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FLAG VECTOR PAIRS, FATNESS, AND THEIR BOUNDS FOR 4-POLYTOPES

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Abstract. Recently Sjöberg and Ziegler showed a remarkable result that completely characterizes the flag vector pair (f_0, f_{03}) of any 4-dimensional polytope. Motivated by their results and techniques, in this paper we show some necessary conditions for other remaining flag vector pairs such as (f_0, f_{02}), (f_{02}, f_{03}), (f_1, f_{02}), and (f_1, f_{03}) to be flag vector pairs of 4-dimensional convex polytopes. Results of this paper give some partial answers to the questions posed by Sjöberg and Ziegler. As an application of the bounds for flag vector pairs (f_1, f_{03}), in this paper we also provide some bounds of the fatness function for certain 4-polytopes as well as 3-polytopes.

1. Introduction and Main Results

For a d-dimensional polytope P, let $f_i = f_i(P)$ denote the number of i-dimensional faces of P. One of the fundamental combinatorial invariants of P is its f-vector, which we are mainly concerned with in this paper. The Euler–Poincaré formula gives the well-known restriction on the f-vectors of polytopes. Another well-known restriction on the f-vectors for simplicial polytopes is the so-called Dehn–Sommerville equations. In [14], McMullen conjectured some characterization of the f-vectors of simplicial polytopes, and then it has been verified by Stanley in [16], and Billera and Lee in [6]. However, currently any complete characterization of the f-vectors of all polytopes with arbitrary dimensions seems to be very much out of reach, whereas Grünbaum, Barnette, and Reay completed the characterization of the f-vector pairs (f_i, f_j) of 4-dimensional polytopes (see [10], [2], and [3]).

There is another combinatorial invariant for convex polytopes, called the flag vector, which is not relatively well-known but obviously generalizes the notion of the f-vector. That is, for any $S \subset \{0, 1, 2, \ldots, d-1\}$, let $f_S =$

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$f_S(P)$ denote the number of chains

$$F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r$$

of faces of P with

$$\{\dim F_1, \dim F_2, \ldots, \dim F_r\} = S.$$

For the sake of simplicity, from now on we use the notation $f_{i_1 i_2 \ldots i_k}(P)$ instead of $f_{\{i_1, i_2, \ldots, i_k\}}(P)$ for any subset $\{i_1, i_2, \ldots, i_k\}$ of $\{0, 1, 2, \ldots, d-1\}$. For instance, $f_{03}(P)$ will mean $f_{\{0,3\}}(P)$. Note that the f-vector of P is then $(f_0, f_1, \ldots, f_{d-1})$, while the flag vector of P is defined to be $(f_S)_{S \subset \{0,1,2,\ldots,d-1\}}$.

The set of all f-vectors of d-dimensional polytopes will be denoted by $\mathcal{F}^d$. Clearly $\mathcal{F}^d$ is a subset of $\mathbb{Z}^d$. Let $\Pi_{i,j}(\mathcal{F}^d)$ denote the projection of the f-vector of $P \in \mathcal{F}^d$ onto the coordinates f_i and f_j. Then $(n, m) \in \Pi_{i,j}(\mathcal{F}^d)$ is called a *polytopal pair* if there is a d-polytope P with $f_i(P) = n$ and $f_j(P) = m$. If $(n, m) \in \Pi_{0,d-1}(\mathcal{F}^d)$, then these pairs must satisfy the well-known upper bound theorem saying

$$m \leq f_{d-1}(C_d(n)), \quad n \leq f_{d-1}(C_d(m)),$$

where $C_d(n)$ denotes a d-dimensional cyclic polytope with n vertices (see [13], [7], and [18, Section 8.4]). For the moment curve in $\mathbb{R}^d$ defined by

$$\alpha : \mathbb{R} \to \mathbb{R}^n, \quad t \mapsto (t, t^2, \ldots, t^d)$$

and for any $n > d$, the standard dth cyclic polytope with n vertices, denoted by $C_d(t_1, t_2, \ldots, t_n)$, is defined as the convex hull in $\mathbb{R}^n$ of n different points $\alpha(t_1)$, $\alpha(t_2)$, …, $\alpha(t_n)$ on the moment curve α such that $t_1 < t_2 < \cdots < t_n$. Cyclic polytopes $C_d(n)$ are precisely those which are combinatorial equivalent to the standard cyclic polytope (see [10] for more details).

In a similar vein, for any two subsets S_1 and S_2 of $\{0, 1, 2, \ldots, d-1\}$, a pair $(f_{S_1}(P), f_{S_2}(P))$, or simply (f_{S_1}, f_{S_2}), of flag numbers of P will be called a *flag vector pair*. More generally, for any k, not necessarily mutually disjoint, subsets $S_1, S_2, \ldots, S_k$ of $\{0, 1, 2 \ldots, d-1\}$, a k-tuple

$$(f_{S_1}(P), f_{S_2}(P), \ldots, f_{S_k}(P))$$

or simply $(f_{S_1}, f_{S_2}, \ldots, f_{S_k})$, of flag numbers of P will be called a *flag vector k-tuple.*

As in the f-vectors, let us denote by $\Pi_{S_1, S_2, \ldots, S_k}$ the projection of the flag vector $(f_S)_{S \subset \{0,1,2,\ldots,d-1\}}$ onto its coordinates $f_{S_1}, f_{S_2}, \ldots, f_{S_k}$. We call $(f_{S_1}, f_{S_2}, \ldots, f_{S_k})$ a *polytopal flag vector k-tuple* if $(f_{S_1}, f_{S_2}, \ldots, f_{S_k})$ belongs to the image of the set of all flag vectors of d-dimensional polytopes under the projection map $\Pi_{S_1, S_2, \ldots, S_k}$, that is, if there is a d-polytope P such that

$$(f_{S_1}(P), f_{S_2}(P), \ldots, f_{S_k}(P)) = (f_{S_1}, f_{S_2}, \ldots, f_{S_k}).$$

In [15], Sjöberg and Ziegler recently showed a remarkable result that completely determines the flag vector pair (f_0, f_{03}) of any 4-dimensional polytopes. In order to obtain such results, they crucially used the work [1] of

Altshuler and Steinberg on 4-polytopes up to 8 vertices. Furthermore, they used stacking, general stacking on cyclic polytopes, facet splitting, truncating, and other techniques for the construction of specific 4-dimensional polytopes.

Our first goal of this paper is to give some necessary conditions for other remaining flag vector pairs such as (f_0, f_{02}), (f_{02}, f_{03}), (f_1, f_{02}), and (f_1, f_{03}) to be flag vector pairs of 4-dimensional convex polytopes. We do not know whether or not the necessary conditions (or their variants) for flag vector pairs given in this paper are also sufficient. However, it can be shown by a case-by-case analysis that there exist some 4-polytopes for certain flag vector pairs satisfying the necessary conditions, which will not be pursued further in this paper (see Remark 1.2 and Subsection 3.3 for some interesting examples).

Recall that a convex polytope P is called *neighborly* (or 2-*neighborly*) if any pair of vertices of P is connected by an edge, forming a complete graph. So any nonneighborly polytope P should have at least one pair of vertices of P which do not form an edge.

Our main results go as follows.

Theorem 1.1. *The flag vector pair* $(f_0, f_{02}) = (f_0(P), f_{02}(P))$ *of a* 4-*polytope* P *satisfies the following two conditions:*

(1) $30 \leq 6f_0 \leq f_{02} \leq 3f_0(f_0-3)$. *Here the lower (resp. upper) bound of* f_{02} *can be achieved by simple (resp. neighborly)* 4-*polytopes.*
(2) *For each* $k \in \{1, 2, 3, 4, 5, 7, 8, 10, 11\}$, *we have*

$$f_{02} \neq 3f_0(f_0-3) - k.$$

Remark 1.2.

(a) The lower and upper bounds given in Theorem 1.1 (1) have been observed by Sjöberg and Ziegler in [15, Subsection 2.7].
(b) We remark that the polytope P obtained by taking the bipyramid over a tetrahedron satisfies

$$(f_0(P), f_1(P), f_{02}(P), f_{03}(P)) = (6, 14, 48, 32),$$

so that we have

$$48 = f_{02}(P) = 3f_0(P)(f_0(P)-3) - 6 = 3 \cdot 6 \cdot 3 - 6.$$

Note also that there exists a 4-polytope P with its flag 4-tuple

$$(f_0(P), f_1(P), f_{02}(P), f_{03}(P)) = (6, 14, 45, 29),$$

so that the flag number $f_{02}(P)$ satisfies

$$45 = f_{02}(P) = 3f_0(P)(f_0(P)-3) - 9 = 3 \cdot 6 \cdot 3 - 9.$$

In fact, the pyramid over triangular bipyramid realizes those flag numbers. See Subsection 3.3 for more details. Also, there exists a 4-polytope P with its flag 4-tuple

$$(f_0(P), f_1(P), f_{02}(P), f_{03}(P)) = (7, 20, 75, 49),$$

so that the flag number $f_{02}(P)$ satisfies

$$75 = f_{02}(P) = 3f_0(P)(f_0(P) - 3) - 9 = 3 \cdot 7 \cdot 4 - 9.$$

The polytope P has the Gale diagram which is the last diagram in [10, Figure 6.3.4]. This polytope has the f-vector $(7, 20, 25, 12)$, and its facets consist of 11 simplices and one bipyramid. Thus every 2-face is a triangle, and $f_{02} = 3f_2$ holds.

Theorem 1.3. *The flag vector pair $(f_{02}, f_{03}) = (f_{02}(P), f_{03}(P))$ of a non-neighborly 4-polytope P satisfies the following inequalities*

$$\frac{2}{3}\left(9 + \sqrt{153 + 12f_{02}}\right) \leq f_{03} \leq \frac{2}{3}f_{02}.$$

A 4-polytope is called 2-*simple* if each edge of the polytope is contained in exactly 3 facets.

Theorem 1.4. *The flag vector pair $(f_1, f_{02}) = (f_1(P), f_{02}(P))$ of a 4-polytope P satisfies the following inequalities*

$$3f_1 \leq f_{02} \leq 6f_1 - 3\left(1 + \sqrt{1 + 8f_1}\right),$$

where the lower (resp. upper) bound of f_{02} can be achieved by 2-simple (resp. neighborly) 4-polytopes.

Theorem 1.5. *The flag vector pair $(f_1, f_{03}) = (f_1(P), f_{03}(P))$ of a non-neighborly 4-polytope P satisfies the following inequalities*

$$f_1 + 1 + \sqrt{1 + 8f_1} \leq f_{03} \leq 4f_1 - 2\left(1 + \sqrt{1 + 8f_1}\right).$$

It would be interesting to find some explicit examples which achieve tight lower and upper bounds given in Theorems 1.1, 1.3, 1.4, and 1.5. Note, however, that for a 4-simplex Δ^4, we have

$$f_0(\Delta^4) = 5, f_1(\Delta^4) = 10, f_{02}(\Delta^4) = 30, f_{03}(\Delta^4) = 20.$$

Hence, a 4-simplex, simple 4-polytopes, and neighborly 4-polytopes provide some examples achieving the lower and upper bounds for the above Theorems 1.1, 1.3, 1.4, and 1.5 in certain cases. Furthermore, the problem of finding exceptional flag vector pairs currently seems to be widely open, and we want to discuss the problem elsewhere (see [15, Subsection 2.7] for some related discussion).

We also remark that some obstructions for flag vector pairs (f_1, f_{04}) of 5-polytopes have been proved in [8]. To be a little more precise, certain bounds of the flag number f_{04} of a 5-polytope have been shown in terms of a given flag number f_1. The upper bounds given in [8] are not optimal, even though the lower bounds are very sharp. On the other hand, recently very sharp and optimal upper and lower bounds for f_1 and f_{04} have been finally obtained in [11].

As an application of the bounds for flag vector pairs (f_1, f_{03}), our second goal of this paper is to partially answer the following question.

Question. *Are there positive constants c and C so that all 4-dimensional convex polytopes P satisfy the inequality*

$$0 < c \leq \frac{f_1(P) + f_2(P)}{f_0(P) + f_3(P)} \leq C?$$

In order to effectively answer the above question, we define the fatness function ϕ_4 by

$$\phi_4 : \mathcal{F}_4 \to \mathbb{R}, \quad P \mapsto \frac{f_1(P) + f_2(P)}{f_0(P) + f_3(P)}.$$

Similarly, in the case of 3-polytopes, we define the fatness function ϕ_3 by

$$\phi_3 : \mathcal{F}_3 \to \mathbb{R}, \quad P \mapsto \frac{f_1(P)}{f_0(P) + f_2(P)}.$$

There are some known results for the values of the fatness function ϕ_3 and ϕ_4. For example, it is known that the 4-simplex has fatness 2, while the 4-cube and the 4-cross polytope have fatness $7/3$. More generally, if P is a simple 4-polytope, then by using the Dehn–Sommerville relations

$$f_2(P) = f_1(P) + f_3(P) - f_0(P), \quad f_1(P) = 2f_0(P)$$

it is easy to obtain the formula for the fatness function ϕ_4, as follows:

$$\phi_4(P) = \frac{f_1(P) + f_2(P)}{f_0(P) + f_3(P)} = \frac{3f_0(P) + f_3(P)}{f_0(P) + f_3(P)} < 3.$$

Since every 4-polytope and its dual have the same fatness by its definition, the same upper bound holds for simplicial 4-polytopes.

On the other hand, it is known that the neighborly cubical 4-polytopes P_n defined by Joswig and Ziegler in [12] have f-vectors

$$\begin{aligned}&(f_0(P_n), f_1(P_n), f_2(P_n), f_3(P_n))\\ &= (2^n, n \times 2^{n-1}, (3n-6)2^{n-2}, (n-2)2^{n-2}).\end{aligned}$$

Hence we can obtain the fatness

$$\phi_4(P_n) = \frac{5n-6}{n+2} < 5$$

which converges to 5 as n goes to ∞. Actually, by a result of Eppstein, Kuperberg, and Ziegler in [9, Theorem 1] there are convex 4-polytopes whose fatness ϕ_4 is greater than 5.048.

In this paper, we want to provide some upper and lower bounds for the fatness function ϕ_4 as well as ϕ_3. More precisely, we have the following

Theorem 1.6. *Let P be a convex 3-polytope. Then the following inequalities hold:*

$$\frac{3}{4} \leq \phi_3(P) < 2.$$

Remark 1.7. The lower bound of Theorem 1.6 has been observed by the referee of this paper, while our original lower bound was $1/2$.

Recall that any d-polytope P satisfies

$$\frac{d}{2} f_0(P) \leq f_1(P) \leq \binom{f_0(P)}{2}. \tag{1.1}$$

This, in particular, implies that any 3-polytope P satisfies $f_1(P) \geq 6$, while any 4-polytope satisfies $f_1(P) \geq 10$.

Theorem 1.8. *Let P be a convex 4-polytope, and let $f_1 = f_1(P)$. Then the following inequalities hold:*

$$\phi_4(P) \geq \frac{2\left(3f_1 + 3 + \sqrt{13 + 4\sqrt{1+8f_1}}\right)}{7f_1 - 3\left(1 + \sqrt{1+8f_1}\right)} > \frac{6}{7}.$$

Theorem 1.9. *Let P be a convex nonneighborly 4-polytope, and let $f_1 = f_1(P)$. Then the following inequalities hold:*

$$\phi_4(P) \geq \frac{(3f_1 + 3 + \sqrt{13 + 4\sqrt{1+8f_1}})}{3f_1 - 1 - \sqrt{1+8f_1}} > 1.$$

As is mentioned in (1.1), in general, $f_0(P)$ and $f_1(P)$ of a d-polytope P satisfy the inequality

$$f_1(P) \leq \binom{f_0(P)}{2}.$$

This implies that if $f_1(P)$ happens to be less than $\binom{f_0(P)}{2}$, then there should be at least one pair of vertices of P which does not form an edge. We call such a pair of vertices a *nonedge*. In particular, any facet of a 4-polytope which is not a simplex should contain at least one nonedge. This is because the only 3-polytope in which every two vertices form an edge is the 3-simplex.

Theorem 1.10. *Let P be a 4-polytope with a unique nonedge. Then the following inequalities hold:*

$$1 < \phi_4(P) < 3.$$

This paper is organized as follows. In Section 2, we collect some notation, definitions, and preliminary facts in order to prove our main theorems given in Section 3. In Section 3, we give some necessary conditions for vector pairs such as (f_0, f_{02}), (f_{02}, f_{03}), (f_1, f_{02}), and (f_1, f_{03}) to be flag vector pairs of 4-dimensional convex polytopes. To be more precise, in Section 3 we give a proof of Theorem 1.1 for a flag vector pair (f_0, f_{02}). To do so, we use a case-by-case analysis. Section 4 is devoted to giving a proof of Theorem 1.3 for a flag vector pair (f_{02}, f_{03}), while Section 5 deals with the case of a flag vector pair (f_1, f_{02}) and there we provide a proof of Theorem 1.4. In Section 6, we give some bounds for a flag vector pair (f_1, f_{03}). Finally, in Section 7 we prove our main Theorems 1.6, 1.8, 1.9, and 1.10, in detail.

2. Preliminaries

This section briefly describes some important theorems necessary for understanding the proofs of our main results given in Section 3. In addition, in this section we set up some notation and definitions used later.

First, we begin with summarizing the well-known facts about the f-vector of convex polytopes, in particular, 4-dimensional polytopes.

Theorem 2.1 ([10, Theorem 10.4.1])**.** *The set of f-vector pairs (f_0, f_3) of 4-polytopes is equal to*

$$\Pi_{0,3}(\mathcal{F}^4) = \left\{ (f_0, f_3) \in \mathbb{Z}^2 \,\middle|\, 5 \leq f_0 \leq \frac{1}{2} f_3(f_3 - 3),\ 5 \leq f_3 \leq \frac{1}{2} f_0(f_0 - 3) \right\}.$$

Theorem 2.2 ([10, Theorem 10.4.2])**.** *The set of f-vector pairs (f_0, f_1) of 4-polytopes is equal to*

$$\Pi_{0,1}(\mathcal{F}^4) = \left\{ (f_0, f_1) \in \mathbb{Z}^2 \,\middle|\, 10 \leq 2f_0 \leq f_1 \leq \frac{1}{2} f_0(f_0 - 1) \right\} - \{(6,12),(7,14),(8,17),(10,20)\}.$$

Theorem 2.3 ([15, Theorem 2.5])**.** *There exists a 4-polytope P with $f_0(P) = f_0$ and $f_{03}(P) = f_{03}$ if and only if the following two conditions hold:*

(1) *f_0 and f_{03} are integers satisfying*

$$20 \leq 4f_0 \leq f_{03} \leq 2f_0(f_0 - 3),$$

and

$$f_{03} \neq 2f_0(f_0 - 3) - k, \quad k \in \{1, 2, 3, 5, 6, 9, 13\}.$$

(2) *(f_0, f_{03}) is not one of the following 18 exceptional pairs*

$$\begin{aligned}
&(6,24),(6,25),(6,28),\\
&(7,28),(7,30),(7,31),(7,33),(7,34),(7,37),(7,40),\\
&(8,33),(8,34),(8,37),(8,40),\\
&(9,37),(9,40),(10,40),(10,43).
\end{aligned}$$

The following generalized Dehn–Sommerville equation plays an important role in the proofs of our main theorems.

Theorem 2.4 ([5, Theorem 2.1])**.** *Let P be a d-polytope, and let S be a subset of $\{0, 1, 2, \ldots, d-1\}$. Let $\{i, k\}$ be a subset of $S \cup \{-1, d\}$ such that $i < k - 1$ and such that there does not exist an integer $j \in S$ with $i < j < k$. Then the following equation holds:*

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup \{j\}}(P) = f_S(P)(1 - (-1)^{k-i-1}).$$

We also need the following theorem of Bayer in [4, Theorems 1.3 and 1.4].

Theorem 2.5. *The flag vector of a 4-polytope P satisfies the inequalities:*

(1) $f_{02}(P) - 3f_2(P) + f_1(P) - 4f_0(P) + 10 \geq 0$.
(2) $-6f_0(P) + 6f_1(P) - f_{02}(P) \geq 0$.

3. Proof of Theorem 1.1: flag vector pairs (f_0, f_{02})

The aim of this section is to give a proof of our main Theorem 1.1. To do so, we first prove a series of results in order to characterize the following set

$$\Pi_{0,02}(\mathcal{F}^4) = \{(f_0(P), f_{02}(P)) \in \mathbb{Z}^2 \mid P \text{ is a 4-polytope}\}.$$

We first begin with the following lemma whose proof easily follows from Theorem 2.4 (or [5, Theorem 2.1]) with $S = \{0\}$, $i = 0$, and $k = 4$. We leave its proof to the reader.

Lemma 3.1. *The flag vector of a* 4*-polytope* P *satisfies the following identity*

$$2f_0(P) - 2f_1(P) + f_{02}(P) - f_{03}(P) = 0.$$

As an immediate consequence, we have the following result that is equivalent to Theorem 1.1 (1).

Proposition 3.2. *The flag vector of a* 4*-polytope* P *satisfies the following inequalities*

$$30 \leq 6f_0(P) \leq f_{02}(P) \leq 3f_0(P)(f_0(P) - 3).$$

Proof. Recall that by a result of Sjöberg and Ziegler ([15, Theorem 2.5] or Theorem 2.3 (1)) we have

$$20 \leq 4f_0 \leq f_{03} \leq 2f_0(f_0 - 3).$$

By combining the above inequalities with the identity given in Lemma 3.1, it is easy to obtain

$$2f_0(f_0 - 3) \geq f_{03} = 2f_0 - 2f_1 + f_{02}.$$

Hence, by Theorem 2.2 we can show

$$\begin{aligned} f_{02} &\leq -2f_0 + 2f_1 + 2f_0(f_0 - 3) \\ &\leq -2f_0 + f_0(f_0 - 1) + 2f_0(f_0 - 3) \\ &= 3f_0(f_0 - 3). \end{aligned} \tag{3.1}$$

Also, it follows from $f_{03} \geq 4f_0$ and $f_1 \geq 2f_0$ that the identity $f_{03} = 2f_0 - 2f_1 + f_{02}$ implies

$$f_{02} \geq 2f_0 + 2f_1 \geq 6f_0 \geq 30. \tag{3.2}$$

By (3.1) and (3.2), we now have

$$30 \leq 6f_0 \leq f_{02} \leq 3f_0(f_0 - 3),$$

which completes the proof. ∎

As a consequence of Proposition 3.2, we have

$$\Pi_{0,02}(\mathcal{F}^4) \subset \left\{(n,m) \in \mathbb{Z}^2 \,|\, 30 \le 6n \le m \le 3n(n-3)\right\}.$$

Note that $f_{02}(P) = 3f_0(P)(f_0(P)-3)$ if and only if P is neighborly, while $f_{02}(P) = 4f_0(P)$ if and only if P is simple (see, e.g., [15, Lemma 2.6]). Thus, if $f_{02}(P) < 3f_0(P)(f_0(P)-3)$, then P is not neighborly, i.e.,

$$f_1(P) \le \frac{1}{2}f_0(P)(f_0(P)-1) - 1 = \binom{f_0(P)}{2} - 1. \tag{3.3}$$

We can complete the proof of Theorem 1.1 (2) by a case-by-case analysis in the following subsections.

3.1. **The case $1 \le k \le 5$.** We begin with the following lemma.

Lemma 3.3. *Assume that $f_{02} = 3f_0(f_0-3) - k$ for some positive integer k. Then we have*

$$\binom{f_0}{2} + \frac{4-k}{2} \le f_1 \le \binom{f_0}{2} - 1. \tag{3.4}$$

Proof. By Theorem 2.3 (or [15, Theorem 2.5]) and Lemma 3.1, we have

$$\begin{aligned} 3f_0(f_0-3) - k = f_{02} &= -2f_0 + 2f_1 + f_{03} \\ &\le -2f_0 + 2f_1 + 2f_0(f_0-3) - 4. \end{aligned}$$

This implies the following inequality

$$2f_1 \ge f_0(f_0-1) + 4 - k.$$

That is, we should have

$$f_1 \ge \frac{f_0(f_0-1)}{2} + \frac{4-k}{2} = \binom{f_0}{2} + \frac{4-k}{2}. \tag{3.5}$$

Note that, since $f_{02}(P)$ is assumed to be equal to $3f_0(P)(f_0(P)-3) - k$ for a positive integer k, P is not neighborly. Thus, it follows from (3.3) that we have

$$f_1(P) \le \binom{f_0(P)}{2} - 1. \tag{3.6}$$

By (3.5) and (3.6), it is now immediate to obtain

$$\binom{f_0}{2} + \frac{4-k}{2} \le f_1 \le \binom{f_0}{2} - 1,$$

as desired. $\square$

Corollary 3.4. *For each $k \in \{1,2,3,4,5\}$, we have*

$$f_{02} \ne 3f_0(f_0-3) - k.$$

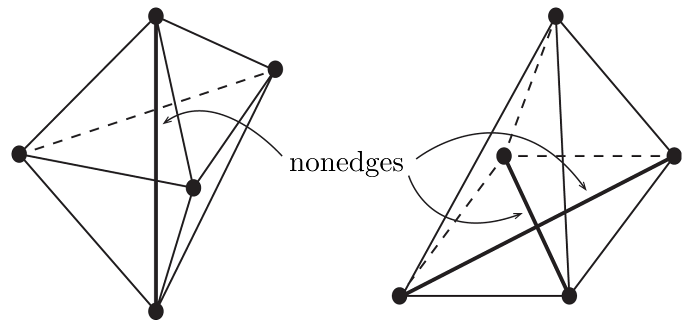

Figure 1. Bipyramid over a triangle with one nonedge and square pyramid with two nonedges.

Proof. We prove it by contradiction. So suppose

$$f_{02} = 3f_0(f_0 - 3) - k.$$

for some positive integer k with $1 \le k \le 5$. Then it follows from (3.4) in Lemma 3.3 that we have

$$\binom{f_0}{2} - 1 < \binom{f_0}{2} - \frac{1}{2} \le \binom{f_0}{2} + \frac{4-k}{2} \le f_1 \le \binom{f_0}{2} - 1.$$

Clearly this is a contradiction. $\square$

3.2. **The case $k = 7$.** Next we want to deal with the case of $k = 7$. In order to exclude the case of $k = 7$ in Theorem 1.1 (2), we first need to recall that if there is a pair of vertices of a polytope not forming an edge, then such a pair of vertices is called a *nonedge* (See Figure 1). Recall that the only 3-polytope in which every two vertices form an edge is the 3-simplex. Hence any 3-polytope which is not a 3-simplex contains a nonedge.

If $f_1 = \binom{f_0}{2} - 1$, there is a unique pair of vertices v_1, v_2 of P which do not form an edge. That is, there is a unique nonedge e in P. If P is not simplicial, there is a facet F, not a 3-simplex, such that F should contain the unique nonedge e. It then should have only five vertices. Further, it is well-known from [10, Section 6.1] (or [15, Section 2]) that there are only two combinatorial types of 3-polytopes with five vertices: the square pyramid and the bipyramid over a triangle. Note that only the bipyramid over a triangle contains a unique nonedge, while the square pyramid contains exactly two nonedges (see Figure 1).

The following lemma holds.

Lemma 3.5. *Let P be a 4-polytope with a unique nonedge, and let t be the number of all tetrahedral facets of P. Then the following statements hold:*

(1) *If the polytope P is not simplicial, then P is a polytope with one bipyramid facet and remaining tetrahedral facets, and f_{02} satisfies*

$$f_{02} = 6t + 9.$$

(2) *If the polytope P is simplicial, then f_{02} satisfies*

$$f_{02} = 6t.$$

Proof. (1) The first statement follows immediately from the fact that among all the 3-polytopes with five vertices only the bipyramid over a triangle contains a unique nonedge. Since in this case every 2-dimensional face of P is a triangle, it is easy to see that

$$f_2 = \frac{4t+6}{2} = 2t+3, \quad f_{02} = 3f_2 = 6t+9.$$

(2) On the other hand, if the polytope P is simplicial, clearly we have $f_2 = 2t$, and thus f_{02} satisfies $f_{02} = 6t$. Hence we are done. □

Now we are ready to deal with the case of $k = 7$ in Theorem 1.1 (2).

Lemma 3.6. *The following statement holds:*

$$f_{02} \neq 3f_0(f_0-3) - 7.$$

Proof. For the proof, suppose that f_{02} is equal to $3f_0(f_0-3)-7$. Then it follows from (3.4) in Lemma 3.3 that we have

$$\binom{f_0}{2} - 2 < \binom{f_0}{2} - \frac{3}{2} \leq f_1 \leq \binom{f_0}{2} - 1.$$

Thus $f_1 = \binom{f_0}{2} - 1$. That is, there is a unique nonedge in P. Thus, it follows from Lemma 3.5 that f_{02} is always equal to 0 mod 3. But $3f_0(f_0-3)-7$ cannot be equal to 0 mod 3. Hence we have

$$f_{02} \neq 3f_0(f_0-3) - 7.$$

This completes the proof of Lemma 3.6 □

3.3. **The case** $k = 8, 9$**.** In this subsection, we want to deal with the cases of $k = 8, 9$ in Theorem 1.1 (2).

Lemma 3.7. *The following statement holds:*

$$f_{02} \neq 3f_0(f_0-3) - 8.$$

Proof. We prove it by contradiction. So suppose that $f_{02} = 3f_0(f_0-3)-8$. Then it follows from (3.4) in Lemma 3.3 that we have

$$\binom{f_0}{2} - 2 = \binom{f_0}{2} - \frac{4}{2} \leq f_1 \leq \binom{f_0}{2} - 1.$$

Thus we have $f_1 = \binom{f_0}{2} - 1$ or $f_1 = \binom{f_0}{2} - 2$.

If $f_1 = \binom{f_0}{2} - 1$, then it follows from Lemma 3.5 that f_{02} is equal to 0 mod 3, while $3f_0(f_0-3)-8$ is not equal to 0 mod 3. Thus we have

$$f_{02} \neq 3f_0(f_0-3) - 8.$$

On the other hand, if $f_1 = \binom{f_0}{2} - 2$, then P has exactly two nonedges. Assume first that P is not simplicial. Let F be any 3-dimensional facet of P. Then F has at most two nonedges. Thus in this case we should have

$$\binom{f_0(F)}{2} - 2 \leq f_1(F) \leq 3f_0(F) - 6,$$

which implies that $4 \leq f_0(F) \leq 5$. So, if F is any nontetrahedral facet of P, then F is a 3-polytope with 5 vertices.

In fact, we can show that the case of $f_1 = \binom{f_0}{2} - 2$ cannot occur at all. To see it, note first that if $f_{02} = 3f_0(f_0 - 3) - 8$ with $f_1 = \binom{f_0}{2} - 2$, then by Lemma 3.1 we have

$$f_{03} = 2f_0(f_0 - 3) + 4 - 8 \equiv 0 \bmod 4.$$

Since each tetrahedral facet has exactly four vertices and P is assumed to be nonsimplicial, this implies that there must be at least four nontetrahedral facets with five vertices in P. Thus P should contain at least four nonedges, which contradicts the assumption that there are only two nonedges in P.

Consequently, in this case we have

$$f_{02} \neq 3f_0(f_0 - 3) - 8.$$

Assume next that P is simplicial. As in Lemma 3.5, it is easy to see that $f_{02} = 6t$. Thus f_{02} is equal to $0 \bmod 3$, which contradicts the fact that $3f_0(f_0 - 3) - 8$ is not equal to $0 \bmod 3$. Thus once again we have $f_{02} \neq 3f_0(f_0 - 3) - 8$.

This completes the proof of Lemma 3.7. □

If $f_1 = \binom{f_0}{2} - 2$, it can be also shown that

$$f_{02} \neq 3f_0(f_0 - 3) - 9.$$

Indeed, suppose that $f_{02} = 3f_0(f_0 - 3) - 9$. Then it follows from Theorem 2.5 (2) and the condition $f_1 = \binom{f_0}{2} - 2$ that we have

$$\begin{aligned} -6f_0 + 6f_1 = -6f_0 + 6\left(\frac{f_0(f_0 - 1)}{2} - 2\right) &= 3f_0(f_0 - 3) - 12 \\ \geq f_{02} = 3f_0(f_0 - 3) - 9, \end{aligned}$$

which is a contradiction.

As mentioned in Remark 1.2, there exists a 4-polytope P with its flag 3-tuple

$$(f_0(P), f_1(P), f_{03}(P)) = (6, 14, 29),$$

so that we have

$$45 = f_{02}(P) = 3f_0(P)(f_0(P) - 3) - 9 = 3 \cdot 6 \cdot 3 - 9,$$

as desired. In fact, the pyramid over a triangular bipyramid realizes the flag 4-tuple

$$(f_0(P), f_1(P), f_{02}(P), f_{03}(P)) = (6, 14, 45, 29).$$

Refer to [15, Table 1] for more details.

3.4. **The case** $k = 10, 11$**.** Finally, in this subsection we want to exclude the cases of $k = 10, 11$ in Theorem 1.1 (2).

Lemma 3.8. *For any* $k \in \{10, 11\}$*, we have*

$$f_{02} \neq 3f_0(f_0 - 3) - k.$$

Proof. For $k = 10$ or 11, suppose that $f_{02} = 3f_0(f_0 - 3) - k$. Then it follows from (3.4) in Lemma 3.3 that we have

$$\binom{f_0}{2} - 4 < \binom{f_0}{2} - \frac{7}{2} \leq f_1 \leq \binom{f_0}{2} - 1.$$

Thus $f_1 = \binom{f_0}{2} - 1$, or $f_1 = \binom{f_0}{2} - 2$, or $f_1 = \binom{f_0}{2} - 3$.

(1) If $f_1 = \binom{f_0}{2} - 1$, then, as in Subsection 3.2, f_{02} is equal to 0 mod 3. Hence we have

$$f_{02} \neq 3f_0(f_0 - 3) - 10, \quad f_{02} \neq 3f_0(f_0 - 3) - 11.$$

(2) If $f_1 = \binom{f_0}{2} - 2$, then it follows from Lemma 3.1 that we have

$$f_{03} = 2f_0(f_0 - 3) + 4 - k \equiv 2 \bmod 4, \quad k = 10, \tag{3.7}$$

and

$$f_{03} = 2f_0(f_0 - 3) + 4 - k \equiv 1 \bmod 4, \quad k = 11. \tag{3.8}$$

This, in particular, implies that P cannot be simplicial.

Assume first that $f_{02} = 3f_0(f_0 - 3) - 10$. Then, as in Subsection 3.3 it follows from (3.7) that there must be at least two nontetrahedral facets with five vertices in P. Since there are only two nonedges in P, either there are exactly two bipyramids with a triangle in P as facets such that each of them contains a unique nonedge or there are exactly two square pyramids as facets whose apices are connected by an edge.

As before, let t denote the number of tetrahedral facets of P. Then we have

$$f_{02} = 3f_2 = 6(t + 3) \text{ or } 6(t + 2).$$

Thus f_{02} is equal to 0 mod 3, while $3f_0(f_0 - 3) - 10$ is not equal to 0 mod 3. This implies that

$$f_{02} \neq 3f_0(f_0 - 3) - 10.$$

Assume next that $f_{02} = 3f_0(f_0 - 3) - 11$. Then, once again as in Subsection 3.3 it follows from (3.8) that there must be at least one, but not two, nontetrahedral facet with five vertices in P. Since there are only two nonedges in P, there should be only one nontetrahedral facet of P which is a bipyramid over a triangle such that one nonedge lies in the bipyramid over a triangle, and the other nonedge does not lie in a facet.

For this case, we have

$$f_{02} = 3f_2 = 6t + 9 \equiv 0 \bmod 3.$$

Since $3f_0(f_0 - 3) - 11$ is not equal to 0 mod 3, we have

$$f_{02} \neq 3f_0(f_0 - 3) - 11.$$

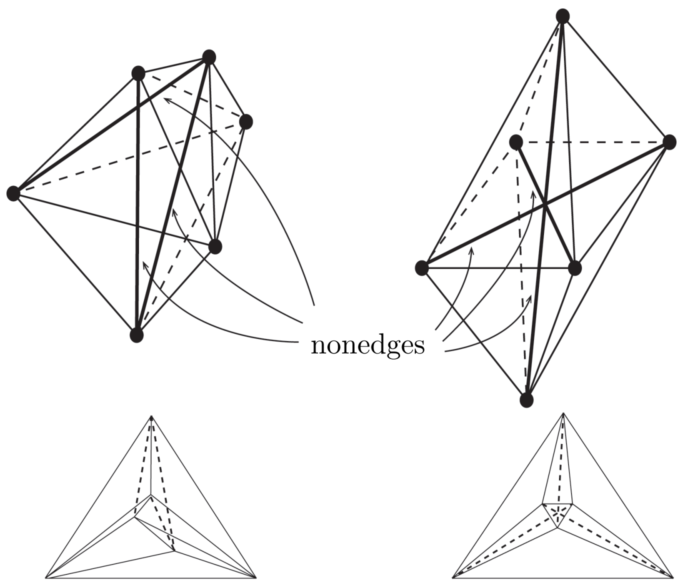

FIGURE 2. Two 3-polytopes with 6 vertices and three nonedges, where the second row shows their corresponding Schlegel diagrams.

(3) It remains to deal with the case of $f_1 = \binom{f_0}{2} - 3$. Assume first that P is not simplicial. Let F be any 3-dimensional facet of P. Then F has at most three nonedges. Hence we have

$$\binom{f_0(F)}{2} - 3 \leq f_1(F) \leq 3f_0(F) - 6, \tag{3.9}$$

which implies that $4 \leq f_0(F) \leq 6$.

(i) Assume that there is a 3-dimensional facet F with $f_0(F) = 6$. Then it follows from (3.9) that F should have 12 edges and three nonedges. Note that there are only two such combinatorially different 3-polytopes F which are both simplicial (see [15, Section 2] and Figure 2).

For this case, we have

$$f_{02} = 3f_2 = 3(2t + 4) = 6t + 12 \equiv \ 0 \bmod 6.$$

Since $3f_0(f_0 - 3) - k$ is not equal to 0 mod 6 for $k \in \{10, 11\}$, we have

$$f_{02} \neq 3f_0(f_0 - 3) - k, \quad k \in \{10, 11\}.$$

(ii) Assume next that there is no facet with six vertices so that every 3-dimensional facet F satisfies $4 \leq f_0(F) \leq 5$. Then any nontetrahedral facet F should have 5 vertices. Since there are exactly three nonedges in P and thus there are at most three nonedges in facets, it is also true that we cannot have more than three nontetrahedral facets in P. Further, in this case we have

$$f_{03} = 2f_0(f_0 - 3) + 6 - k \equiv 0 \bmod 4, \quad k = 10,$$

and

$$f_{03} = 2f_0(f_0 - 3) + 6 - k \equiv 3 \bmod 4, \quad k = 11.$$

This, in particular, implies that the nontetrahedral facets of P cannot consist of two square pyramids with two nonedges coming from their common square and a third nonedge coming from their apices. So we have two cases to consider:

(a) The nontetrahedral facets of P are three bipyramids over a triangle.
(b) The nontetrahedral facets of P are two square pyramids and one bipyramid over a triangle.

In the case of (a), we have

$$f_2 = \frac{4t + 18}{2} = 2t + 9.$$

Thus we have

$$f_{02} = 3f_2 = 3(2t + 9) = 6t + 27 \equiv 3 \bmod 6.$$

This implies that

$$f_{02} \neq 3f_0(f_0 - 3) - k, \quad k \in \{10, 11\}.$$

Similarly, in the case of (b), we have

$$f_2 = \frac{4t + 16}{2} = 2t + 8.$$

Thus we have

$$f_{02} = 3f_2 = 3(2t + 8) = 6t + 24 \equiv 0 \bmod 6.$$

Once again this implies that

$$f_{02} \neq 3f_0(f_0 - 3) - k, \quad k \in \{10, 11\}.$$

Assume next that P is simplicial. Then we have $f_{02} = 6t$, which is equal to 0 mod 6. Hence

$$f_{02} \neq 3f_0(f_0 - 3) - k, \quad k \in \{10, 11\}.$$

This completes the proof of Lemma 3.8. □

To sum up all of the previous results, we can obtain the following theorem (Theorem 1.1).

Theorem 3.9. *The flag vector pair* $(f_0, f_{02}) = (f_0(P), f_{02}(P))$ *of a 4-polytope* P *satisfies the following two conditions:*

(1) $30 \leq 6f_0 \leq f_{02} \leq 3f_0(f_0 - 3)$. *Here the lower (resp. upper) bound of* f_{02} *can be achieved by simple (resp. neighborly) 4-polytopes.*
(2) *For each* $k \in \{1, 2, 3, 4, 5, 7, 8, 10, 11\}$, *we have*

$$f_{02} \neq 3f_0(f_0 - 3) - k.$$

4. Proof of Theorem 1.3: flag vector pairs (f_{02}, f_{03})

The aim of this section is to give a proof of Theorem 1.3.

Theorem 4.1. *The flag vector pair $(f_{02}, f_{03}) = (f_{02}(P), f_{03}(P))$ of a nonneighborly 4-polytope P satisfies the following inequalities*

$$\frac{2}{3}\left(9+\sqrt{153+12f_{02}}\right) \leq f_{03} \leq \frac{2}{3}f_{02}.$$

Proof. To see it, recall first from Theorem 2.5 (2) that we have

$$f_0 - f_1 \leq -\frac{1}{6}f_{02}.$$

By Lemma 3.1, it is easy to obtain

$$f_{02} = f_{03} - 2(f_0 - f_1) \geq f_{03} + \frac{1}{3}f_{02}.$$

Thus we have $f_{03} \leq \frac{2}{3}f_{02}$.

On the other hand, if P is nonneighborly, then it follows from Theorem 1.1 that we have

$$f_{02} \leq 3f_0(f_0 - 3) - 6.$$

That is, $f_0^2 - 3f_0 - 2 - \frac{1}{3}f_{02} \geq 0$. Thus, it is not difficult to obtain

$$f_0 \geq \frac{1}{6}\left(9+\sqrt{153+12f_{02}}\right).$$

Since $f_{03} \geq 4f_0$, this implies

$$f_{03} \geq \frac{2}{3}\left(9+\sqrt{153+12f_{02}}\right).$$

This completes the proof of Theorem 4.1. □

5. Proof of Theorem 1.4: flag vector pairs (f_1, f_{02})

The aim of this section is to give a proof of our main Theorem 1.4. Recall that a 4-polytope is 2-simple if each edge of the polytope is contained in exactly 3 facets.

Theorem 5.1. *The flag vector pair $(f_1, f_{02}) = (f_1(P), f_{02}(P))$ of a 4-polytope P satisfies the following inequalities*

$$3f_1 \leq f_{02} \leq 6f_1 - 3\left(1+\sqrt{1+8f_1}\right),$$

where the lower (resp. upper) bound of f_{02} can be achieved by 2-simple (resp. neighborly) 4-polytopes.

Proof. For the proof, recall first from Theorem 2.2 that we have

$$2f_1 \leq f_0(f_0 - 1), \text{ i.e., } f_0^2 - f_0 - 2f_1 \geq 0.$$

Thus it is easy to obtain

$$f_0 \geq \frac{1}{2}\left(1+\sqrt{1+8f_1}\right). \tag{5.1}$$

Also, it follows from Theorem 2.5 (2) that we have

$$f_{02} \leq -6f_0 + 6f_1 \leq -3\left(1+\sqrt{1+8f_1}\right) + 6f_1.$$

Since each edge is contained in at least three facets for 4-polytopes, we should have

$$f_{12} \geq 3f_1. \tag{5.2}$$

Thus it follows from (5.2) and the identity $f_{02} = f_{12}$ that we have $f_{02} \geq 3f_1$.

Consequently, we can obtain

$$3f_1 \leq f_{02} \leq 6f_1 - 3\left(1+\sqrt{1+8f_1}\right),$$

where the lower (resp. upper) bound of f_{02} can be achieved by 2-simple (resp. neighborly) 4-polytopes. This completes the proof of Theorem 5.1. □

6. Proof of Theorem 1.5: flag vector pairs (f_1, f_{03})

The aim of this section is to give a proof of our main Theorem 1.5. To do so, we first begin with the following theorem.

Theorem 6.1. *The flag vector pair $(f_1, f_{03}) = (f_1(P), f_{03}(P))$ of a 4-polytope P satisfies the following inequalities*

$$f_1 + 1 + \sqrt{1+8f_1} \leq f_{03} \leq 5f_1 - 3\left(1+\sqrt{1+8f_1}\right),$$

where the lower (resp. upper) bound of f_{02} can be achieved by 2-simple (resp. neighborly) 4-polytopes.

Proof. By the identity

$$f_{03} = 2f_0 - 2f_1 + f_{02},$$

it follows from (5.1) and Theorem 5.1 that we have

$$\begin{aligned} f_{03} &= 2f_0 - 2f_1 + f_{02} \\ &\geq 1 + \sqrt{1+8f_1} - 2f_1 + 3f_1 \\ &= f_1 + 1 + \sqrt{1+8f_1}. \end{aligned}$$

Moreover, it is also true that

$$\begin{aligned} f_{03} &= 2f_0 - 2f_1 + f_{02} \\ &\leq f_1 - 2f_1 + 6f_1 - 3(1+\sqrt{1+8f_1}) \\ &= 5f_1 - 3(1+\sqrt{1+8f_1}). \end{aligned}$$

This completes the proof of Theorem 6.1. □

Finally we close this section with the following theorem which improves the upper bound for f_{03} given in Theorem 6.1 in certain cases.

Theorem 6.2. *The flag vector pair $(f_1, f_{03}) = (f_1(P), f_{03}(P))$ of a non-neighborly 4-polytope P satisfies the following inequalities*

$$f_1 + 1 + \sqrt{1+8f_1} \leq f_{03} \leq 4f_1 - 2\left(1 + \sqrt{1+8f_1}\right).$$

Proof. To show it, we crucially make use of Theorems 4.1 and 5.1. Indeed, it follows from Theorems 4.1 and 5.1 that we have

$$\begin{aligned} f_{03} \leq \frac{2}{3} f_{02} &\leq \frac{2}{3}\left(6f_1 - 3\left(1 + \sqrt{1+8f_1}\right)\right) \\ &= 4f_1 - 2\left(1 + \sqrt{1+8f_1}\right). \end{aligned}$$

This together with Theorem 6.1 completes the proof. □

Notice that for any $f_1 \geq 10$, which is always true for any 4-polytopes, we have

$$4f_1 - 2\left(1 + \sqrt{1+8f_1}\right) \leq 5f_1 - 3\left(1 + \sqrt{1+8f_1}\right).$$

Therefore, for any nonneighborly 4-polytope P (so, $f_1 \geq 11$) Theorem 6.2 gives better upper bound for f_{03} in terms of f_1 than those given in Theorem 6.1.

7. Proofs of Theorems 1.6, 1.8, 1.9, and 1.10

The aim of this section is to give proofs of Theorems 1.6, 1.8, 1.9, and 1.10. For simplicity, as above we denote by $(f_0, f_1, \ldots, f_{d-1})$ the f-vector of d-polytope unless there is any confusion.

To do so, we begin with the proof of Theorem 1.6.

Theorem 7.1. *Let P be a convex 3-polytope. Then the following inequalities hold:*

$$\frac{3}{4} \leq \phi_3(P) < 2.$$

Proof. Since P is a convex 3-polytope, by a result of Steinitz in [17] it satisfies the following relations:

- $f_1 = f_0 + f_2 - 2$ (Euler–Poincaré equation).
- $f_2 \leq 2f_0 - 4$ and $f_0 \leq 2f_2 - 4$ with equality for simplicial and simple 3-polytopes, respectively.
- $\frac{3}{2} f_0 \leq f_1 \leq 3f_0 - 6$.

Hence we have

$$\phi_3(P) = \frac{f_1}{f_0 + f_2} = \frac{f_0 + f_2 - 2}{f_0 + f_2} = 1 - \frac{2}{f_0 + f_2} \geq \frac{3}{4}, \tag{7.1}$$

where we make use of the inequality $f_0 + f_2 \geq 8$ in order to obtain the last inequality.

On the other hand, it is also easy to obtain

$$\phi_3(P) = \frac{f_1}{f_0 + f_2} \leq \frac{6f_0 - 12}{3f_0 + 4} = 2 - \frac{20}{3f_0 + 4} < 2. \tag{7.2}$$

Therefore, it follows from (7.1) and (7.2) that we have

$$\frac{3}{4} \leq \phi_3(P) < 2,$$

as desired. □

Theorem 7.2. *Let P be a convex 4-polytope. Then the following inequalities hold:*

$$\phi_4(P) \geq \frac{2\left(3f_1 + 3 + \sqrt{13 + 4\sqrt{1+8f_1}}\right)}{7f_1 - 3\left(1 + \sqrt{1+8f_1}\right)} > \frac{6}{7}.$$

Proof. For the proof, first note that by Euler–Poincaré equation we have

$$f_2 = -f_0 + f_1 + f_3.$$

Thus the fatness function ϕ_4 satisfies

$$\phi_4(P) = \frac{f_1 + f_2}{f_0 + f_3} = \frac{-f_0 + 2f_1 + f_3}{f_0 + f_3}. \tag{7.3}$$

Since each 3-dimensional face has at least four vertices and so $f_{03} \geq 4f_3$ (or by the lower bound in Theorem 2.3 (1) and its duality), it follows from (7.3) that we have

$$\phi_4(P) \geq \frac{-f_0 + 2f_1 + f_3}{f_0 + \frac{1}{4}f_{03}}.$$

By using the inequality $f_{03} \leq 5f_1 - 3\left(1 + \sqrt{1+8f_1}\right)$ from Theorem 6.1 once again, it is easy to obtain

$$f_0 + \frac{1}{4}f_{03} \leq f_0 + \frac{5}{4}f_1 - \frac{3}{4}\left(1 + \sqrt{1+8f_1}\right).$$

This together with $f_1 \geq 2f_0$ implies

$$\phi_4(P) \geq \frac{\frac{3}{2}f_1 + f_3}{\frac{7}{4}f_1 - \frac{3}{4}\left(1 + \sqrt{1+8f_1}\right)}. \tag{7.4}$$

On the other hand, it follows from [10, Theorem 10.4.1] that we have

$$f_3^2 - 3f_3 - 2f_0 \geq 0 \text{ and } f_0^2 - f_0 - 2f_1 \geq 0,$$

which implies

$$f_3 \geq \frac{3 + \sqrt{9 + 8f_0}}{2} \text{ and } f_0 \geq \frac{1 + \sqrt{1+8f_1}}{2}.$$

Consequently, f_3 satisfies

$$f_3 \geq \frac{3 + \sqrt{13 + 4\sqrt{1+8f_1}}}{2}. \tag{7.5}$$

By combining (7.4) with (7.5), we can obtain

$$\phi_4(P) \geq \frac{2\left(3f_1 + 3 + \sqrt{13 + 4\sqrt{1+8f_1}}\right)}{7f_1 - 3\left(1 + \sqrt{1+8f_1}\right)}, \quad f_1 \geq 10.$$

Now, let $f(x)$ be a function given by

$$f(x) = \frac{(6x + 6 + 2\sqrt{13 + 4\sqrt{1 + 8x}})}{7x - 3 - 3\sqrt{1 + 8x}}, \quad x \geq 10.$$

Then f is a decreasing function and satisfies $f(10) = 2$ and $6/7 < f(x) \leq 2$. So in this case the lower bound for the fatness function ϕ_4 is bounded from above by 2 and bounded from below by 6/7.

This completes the proof of Theorem 7.2. $\square$

Theorem 7.3. *Let P be a convex nonneighborly 4-polytope. Then the following inequalities hold:*

$$\phi_4(P) \geq \frac{(3f_1 + 3 + \sqrt{13 + 4\sqrt{1 + 8f_1}})}{3f_1 - 1 - \sqrt{1 + 8f_1}} > 1, \quad f_1 \geq 10.$$

Proof. The proof of Theorem 7.3 is very similar to that of Theorem 7.2 with the inequality

$$f_{03} \leq 4f_1 - 2(1 + \sqrt{1 + 8f_1})$$

stated in Theorem 6.2. So we leave its detailed proof to the reader. $\square$

Remark 7.4. Let $f(x)$ be a function given by

$$f(x) = \frac{(3x + 3 + \sqrt{13 + 4\sqrt{1 + 8x}})}{3x - 1 - \sqrt{1 + 8x}}, \quad x \geq 10.$$

Then f is a decreasing function and satisfies $f(10) = 2$ and $1 < f(x) \leq 2$. So in this case the lower bound for the fatness function ϕ_4 is bounded from above by 2 and bounded from below by 1.

Theorem 7.5. *Let P be a 4-polytope with a unique nonedge. Then the following inequalities hold:*

$$1 < \phi_4(P) < 3.$$

Proof. For the proof, we make use of the fact from Lemma 3.5 that if P is not simplicial, then P should be a 4-polytope with only one bipyramid facet and remaining tetrahedron facets. As before, let t denote the number of all tetrahedral facets of P.

Assume first that P is not simplicial. Then we have

$$f_3(P) = 1 + t, \ f_2(P) = 2t + 3, \ f_{03}(P) = 5 + 4t.$$

Thus it follows from the relation

$$\begin{aligned} 6t + 9 = f_{02}(P) &= -2f_0(P) + 2f_1(P) + f_{03}(P) \\ &= -2f_0(P) + 2f_1(P) + 5 + 4t \end{aligned}$$

that we have

$$f_1(P) = f_0(P) + t + 2.$$

Hence, since $f_0 \geq 5$ and $t \geq 5$ by Theorem 2.1, it is straightforward to show that

$$\begin{aligned}1 < \phi_4(P) &= \frac{f_1+f_2}{f_0+f_3} = \frac{f_0+3t+5}{f_0+1+t}\\ &= 1+\frac{2t+4}{f_0+1+t} \leq 1+\frac{2t+4}{t+6} = 3-\frac{8}{t+6}\\ &< 3\end{aligned}$$

Assume next that P is simplicial. Then, it is easy to obtain

$$f_3(P)=t,\ f_2(P)=2t,\ f_{03}(P)=4t,\ f_1(P)=f_0(P)+t.$$

Thus, since $f_0 \geq 5$ and $t \geq 5$, once again we have

$$\begin{aligned}1 < \phi_4(P) &= \frac{f_1+f_2}{f_0+f_3} = \frac{f_0+3t}{f_0+t}\\ &= 1+\frac{2t}{f_0+t} \leq 1+\frac{2t}{t+5}\\ &< 3\end{aligned}$$

This completes the proof of Theorem 7.5. □

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