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A NOTE ON THE FULLY DEGENERATE BELL POLYNOMIALS OF THE SECOND KIND

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ABSTRACT. In this paper, the authors study a new degenerating approach to the Bell polynomials which are called fully degenerate Bell polynomials of the second kind. We establish some identities from the fully degenerate Bell polynomials of the second kind and give explicit relations to special numbers and polynomials.

1. Introduction

In combinatorial mathematics, the Bell polynomials are used in the study of partitions (see [1, 2, 4, 5, 8]). The Bell polynomials $\mathrm{Bel}_n(x)$ are defined by the generating function to be [7, 8, 12, 14, 21, 24, 25]

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \mathrm{Bel}_n(x)\frac{t^n}{n!}.$$

When $x = 1$, $\mathrm{Bel}_n = \mathrm{Bel}_n(1)$ are called the n-th Bell numbers.

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by (see [3, 6, 9, 13])

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda}\frac{t^n}{n!}, \quad e_\lambda(t) = e_\lambda^1(t),$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, for $n \geq 1$.

In [9], the first works on the degenerate Bell numbers and polynomials are done by Kim and Kim

$$e_\lambda^{x(e_\lambda(t)-1)}(1) = \sum_{n=0}^{\infty} \mathrm{Bel}_{n,\lambda}(x)\frac{t^n}{n!}.$$

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Later in [14], Kim et al. introduced the partially degenerate Bell polynomials $\mathrm{bel}_{n,\lambda}(x)$, which are given by the generating function

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \mathrm{bel}_{n,\lambda}(x)\frac{t^n}{n!}.$$

In [6], Dolgy et al. defined and studied the fully degenerate Bell polynomials $\mathrm{B}_{n,\lambda}(x)$ by the generating function

$$e_\lambda(x(e_\lambda(t)-1)) = (1+\lambda x((1+\lambda t)^{1/\lambda}-1))^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}(x)\frac{t^n}{n!}.$$

Kim et al. defined in [18, 19] the new type degenerate Bell polynomials, $\mathrm{B}_n(x|\lambda)$, $n \geq 0$ by

$$e_\lambda^x(e^t-1) = \big(1+\lambda(e^t-1)\big)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathrm{B}_n(x|\lambda)\frac{t^n}{n!}.$$

When $x = 1$, $\mathrm{B}_n(1|\lambda)$ are called the new type degenerate Bell numbers. The authors obtain several expressions of identities on those numbers and polynomials, (see [18, 19]). Recently in [23], authors considered a new degenerating approach to the Bell polynomials which were called the fully degenerate Bell polynomials of the second kind. We gave some identities from the generating function and especially by using the differential equations on those polynomials. In this paper, we continuously study a new degenerating approach to the Bell polynomials which are called fully degenerate Bell polynomials of the second kind. We give explicit identities from the generating function and relate our polynomials to special numbers and polynomials.

As is well known, for $k \geq 0$, the Stirling numbers of the first kind $S_1(n,k)$, the Stirling numbers of the second kind $S_2(n,k)$ and the central factorial numbers of the second kind $T(n,k)$ are defined by the generating functions (see [6, 7, 8, 11, 13])

$$\begin{aligned}
\frac{1}{k!}(\log(1+t))^k &= \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!},\\
\frac{1}{k!}(e^t-1)^k &= \sum_{k=0}^{n} S_2(n,k)\frac{t^n}{n!},\\
\frac{1}{k!}(e^{\frac{1}{2}}-e^{-\frac{1}{2}})^k &= \sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!}.
\end{aligned}$$

Recall from [6, 10, 11, 15, 17, 22] that the degenerate Stirling numbers of the first kind $S_{1,\lambda}(n,k)$, the degenerate λ-Stirling numbers of the second kind $S_{2,\lambda}(n,k)$ and the degenerate central factorial numbers of the second kind

$T_{2,\lambda}(n,k)$ are generated by

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}, \tag{1.1}$$

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{k=0}^{n} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \tag{1.2}$$

$$\frac{1}{k!}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^k = \sum_{n=k}^{\infty} T_{2,\lambda}(n,k)\frac{t^n}{n!}, \tag{1.3}$$

where k is a nonnegative integer. It is clear that

$$\lim_{\lambda\to 0} S_{1,\lambda}(n,k) = S_1(n,k), \qquad \lim_{\lambda\to 0} S_{2,\lambda}(n,k) = S_2(n,k),$$
$$\lim_{\lambda\to 0} T_{2,\lambda}(n,k) = T(n,k).$$

It is common knowledge that the Bernoulli polynomials $\mathrm{B}_n(x)$, the Euler polynomials $\mathrm{E}_n(x)$ and the Cauchy polynomials $C_n(x)$ are given by

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} \mathrm{B}_n(x)\frac{t^n}{n!},$$
$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} \mathrm{E}_n(x)\frac{t^n}{n!},$$
$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} C_n(x)\frac{t^n}{n!}.$$

In view of these, for any nonzero $\lambda \in \mathbb{R}$, the degenerate polynomials are given by the generating functions

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}, \tag{1.4}$$

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}, \tag{1.5}$$

$$\frac{t}{\log_\lambda(1+t)}(1+t)^x = \sum_{n=0}^{\infty} C_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.6}$$

Among them, the polynomials $\beta_{n,\lambda}(x), \mathcal{E}_{n,\lambda}(x)$, and $C_{n,\lambda}(x)$ are called the Carlitz's degenerate Bernoulli polynomials, the degenerate Euler polynomials and the degenerate Cauchy polynomials, respectively. The Carlitz's degenerate Bernoulli numbers $\beta_{n,\lambda}$, the degenerate Euler numbers $\mathcal{E}_{n,\lambda}$ and the degenerate Cauchy numbers $C_{n,\lambda}$ are expressed by means of these polynomials, as follows:

$$\beta_{n,\lambda} = \beta_{n,\lambda}(0), \quad \mathcal{E}_{n,\lambda} = \mathcal{E}_{n,\lambda}(0), \quad C_{n,\lambda} = C_{n,\lambda}(0).$$

We obtain identities involving the fully degenerate Bell polynomials of the second kind, the Carlitz's degenerate Bernoulli polynomials, and the λ-Stirling numbers of the second kind. We also have two identities involving those fully degenerate Bell polynomials of the second kind, the degenerate Euler polynomials and the λ-Stirling numbers of the second kind. In addition, we can find an identity involving those fully degenerate Bell polynomials of the second kind, the Cauchy polynomials, and the degenerate λ-Stirling numbers of the second kind. As an application, we can express some identities involving the degenerate Bell polynomials of the second kind, the degenerate Fubini polynomials, the degenerate Stirling numbers of the first and the second kind, and the degenerate derangement numbers and polynomials.

2. Some identities of fully degenerate Bell polynomials of the second kind

In this section, we establish some identities of the degenerate Bell polynomials of the second kind. Specially we relate our polynomials to the Carlitz's degenerate Bernoulli polynomials, the degenerate Euler polynomials, and the degenerate Stirling numbers of the first kind and the second kind.

We recall the fully degenerate Bell polynomials of the second kind in [23], denoted by $\mathrm{B}^*_{n,\lambda}(x)$, by the generating function

$$\begin{aligned} e^x_\lambda(e_\lambda(t)-1) &= (1+\lambda(e_\lambda(t)-1))^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} \mathrm{B}^*_{n,\lambda}(x)\frac{t^n}{n!}. \end{aligned} \tag{2.1}$$

When $x=1$, $\mathrm{B}^*_{n,\lambda} = \mathrm{B}^*_{n,\lambda}(1)$ are called the fully degenerate Bell numbers of the second kind, which are the same as the degenerate Bell numbers in [6].

In [23], it is listed that for $n \geq 0$

$$\mathrm{B}^*_{n,\lambda}(x) = \sum_{l=0}^{n} (x)_{l,\lambda} S_{2,\lambda}(n,l) \tag{2.2}$$

and

$$(x)_{n,\lambda} = \sum_{m=0}^{n} \mathrm{B}^*_{m,\lambda}(x) S_{1,\lambda}(n,m).$$

In [15, 22], we have an identity on the λ-Stirling numbers of the second kind, which are defined in (1.2).

$$S_{2,\lambda}(n,m) = \sum_{k=0}^{n} \lambda^{n-k} S_1(n,k) S_2(k,m). \tag{2.3}$$

We can rewrite the right hand side of (2.3) as follows:

$$S_{2,\lambda}(n,m)=\sum_{k=0}^{n-1}\lambda^{n-k}S_1(n,k)S_2(k,m)+S_2(n,m). \tag{2.4}$$

It follows that
(2.5)

$$\begin{aligned}
S_{2,\lambda}(n,m)=&\,\lambda\sum_{k=0}^{n-1}-(n-1)\lambda^{(n-1)-k}S_1(n-1,k)S_2(k,m)\\
&+\sum_{k=1}^{n-1}\lambda^{n-k}S_1(n-1,k-1)mS_2(k-1,m)\\
&+\sum_{k=1}^{n-1}(\lambda^{n-k}S_1(n-1,k-1)S_2(k-1,m-1))+S_2(n,m)\\
=&\,\lambda\sum_{k=0}^{n-1}-(n-1)\lambda^{(n-1)-k}S_1(n-1,k)S_2(k,m)\\
&+\sum_{k=0}^{n-1}(m\lambda^{n-1-k}S_1(n-1,k)mS_2(k,m))-mS_2(n-1,m)\\
&+\sum_{k=0}^{n-1}\lambda^{n-1-k+1}(S_1(n-1,k-1)S_2(k-1,m))\\
&-S_2(n-1,m-1)+S_2(n,m)\\
=&-\lambda(n-1)S_{2,\lambda}(n-1,m)+mS_{2,\lambda}(n-1,m)-mS_2(n-1,m)\\
&+S_{2,\lambda}(n-1,m-1)-S_2(n-1,m-1)+S_2(n,m).
\end{aligned}$$

Using the recurrence relation of the Stirling numbers of the second kind in (2.5), we have the recurrence relation of the λ-Stirling numbers of the second kind, which is proved in the [17, Theorem 1].

Theorem 2.1. *For $n\geq 0$, we have*

$$S_{2,\lambda}(n,m)=\left(-\lambda(n-1)+m\right)S_{2,\lambda}(n-1,m)+S_{2,\lambda}(n-1,m-1).$$

Applying (2.4) in (2.2) for $n\geq 1$, we have

$$\begin{aligned}
\mathrm{B}^*_{n,\lambda}(x)&=\sum_{m=0}^{n}(x)_{m,\lambda}S_{2,\lambda}(n,m)\\
&=\sum_{m=0}^{n}(x)_{m,\lambda}S_2(n,m)+\sum_{m=0}^{n}\sum_{k=0}^{n-1}(x)_{m,\lambda}\lambda^{n-k}S_1(n,k)S_2(k,m) \qquad (2.6)\\
&=\mathrm{Bel}_{n,\lambda}(x|\lambda)+\sum_{k=0}^{n-1}\sum_{m=0}^{n}\lambda^{n-k}S_1(n,k)S_2(k,m)(x)_{m,\lambda}.
\end{aligned}$$

So we describe the differences between degenerate Bell polynomials in [18, 19] and fully degenerate Bell polynomials of the second kind.

Remark: The identity (2.6) can be written

$$\mathrm{B}^*_{n,\lambda}(x) = \mathrm{Bel}_{n,\lambda}(x|\lambda) + \sum_{j=1}^{n} \lambda^j g_j(x),$$

where

$$g_j(x) = \sum_{m=0}^{n-j} S_1(n, n-j) S_2(n-j, m)(x)_{m,\lambda}$$

and

$$\deg g_j(x) = n - j.$$

Now we apply the ideas in [9] to our degenerate Bell polynomials of the second kind. We can give some explicit expressions for those polynomials to special numbers and polynomials.

Replacing t by $e_\lambda(t) - 1$ in (1.4) gives

$$\begin{aligned}\frac{e_\lambda(t)-1}{e_\lambda(e_\lambda(t)-1)-1} e^x_\lambda(e_\lambda(t)-1) &= \sum_{m=0}^{\infty} \beta_{m,\lambda}(x) \frac{1}{m!}(e_\lambda(t)-1)^m \\ &= \sum_{m=0}^{\infty} \beta_{m,\lambda}(x) \sum_{n=m}^{\infty} S_{2,\lambda}(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \beta_{m,\lambda}(x) S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

From (2.1) it can be deduced that

$$\begin{aligned}\frac{e_\lambda(t)-1}{e_\lambda(e_\lambda(t)-1)-1} e^x_\lambda(e_\lambda(t)-1) &= \sum_{m=0}^{\infty} \beta_{m,\lambda} \frac{1}{m!}(e_\lambda(t)-1)^m \sum_{l=0}^{\infty} \mathrm{B}^*_{l,\lambda}(x)\frac{t^l}{l!} \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{k} \beta_{m,\lambda} S_{2,\lambda}(k,m)\frac{t^k}{k!} \sum_{l=0}^{\infty} \mathrm{B}^*_{l,\lambda}(x)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{m=0}^{k} \beta_{m,\lambda} S_{2,\lambda}(k,m)\, \mathrm{B}^*_{n-k,\lambda}(x) \right) \frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

Therefore, by comparing the coefficients on both sides of (2.7) and (2.8), we obtain a theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} \beta_{m,\lambda}(x) S_{2,\lambda}(n,m) = \sum_{k=0}^{n} \binom{n}{k} \sum_{m=0}^{k} \beta_{m,\lambda} S_{2,\lambda}(k,m)\, \mathrm{B}^*_{n-k,\lambda}(x).$$

Let us replace t by $e_\lambda(t)-1$ in (1.5). Then we get

$$\begin{aligned}\frac{2}{e_\lambda(e_\lambda(t)-1)+1}e_\lambda^x(e_\lambda(t)-1) &= \sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}(x)\frac{1}{m!}(e_\lambda(t)-1)^m \\ &= \sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}(x)\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\mathcal{E}_{m,\lambda}(x)S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.9}$$

The left hand side of (2.9) is also given by

$$\begin{aligned}\frac{2}{e_\lambda(e_\lambda(t)-1)+1}e_\lambda^x(e_\lambda(t)-1) &= \sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}\frac{(e_\lambda(t)-1)^m}{m!}\sum_{l=0}^{\infty}\mathrm{B}^*_{l,\lambda}(x)\frac{t^l}{l!} \\ &= \sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}\sum_{k=m}^{\infty}S_{2,\lambda}(k,m)\frac{t^k}{k!}\sum_{l=0}^{\infty}\mathrm{B}^*_{l,\lambda}(x)\frac{t^l}{l!} \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{k}\mathcal{E}_{m,\lambda}S_{2,\lambda}(k,m)\frac{t^k}{k!}\sum_{l=0}^{\infty}\mathrm{B}^*_{l,\lambda}(x)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k}\mathcal{E}_{m,\lambda}S_{2,\lambda}(k,m)\,\mathrm{B}^*_{n-k,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

Therefore, from (2.9) and (2.10), we obtain the following.

Theorem 2.4. *For $n\geq 0$, we have*

$$\sum_{m=0}^{n}\mathcal{E}_{m,\lambda}(x)S_{2,\lambda}(n,m)=\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k}\mathcal{E}_{m,\lambda}S_{2,\lambda}(k,m)\,\mathrm{B}^*_{n-k,\lambda}(x).$$

From (1.4), the generating function of the Carlitz's degenerate Bernoulli polynomials $\beta_{l,\lambda}(x)$ can be rewritten as

$$te_\lambda^x(t)=\sum_{l=0}^{\infty}\beta_{l,\lambda}(x)\frac{t^l}{l!}(e_\lambda(t)-1). \tag{2.11}$$

By replacing t by $e_\lambda(t)-1$ in (2.11), we obtain

$$(e_\lambda(t)-1)e_\lambda^x(e_\lambda(t)-1)=\sum_{l=0}^{\infty}\beta_{l,\lambda}(x)\frac{(e_\lambda(t)-1)^l}{l!}(e_\lambda(e_\lambda(t)-1)-1). \tag{2.12}$$

The right hand side of (2.12) is equal to

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{l=0}^{m}\beta_{l,\lambda}(x)S_{2,\lambda}(m,l)\frac{t^m}{m!}\sum_{k=1}^{\infty}\mathrm{B}^*_{k,\lambda}\frac{t^k}{k!}\\ =\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\binom{n}{k}\sum_{l=0}^{n-k}\beta_{l,\lambda}(x)S_{2,\lambda}(n-k,l)\,\mathrm{B}^*_{k,\lambda}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.13}$$

On the other hand, the left hand side of (2.12) is equal to

$$\sum_{k=1}^{n}(1)_{k,\lambda}\frac{t^k}{k!}\sum_{m=0}^{\infty}\mathrm{B}^*_{m,\lambda}(x)\frac{t^m}{m!}=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\binom{n}{k}(1)_{k,\lambda}\,\mathrm{B}^*_{n-k,\lambda}(x)\frac{t^n}{n!}.\tag{2.14}$$

Therefore, by equating (2.13) and (2.14), we obtain the following.

Theorem 2.5. *For $n \geq 0$, we have*

$$\sum_{k=1}^{n}\binom{n}{k}\sum_{l=0}^{n-k}\beta_{l,\lambda}(x)S_{2,\lambda}(n-k,l)\,\mathrm{B}^*_{k,\lambda}=\sum_{k=1}^{n}\binom{n}{k}(1)_{k,\lambda}\,\mathrm{B}^*_{n-k,\lambda}(x).$$

Setting $x = 0$ in Theorem 2.5 reveals the following.

Corollary 2.6. *For any $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n}\sum_{l=0}^{n-k}\binom{n}{k}\beta_{l,\lambda}S_{2,\lambda}(n-k,l)\,\mathrm{B}^*_{k,\lambda}=1.$$

From (1.5), the generating function of the degenerate Euler polynomials $\mathcal{E}_{l,\lambda}$ can be formulated as

$$2=\sum_{l=0}^{\infty}\mathcal{E}_{l,\lambda}\frac{t^l}{l!}(e_\lambda(t)+1)$$

which implies that

$$\begin{aligned}2&=\sum_{l=0}^{\infty}\mathcal{E}_{l,\lambda}\frac{1}{l!}(e_\lambda(t)-1)^l(e_\lambda(e_\lambda(t)-1)+1)\\&=\sum_{k=0}^{\infty}\sum_{l=0}^{k}\mathcal{E}_{l,\lambda}S_{2,\lambda}(k,l)\frac{t^k}{k!}\left(\sum_{m=0}^{\infty}\mathrm{B}^*_{m,\lambda}\frac{t^m}{m!}+1\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}\mathcal{E}_{l,\lambda}S_{2,\lambda}(k,l)\,\mathrm{B}^*_{n-k,\lambda}+\sum_{k=0}^{n}\mathcal{E}_{k,\lambda}S_{2,\lambda}(n,k)\right)\frac{t^n}{n!}.\end{aligned}$$

Consequently, it follows that

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}\mathcal{E}_{l,\lambda}S_{2,\lambda}(k,l)\,\mathrm{B}^*_{n-k,\lambda}+\sum_{k=0}^{n}\mathcal{E}_{k,\lambda}S_{2,\lambda}(n,k)=\begin{cases}2, & \text{for } n=0\\ 0, & \text{for } n\geq 1.\end{cases}$$

Therefore, we obtain the following.

Theorem 2.7. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}\mathcal{E}_{l,\lambda}S_{2,\lambda}(k,l)\,\mathrm{B}^{*}_{n-k,\lambda}=-\sum_{k=0}^{n}\mathcal{E}_{k,\lambda}S_{2,\lambda}(n,k).$$

In other words, for $n \geq 0$, we have

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}\mathcal{E}_{l,\lambda}S_{2,\lambda}(k,l)\,\mathrm{B}^{*}_{n-k,\lambda}+\sum_{k=0}^{n}\mathcal{E}_{k,\lambda}S_{2,\lambda}(n,k)=2\,\mathrm{B}^{*}_{n,\lambda}(0).$$

Let us replace t by $e_\lambda(e_\lambda(t)-1)-1$ in (1.6). Then we have

(2.15)

$$\frac{e_\lambda(e_\lambda(t)-1)-1}{e_\lambda(t)-1}e_\lambda^{x}(e_\lambda(e_\lambda(t)-1))=\sum_{m=0}^{\infty}C_{m,\lambda}(x)\frac{1}{m!}(e_\lambda(e_\lambda(t)-1)-1)^{m}.$$

This implies that

$$\begin{aligned}&\sum_{m=0}^{\infty}C_{m,\lambda}(x)\frac{1}{m!}(e_\lambda(e_\lambda(t)-1)-1)^{m}\\&=\sum_{m=0}^{\infty}C_{m,\lambda}(x)\sum_{k=m}^{\infty}S_{2,\lambda}(k,m)\frac{1}{k!}(e_\lambda(t)-1)^{k}\\&=\sum_{k=0}^{\infty}\sum_{m=0}^{k}C_{m,\lambda}(x)S_{2,\lambda}(k,m)\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^{n}}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}C_{m,\lambda}(x)S_{2,\lambda}(k,m)S_{2,\lambda}(n,k)\right)\frac{t^{n}}{n!}.\end{aligned}\tag{2.16}$$

As a result, by (2.15) and (2.16), it follows that

$$\begin{aligned}&\frac{e_\lambda(e_\lambda(t)-1)-1}{e_\lambda(t)-1}e_\lambda^{x}(e_\lambda(e_\lambda(t)-1))\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}C_{m,\lambda}(x)S_{2,\lambda}(k,m)S_{2,\lambda}(n,k)\right)\frac{t^{n}}{n!}.\end{aligned}\tag{2.17}$$

On the other hand,

$$
\begin{aligned}
&\frac{e_\lambda(e_\lambda(t)-1)-1}{e_\lambda(t)-1}e_\lambda^x(e_\lambda(e_\lambda(t)-1))\\
&=\frac{e_\lambda^{x+1}(e_\lambda(t)-1)-e_\lambda^x(e_\lambda(t)-1)}{e_\lambda(t)-1}\\
&=\left(\frac{t}{e_\lambda(t)-1}\right)\frac{1}{t}\left(e_\lambda^{x+1}(e_\lambda(t)-1)-e_\lambda^x(e_\lambda(t)-1)\right)\\
&=\left(\sum_{k=0}^\infty \beta_{k,\lambda}\frac{t^k}{k!}\right)\sum_{m=0}^\infty \frac{1}{t}\left(\mathrm{B}^*_{m,\lambda}(x+1)-\mathrm{B}^*_{m,\lambda}(x)\right)\frac{t^m}{m!} \qquad (2.18)\\
&=\left(\sum_{k=0}^\infty \beta_{k,\lambda}\frac{t^k}{k!}\right)\sum_{m=0}^\infty \left(\frac{\mathrm{B}^*_{m+1,\lambda}(x+1)-\mathrm{B}^*_{m+1,\lambda}(x)}{m+1}\right)\frac{t^m}{m!}\\
&=\sum_{n=0}^\infty\left(\sum_{m=0}^n\binom{n}{m}\left(\frac{\mathrm{B}^*_{m+1,\lambda}(x+1)-\mathrm{B}^*_{m+1,\lambda}(x)}{m+1}\right)\beta_{n-m,\lambda}\right)\frac{t^n}{n!}.
\end{aligned}
$$

Therefore, by (2.17) and (2.18), we obtain the following.

Theorem 2.8. *For $n \geq 0$, we have*

$$
\begin{aligned}
&\sum_{k=0}^n\sum_{m=0}^k C_{m,\lambda}(x)S_{2,\lambda}(k,m)S_{2,\lambda}(n,k)\\
&=\sum_{m=0}^n\binom{n}{m}\left(\frac{\mathrm{B}^*_{m+1,\lambda}(x+1)-\mathrm{B}^*_{m+1,\lambda}(x)}{m+1}\right)\beta_{n-m,\lambda}.
\end{aligned}
$$

Letting $x=0$ in Theorem 2.8 leads to

Corollary 2.9. *For $n \geq 0$, we have*

$$
\sum_{k=0}^n\sum_{m=0}^k C_{m,\lambda}S_{2,\lambda}(k,m)S_{2,\lambda}(n,k)=\sum_{m=0}^n\binom{n}{m}\frac{\mathrm{B}^*_{m+1,\lambda}}{m+1}\beta_{n-m,\lambda}.
$$

3. Further identities of the fully degenerate Bell numbers and polynomials of the second kind

In this section, we further investigate the fully degenerate Bell polynomials of the second kind. We will discover or recover more identities on those polynomials related to the degenerate central factorial numbers and the degenerate Stirling numbers of the second kind. Finally, we establish the Dovinski-like theorem on the fully degenerate Bell polynomials of the second kind.

From (1.3) and (2.1), we have the following observation.

$$\begin{aligned}e_\lambda^x(e_\lambda(t)-1) &= \sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{1}{m!}(e_\lambda(t)-1)^m \\ &= \sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{1}{m!}(e_\lambda^{1/2}(t)-e_\lambda^{-1/2}(t))^m e_\lambda^{m/2}(t) \\ &= \sum_{m=0}^{\infty}(x)_{m,\lambda}\sum_{n=m}^{\infty}\sum_{k=m}^{n}\binom{n}{k}T_{2,\lambda}(k,m)\left(\frac{m}{2}\right)_{n-k,\lambda}\frac{t^n}{n!}.\end{aligned}$$

Therefore, we can derive the following.

Theorem 3.1. *For $n \geq 0$, we have*

$$\mathrm{B}^*_{n,\lambda}(x) = \sum_{m=0}^{n}\sum_{k=m}^{n}\binom{n}{k}(x)_{m,\lambda}T_{2,\lambda}(k,m)\left(\frac{m}{2}\right)_{n-k,\lambda}.$$

In particular, when λ approaches zero in Theorem 3.1, we obtain the known identity in [18]

$$\mathrm{B}_n(x) = \sum_{m=0}^{n}\sum_{k=m}^{n}\binom{n}{k}x^m T_2(k,m)\left(\frac{m}{2}\right)^{n-k}.$$

On the other hand, by (1.2) and (2.1), it follows that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{B}^*_{n,\lambda}(x)\frac{t^n}{n!} &= e_\lambda^x(e_\lambda(t)-1) = e^{\frac{x}{\lambda}\log(1+\lambda(e_\lambda(t)-1))} \\ &= \sum_{m=0}^{\infty}\left(\frac{x}{\lambda}\right)^m\frac{1}{m!}\log\left(1+\lambda(e_\lambda(t)-1)\right)^m \\ &= \sum_{m=0}^{\infty}\left(\frac{x}{\lambda}\right)^m\sum_{l=m}^{\infty}S_1(l,m)\lambda^l\frac{(e_\lambda(t)-1)^l}{l!} \\ &= \sum_{m=0}^{\infty}\left(\frac{x}{\lambda}\right)^m\sum_{l=m}^{\infty}S_1(l,m)\lambda^l\sum_{n=l}^{\infty}S_{2,\lambda}(n,l)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}x^m\lambda^{l-m}S_1(l,m)S_{2,\lambda}(n,l)\right)\frac{t^n}{n!}.\end{aligned}$$

Equating coefficients of $t^n/n!$ on both sides of this yields the following.

Theorem 3.2. *For $n \geq 0$, we have*

$$\mathrm{B}^*_{n,\lambda}(x) = \sum_{l=0}^{n}\sum_{m=0}^{l}x^m\lambda^{l-m}S_1(l,m)S_{2,\lambda}(n,l)$$

and

$$\mathrm{B}^*_{n,\lambda} = \sum_{l=0}^{n}\sum_{m=0}^{l}\lambda^{l-m}S_1(l,m)S_{2,\lambda}(n,l).$$

Specially, if λ approaches to 0, we have a very interesting identity:

$$\mathrm{B}_n = \sum_{m=0}^{n} S_2(n,m).$$

Using the generating function of the fully degenerate Bell polynomials of the second kind, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathrm{B}^*_{n,\lambda}(x)\frac{t^n}{n!} &= e_\lambda^x(e_\lambda(t)-1) \\
&= \sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{1}{l!}(e_\lambda(t)-1)^l \\
&= \sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{1}{l!}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}e_\lambda^m(t) \\
&= \sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{1}{l!}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}\sum_{n=0}^{\infty}(m)_{n,\lambda}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{1}{l!}\binom{l}{m}(-1)^{l-m}(m)_{n,\lambda}(x)_{l,\lambda}\right)\frac{t^n}{n!}.
\end{aligned}$$

Accordingly, comparing both sides of this leads to the the following

Theorem 3.3. *For $n \geq 0$, we have*

$$\mathrm{B}^*_{n,\lambda}(x) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{1}{l!}\binom{l}{m}(-1)^{l-m}(m)_{n,\lambda}(x)_{l,\lambda}.$$

In particular,

$$\mathrm{B}^*_{n,\lambda} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{1}{l!}\binom{l}{m}(-1)^{l-m}(m)_{n,\lambda}(1)_{l,\lambda}.$$

From Theorem 3.3, it is worthy to note that

$$\begin{aligned}
\lim_{\lambda\to 0}\mathrm{B}^*_{n,\lambda}(x) &= \sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{1}{l!}\binom{l}{m}(-1)^{l-m}m^n x^l \\
&= e^{-x}\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k \\
&= e^{x(e^t-1)} \\
&= \mathrm{B}_n(x).
\end{aligned}$$

By using Theorem 3.2 or Theorem 3.3, we can show that the first five fully degenerate Bell polynomials of the second kind $\mathrm{B}^*_{n,\lambda}(x)$ for $1 \leq n \leq 5$

are as follows:

$$
\begin{aligned}
\mathrm{B}^*_{1,\lambda}(x) &= x,\\
\mathrm{B}^*_{2,\lambda}(x) &= (x^2+x)-2x\lambda,\\
\mathrm{B}^*_{3,\lambda}(x) &= (x^3+3x^2+x)+(-6x^2-6x)\lambda+7x\lambda^2,\\
\mathrm{B}^*_{4,\lambda}(x) &= (x^4+6x^3+7x^2+x)+(-12x^3-36x^2-13x)\lambda\\
&\quad+(40x^2+41x)\lambda^2-35x\lambda^3,\\
\mathrm{B}^*_{5,\lambda}(x) &= (x^5+10x^4+25x^3+15x^2+x)\\
&\quad+(-20x^4-120x^3-145x^2-25x)\lambda\\
&\quad+(130x^3+395x^2+155x)\lambda^2\\
&\quad+(-315x^2-335x)\lambda^3+228x\lambda^4.
\end{aligned}
\tag{3.1}
$$

Figure 1 shows the graphs of the degenerate Bell polynomials $\mathrm{B}_n(x|\lambda)$ and the fully degenerate Bell polynomials of the second kind $\mathrm{B}_{n,\lambda}(x)$ for $\lambda = 0.5, 0.3$ and 0.1.

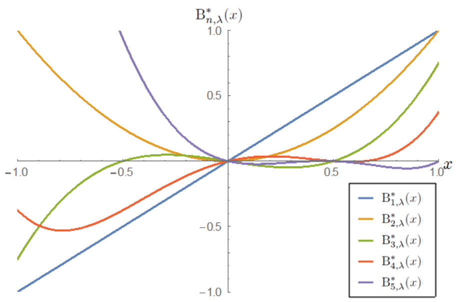
We observe that

$$
\begin{aligned}
\frac{\partial}{\partial t}\sum_{n=0}^{\infty}\mathrm{B}^*_{n,\lambda}(x)\frac{t^n}{n!} &= \sum_{n=1}^{\infty}\mathrm{B}^*_{n,\lambda}(x)\frac{t^{n-1}}{(n-1)!}\\
&= \sum_{n=0}^{\infty}\mathrm{B}^*_{n+1,\lambda}(x)\frac{t^n}{n!}\\
&= \frac{\partial}{\partial t}\left(1+\lambda(e_\lambda(t)-1)\right)^{\frac{x}{\lambda}}\\
&= xe_\lambda^{1-\lambda}(t)\left(1+\lambda(e_\lambda(t)-1)\right)^{\frac{x-\lambda}{\lambda}}\\
&= x\sum_{l=0}^{\infty}(x-\lambda)_{l,\lambda}\frac{1}{l!}\left(e_\lambda(t)-1\right)^l\sum_{m=0}^{\infty}(1-\lambda)_{m,\lambda}\frac{t^m}{m!}\\
&= \sum_{n=0}^{\infty}\left(x\sum_{k=0}^{m}\sum_{l=0}^{k}\binom{n}{k}(x-\lambda)_{l,\lambda}(1-\lambda)_{n-k,\lambda}S_{2,\lambda}(k,l)\right)\frac{t^n}{n!}.
\end{aligned}
$$

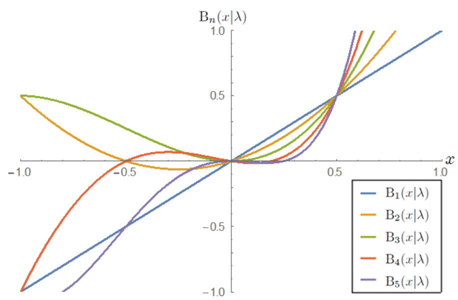
Therefore, we obtain the following.

Theorem 3.4. *(Dovinski like Theorem) For $n \geq 0$, we have*

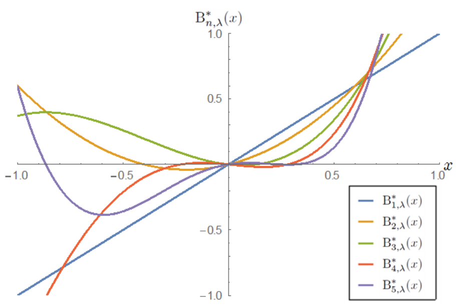
$$
\begin{aligned}
\mathrm{B}^*_{n+1,\lambda}(x) &= x\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}(x-\lambda)_{l,\lambda}S_{2,\lambda}(k,l)(1-\lambda)_{n-k,\lambda}\\
&= x\sum_{k=0}^{n}\binom{n}{k}\mathrm{B}^*_{k,\lambda}(x-\lambda)(1-\lambda)_{n-k,\lambda}.
\end{aligned}
$$



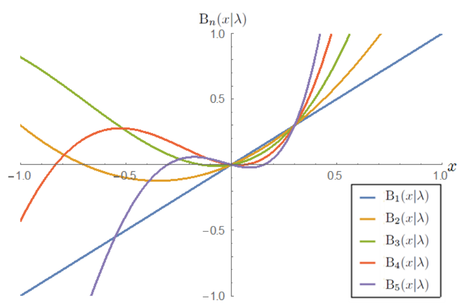
(A) $\mathrm{B}^*_{n,0.5}(x)$



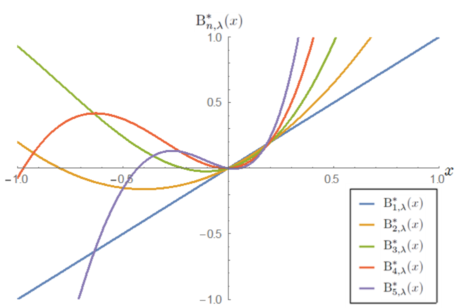
(B) $\mathrm{B}_n(x|0.5)$



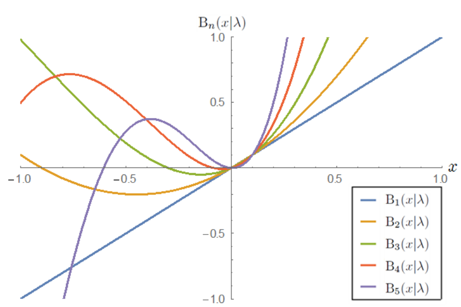
(C) $\mathrm{B}^*_{n,0.3}(x)$



(D) $\mathrm{B}_n(x|0.3)$



(E) $\mathrm{B}^*_{n,0.1}(x)$



(F) $\mathrm{B}_n(x|0.1)$

FIGURE 1. Graphs of the degenerate Bell polynomials $\mathrm{B}_n(x|\lambda)$ and the fully degenerate Bell polynomials of the second kind $\mathrm{B}_{n,\lambda}(x)$ for $\lambda = 0.5, 0.3$ and 0.1.

Remark: Letting $\lambda \to 0$ in Theorem 3.4 reduces to

$$\begin{aligned}
\mathrm{B}_{n+1}(x) &= \lim_{\lambda\to 0} \mathrm{B}^*_{n+1,\lambda}(x) \\
&= x \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k} x^l S_2(k,l) \\
&= x \sum_{k=0}^{n} \binom{n}{k} \mathrm{B}_k(x).
\end{aligned}$$

This is the well-known Dovinski's Theorem, which can be found in [6, 7, 8].

We want to study some applications of the fully degenerate Bell polynomials of the second kind to give some identities related to the degenerate derangement polynomials and the degenerate Fubini polynomials, which are introduced in the paper [16, 20].

The degenerate derangement polynomials $d_{n,\lambda}(x)$, and the degenerate Fubini polynomials $F_{n,\lambda}(x)$ are given by

$$\frac{1}{1-t}e_{\lambda}^{x-1}(t)=\sum_{n=0}^{\infty}d_{n,\lambda}(x)\frac{t^n}{n!}, \tag{3.2}$$

$$\frac{1}{1-x(e_{\lambda}(t)-1)}=\sum_{n=0}^{\infty}F_{n,\lambda}(x)\frac{t^n}{n!}. \tag{3.3}$$

Replacing t by $e_{\lambda}(t)-1$ in (3.2) results in

$$\begin{aligned}\frac{1}{2-e_{\lambda}(t)}e_{\lambda}^{x-1}(e_{\lambda}(t)-1)&=\sum_{l=0}^{\infty}d_{l,\lambda}(x)\frac{1}{l!}(e_{\lambda}(t))^l\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}S_{2,\lambda}(n,l)d_{l,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.4}$$

The left hand side of (3.4) can be expanded into

$$\begin{aligned}\frac{1}{2-e_{\lambda}(t)}e_{\lambda}^{x-1}(e_{\lambda}(t)-1)&=\sum_{l=0}^{\infty}F_{l,\lambda}(1)\frac{t^l}{l!}\sum_{m(=0}^{\infty}\mathrm{B}^{*}_{m,\lambda}(x-1)\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}F_{l,\lambda}(1)\,\mathrm{B}^{*}_{n-l,\lambda}(x-1)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.5}$$

Therefore by (3.4) and (3.5), we obtain a relation between the degenerate derangement polynomials via Fubini numbers and the degenerate Bell polynomials of the second kind.

Theorem 3.6. *For $n\geq 0$, we have*

$$\sum_{l=0}^{\infty}S_{2,\lambda}(n,l)d_{l,\lambda}(x)=\sum_{l=0}^{n}\binom{n}{l}F_{l,\lambda}(1)\,\mathrm{B}^{*}_{n-l,\lambda}(x-1).$$

Now we replace t by $\log_{\lambda}(1+t)$ in (3.3) with $x=1$, we get

$$\begin{aligned}\frac{1}{1-t}&=\sum_{l=0}^{\infty}F_{l,\lambda}(1)\frac{1}{l!}\left(\log_{\lambda}(1+t)\right)^l\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}F_{l,\lambda}(1)S_{1,\lambda}(n,l)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.6}$$

On the other hand, the left hand side of (3.6) can be represented by the degenerate derangement polynomials as

$$
\begin{aligned}
\frac{1}{1-t} &= \left(\frac{1}{1-t}e_\lambda^{x-1}(t)\right)e_\lambda^{1-x}(t) \\
&= \left(\sum_{l=0}^{\infty} d_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(1-x)_{m,\lambda}\frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}d_{l,\lambda}(x)(1-x)_{n-l,\lambda}\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.7}
$$

By (3.2), (3.6) and (3.7), we have a result.

Theorem 3.7. *For $n \geq 0$, we have*

$$\sum_{l=0}^{\infty} F_{l,\lambda}(1)S_{1,\lambda}(n,l) = \sum_{l=0}^{n}\binom{n}{l}d_{l,\lambda}(x)(1-x)_{n-l,\lambda}.$$

We represent (3.7) in a slightly different form as

$$
\begin{aligned}
\frac{1}{1-t} &= \left(\frac{1}{1-t}e_\lambda^{x}(e_\lambda(t)-1)\right)e_\lambda^{x}(e_\lambda(t)-1) \\
&= \left(\sum_{k=0}^{\infty} k!\frac{t^k}{k!}\sum_{l=0}^{\infty}\mathrm{B}^*_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(-x)_{m,\lambda}\frac{(e_\lambda(t)-1)^m}{m!}\right) \\
&= \sum_{l=0}^{\infty}\left(\sum_{k=0}^{l}\binom{l}{k}(l-k)!\,\mathrm{B}^*_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(-x)_{m,\lambda}\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{t^n}{n!}\right) \\
&= \sum_{l=0}^{\infty}\left(\sum_{k=0}^{l}\binom{l}{k}(l-k)!\,\mathrm{B}^*_{l,\lambda}(x)\frac{t^l}{l!}\right)\sum_{j=0}^{\infty}\left(\sum_{m=0}^{j}(-x)_{m,\lambda}S_{2,\lambda}(j,m)\right)\frac{t^j}{j!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{l}\sum_{m=0}^{n-l}\binom{l}{k}(l-k)!\,\mathrm{B}^*_{l,\lambda}(x)(-x)_{m,\lambda}S_{2,\lambda}(n-l,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.8}
$$

Thus by (3.2), (3.6) and (3.8), we obtain the following result.

Theorem 3.8. *For $n \geq 0$, we have*

$$\sum_{l=0}^{\infty} F_{l,\lambda}(1)S_{1,\lambda}(n,l) = \sum_{k=0}^{l}\sum_{m=0}^{n-l}\binom{l}{k}(l-k)!\,\mathrm{B}^*_{l,\lambda}(x)(-x)_{m,\lambda}S_{2,\lambda}(n-l,m).$$

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