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GEOMETRIC POLYNOMIALS AND INTEGER PARTITIONS

MIRCEA MERCA

ABSTRACT. In this paper, we show that the geometric polynomials can be expressed as sums over integer partitions in two different ways. New formulas involving geometric numbers, Bernoulli numbers, and Genocchi numbers are derived in this context.

1. Introduction

The geometric polynomials (also known as Fubini polynomials) are defined as follows (see [22, 24, 26]):

$$\omega_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! x^k, \tag{1.1}$$

where $\left\{ {n \atop k} \right\}$ are the Stirling numbers of the second kind. Recall that the Stirling numbers of the second kind count the number of ways to partition a set of n objects into k nonempty subsets. In terms of partitions of an n-set, $\left\{ {n \atop k} \right\} k!$ is the number of distinct ordered partitions with k subsets. For example, $\left\{ {3 \atop 2} \right\} = 3$ because the set $\{1,2,3\}$ can be partitioned into two subsets in three ways: $\big\{\{1,2\},\{3\}\big\}$, $\big\{\{1,3\},\{2\}\big\}$ and $\big\{\{1\},\{2,3\}\big\}$. The ordered partitions $\big\{\{1,2\},\{3\}\big\}$, $\big\{\{3\},\{1,2\}\big\}$, $\big\{\{1,3\},\{2\}\big\}$, $\big\{\{2\},\{1,3\}\big\}$, $\big\{\{1\},\{2,3\}\big\}$, and $\big\{\{2,3\},\{1\}\big\}$ are counted by $\left\{ {3 \atop 2} \right\} 2! = 6$. The first few geometric polynomials are:

$$\begin{aligned}
\omega_0(x) &= 1,\\
\omega_1(x) &= x,\\
\omega_2(x) &= x + 2x^2,\\
\omega_3(x) &= x + 6x^2 + 6x^3,\\
\omega_4(x) &= x + 14x^2 + 36x^3 + 24x^4.
\end{aligned}$$

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The geometric polynomials first appeared in Euler's book [13, Part 2, §172]. They describe the action of the derivative operator $\left(x\frac{d}{dx}\right)^m$, $m = 0,1,2,\dots$ on the function $1/(1-x)$,

$$\left(x\frac{d}{dx}\right)^m \frac{1}{1-x} = \sum_{k=0}^{\infty} k^m x^k = \frac{1}{1-x} F_m\left(\frac{1}{1-x}\right), \qquad |x|<1.$$

The use of geometric polynomials by Euler was pointed out by K. Boyadzhiev in [6] and H. Gould mentions these polynomials in [14]. Gould refers to the book of I. J. Schwatt [24].

In the last two decades, the geometric polynomials have received considerable attention as an effective tool in different topics in analysis, combinatorics and number theory [4, 5, 7, 8, 10, 12, 11, 16, 17, 19, 20, 22]. In [10] the authors considered Euler–Seidel matrices method and obtained some fundamental properties of the geometric polynomials as the following linear recurrence relations:

$$\omega_{n+1}(x) = x\sum_{k=0}^{n} \binom{n+1}{k} \omega_k(x), \tag{1.2}$$

$$\omega_{n+1}(x) = \frac{x}{1+x}\sum_{k=0}^{n} \binom{n+1}{k} \big(\omega_k(x)+\omega_{k+1}(x)\big),$$

$$\omega_{n+1}(x) = x\frac{d}{dx}\big(\omega_n(x)+x\omega_n(x)\big).$$

In this paper, using the exponential generating functions of the geometric polynomials [4], i.e.,

$$\sum_{n=0}^{\infty} \omega_n(x)\frac{z^n}{n!} = \frac{1}{1-x(e^z-1)}, \tag{1.3}$$

we shall establish formulas for $\omega_n(x)$ or $\omega_{n-1}(x)$ as sums over all the unrestricted integer partitions of n. Recall [1] that a partition of a positive integer n is a weakly decreasing sequence of positive integers

$$\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_k > 0$$

whose sum is n,

$$\lambda_1+\lambda_2+\dots+\lambda_k = n.$$

The positive integers in the sequence are called parts. To make formulas more concise, we pad the sequence of parts with zeros to obtain n nonnegative parts.

Theorem 1.1. *For $n>0$,*

$$\omega_n(x) = \sum_{\substack{\lambda_1+\lambda_2+\dots+\lambda_n=n \\ \lambda_1\geqslant\lambda_2\geqslant\dots\geqslant\lambda_n\geqslant 0}} \frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_n}{0} n!}{1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}} x^{\lambda_1}.$$

For example, the case $n=4$ of this theorem reads as follows

$$\begin{aligned}\omega_4(x) &= \frac{\binom{4}{0}4!}{1^4}x^4 + \frac{\binom{3}{1}\binom{1}{0}4!}{1^3 2^1}x^3 + \frac{\binom{2}{2}\binom{2}{0}4!}{1^2 2^2}x^2 \\ &\quad + \frac{\binom{2}{1}\binom{1}{1}\binom{1}{0}4!}{1^2 2^1 3^1}x^2 + \frac{\binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{1}{0}4!}{1^1 2^1 3^1 4^1}x \\ &= 24x^4 + 36x^3 + 6x^2 + 8x^2 + x \\ &= 24x^4 + 36x^3 + 14x^2 + x,\end{aligned}$$

because the partitions in question are:

$$4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1. \tag{1.4}$$

Theorem 1.2. *For $n > 1$,*

$$\omega_{n-1}(x) = \frac{1}{x+1} \cdot \sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_n=n \\ \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0}} \frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_n}{0}n!}{1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}}\frac{x^{\lambda_1}}{\lambda_1}.$$

For example, using (1.4) and Theorem 1.2, we can write:

$$\begin{aligned}\omega_3(x) &= \frac{1}{x+1}\left(\frac{\binom{4}{0}4!}{1^4}\frac{x^4}{4} + \frac{\binom{3}{1}\binom{1}{0}4!}{1^3 2^1}\frac{x^3}{3} + \frac{\binom{2}{2}\binom{2}{0}4!}{1^2 2^2}\frac{x^2}{2}\right. \\ &\quad \left. + \frac{\binom{2}{1}\binom{1}{1}\binom{1}{0}4!}{1^2 2^1 3^1}\frac{x^2}{2} + \frac{\binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{1}{0}4!}{1^1 2^1 3^1 4^1}x\right) \\ &= \frac{6x^4 + 12x^3 + 7x^2 + x}{x+1} \\ &= \frac{(x+1)(6x^3 + 6x^2 + x)}{x+1} \\ &= 6x^3 + 6x^2 + x.\end{aligned}$$

Related to Theorems 1.1 and 1.2, we remark that the coefficient

$$\frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_n}{0}n!}{1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}}$$

is the number of preferential arrangements associated with an integer partition of n and can be seen in the On-Line Encyclopedia of Integer Sequence [25, A049019].

For $x = 1$ in (1.1), we get nth geometric number (ordered Bell number or Fubini number):

$$\omega_n = \omega_n(1) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k!.$$

These numbers were called Fubini numbers by Louis Comtet, because they count the number of different ways to rearrange the ordering of sums or integrals in Fubini's theorem [22]. On the other hand, the nth geometric number counts the distinct ordered partitions of an n-set. These numbers have been studied by many authors. Various characterizations of these numbers can be found in the literature [2, 3, 9, 10, 15, 18, 21, 23, 27]. Taking into account Theorems 1.1 and 1.2, we can write the following decompositions of the geometric numbers.

Corollary 1.3. *For $n > 0$,*

$$\omega_n = \sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_n=n \\ \lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n\geqslant 0}} \frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_n}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}} \cdot n!.$$

Corollary 1.4. *For $n > 1$,*

$$\omega_{n-1} = \sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_n=n \\ \lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n\geqslant 0}} \frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_n}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}} \frac{n!}{2\lambda_1}.$$

Using (1.1) and Theorem 1.1, we easily deduce the following formula for the Stirling numbers of the second kind as a sum over the partitions of n in which the largest part has size k.

Corollary 1.5. *For $n, k > 0$,*

$$\left\{ n \atop k \right\} = \sum_{\substack{k+\lambda_2+\cdots+\lambda_n=n \\ k\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n\geqslant 0}} \frac{\binom{k}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_n}{0}}{1^{k}2^{\lambda_2}\cdots n^{\lambda_n}} \cdot \frac{n!}{k!}.$$

According to [17, Theorem 1.2] and [4, eq. (3.29)], we have the following relations for $n > 0$:

$$\int_{-1}^{0} \omega_n(x)dx = B_n \tag{1.5}$$

and

$$\omega_n\left(-\frac{1}{2}\right) = \frac{G_{n+1}}{n+1}, \tag{1.6}$$

where B_n is the nth Bernoulli number and G_n is the nth Genocchi number. Recall that these numbers are defined by the exponential generating functions

$$\sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1}, \qquad |z| < 2\pi$$

and

$$\sum_{n=0}^{\infty} G_n \frac{z^n}{n!} = \frac{2z}{e^z + 1}, \qquad |z| < \pi.$$

Over the years, the works of Bernoulli numbers and Genocchi numbers and their combinatorial relations have received much attention. Using (1.5), (1.6), and Theorem 1.1, we derive new formulas for B_n and G_{n+1} as sums over all the integer partitions of n.

Corollary 1.6. *For $n > 0$,*

$$B_n = \sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_n=n\\ \lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n\geqslant 0}} (-1)^{\lambda_1} \frac{n!}{\lambda_1+1} \frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_n}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}}$$

and

$$G_{n+1} = \sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_n=n\\ \lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n\geqslant 0}} (-1)^{\lambda_1} \frac{(n+1)!}{2^{\lambda_1}} \frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_n}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}}.$$

The case $n = 4$ of this corollary reads as follows:

$$\begin{aligned} B_4 &= \frac{4!}{5}\cdot\frac{1}{1^4} - \frac{4!}{4}\cdot\frac{3}{1^3 2^1} + \frac{4!}{3}\cdot\frac{1}{1^2 2^2} + \frac{4!}{4}\cdot\frac{2}{1^2 2^1 3^1} - \frac{4!}{2}\cdot\frac{1}{1^1 2^1 3^1 4^1} \\ &= \frac{24}{5} - 9 + 2 + \frac{8}{3} - \frac{1}{2} \\ &= -\frac{1}{30} \end{aligned}$$

and

$$\begin{aligned} G_5 &= \frac{5!}{2^4}\cdot\frac{1}{1^4} - \frac{5!}{2^3}\cdot\frac{3}{1^3 2^1} + \frac{5!}{2^2}\cdot\frac{1}{1^2 2^2} + \frac{5!}{2^2}\cdot\frac{2}{1^2 2^1 3^1} - \frac{5!}{2^1}\cdot\frac{1}{1^1 2^1 3^1 4^1} \\ &= \frac{15}{2} - \frac{45}{2} + \frac{15}{2} + 10 - \frac{5}{2} \\ &= 0. \end{aligned}$$

Another decomposition for the nth Genocchi number can be easily obtained if we consider Theorem 1.2 and the relation (1.6).

Corollary 1.7. *For $n > 1$,*

$$G_n = \sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_n=n\\ \lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n\geqslant 0}} (-1)^{\lambda_1} \frac{n\cdot n!}{2^{\lambda_1-1}\cdot\lambda_1} \frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_n}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}}.$$

For example, the case $n = 4$ of Corollary 1.7 can be written as:

$$\begin{aligned} G_4 &= \frac{4\cdot 4!}{2^3\cdot 4}\cdot\frac{1}{1^4} - \frac{4\cdot 4!}{2^2\cdot 3}\cdot\frac{3}{1^3 2^1} + \frac{4\cdot 4!}{2^1\cdot 2}\cdot\frac{1}{1^2 2^2} \\ &\qquad + \frac{4\cdot 4!}{2^1\cdot 2}\cdot\frac{2}{1^2 2^1 3^1} - \frac{4\cdot 4!}{2^0\cdot 1}\cdot\frac{1}{1^1 2^1 3^1 4^1} \\ &= 3 - 12 + 6 + 8 - 4 \\ &= 1. \end{aligned}$$

The rest of this paper is organized as follows. In the next two sections we will prove Theorems 1.1 and 1.2. In Section 4 we will provide a decomposition of the geometric polynomial $\omega_{n+1}(x)$ as a sum over the weak compositions of n into exactly 3 parts.

2. Proof of Theorem 1.1

Using the generating functions for the geometric polynomials (1.3) and the exponential series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad |z| < 1, \tag{2.1}$$

we can write

$$\begin{aligned}
&\sum_{n=0}^{\infty} \omega_n(x) \frac{z^n}{n!} \\
&= \frac{1}{1 - x(e^z - 1)} \\
&= \left(1 - x \sum_{n=1}^{\infty} \frac{z^n}{n!}\right)^{-1} \\
&= 1 + \sum_{j=1}^{\infty} \left(x \sum_{n=1}^{\infty} \frac{z^n}{n!}\right)^j \\
&= 1 + \sum_{j=1}^{\infty} x^j \sum_{n=1}^{\infty} \left(\sum_{\substack{t_1+2t_2+\cdots+nt_n=n \\ t_1+t_2+\cdots+t_n=j}} \binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n} \prod_{i=1}^{n} \frac{1}{i!^{t_i}} \right) z^n \\
&= 1 + \sum_{n=1}^{\infty} \left(\sum_{t_1+2t_2+\cdots+nt_n=n} x^{t_1+t_2+\cdots+t_n} \binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n} \times \right. \\
&\qquad\qquad \left. \times \prod_{i=1}^{n} \frac{1}{i^{t_i+t_2+\cdots+t_n}} \right) z^n.
\end{aligned}$$

Thus we deduce that

$$\frac{\omega_n(x)}{n!} = \sum_{t_1+2t_2+\cdots+nt_n=n} x^{t_1+t_2+\cdots+t_n} \binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n} \prod_{i=1}^{n} \frac{1}{i^{t_i+t_2+\cdots+t_n}}.$$

We see that this decomposition of $\omega_n(x)/(n!)$ is a sum over all the partitions of n. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the conjugate partition of the partition

$$t_1 + 2t_2 + \cdots + nt_n = n.$$

It is clear that $\lambda_i = t_i + t_{i+1} + \cdots + t_n$. In this way, we obtain

$$\frac{\omega_n(x)}{n!} = \sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_n=n \\ \lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n\geqslant 0}} \frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_n}{0}}{1^{\lambda_1}2^{\lambda_2}\ldots n^{\lambda_n}} x^{\lambda_1}.$$

This concludes the proof.

3. Proof of Theorem 1.2

Taking into account the exponential series (2.1) and the logarithmic series

$$\ln(1+z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}z^n, \qquad |z|<1,$$

we can write

$$\begin{aligned}
&\frac{d}{dz}\ln\big(1-x(e^z-1)\big)\\
&\quad=\frac{d}{dz}\ln\left(1-x\sum_{n=1}^{\infty}\frac{z^n}{n!}\right)\\
&\quad=-\frac{d}{dz}\sum_{j=1}^{\infty}\frac{x^j}{j}\left(\sum_{n=1}^{\infty}\frac{z^n}{n!}\right)^j\\
&\quad=-\frac{d}{dz}\sum_{j=1}^{\infty}\frac{x^j}{j}\sum_{n=1}^{\infty}\left(\sum_{\substack{t_1+t_2+\cdots+t_n=j\\ t_1+2t_2+\cdots+nt_n=n}}\binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n}\prod_{i=1}^{n}\left(\frac{1}{i!}\right)^{t_i}\right)z^n\\
&\quad=-\frac{d}{dz}\sum_{n=1}^{\infty}\left(\sum_{t_1+2t_2+\cdots+nt_n=n}\frac{x^{t_1+t_2+\cdots+t_n}}{t_1+t_2+\cdots+t_n}\binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n}\times\right.\\
&\qquad\qquad\left.\times\prod_{i=1}^{n}\left(\frac{1}{i}\right)^{t_i+t_{i+1}+\cdots+t_n}\right)z^n\\
&\quad=-\frac{d}{dz}\sum_{n=1}^{\infty}\left(\sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_n=n\\ \lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n\geqslant0}}\frac{x^{\lambda_1}}{\lambda_1}\frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_n}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}}\right)z^n\\
&\quad=-\sum_{n=1}^{\infty}\left(\sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_n=n\\ \lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n\geqslant0}}\frac{n\cdot x^{\lambda_1}}{\lambda_1}\frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_n}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}}\right)z^{n-1}
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dz}\ln\big(1-x(e^z-1)\big)&=-\frac{xe^z}{1-x(e^z-1)}\\
&=-x\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\omega_n(x)\frac{z^n}{n!}\right)\\
&=-x\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{\omega_k(x)}{k!(n-k)!}\right)z^n
\end{aligned}$$

$$
\begin{aligned}
&= -x \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \omega_k(x) \right) \frac{z^n}{n!} \\
&= -x - \sum_{n=1}^{\infty} \left(x \sum_{k=0}^{n-1} \binom{n}{k} \omega_k(x) + x\omega_n(x) \right) \frac{z^n}{n!} \\
&= -x - \sum_{n=1}^{\infty} \big(\omega_n(x) + x\omega_n(x)\big) \frac{z^n}{n!}, \qquad (3.1)
\end{aligned}
$$

where we have invoked (1.2). Thus we deduce that

$$
\frac{(1+x)\omega_{n-1}(x)}{(n-1)!} = \sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_n=n \\ \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0}} \frac{n \cdot x^{\lambda_1}}{\lambda_1} \frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_n}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}}
$$

and the theorem is proved.

4. Convolutions over weak compositions

Recall that a composition of an integer n is a way of writing n as the sum of a sequence of positive integers. A weak composition of an integer n is similar to a composition of n, but allowing terms of the sequence to be zero. In this section, we show that the geometric polynomials $\omega_n(x)$ and $\omega_{n+1}(x)$ can be written as a summation over all weak compositions of n into exactly 3 parts.

Theorem 4.1. *For $n \geqslant 0$,*

$$
\omega_{n+1}(x) = x \sum_{a+b+c=n} \binom{n}{a,b,c} \omega_a(x)\omega_b(x),
$$

where a, b, c are nonnegative integers.

Proof. Using the Cauchy product of two power series, we obtain

$$
\begin{aligned}
\left(1 - x\sum_{n=1}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \omega_n(x)\frac{z^n}{n!}\right) &= 1 + \sum_{n=1}^{\infty} \left(\frac{\omega_n(x)}{n!} - x \sum_{k=0}^{n-1} \frac{\omega_k(x)}{k!(n-k!)} \right) z^n \\
&= 1 + \sum_{n=1}^{\infty} \left(\omega_n(x) - x \sum_{k=0}^{n-1} \binom{n}{k} \omega_k(x) \right) \frac{z^n}{n!} \\
&= 1,
\end{aligned}
$$

where we have invoked (1.2). According to the proof of Theorem 1.2, we can write

$$
\begin{aligned}
x + \sum_{n=1}^{\infty} (1+x)\omega_n(x) \frac{z^n}{n!} &= \frac{d}{dz} \ln \left(1 - x \sum_{n=1}^{\infty} \frac{z^n}{n!} \right)^{-1} \\
&= \frac{d}{dz} \ln \left(\sum_{n=0}^{\infty} \omega_n(x) \frac{z^n}{n!} \right)
\end{aligned}
$$

$$= \left(\sum_{n=1}^{\infty} \omega_n(x) \frac{z^{n-1}}{(n-1)!}\right) \left(\sum_{n=0}^{\infty} \omega_n(x) \frac{z^n}{n!}\right)^{-1}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} \omega_{n+1}(x) \frac{z^n}{n!} &= \left(x + \sum_{n=1}^{\infty} (1+x) \omega_n(x) \frac{z^n}{n!}\right) \left(\sum_{n=0}^{\infty} \omega_n(x) \frac{z^n}{n!}\right) \\
&= \left(-1 + \sum_{n=0}^{\infty} (1+x) \omega_n(x) \frac{z^n}{n!}\right) \left(\sum_{n=0}^{\infty} \omega_n(x) \frac{z^n}{n!}\right) \\
&= -\sum_{n=0}^{\infty} \omega_n(x) \frac{z^n}{n!} + (1+x) \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{\omega_k(x)\omega_{n-k}(x)}{k!(n-k)!}\right) z^n.
\end{aligned}$$

Thus we deduce that

$$\frac{\omega_n(x)}{(n-1)!} = -\frac{\omega_{n-1}(x)}{(n-1)!} + (1+x) \sum_{k=0}^{n-1} \frac{\omega_k(x)\omega_{n-1-k}(x)}{k!(n-1-k)!}$$

or

$$\frac{n\omega_n(x)}{n!} = -\frac{\omega_{n-1}(x)}{(n-1)!} + (1+x) \sum_{k=1}^{n} \frac{\omega_{k-1}(x)\omega_{n-k}(x)}{(k-1)!(n-k)!}$$

or

$$\frac{n\omega_n(x)}{n!} = x\frac{\omega_{n-1}(x)}{(n-1)!} + \sum_{k=2}^{n} \frac{(1+x)\omega_{k-1}(x)}{(k-1)!} \frac{\omega_{n-k}(x)}{(n-k)!}. \tag{4.1}$$

On the other hand, by (3.1) we see that

$$\sum_{k=1}^{n} \frac{x}{(k-1)!} \frac{\omega_{n-k}(x)}{(n-k)!} = \begin{cases} \dfrac{(1+x)\omega_{n-1}(x)}{(n-1)!}, & \text{for } n > 1, \\ x, & \text{for } n = 1. \end{cases} \tag{4.2}$$

Taking into account (4.1) and (4.2), we obtain

$$\frac{n\omega_n(x)}{n!} = \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{x}{(j-1)!} \frac{\omega_{k-j}(x)}{(k-j)!} \frac{\omega_{n-k}(x)}{(n-k)!}.$$

After a little manipulation we arrive at our identity. □

For example, the weak compositions of 3 are:

0+0+3, 0+1+2, 0+2+1, 1+1+1,
0+3+0, 1+0+2, 2+0+1,
3+0+0, 1+2+0, 2+1+0.

Thus the case $n = 3$ of Theorem 4.1 reads as follows:

$$\begin{aligned}
\omega_4(x) = x \Bigg(&\binom{3}{0,0,3} \omega_0(x)\omega_0(x) + 2\binom{3}{0,1,2} \omega_0(x)\omega_1(x) \\
&+ 2\binom{3}{0,2,1} \omega_0(x)\omega_2(x) + 2\binom{3}{0,3,0} \omega_0(x)\omega_3(x)
\end{aligned}$$

$$
\begin{aligned}
&+\binom{3}{1,1,1}\omega_1(x)\omega_1(x)+2\binom{3}{1,2,0}\omega_1(x)\omega_2(x)\Big)\\
&= x\big(1+6x+6(x+2x^2)+2(x+6x^2+6x^3)+6x^2+6x(x+2x^2)\big)\\
&= x+14x^2+36x^3+24x^4.
\end{aligned}
$$

Remark: Theorem 4.1 can be written in the following equivalent form: for $n \geqslant 0$,

$$
n\omega_n(x) = x \sum_{a+b+c=n} \binom{n}{a,b,c} a\omega_b(x)\omega_c(x),
$$

where a, b, c are nonnegative integers.

As consequences of Theorem 4.1, we remark the following convolution identities involving geometric numbers and Genocchi numbers. These identities seem to be new.

Corollary 4.3. *For $n \geqslant 0$,*

$$
\omega_{n+1} = \sum_{a+b+c=n} \binom{n}{a,b,c} \omega_a\omega_b,
$$

where a, b, c are nonnegative integers.

Corollary 4.4. *For $n > 1$,*

$$
G_n = \frac{1}{2-2n} \sum_{a+b+c=n} \binom{n}{a,b,c} G_aG_b,
$$

where a, b, c are nonnegative integers.

Remark: Taking into account that $G_n = 2(1-2^n)B_n$, the last corollary can be rewritten in terms of the Bernoulli numbers as follows: for $n > 1$,

$$
B_n = \frac{1}{1-n} \sum_{a+b+c=n} \binom{n}{a,b,c} \frac{(1-2^a)(1-2^b)}{1-2^n} B_aB_b,
$$

where a, b, c are nonnegative integers.

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Department of Mathematics, University of Craiova
200585 Craiova, Romania

Academy of Romanian Scientists, Ilfov 3, Sector 5, Bucuresti, Romania
E-mail address: `mircea.merca@profinfo.edu.ro`