c)$-series for $\lambda,\ \rho\in\mathbb{Z}$ is always explicitly evaluable in the $\Omega_0^0(a',c')$-series with the number of terms at most $(1+|\rho|)\times\big(1+|\lambda|+|\rho|\big)$.

Throughout the paper, we shall utilize the following notation. Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of integers and natural numbers with $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. For two indeterminates x and q, define the shifted factorials by $(x;q)_0 = \langle x;q\rangle_0 = 1$ and

$$\left.\begin{aligned}(x;q)_n &= (1-x)(1-qx)\cdots(1-q^{n-1}x)\\ \langle x;q\rangle_n &= (1-x)(1-x/q)\cdots(1-q^{1-n}x)\end{aligned}\right\}\quad\text{for}\quad n\in\mathbb{N}.$$

The rising factorial of negative order can be expressed as

$$(x;q)_{-n} = \frac{1}{(q^{-n}x;q)_n} = q^{\binom{n}{2}}\frac{(-q/x)^n}{(q/x;q)_n}\quad\text{where}\quad n\in\mathbb{N}.$$

The product and fraction of shifted factorials are abbreviated respectively to

$$\begin{aligned}[\,\alpha,\ \beta,\ \cdots,\ \gamma;q\,]_n &= (\alpha;q)_n\,(\beta;q)_n\cdots(\gamma;q)_n\,,\\ \left[\begin{matrix}\alpha,\ \beta,\ \cdots,\ \gamma\\ A,\ B,\ \cdots,\ C\end{matrix}\,\middle|\, q\right]_n &= \frac{(\alpha;q)_n\,(\beta;q)_n\cdots(\gamma;q)_n}{(A;q)_n\,(B;q)_n\cdots(C;q)_n}.\end{aligned}$$

Following Bailey [3] and Gasper–Rahman [9], the basic hypergeometric series (shortly as q-series) is defined by

$${}_{1+p}\phi_p\left[\begin{matrix}a_0,\,a_1,\,\cdots,\,a_p\\ b_1,\,\cdots,\,b_p\end{matrix}\,\middle|\, q;z\right] = \sum_{n=0}^{\infty}\left[\begin{matrix}a_0,\,a_1,\,\cdots,\,a_p\\ q,\ b_1,\,\cdots,\,b_p\end{matrix}\,\middle|\, q\right]_n z^n.$$

This series terminates if one of its numerator parameters is of the form q^{-m} with $m\in\mathbb{N}_0$. Otherwise, the series is said to be nonterminating. In the latter case, the base q will be restricted, for convenience, to $|q|<1$.

We shall organize the paper in the following manner. In the next section, we prove two theorems that transform the Ω_λ^ρ-series into the Ω_λ^0-series. Then the Ω_λ^0-series will be explicitly evaluated in Section 3. Finally, the paper will end up with ten examples as applications.

2. Reduction Formulae from Ω_λ^ρ to Ω_λ^0

By applying the series rearrangement and the q-binomial theorem

$$(x;q)_m = \sum_{k=0}^{m} q^{\binom{k}{2}} \begin{bmatrix} m \\ k \end{bmatrix} (-x)^k$$

we shall derive, in this section, two transformation formulae that express the Ω_λ^ρ-series in terms of the Ω_λ^0-series.

2.1. $\rho \geq 0$**.** By inserting the binomial relation in the Ω_λ^ρ-series

$$(q^{n-\rho}c;q)_\rho = \sum_{k=0}^{\rho} (-c)^{\rho-k} \begin{bmatrix} \rho \\ k \end{bmatrix} q^{\binom{\rho-k}{2}+(n-\rho)(\rho-k)}$$

we can reformulate the following double series

$$\begin{aligned}
\Omega_\lambda^\rho(a,c) &= \sum_{n=0}^{\infty} \frac{(a;q)_n(c;q)_n}{(q;q)_n(q^{1+\rho}a/c;q)_n} \Big(\frac{q^{1+\lambda}}{-c}\Big)^n \\
&\quad \times \sum_{k=0}^{\rho} \frac{(-c)^{\rho-k}}{(q^{n-\rho}c;q)_\rho} \begin{bmatrix} \rho \\ k \end{bmatrix} q^{\binom{\rho-k}{2}+(n-\rho)(\rho-k)} \\
&= \sum_{k=0}^{\rho} \frac{(-c)^{\rho-k}}{(q^{-\rho}c;q)_\rho} \begin{bmatrix} \rho \\ k \end{bmatrix} q^{\binom{\rho-k}{2}-\rho(\rho-k)} \\
&\quad \times \sum_{n=0}^{\infty} \frac{(a;q)_n(q^{-\rho}c;q)_n}{(q;q)_n(q^{1+\rho}a/c;q)_n} \Big(\frac{q^{1+\lambda+\rho-k}}{-c}\Big)^n.
\end{aligned}$$

Writing the last sum as $\Omega_{\lambda-k}^0(a,q^{-\rho}c)$, we derive the following reduction formula.

Theorem 2.1 ($\lambda,\ \rho \in \mathbb{Z}$ with $\rho \geq 0$)**.**

$$\Omega_\lambda^\rho(a,c) = \sum_{k=0}^{\rho} q^{\binom{k}{2}} \begin{bmatrix} \rho \\ k \end{bmatrix} \frac{(-q/c)^k}{(q/c;q)_\rho} \Omega_{\lambda-k}^0(a,q^{-\rho}c).$$

2.2. $\rho \leq 0$**.** Instead, by putting another binomial relation inside the Ω_λ^ρ-series

$$(q^{1+n+\rho}a/c;q)_{-\rho} = \sum_{k=0}^{-\rho} \Big(\frac{-a}{c}\Big)^k \begin{bmatrix} -\rho \\ k \end{bmatrix} q^{\binom{k+1}{2}+k\rho+kn}$$

we can analogously manipulate the following double series

$$\begin{aligned}\Omega_\lambda^\rho(a,c) =& \sum_{n=0}^{\infty} \frac{(a;q)_n(c;q)_n}{(q;q)_n(q^{1+\rho}a/c;q)_n}\Big(\frac{q^{1+\lambda}}{-c}\Big)^n \\ &\times \sum_{k=0}^{-\rho}\Big(\frac{-a}{c}\Big)^k \begin{bmatrix}-\rho\\ k\end{bmatrix} \frac{q^{\binom{k+1}{2}+k\rho+kn}}{(q^{1+n+\rho}a/c;q)_{-\rho}} \\ =& \sum_{k=0}^{-\rho}\Big(\frac{-a}{c}\Big)^k \begin{bmatrix}-\rho\\ k\end{bmatrix} \frac{q^{\binom{k+1}{2}+k\rho}}{(q^{1+\rho}a/c;q)_{-\rho}} \sum_{n=0}^{\infty} \frac{(a;q)_n(c;q)_n}{(q;q)_n(qa/c;q)_n}\Big(\frac{q^{1+\lambda+k}}{-c}\Big)^n.\end{aligned}$$

Writing the last sum as $\Omega^0_{\lambda+k}(a,c)$, we derive another reduction formula.

Theorem 2.2 ($\lambda,\ \rho \in \mathbb{Z}$ with $\rho \le 0$)**.**

$$\Omega_\lambda^\rho(a,c) = \sum_{k=0}^{-\rho} q^{\binom{k}{2}} \begin{bmatrix}-\rho\\ k\end{bmatrix} \frac{(-q^{1+\rho}a/c)^k}{(q^{1+\rho}a/c;q)_{-\rho}} \Omega^0_{\lambda+k}(a,c).$$

3. Reduction Formulae from Ω_λ^0 to Ω_0^0

To remove the λ-parameter, we start from the following linearization lemma, which is a reformulation of the q-Saalschütz summation formula (cf. [9, II.12])

$$ {}_3\phi_2\left[\begin{matrix} q^{-m},\ a,\ b \\ c, q^{1-m}ab/c \end{matrix} \,\Big|\, q;q\right] = \left[\begin{matrix} c/a, c/b \\ c, c/ab\end{matrix}\,\Big|\, q\right]_m. \tag{3.1}$$

Lemma 3.1 (Linear representation)**.** *Let x be a variable and m a natural number. Then for three indeterminates $\{u,v,w\}$, the following linear representation formula holds*

$$(wx;q)_m = \sum_{k=0}^{m} (ux;q)_{m-k}\langle vx;q\rangle_k\, \mathcal{E}_m^k(u,v,w), \tag{3.2}$$

where the connection coefficients $\{\mathcal{E}_m^k(u,v,w)\}$ are independent of x and given by

$$\mathcal{E}_m^k(u,v,w) = q^{\binom{k}{2}} \begin{bmatrix}m\\ k\end{bmatrix} \frac{(w/u;q)_k(w/v;q)_m}{(w/v;q)_k(u/v;q)_m}\Big(-\frac{u}{v}\Big)^k. \tag{3.3}$$

Proof. Recall the following three relations

$$\begin{aligned}\begin{bmatrix}m\\ k\end{bmatrix} =& (-1)^k \frac{(q^{-m};q)_k}{(q;q)_k} q^{mk-\binom{k}{2}}, \\ (ux;q)_{m-k} =& \Big(\frac{-q}{ux}\Big)^k \frac{(ux;q)_m}{(q^{1-m}/ux;q)_k} q^{\binom{k}{2}-mk}, \\ \langle vx;q\rangle_k =& (-vx)^k(1/vx;q)_k q^{-\binom{k}{2}}.\end{aligned}$$

Substituting (3.3) into (3.2), we confirm the lemma by simplifying the finite sum

$$\sum_{k=0}^{m}(ux;q)_{m-k}\langle vx;q\rangle_k\begin{bmatrix}m\\k\end{bmatrix}q^{\binom{k}{2}}\frac{(w/u;q)_k(w/v;q)_m}{(w/v;q)_k(u/v;q)_m}\Big(-\frac{u}{v}\Big)^k$$
$$=\frac{(ux;q)_m(w/v;q)_m}{(u/v;q)_m}\sum_{k=0}^{m}q^k\left[\begin{matrix}q^{-m},1/vx,w/u,\\ q,q^{1-m}/ux,w/v\end{matrix}\,\Big|\,q;q\right]_k=(wx;q)_m,$$

where the last sum has been evaluated by means of (3.1). □

In addition, we have, according to (1.2), the expression

$$\Omega_\lambda^0(a,c)=\left[\begin{matrix}q/c,-q^{1+\lambda}a/c\\ qa/c,-q^{1+\lambda}/c\end{matrix}\,\Big|\,q\right]_\infty {}_2\phi_1\left[\begin{matrix}a,\ -q^\lambda c\\ -q^{1+\lambda}a/c\end{matrix}\,\Big|\,q;q/c\right]. \tag{3.4}$$

Therefore, in order to evaluate $\Omega_\lambda^0(a,c)$, it suffices to find explicit formulae for the rightmost nonterminating ${}_2\phi_1$-series.

3.1. $\lambda\geq 0$. Specifying in Lemma 3.1 by

$$\left.\begin{matrix}m\to\lambda\\ x\to q^n\end{matrix}\right\}\quad\text{and}\quad\left\{\begin{matrix}u\to a\\ v\to -q^\lambda a/c\\ w\to -c\end{matrix}\right.$$

we get the equation

$$(-q^nc;q)_\lambda=\sum_{k=0}^{\lambda}(q^na;q)_k\langle -q^{n+\lambda}a/c;q\rangle_{\lambda-k}\mathcal{E}_\lambda^{\lambda-k}(a,-q^\lambda a/c,-c).$$

By inserting this relation in the ${}_2\phi_1$-series displayed in (3.4), we can reformulate the double sum

$${}_2\phi_1\left[\begin{matrix}a,\ -q^\lambda c\\ -q^{1+\lambda}a/c\end{matrix}\,\Big|\,q;q/c\right]=\sum_{n=0}^{\infty}\Big(\frac{q}{c}\Big)^n\left[\begin{matrix}a,\ -q^\lambda c\\ q,-q^{1+\lambda}a/c\end{matrix}\,\Big|\,q\right]_n$$
$$\times\sum_{k=0}^{\lambda}\frac{\mathcal{E}_\lambda^{\lambda-k}(a,-q^\lambda a/c,-c)}{(-q^nc;q)_\lambda}(q^na;q)_k\langle -q^{n+\lambda}a/c;q\rangle_{\lambda-k}.$$

Interchanging the summation order and then applying the equalities

$$\frac{(-q^\lambda c;q)_n}{(-q^nc;q)_\lambda}=\frac{(-c;q)_n}{(-c;q)_\lambda},$$
$$(a;q)_n(q^na;q)_k=(a;q)_k(q^ka;q)_n,$$
$$\frac{\langle -q^{n+\lambda}a/c;q\rangle_{\lambda-k}}{(-q^{1+\lambda}a/c;q)_n}=\frac{(-qa/c;q)_\lambda}{(-qa/c;q)_k(-q^{1+k}a/c;q)_n};$$

we can reformulate the double sum as

$$
{}_2\phi_1\left[\begin{matrix} a,\ -q^{\lambda}c \\ -q^{1+\lambda}a/c \end{matrix} \,\Big|\, q; q/c\right] = \sum_{k=0}^{\lambda} \frac{(a;q)_k(-qa/c;q)_\lambda}{(-qa/c;q)_k(-c;q)_\lambda}\mathcal{E}_\lambda^{\lambda-k}(a,-q^\lambda a/c,-c)
\times \sum_{n=0}^{\infty}\left[\begin{matrix} q^k a,\ -c \\ q, -q^{1+k}a/c \end{matrix} \,\Big|\, q\right]_n \Big(\frac{q}{c}\Big)^n.
$$

Writing the last series as $\Omega_0^0(q^k a,-c)$, we get the following expression:

$$
\Omega_\lambda^0(a,c) = \left[\begin{matrix} q/c,\ -qa/c \\ qa/c, -q^{1+\lambda}/c \end{matrix} \,\Big|\, q\right]_\infty
\times \sum_{k=0}^{\lambda} \mathcal{E}_\lambda^{\lambda-k}(a,-q^\lambda a/c,-c)\frac{(a;q)_k\ \Omega_0^0(q^k a,-c)}{(-qa/c;q)_k(-c;q)_\lambda}.
$$

According to (3.3), the $\mathcal{E}$-coefficient can explicitly be restated as

$$
\begin{aligned}
\mathcal{E}_\lambda^{\lambda-k}(a,-q^\lambda a/c,-c) &= q^{\binom{\lambda-k}{2}}\begin{bmatrix}\lambda\\k\end{bmatrix}\frac{(-c/a;q)_{\lambda-k}(q^{-\lambda}c^2/a;q)_\lambda}{(q^{-\lambda}c^2/a;q)_{\lambda-k}(-q^{-\lambda}c;q)_\lambda}\big(q^{-\lambda}c\big)^{\lambda-k}\\
&= q^{\binom{\lambda-k}{2}}\begin{bmatrix}\lambda\\k\end{bmatrix}\frac{\langle q^{-1}c^2/a;q\rangle_k(-c/a;q)_\lambda}{\langle -q^{\lambda-1}c/a;q\rangle_k(-q^{-\lambda}c;q)_\lambda}\big(q^{-\lambda}c\big)^{\lambda-k}\\
&= q^{k-\binom{\lambda+1}{2}}\frac{(q^{-\lambda};q)_k(qa/c^2;q)_k(-c/a;q)_\lambda}{(q;q)_k)(-q^{1-\lambda}a/c;q)_k(-q^{-\lambda}c;q)_\lambda}c^\lambda\\
&= q^{k+\binom{\lambda}{2}}\frac{(q^{-\lambda};q)_k(qa/c^2;q)_k(-q^{1-\lambda}a/c;q)_\lambda}{(q;q)_k)(-q^{1-\lambda}a/c;q)_k(-q/c;q)_\lambda}\Big(\frac{c}{a}\Big)^\lambda.
\end{aligned}
$$

This enables us to make further simplifications

$$
\begin{aligned}
\Omega_\lambda^0(a,c) &= \left[\begin{matrix} q/c,\ -qa/c \\ qa/c, -q^{1+\lambda}/c \end{matrix} \,\Big|\, q\right]_\infty \sum_{k=0}^{\lambda}\frac{(a;q)_k\ \Omega_0^0(q^k a,-c)}{(-qa/c;q)_k(-c;q)_\lambda}\\
&\quad\times q^{k+\binom{\lambda}{2}}\frac{(q^{-\lambda};q)_k(qa/c^2;q)_k(-q^{1-\lambda}a/c;q)_\lambda}{(q;q)_k)(-q^{1-\lambda}a/c;q)_k(-q/c;q)_\lambda}\Big(\frac{c}{a}\Big)^\lambda\\
&= \left[\begin{matrix} q/c,\ -qa/c \\ qa/c, -q^{1+\lambda}/c \end{matrix} \,\Big|\, q\right]_\infty \frac{(-q^{1-\lambda}a/c;q)_\lambda}{(-q^{1-\lambda}/c;q)_{2\lambda}}\\
&\quad\times\sum_{k=0}^{\lambda}\frac{q^k}{a^\lambda}\frac{(q^{-\lambda};q)_k(a;q)_k(qa/c^2;q)_k\ \Omega_0^0(q^k a,-c)}{(q;q)_k)(-qa/c;q)_k(-q^{1-\lambda}a/c;q)_k}.
\end{aligned}
$$

The final expression is highlighted as the following theorem.

Theorem 3.2 ($\lambda\in\mathbb{Z}$ with $\lambda\geq 0$)**.**

$$
\Omega_\lambda^0(a,c) = \left[\begin{matrix} q/c,\ -q^{1-\lambda}a/c \\ qa/c, -q^{1-\lambda}/c \end{matrix} \,\Big|\, q\right]_\infty
\times\sum_{k=0}^{\lambda}\frac{q^k}{a^\lambda}\left[\begin{matrix} q^{-\lambda},\ a,\ qa/c^2 \\ q, -qa/c, -q^{1-\lambda}a/c \end{matrix} \,\Big|\, q\right]_k \Omega_0^0(q^k a,-c).
$$

The formula in this theorem is explicit because the $\Omega_0^0(q^k a, -c)$ series can be evaluated by (1.1) as

$$\Omega_0^0(q^k a, -c) = {}_2\phi_1\left[\begin{matrix} q^k a, \ -c \\ -q^{1+k} a/c \end{matrix} \,\Big|\, q; q/c\right] = \frac{(q^{1+k}a; q^2)_\infty}{(q^{1+k}a/c^2; q^2)_\infty}\left[\begin{matrix} q^{1+k}a/c^2, -q \\ -q^{1+k}a/c, q/c \end{matrix} \,\Big|\, q\right]_\infty .$$

3.2. $\lambda \le 0$. Alternatively, specifying in Lemma 3.1 by

$$\left.\begin{matrix} m \to -\lambda \\ x \to q^n \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} u \to -q^\lambda c \\ v \to 1 \\ w \to -q^{1+\lambda}a/c \end{matrix}\right.$$

we have the following equation

$$(-q^{1+n+\lambda}a/c; q)_{-\lambda} = \sum_{k=0}^{-\lambda} \langle q^n; q\rangle_k (-q^{n+\lambda}c; q)_{-\lambda-k} \mathcal{E}_{-\lambda}^k(-q^\lambda c, 1, -q^{1+\lambda}a/c).$$

By putting this relation inside the ${}_2\phi_1$-series displayed in (3.4), we get the double sum

$$\begin{aligned} {}_2\phi_1\left[\begin{matrix} a, \ -q^\lambda c \\ -q^{1+\lambda}a/c \end{matrix} \,\Big|\, q; q/c\right] &= \sum_{n=0}^{\infty} \left(\frac{q}{c}\right)^n \left[\begin{matrix} a, \ -q^\lambda c \\ q, -q^{1+\lambda}a/c \end{matrix} \,\Big|\, q\right]_n \\ &\times \sum_{k=0}^{-\lambda} \frac{\mathcal{E}_{-\lambda}^k(-q^\lambda c, 1, -q^{1+\lambda}a/c)}{(-q^{1+n+\lambda}a/c; q)_{-\lambda}} \langle q^n; q\rangle_k (-q^{n+\lambda}c; q)_{-\lambda-k}. \end{aligned}$$

Exchanging the summation order first and then making use of the equalities

$$\begin{aligned} &\frac{\langle q^n; q\rangle_k}{(q;q)_n} = \frac{1}{(q;q)_{n-k}}, \\ &(-q^\lambda c; q)_n (-q^{n+\lambda}c; q)_{-\lambda-k} = \frac{(-c;q)_{n-k}}{(-c;q)_\lambda}, \\ &\frac{1}{(-q^{1+\lambda}a/c; q)_n (-q^{1+n+\lambda}a/c; q)_{-\lambda}} = \frac{1}{(-q^{1+\lambda}a/c; q)_{-\lambda}(-qa/c; q)_n}; \end{aligned}$$

we can manipulate the double sum expression below

$$\begin{aligned}
&{}_2\phi_1\left[\begin{matrix} a,\ -q^{\lambda}c \\ -q^{1+\lambda}a/c \end{matrix}\,\middle|\, q; q/c\right]\\
&= \sum_{k=0}^{-\lambda} \frac{\mathcal{E}^k_{-\lambda}(-q^{\lambda}c, 1, -q^{1+\lambda}a/c)}{(-c;q)_{\lambda}(-q^{1+\lambda}a/c;q)_{-\lambda}} \sum_{n=k}^{\infty} \frac{(a;q)_n(-c;q)_{n-k}}{(-qa/c;q)_n(q;q)_{n-k}}\Big(\frac{q}{c}\Big)^n\\
&= \sum_{k=0}^{-\lambda}\Big(\frac{q}{c}\Big)^k \frac{(a;q)_k(-qa/c;q)_{\lambda}}{(-c;q)_{\lambda}(-qa/c;q)_k}\mathcal{E}^k_{-\lambda}(-q^{\lambda}c, 1, -q^{1+\lambda}a/c)\\
&\times \sum_{n=0}^{\infty}\left[\begin{matrix} q^k a,\ -c \\ q, -q^{1+k}a/c \end{matrix}\,\middle|\, q\right]_n \Big(\frac{q}{c}\Big)^n,
\end{aligned}$$

where the last line is justified by the replacement $n \to n+k$. Observing that the last sum with respect to n is again $\Omega_0^0(q^k a, -c)$, we get the expression

$$\begin{aligned}
\Omega^0_{\lambda}(a,c) = \left[\begin{matrix} q/c,\ -qa/c \\ qa/c, -q^{1+\lambda}/c \end{matrix}\,\middle|\, q\right]_{\infty} \sum_{k=0}^{-\lambda}\Big(\frac{q}{c}\Big)^k \frac{(a;q)_k\,\Omega^0_0(q^k a, -c)}{(-qa/c;q)_k(-c;q)_{\lambda}}\\
\times \mathcal{E}^k_{-\lambda}(-q^{\lambda}c, 1, -q^{1+\lambda}a/c).
\end{aligned}$$

Replacing the $\mathcal{E}$-coefficient by (3.3), we can simplify further the above sum

$$\begin{aligned}
&\sum_{k=0}^{-\lambda}\Big(\frac{q}{c}\Big)^k \frac{(a;q)_k\,\Omega^0_0(q^k a, -c)}{(-qa/c;q)_k(-c;q)_{\lambda}}\mathcal{E}^k_{-\lambda}(-q^{\lambda}c, 1, -q^{1+\lambda}a/c)\\
=&\sum_{k=0}^{-\lambda}\frac{(a;q)_k\,\Omega^0_0(q^k a, -c)}{(-qa/c;q)_k(-c;q)_{\lambda}}\begin{bmatrix}-\lambda\\ k\end{bmatrix}\frac{(qa/c^2;q)_k(-q^{1+\lambda}a/c;q)_{-\lambda}}{(-q^{1+\lambda}a/c;q)_k(-q^{\lambda}c;q)_{-\lambda}}q^{\binom{k+1}{2}+k\lambda}\\
=&\sum_{k=0}^{-\lambda}\frac{\Omega^0_0(q^k a, -c)}{(-qa/c;q)_{\lambda}}\begin{bmatrix}-\lambda\\ k\end{bmatrix}\frac{(a;q)_k(qa/c^2;q)_k}{(-qa/c;q)_k(-q^{1+\lambda}a/c;q)_k}q^{\binom{k+1}{2}+k\lambda}\\
=&\sum_{k=0}^{-\lambda}\frac{\Omega^0_0(q^k a, -c)}{(-qa/c;q)_{\lambda}}\frac{(q^{\lambda};q)_k(a;q)_k(qa/c^2;q)_k}{(q;q)_k(-qa/c;q)_k(-q^{1+\lambda}a/c;q)_k}(-q)^k.
\end{aligned}$$

Consequently, we establish another explicit formula.

Theorem 3.3 ($\lambda \in \mathbb{Z}$ with $\lambda \le 0$)**.**

$$\begin{aligned}
\Omega^0_{\lambda}(a,c) = &\left[\begin{matrix} q/c,\ -q^{1+\lambda}a/c \\ qa/c, -q^{1+\lambda}/c \end{matrix}\,\middle|\, q\right]_{\infty}\\
&\times \sum_{k=0}^{-\lambda}(-q)^k\left[\begin{matrix} q^{\lambda},\ a,\ qa/c^2 \\ q, -qa/c, -q^{1+\lambda}a/c \end{matrix}\,\middle|\, q\right]_k \Omega^0_0(q^k a, -c).
\end{aligned}$$

4. Conclusive Theorem and Examples

Summing up the results shown in the previous two sections, we can evaluate the Ω_λ^ρ-series, for any given pair of integers λ and ρ, by carrying out the following procedure:

- *Step A*. If $\rho \neq 0$, we first transform the Ω_λ^ρ-series into the $\Omega_{\lambda'}^0$-series by making use of Theorems 2.1 and 2.2 and then go to *Step B*.
- *Step B*. If $\rho = 0$, we evaluate the Ω_λ^0-series by means of Theorems 3.2 and 3.3.

Therefore, we have confirmed the following conclusive theorem.

Theorem 4.1. *For any given pair of integers λ and ρ, the Ω_λ^ρ-series can be explicitly expressed as a linear combination of the Ω_0^0-series with the number of terms at most* $(1+|\rho|)(1+|\lambda|+|\rho|)$.

The afore-described procedure is realized by appropriately devised *Mathematica* commands, that are executed to produce several closed formulae of the Ω_λ^ρ-series for small integers λ and ρ. Ten remarkable examples are recorded as follows.

Example 4.2 ($\lambda = 1$ and $\rho = 0$)**.**

$$ {}_2\phi_1\left[\begin{matrix} a,\ c \\ qa/c \end{matrix} \,\Big|\, q; -q^2/c\right] = \left[\begin{matrix} qa/c^2, -q \\ qa/c, -1/c \end{matrix} \,\Big|\, q\right]_\infty \times \left\{ \frac{(1+a/c)(qa;q^2)_\infty}{a(qa/c^2;q^2)_\infty} - \frac{(a;q^2)_\infty}{a(q^2a/c^2;q^2)_\infty} \right\}. $$

Example 4.3 ($\lambda = -1$ and $\rho = 0$)**.**

$$ {}_2\phi_1\left[\begin{matrix} a,\ c \\ qa/c \end{matrix} \,\Big|\, q; -1/c\right] = \left[\begin{matrix} qa/c^2, -q \\ qa/c, -1/c \end{matrix} \,\Big|\, q\right]_\infty \times \left\{ \frac{(1+a/c)(qa;q^2)_\infty}{(qa/c^2;q^2)_\infty} + \frac{(a;q^2)_\infty}{(q^2a/c^2;q^2)_\infty} \right\}. $$

Example 4.4 ($\lambda = 0$ and $\rho = -1$: Kim et al. [10, eq. (3.2)])**.**

$$ {}_2\phi_1\left[\begin{matrix} a,\ c \\ a/c \end{matrix} \,\Big|\, q; -q/c\right] = \left[\begin{matrix} a/c^2, -q \\ a/c, -1/c \end{matrix} \,\Big|\, q\right]_\infty \left\{ \frac{(qa;q^2)_\infty}{(qa/c^2;q^2)_\infty} + \frac{(a;q^2)_\infty}{c(a/c^2;q^2)_\infty} \right\}. $$

Example 4.5 ($\lambda = -1$ and $\rho = -1$)**.**

$$ {}_2\phi_1\left[\begin{matrix} a,\ c \\ a/c \end{matrix} \,\Big|\, q; -1/c\right] = \left[\begin{matrix} a/c^2, -q \\ a/c, -1/c \end{matrix} \,\Big|\, q\right]_\infty \left\{ \frac{(qa;q^2)_\infty}{(qa/c^2;q^2)_\infty} + \frac{(a;q^2)_\infty}{(a/c^2;q^2)_\infty} \right\}. $$

Example 4.6 ($\lambda = -1$ and $\rho = -2$)**.**

$$ {}_2\phi_1\left[\begin{matrix} a,\ c \\ a/cq \end{matrix} \,\Big|\, q; -1/c\right] = \left[\begin{matrix} a/c^2q, -q \\ a/cq, -1/c \end{matrix} \,\Big|\, q\right]_\infty \times \left\{ \frac{(1-a/cq)(qa;q^2)_\infty}{(a/c^2q;q^2)_\infty} + \frac{(a;q^2)_\infty}{(a/c^2;q^2)_\infty} \right\}. $$

Example 4.7 ($\lambda=0$ and $\rho=1$: Kim et al. [10, eq. (3.1)]).

$$
{}_2\phi_1\left[\begin{matrix}a,\ c\\ q^2a/c\end{matrix}\,\Big|\, q; -q/c\right]=\frac{1}{1-q/c}\left[\begin{matrix}q^2a/c^2, -q\\ q^2a/c, -q/c\end{matrix}\,\Big|\, q\right]_\infty \times\left\{\frac{(qa;q^2)_\infty}{(q^3a/c^2;q^2)_\infty}-\frac{q(a;q^2)_\infty}{c(q^2a/c^2;q^2)_\infty}\right\}.
$$

Example 4.8 ($\lambda=1$ and $\rho=1$).

$$
{}_2\phi_1\left[\begin{matrix}a,\ c\\ q^2a/c\end{matrix}\,\Big|\, q; -q^2/c\right]=\frac{a^{-1}}{1-q/c}\left[\begin{matrix}q^2a/c^2, -q\\ q^2a/c, -q/c\end{matrix}\,\Big|\, q\right]_\infty \times\left\{\frac{(qa;q^2)_\infty}{(q^3a/c^2;q^2)_\infty}-\frac{(a;q^2)_\infty}{(q^2a/c^2;q^2)_\infty}\right\}.
$$

Example 4.9 ($\lambda=1$ and $\rho=2$).

$$
{}_2\phi_1\left[\begin{matrix}a,\ c\\ q^3a/c\end{matrix}\,\Big|\, q; -q^2/c\right]=\frac{1}{a(q/c;q)_2}\left[\begin{matrix}q^3a/c^2, -q\\ q^3a/c, -q^2/c\end{matrix}\,\Big|\, q\right]_\infty \times\left\{\frac{(1-qa/c)(qa;q^2)_\infty}{(q^3a/c^2;q^2)_\infty}-\frac{(a;q^2)_\infty}{(q^4a/c^2;q^2)_\infty}\right\}.
$$

Example 4.10 ($\lambda=1$ and $\rho=-1$).

$$
{}_2\phi_1\left[\begin{matrix}a,\ c\\ a/c\end{matrix}\,\Big|\, q; -q^2/c\right]=\left[\begin{matrix}a/c^2, -q\\ a/c, -1/cq\end{matrix}\,\Big|\, q\right]_\infty \times\left\{\frac{(q-qc+a/c+qa/c)(a;q^2)_\infty}{qac(a/c^2;q^2)_\infty}-\frac{(q-qc-a-qa)(qa;q^2)_\infty}{qac(qa/c^2;q^2)_\infty}\right\}.
$$

Example 4.11 ($\lambda=-1$ and $\rho=1$).

$$
{}_2\phi_1\left[\begin{matrix}a,\ c\\ q^2a/c\end{matrix}\,\Big|\, q; -1/c\right]=\left[\begin{matrix}q^2a/c^2, -q\\ q^2a/c, -1/c\end{matrix}\,\Big|\, q\right]_\infty \times\left\{\frac{(q-a-c-qa)(qa;q^2)_\infty}{(q-c)(q^3a/c^2;q^2)_\infty}+\frac{(q-c+qa/c+q^2a/c)(a;q^2)_\infty}{(q-c)(q^2a/c^2;q^2)_\infty}\right\}.
$$

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