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TOTAL DOMINATOR TOTAL COLORING OF A GRAPH

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ABSTRACT. Here, we initiate to study the total dominator total coloring of a graph which is a total coloring of the graph such that each object of the graph is adjacent or incident to every object of some color class. In more detailes: In section 2 we present some tight lower and upper bounds for the total dominator total chromatic number of a graphs in terms of some parameters such as order, size, the total dominator chromatic and total domination numbers of the graph and its line graph. In section 3 we restrict our attention to trees and present a Nordhaus-Gaddum-like relation for them, and finally in last section we show that there exist graphs that their total dominator total chromatic numbers are equal to their orders.

1. INTRODUCTION

All graphs considered here are nonempty, finite, undirected and simple. For standard graph theory terminology not given here we refer to [20]. Let $G = (V, E)$ be a graph with the *vertex set* V of *order* $n(G)$ and the *edge set* E of *size* $m(G)$. The *open neighborhood* and the *closed neighborhood* of a vertex $v \in V$ are $N_G(v) = \{u \in V \mid uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. The *degree* of a vertex v is also $deg_G(v) = |N_G(v)|$. The *minimum* and *maximum degree* of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. If $\delta(G) = \Delta(G) = k$, then G is called *k-regular*. For two vertices u and v in a connected graph G the *distance* between u and v is the minimum length of a shortest (u, v)-path in G and is denoted by $d(u, v)$. The maximum distance among all pairs of vertices of G is the *diameter* of G, which is denoted by *diam(G)*. A *Hamiltonian path* in a graph G is a path which contains every vertex of G. An *independent set* of G is a subset of vertices of G, no two of which are adjacent. And a *maximum independent set* is an independent set of the largest cardinality in G. This cardinality is called the *independence number* of G, and is denoted by $\alpha(G)$. Also a *mixed independent set* of G is a subset of $V \cup E$, no two objects of which are adjacent or incident, and a

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maximum mixed independent set is a mixed independent set of the largest cardinality in G. This cardinality is called the *mixed independence number* of G, and is denoted by $\alpha_{mix}(G)$. Two isomorphic graphs G and H are shown by $G \cong H$.

We write K_n, C_n and P_n for a *complete graph*, a *cycle* and a *path* of order n, respectively, and $K_{m,n}$ is a a *bipartite complete graph* of order $m+n$ while $G[S]$ denote the *induced subgraph* of G by a vertex set S. A complete bipartite graph $K_{1,n}$ is called a *star*. The *line graph* $L(G)$ of G is a graph with the vertex set $E(G)$ and two vertices of $L(G)$ are adjacent when they are incident in G. The *total graph* $T(G)$ of a graph $G=(V,E)$ is the graph whose vertex set is $V \cup E$ and two vertices are adjacent whenever they are either adjacent or incident in G [1]. It is obvious that if G has order n and size m, then $T(G)$ has order $n+m$ and size $3m+|E(L(G))|$, and also $T(G)$ contains both G and $L(G)$ as two induced subgraphs and it is the largest graph formed by adjacent and incidence relation between graph elements. In this paper, by assumption $V=\{v_1,v_2,\ldots,v_n\}$, we use the notations $V(T(G))=V\cup\mathcal{E}$ where $\mathcal{E}=\{e_{ij} \mid v_iv_j \in E\}$, and $E(T(G))=\{v_ie_{ij}, v_je_{ij} \mid v_iv_j \in E\} \cup E \cup E(L(G))$. obviously $deg_{T(G)}(v_i)=2deg_G(v_i)$ and $deg_{T(G)}(e_{ij})=deg_G(v_i)+deg_G(v_j)$. So if G is k-regular, then $T(G)$ is $2k$-regular. Also we have $\alpha_{mix}(G)=\alpha(T(G))$.

Here, we fix a notation for the vertex set and the edge set of a line and total of a graph which we use thorough this paper. For a graph $G=(V,E)$ with the vertex set $V=\{v_i| \ 1\le i\le n\}$, we have $V(L(G))=\mathcal{E}$ and $E(L(G))=\{e_{ij}e_{ik} \mid e_{ij},e_{ik}\in\mathcal{E} \text{ and } j\neq k\}$, $V(T(G))=V\cup\mathcal{E}$ and $E(T(G))=E\cup E(L(G))\cup\{e_{ij}v_i, e_{ij}v_j \mid e_{ij}\in\mathcal{E}\}$, where $\mathcal{E}=\{e_{ij} \mid v_iv_j\in E\}$. In Figure 1, a graph G and its total graph are shown for an example.

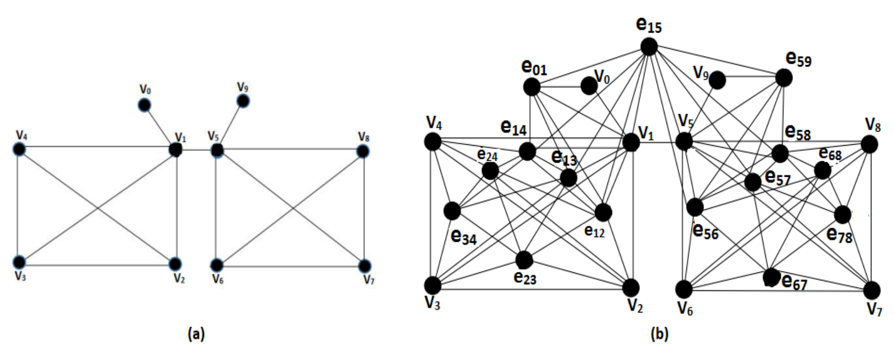

FIGURE 1. The illustration of G (left) and $T(G)$ (right).

Domination: Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [7, 8]. Also, recently two new books [9, 10] are written on this topics by Haynes, Hedetniemi, and Henning.

A famous type of domination is total domination, and the literature on this subject has been surveyed and detailed in the recent book [11]. A *total dominating set*, briefly TDS, S of a graph $G = (V, E)$ is a subset of the vertex set of G such that for each vertex v, $N_G(v) \cap S \neq \emptyset$. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a TDS of G. Similarly, a subset $S \subseteq V \cup E$ of a graph G is called a *total mixed dominating set*, briefly TMDS, of G if each object of $V \cup E$ is either adjacent or incident to an object of S, and the *total mixed domination number* $\gamma_{tm}(G)$ of G is the minimum cardinality of a TMDS [18]. A min-TDS/min-TMDS of G denotes a TDS/TMDS of G with minimum cardinality. Also we agree that *a vertex v dominates an edge e* or *an edge e dominates a vertex v* mean $v \in e$. Similarly, we agree that *an edge dominates another edge* means they have a common vertex. The next theorem can be easily proved.

Theorem 1.1 (Kazemnejad, Kazemi, and Moradi, [18]). *For any graph G without isolate vertex,*

$$\gamma_{tm}(G) = \gamma_t(T(G)).$$

Graph Coloring: Graph coloring is used as a model for a vast number of practical problems involving allocation of scarce resources (e.g., scheduling problems), and has played a key role in the development of graph theory and, more generally, discrete mathematics and combinatorial optimization. Graph colorability is NP-complete in the general case, although the problem is solvable in polynomial time for many classes [5]. A *proper coloring* of a graph G is a function from the vertices of the graph to a set of colors such that any two adjacent vertices have different colors, and the minimum number of colors needed in a proper coloring of a graph is called the *chromatic number* $\chi(G)$ of G. In a simlar way, a *total coloring* of G assigns a color to each vertex and to each edge so that colored objects have different colors when they are adjacent or incident, and the minimum number of colors needed in a total coloring of a graph is called the *total chromatic number* $\chi_T(G)$ of G [20]. The Total Coloring Conjecture (Behzad, 1965) states that:

Behzad's Conjecture. *For every simple graph G, $\chi_T(G) \leq \Delta(G) + 2$.*

A *color class* in a coloring of a graph is a set consisting of all those objects assigned the same color. For simply, if f is a coloring of G with the color classes $V_1, V_2, \ldots, V_\ell$, we write $f = (V_1, V_2, \ldots, V_\ell)$. Motivated by the relation between coloring and total dominating, the concept of total dominator coloring in graphs introduced in [14] by Kazemi, and extended in [12, 13, 15, 16, 17]. The reader can study section 4 of Part I from the book [10] for more information on this concept.

Definition 1.2 (Kazemi, [14]). A *total dominator coloring,* briefly TDC, of a graph G with a positive minimum degree is a proper coloring of G in which each vertex of the graph is adjacent to every vertex of some color class. The *total dominator chromatic number* $\chi_d^t(G)$ of G is the minimum number of color classes in a TDC of G.

Here, we initiate to study a new type of coloring called total dominator total coloring of a graph which is obtained from the concept of total dominator coloring of a graph by replacing total coloring of a graph instead of (vertex) coloring of it.

Definition 1.3. A *total dominator total coloring,* briefly TDTC, of a graph G with a positive minimum degree is a total coloring of G in which each object of the graph is adjacent or incident to every object of some color class. The *total dominator total chromatic number* $\chi_d^{tt}(G)$ of G is the minimum number of color classes in a TDTC of G.

Next theorem can be easily proved.

Theorem 1.4. *For any graph G with no isolated vertex, $\chi_d^{tt}(G) = \chi_d^t(T(G))$.*

For any TDC (TDTC) $f = (V_1, V_2, \ldots, V_\ell)$ of a graph G, a vertex (an object) v is called a *common neighbor* of V_i or we say V_i *totally dominates* v, and we write $v \succ_t V_i$, if vertex (object) v is adjacent (adjacent or incident) to every vertex (object) in V_i. Otherwise we write $v \not\succ_t V_i$. Also v is called a *private neighbor of* V_i *with respect to* f if $v \succ_t V_i$ and $v \not\succ_t V_j$ for all $j \neq i$. The set of all common neighbors of V_i with respect to f is called the *common neighborhood* of V_i in G and denoted by $CN_{G,f}(V_i)$ or simply by $CN(V_i)$. Also every TDC or TDTC of G with $\chi_d^t(G)$ or $\chi_d^{tt}(G)$ colors is called respectively a *min*-TDC or a *min*-TDTC. For an examples see Figure 2.

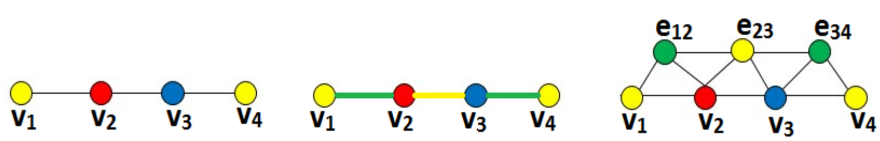

FIGURE 2. A min-TDC of P_4 (left), a min-TDTC of P_4 (middle) and a min-TDC of $T(P_4)$ (right).

Goal: As we have mentioned before, here we initiate to study the total dominator total coloring of a graph which is a total coloring of the graph such that each object of the graph is adjacent or incident to every object of some color class. In more detailes: In section 2 we present some tight lower and upper bounds for the total dominator total chromatic number of a graphs in terms of some parameters such as order, size, the total dominator chromatic and total domination numbers of the graph and its line graph. In section 3 we restrict our attention to trees and present a Nordhaus-Gaddum-like relation for them, and finally in the last section we show that there exist graphs that their total dominator total chromatic numbers are equal to their orders. The following theorems are useful for our investigation.

Theorem 1.5 (Kazemi, [14]). *For any connected graph G of order n with $\delta(G) \geq 1$,*

$$\max\{\chi(G), \gamma_t(G), 2\} \leq \chi_d^t(G) \leq n.$$

Furthermore, $\chi_d^t(G) = 2$ if and only if G is a complete bipartite graph, or $\chi_d^t(G) = n$ if and only if G is a complete graph.

Theorem 1.6 (Kazemi, [14]). *For any connected graph G with $\delta(G) \geq 1$,*

$$\chi_d^t(G) \leq \gamma_t(G) + \min_S \chi(G[V(G) - S]), \tag{1.1}$$

where $S \subseteq V(G)$ is a min-TDS of G. And so $\chi_d^t(G) \leq \gamma_t(G) + \chi(G)$.

Theorem 1.7 (Harary, [6]). *For any nonempty graph G,*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Theorem 1.8 (König's theorem, [3]). *For any nonempty bipartite graph G,*

$$\chi'(G) = \Delta(G).$$

Theorem 1.9 (Behzad, Chartrand, and Cooper, [2]). *For any complete graph K_n of order at least 2,*

$$\chi_T(K_n) = 2\left\lceil \frac{n+1}{2} \right\rceil - 1.$$

Theorem 1.10 (Kazemnejad, Kazemi, and Moradi, [18]). *For any complete graph K_n of order $n \geq 2$,*

$$\gamma_{tm}(K_n) = \left\lceil \frac{5n}{3} \right\rceil - n.$$

Theorem 1.11 (Kazemnejad, Kazemi, and Moradi, [18]). *For any graph G of order $n \geq 2$ which has a Hamiltonian path, $\gamma_{tm}(G) \leq \lceil \frac{5n}{3} \rceil - n$.*

Theorem 1.12 (Kazemnejad, Kazemi, and Moradi, [18]). *For any tree $\mathbb{T}$ of order $n \geq 3$, $\gamma_{tm}(\mathbb{T}) \leq \lfloor \frac{2n}{3} \rfloor$.*

Proof. Let $\mathbb{T}$ be a tree with the vertex set V. It is sufficient to prove $\gamma_t(T(\mathbb{T})) \leq \lfloor \frac{2n}{3} \rfloor$. Then $V(T(\mathbb{T})) = V \cup \mathcal{E}$. Choose a leaf v of $\mathbb{T}$ and label each vertex of $\mathbb{T}$ with its distance from v to modolu 3. This partitions V to the three independent sets A_0, A_1 and A_2 where $A_i = \{u \in V \mid d_{\mathbb{T}}(u,v) \equiv i \pmod 3\}$ for $0 \leq i \leq 2$. Then by the pigeonhole principle at least one of them, say A_0, contains at least one third of the vertices of $\mathbb{T}$, and so $|A_1 \cup A_2| \leq \lfloor \frac{2n}{3} \rfloor$. We see that every nonleaf vertex and every leaf $v_i \in V(\mathbb{T}) - A_1 \cup A_2$ is adjacent to some vertex in $A_1 \cup A_2$. If needed, we replace every leaf $v_i \in A_1 \cup A_2$ by a vertex from $N_{T(\mathbb{T})}(v_i) - A_1 \cup A_2$. The given set S is a TDS of $T(\mathbb{T})$. Because obviously $N(v_i) \cap S \neq \emptyset$ for each $v_i \in V(\mathbb{T})$, and $\{v_i, v_j\} \cap S \neq \emptyset$ for each $v_iv_j \in E(\mathbb{T})$ (because $d_{\mathbb{T}}(v, v_i) \not\equiv d_{\mathbb{T}}(v, v_j) \pmod 3$), and so every $e_{ij} \in \mathcal{E}$ is dominated by $v_i \in S$ or $v_j \in S$. So $\gamma_t(T(\mathbb{T})) \leq |S| \leq \lfloor \frac{2n}{3} \rfloor$. □

2. Some bounds

Since the problem of finding total dominator chromatic number of a graph is NP-complete [14], the problem of finding total dominator total chromatic number of a graph is also NP-complete by Theorem 1.4. Also since for any graph G with no isolated vertex of order n and size m, $\gamma_t(G) \leq \chi_d^t(G) \leq n$, by Theorem 1.5, and every TDTC is a total coloring, and also $\chi_d^{tt}(G) = \chi_d^t(T(G)) = n+m$ if and only if $G \cong K_2$, we have the next theorem.

Theorem 2.1. *For any connected graph G of order $n \geq 3$ and size m,*

$$\max\{\chi_T(G), \gamma_{tm}(G)\} \leq \chi_d^{tt}(G) \leq n+m-1.$$

Also since for any graph $G = G_1 + G_2 + \cdots + G_\omega$ with (connected) components G_1, G_2, …, G_ω which has no isolated vertex,

$$\max_{1\leq i\leq\omega} \chi_d^t(G_i) + 2\omega - 2 \leq \chi_d^t(G) \leq \sum_{i=1}^{\omega} \chi_d^t(G_i),$$

from [14], and since $T(G) = T(G_1) + T(G_2) + \cdots + T(G_\omega)$, we have the next theorem.

Theorem 2.2. *For any graph G with (connected) components $G_1, G_2, \ldots, G_\omega$ which has no isolate vertex,*

$$\max_{1\leq i\leq\omega} \chi_d^{tt}(G_i) + 2\omega - 2 \leq \chi_d^{tt}(G) \leq \sum_{i=1}^{\omega} \chi_d^{tt}(G_i).$$

Therefore, it is sufficient to verify the total dominator total chromatic number of connected graphs. Next theorem gives some bounds for the total dominator total chromatic number of a connected graph in terms of the total dominator chromatic numbers of the graph and its line graph.

Theorem 2.3. *For any connected graph G of order at least 3 and with no isolated vertex,*

$$\max\{\chi_d^t(G), \chi_d^t(L(G))\} \leq \chi_d^{tt}(G) \leq \chi_d^t(L(G)) + \chi_d^t(G).$$

And the bounds are tight.

Proof. Let $G = (V, E)$ be a connected graph of order $n \geq 3$ with $\delta(G) \geq 1$ and the vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and let $\mathcal{E} = \{e_{ij} \mid v_iv_j \in E\}$. Let also $f = (V_1, \ldots, V_\ell)$ be a min-TDTC of G. To prove $\chi_d^t(G) \leq \chi_d^{tt}(G)$, without loss of generality, we may assume $V_i \cap V \neq \emptyset$ if and only if $1 \leq i \leq m$ for some $1 \leq m \leq \ell$. For every $1 \leq k \leq m$ or every $m < k \leq \ell$ such that for every vertex $v_i \in V$, $v_i \not\succ_t V_k$, we set $W_k = V_k - \mathcal{E}$. Notice that if $v_i \succ_t V_k$ for some $v_i \in V$ and some $m < k \leq \ell$, then $V_k = \{e_{ij} \mid \text{for some } j \neq i\}$, and since V_k is a (mixed) independent set, we have $V_k = \{e_{ij}\}$ for some $j \neq i$. On the other hand $e_{ij} \succ_t V_{k'}$ for some $k' \neq k$ implies $V_{k'} \subseteq \{v_i, e_{j\ell}\}$ for some $\ell \neq i$ (or similarly $V_{k'} \subseteq \{v_j, e_{i\ell}\}$ for some $\ell \neq j$). Then we set $W_k = \{v_i\}$ and $W_{k'} = \{v_j\}$. Therefore the coloring function $(W_1, \ldots, W_\ell)$ is a TDC of G, and so $\chi_d^t(G) \leq \ell = \chi_d^{tt}(G)$.

In a similar way, to prove $\chi_d^t(L(G)) \leq \chi_d^{tt}(G)$, without loss of generality, we may assume $V_i \cap \mathcal{E} \neq \emptyset$ if and only if $1 \leq i \leq m$ for some $1 \leq m \leq \ell$. For every $1 \leq k \leq m$ or every $m < k \leq \ell$ such that for every vertex $e_{ij} \in \mathcal{E}$, $e_{ij} \not\succ_t V_k$, we set $W_k = V_k - V$. Notice that if $e_{ij} \succ_t V_k$ for some $e_{ij} \in \mathcal{E}$ and some $m < k \leq \ell$, then $V_k \subset \{v_i, v_j\}$, by $e_{ij} = v_iv_j \in E(G)$. So, without loss of generality, we may assume $V_k = \{v_i\}$. If $deg_G(v_i) \geq 2$, then $v_iv_\ell \in E$ for some $\ell \neq j$, and in this case we set $W_k = \{e_{i\ell}\}$. Otherwise, since $\{e_{ip} \mid v_p \in N_G(v_i)\} = \{e_{ij}\}$, we set $W_k = \{e_{j\ell}\}$ where $v_\ell \in N_G(v_j)$. Then the function $(W_1, \ldots, W_\ell)$ is a TDC of $L(G)$, and so $\chi_d^t(L(G)) \leq \ell = \chi_d^{tt}(G)$. Therefore we have proved $\max\{\chi_d^t(G), \chi_d^t(L(G))\} \leq \chi_d^{tt}(G)$. Since $\chi_d^t(K_3) = \chi_d^t(L(K_3)) = \chi_d^{tt}(K_3) = 3$, the lower bound is tight for K_3.

The upper bound is proved by considering this fact that for any min-TDC $(V_1, \ldots, V_p)$ of G and any min-TDC $(V_1', \ldots, V_q')$ of $L(G)$, the coloring function $(V_1, \ldots, V_p, V_1', \ldots, V_q')$ is a TDTC of G. The upper bound is tight for the cycle C_4, because of $\chi_d^t(C_4) = \chi_d^t(L(C_4)) = 2$ (because $L(C_4) \cong C_4$), and $\chi_d^{tt}(C_4) = 4$ (because for any min-TDTC $f = (V_1, V_2, \ldots, V_\ell)$ of C_4, $8 = |V \cup \mathcal{E}| = \sum_{i=1}^{\ell} |V_i| \leq \ell \cdot \alpha_{mix}(C_4) = 2\ell$ implies $\ell \geq 4$, and the coloring function $(\{e_{12}, e_{34}\}, \{e_{23}, e_{14}\}, \{v_1, v_3\}, \{v_2, v_4\})$ is a TDTC of C_4). □

Next theorem gives an upper bound for the total dominator chromatic number of a connected graph in terms of the total domination numbers of the graph and its line graph.

Theorem 2.4. *Let G be a connected graph with $\delta(G) \geq 1$. Then we have the following tight bound*

$$\chi_d^{tt}(G) \leq \chi(T(G) - S_1 \cup S_2) + \gamma_t(G) + \gamma_t(L(G)),$$

where S_1 is a min-TDS of G and S_2 is a min-TDS of $L(G)$.

Proof. Let S_1 be a min-TDS of G and S_2 be a min-TDS of $L(G)$. Color $T(G) - S_1 \cup S_2$ with minimum colors, and assign $|S_1| + |S_2|$ new colors to the $|S_1| + |S_2|$ vertices of $S_1 \cup S_2$. Since $S_1 \cup S_2$ is a TDS of $T(G)$, this coloring of $T(G)$ is TDC, and so $\chi_d^{tt}(G) = \chi_d^t(T(G)) \leq \chi(T(G) - S_1 \cup S_2) + \gamma_t(G) + \gamma_t(L(G))$.

Let G be the graph given in Figure 3. Obviously $S_1 = \{v_1, v_5\}$ is a min-TDS of G and $S_2 = \{e_{12}, e_{23}, e_{56}, e_{67}\}$ is a min-TDS of $L(G)$. Since $T(G) - S_1 \cup S_2$ can be easily colored by 4 colors and the subgraph of $T(G) - S_1 \cup S_2$ induced by $\{v_4, e_{14}, e_{24}, e_{34}\}$ is isomorphic to K_4, we have $\chi(T(G) - (S_1 \cup S_2)) = 4$, and so

$$\chi_d^{tt}(G) = \chi_d^t(T(G)) \leq \chi(T(G) - (S_1 \cup S_2)) + \gamma_t(G) + \gamma_t(L(G)) = 10. \tag{2.1}$$

Now let $f = (V_1, \ldots, V_\ell)$ be a TDC of $T(G)$. Since $v_0 \succ_t V_k$ implies $V_k = \{v_1\}$ or $V_k = \{e_{01}\}$, and also $v_9 \succ_t V_t$ implies $V_t = \{v_5\}$ or $V_t = \{e_{59}\}$, we assume $V_k = \{v_1\}$ and $V_t = \{v_5\}$, and then we choose set $A = \{e_{01}, e_{12}, e_{13}, e_{14}, e_{15}, e_{56}, e_{57}, e_{58}, e_{59}\}$ (notice: if $V_k = \{e_{01}\}$ or $V_t = \{e_{59}\}$, then we replace e_{01} by v_1 or e_{59} by v_5 in A, respectively). Since the subgraph of $T(G)$ induced by A is in fact two copies of K_5 which have only e_{15} in common, we can color it by minimum 5 colors other than the colors of v_1 and v_5. Let $f(v_1) = 1$, $f(v_5) = 2$, $f(e_{12}) = f(e_{56}) = 3$, $f(e_{13}) = f(e_{57}) = 4$, $f(e_{14}) = f(e_{58}) = 5$, $f(e_{01}) = f(e_{59}) = 6$, $f(e_{15}) = 7$. Obviously $e_{ij} \in \mathcal{E} - A$ implies $e_{ij} \not\succ_t V_k$ for every $1 \leq k \leq 7$. Since the number of indices of the vertices in the set $B = \{e_{23}, e_{24}, e_{34}\}$ is three, we obtain $|\{k \mid e_{ij} \succ_t V_k, \text{ for every } e_{ij} \in B\}| \geq 2$, that means $\ell \geq 9$. On the other hand, we have $e_{67} \not\succ_t V_k$ for every $1 \leq k \leq 9$, because $N(e_{67}) \cap N(e_{ij}) = \emptyset$ for each $e_{ij} \in B$. So $\ell = 10$ by (2.1). □

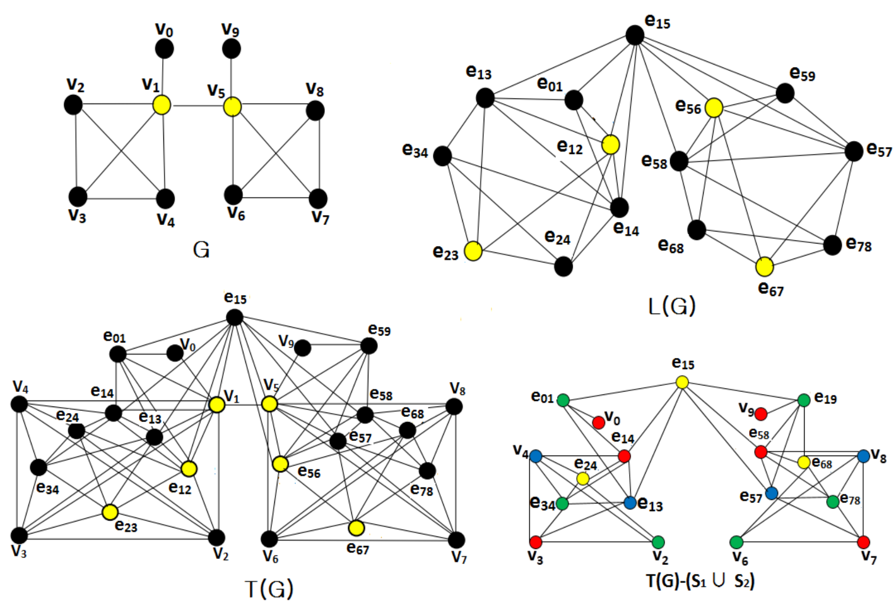

FIGURE 3. The illustration of G, $L(G)$, $T(G)$ and $T(G) - S_1 \cup S_2$.

Now, we establish an upper bounds on the total dominator total chromatic number of the complete graphs and then a family of graphs. First, a lemma.

Lemma 2.5. *For any complete graph K_n of order $n \geq 2$, $\alpha_{mix}(K_n) = \lceil \frac{n}{2} \rceil$.*

Proof. Let K_n be the complete graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and let $\mathcal{E} = \{e_{ij} \mid 1 \leq i < j \leq n\}$. Let S be a mixed independent set of K_n. Since $|S \cap V| \leq 1$ and so $\{p, q\} \cap \{r, k\} = \emptyset$ for every $e_{pq}, e_{rk} \in S$, we conclude $|S| \leq \lceil \frac{n}{2} \rceil$. On the other hand, since the sets $\{e_{(2i-1)(2i)} \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ when n is even, and $\{v_n, e_{(2i-1)(2i)} \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ when n is odd, are two independent sets with cardinality $\lceil \frac{n}{2} \rceil$, we obtain $\alpha_{mix}(K_n) = \lceil \frac{n}{2} \rceil$. □

Proposition 2.6. *For any complete graph K_n of order $n \geq 2$,*

$$n + \epsilon \leq \chi_d^{tt}(K_n) \leq \left\lceil \frac{5n}{3} \right\rceil,$$

where ϵ is 1 when n is even, and is zero otherwise.

Proof. Let K_n be the complete graph with the vertex set $V = \{v_1, v_2, \dots, v_n\}$, and let $\mathcal{E} = \{e_{ij} \mid 1 \le i < j \le n\}$. We first prove the lower bound. Let $f = (V_1, \dots, V_\ell)$ be a TDTC of K_n. Then, by the partition $V \cup \mathcal{E} = \bigcup_{i=1}^{\ell} V_i$ and Lemma 2.5, we have

$$\begin{aligned}\frac{n(n+1)}{2} &= \sum_{i=1}^{\ell} |V_i| \\ &\le \ell \cdot \alpha_{mix}(K_n) \\ &= \ell \left\lceil \frac{n}{2} \right\rceil,\end{aligned}$$

which implies $\chi_d^{tt}(K_n) = \chi_d^t(T(K_n)) \ge n + \epsilon$ in which $\epsilon = 1$ for even n and $\epsilon = 0$ otherwise. For the upper bound, first we have

$$\chi_d^{tt}(K_n) \le \begin{cases} \lceil \frac{5n}{3} \rceil & \text{if } n \text{ is odd,} \\ \lceil \frac{5n}{3} \rceil + 1 & \text{if } n \text{ is even,} \end{cases}$$

by Theorems 1.6, 1.9, 1.10. So we may assume that n is even. Let $f = (V_1, \dots, V_{n+1})$ be a proper coloring of $T(K_n)$ with minimum number of colors such that $v_i \in V_i$ for $1 \le i \le n$. Then $|V_i| = \frac{n}{2}$ for each $1 \le i \le n+1$ and $V_{n+1} \subseteq \mathcal{E}$. From the proof of Theorem 1.10, we know that the sets

$$\begin{array}{ll} S_0 = \{e_{(3i+1)(3i+2)}, e_{(3i+2)(3i+3)} \mid 0 \le i \le \lfloor \frac{n}{3} \rfloor - 1\} & \text{if } n \equiv 0 \pmod 6, \\ S_1 = S_0 \cup \{e_{(n-1)n}\} & \text{if } n \equiv 4 \pmod 6, \\ S_2 = S_0 \cup \{e_{(n-2)(n-1)}, e_{(n-1)n}\} & \text{if } n \equiv 2 \pmod 6, \end{array}$$

are min-TDSs of $T(K_n)$. Since the complete graph K_n is a subgraph of $T(K_n) - S_i$ for each i, we have $\chi(T(K_n) - S_i) \ge n$ for each i. Let $h = (V'_1, \dots, V'_\ell)$ be a proper coloring of $T(K_n) - S_i$ for each i. Then $|V'_j| \le \frac{n}{2}$ for each j. Similar to the proof of Lemma 2.5, since every independent set of $T(K_n) - S_i$ has cardinality at most $\lceil \frac{n}{2} \rceil$, and $\{e_{i(n/2+i)} \mid 1 \le i \le \frac{n}{2}\}$ is an independent set of $T(K_n) - S_i$, we obtain $\alpha(T(K_n) - S_i) = \lceil \frac{n}{2} \rceil$. On the other hand, by knowing $|V(T(K_n) - S_i)| = \frac{3n^2 - n - 2i}{6}$ when $n \equiv i \pmod 3$ and $0 \le i \le 2$, we have $|V(T(K_n) - S_i)| \le n \lceil \frac{n}{2} \rceil$ and so $\chi(T(K_n) - S_i) = n$ for $0 \le i \le 2$. Therefore, by Theorem 1.6, $\chi_d^t(T(K_n)) \le \chi(T(K_n) - S) + \gamma_t(T(K_n)) = \lceil \frac{5n}{3} \rceil$ in which S is a min-TDS of $T(K_n)$, and this completes our proof. $\square$

Since every connected graph G of order $n \ge 2$ is a subgraph of a complete graph K_n, obviously $\chi_T(G) \le \chi_T(K_n)$. Similar to the proof of Proposition 2.6, the following theorem can be proved.

Theorem 2.7. *For any graph G of order $n \ge 2$ and with the total mixed domination number at most $\lceil \frac{5n}{3} \rceil - n$, $\chi_d^{tt}(G) \le \lceil \frac{5n}{3} \rceil$.*

By Theorem 1.11, every graph which has a Hamiltonian path satisfies in Theorem 2.7.

3. Trees

3.1. **Total dominator total chromatic number of a tree.** Here, we calculate the total dominator total chromatic number of a tree of order at most 4 or diameter at most 3, and give tight lower and upper bounds for the total dominator total chromatic number of a tree of order $n \geq 5$.

Theorem 3.1. *For any tree $\mathbb{T}$ of order $n \geq 2$,*

$$\chi_d^{tt}(\mathbb{T}) = \begin{cases} 3 & n = 2, \\ n & n = 3, 4, \end{cases}$$

and if $n \geq 5$, we have the tight bounds

$$5 \leq \chi_d^{tt}(\mathbb{T}) \leq \left\lfloor \frac{2n}{3} \right\rfloor + \Delta(\mathbb{T}) + 1.$$

Proof. Let $f = (V_1, V_2, \ldots, V_\ell)$ be a min-TDC of $T(\mathbb{T})$ in which $\mathbb{T} = (V, E)$ is a tree of order $n \geq 2$, and so $V(T(\mathbb{T})) = V \cup \mathcal{E}$ where $\mathcal{E} = \{e_{ij} \mid v_i v_j \in E(\mathbb{T})\}$. First consider $2 \leq n \leq 4$. Then $\mathbb{T} \in \{P_2, P_3, P_4, K_{1,3}\}$. Let $\mathbb{T} = P_n : v_1 v_2 \cdots v_n$ when $2 \leq n \leq 4$. Then $V(T(P_n)) = V \cup \mathcal{E}$ where $\mathcal{E} = \{e_{i(i+1)} | 1 \leq i \leq n-1\}$. Since $T(P_2)$ is isomorphic to K_3, and also K_3 is a subgraph of $T(P_3)$ and $(\{v_1, e_{23}\}, \{v_3, e_{12}\}, \{v_2\})$ is a TDC of $T(P_3)$, we have $\chi_d^{tt}(P_n) = \chi_d^t(T(P_n)) = 3$ for $n = 2, 3$. Since the subgraph of $T(P_4)$ induced by $\{v_1, v_2, e_{12}\}$ is isomorphic to a complete graph of order 3, we may assume $f(v_1) = 1$, $f(v_2) = 2$ and $f(e_{12}) = 3$. Then, $v_4 \not\succ_t V_i$ for $1 \leq i \leq 3$ implies $\ell \geq 4$. Now since $(\{v_2\}, \{v_3\}, \{v_1, e_{23}, v_4\}, \{e_{12}, e_{34}\})$ is a TDC of $T(P_4)$, we have $\chi_d^{tt}(P_4) = 4$. In the next step, let $\mathbb{T} = K_{1,3} = (V, E)$ where $V = \{v_i \mid 0 \leq i \leq 3\}$ and $E = \{v_0 v_i \mid 1 \leq i \leq 3\}$. Then on one hand, this fact that the subgraph of $T(K_{1,3})$ induced by $\{e_{0i} \mid 1 \leq i \leq 3\} \cup \{v_0\}$ is a complete graph of order 4 implies $\ell \geq 4$, and on the other hand, since $(\{v_0\}, \{e_{01}, v_3\}, \{e_{02}, v_1\}, \{e_{03}, v_2\})$ is a TDC of $T(K_{1,3})$, we obtain $\chi_d^{tt}(K_{1,3}) = 4$. Therefore we continue our proof when $n \geq 5$. For the lower bound, since the subgraph H_{v_i} of $T(\mathbb{T})$ induced by $\{v_i\} \cup \{e_{ij} \mid v_j \in N_{\mathbb{T}}(v_i)\}$ is a complete graph of order $1 + deg_{\mathbb{T}}(v_i)$, we have done when $\Delta(\mathbb{T}) \geq 4$. Thus $\Delta(\mathbb{T}) \leq 3$. If $\Delta(\mathbb{T}) = deg_{\mathbb{T}}(v_i) = 3$ for some v_i, then $H_{v_i} \cong K_4$, and by assumption $f(V(H_{v_i})) = \{1, 2, 3, 4\}$, $n \geq 5$ implies that there exists a vertex $v_j \notin N_{T(\mathbb{T})}(v_i)$ which is not totally dominated by V_i when $1 \leq i \leq 4$. So $\ell \geq 5$, as desired. Finally, let $\Delta(\mathbb{T}) = 2$. Then $\mathbb{T} = P_n : v_1 v_2 \cdots v_n$ is a path of order $n \geq 5$. Since v_1 is totally dominated only by $\{v_2\}$ or $\{e_{12}\}$, and v_n is totally dominated only by $\{v_{n-1}\}$ or $\{e_{(n-1)n}\}$, and since H_{v_3} is a subgraph of $T(\mathbb{T}) - (V_k \cup V_m)$, we have $\ell \geq 5$, as desired. The lower bound is tight for $\mathbb{T} = P_5$. Because $(\{v_2\}, \{v_3\}, \{v_4\}, \{v_1, e_{23}, e_{45}\}, \{e_{12}, e_{34}, v_5\})$ is a TDC of $T(P_5)$.

To prove the upper bound, first

$$\chi_d^{tt}(\mathbb{T}) = \chi_d^t(T(\mathbb{T})) \leq \gamma_{tm}(\mathbb{T}) + \min_S \chi(T(\mathbb{T})[V \cup \mathcal{E} - S])$$

where $S \subseteq V(T(\mathbb{T}))$ is a min-TDS of $T(\mathbb{T})$ by (1.1). Let $T(\mathbb{T})[V \cup \mathcal{E} - S]$ be the subgraph of $T(\mathbb{T})$ induced by $V \cup \mathcal{E} - S = \mathcal{E} \cup A_0$ where S is the min-TDS of $T(\mathbb{T})$ in the proof of Theorem 1.12. Then

$$\begin{array}{rcll} \chi(T(\mathbb{T})[V \cup \mathcal{E} - S]) & \leq & \chi(T(\mathbb{T})[\mathcal{E}]) + \chi(T(\mathbb{T})[A_0]) & \\ & = & \chi(L(\mathbb{T})) + 1 & (A_0 \text{ is independent}) \\ & = & \chi^{'}(\mathbb{T}) + 1 & \\ & = & \Delta(\mathbb{T}) + 1. & (\text{König's Theorem}) \end{array}$$

On the other hand, since $\gamma_{tm}(\mathbb{T}) \leq \lfloor \frac{2n}{3} \rfloor$ by Theorem 1.12, we have completed our proof.

The upper bound is tight for path P_7 with vertex set $V = \{v_1, v_2, \ldots, v_7\}$ and edge set $E = \{v_i v_{i+1} \mid 1 \leq i \leq 6\}$. Let $f = (V_1, V_2, \ldots, V_\ell)$ be a TDC of $T(P_7)$ where $V(T(P_7)) = V \cup \mathcal{E}$ and $\mathcal{E} = \{e_{i(i+1)} | 1 \leq i \leq 6\}$. Let $v_1 \succ_t V_k$ for some k. Then $V_k = \{w\}$ where $w \in \{v_2, e_{12}\}$, and so $w \succ_t V_m$ for some $m \neq k$ (because either $V_m \subseteq \{v_1, v_3, e_{12}, e_{23}\}$ if $w = v_2$ or $V_m \subseteq \{v_1, v_2, e_{23}\}$ if $w = e_{12}$). Since a similar result holds by consider v_7 instead of v_1, we conclude that the number of V_k such that $v_i \succ_t V_k$ for some $v_i \in V - \{v_4\}$ is at least four, and that V_ks do not contain vertices v_4, e_{34}, e_{45}. Since the subgraph of $T(P_7)$ induced by $\{v_4, e_{34}, e_{45}\}$ is a complete graph, we conclude $\ell \geq 7$. On the other hand, since $(\{v_1, v_4, v_7, e_{23}, e_{56}\}, \{e_{12}, e_{34}, e_{67}\}, \{e_{45}\}, \{v_2\}, \{v_3\}, \{v_5\}, \{v_6\})$ is a TDC of $T(P_7)$, we have $\chi_d^{tt}(P_7) = 7$. □

Theorem 3.2. *For any nonempty tree* $\mathbb{T}$, *diam*$(\mathbb{T}) \leq 3$ *if and only if*

$$\chi_d^{tt}(\mathbb{T}) = \begin{cases} \Delta(\mathbb{T}) + 2 & \text{if } diam(\mathbb{T}) = 1, 3, \\ \Delta(\mathbb{T}) + 1 & \text{if } diam(\mathbb{T}) = 2. \end{cases}$$

Proof. Let $\mathbb{T}$ be a tree of order at least 2. Since $diam(\mathbb{T}) = 1$ implies $\mathbb{T} \cong K_2$ and $T(\mathbb{T}) \cong K_3$, and so $\chi_d^{tt}(\mathbb{T}) = \chi_d^t(K_3) = \chi(K_3) = 3 = \Delta(\mathbb{T}) + 2$, in the first step, we assume $diam(\mathbb{T}) = 2$. Then $n \geq 3$ and $\mathbb{T} \cong K_{1,n-1}$. Let $V(K_{1,n-1}) = \{v_i \mid 0 \leq i \leq n-1\}$ in which $deg(v_0) = n-1$. Since $T(\mathbb{T})[\{e_{0i} \mid 1 \leq i \leq n-1\} \cup \{v_0\}] \cong K_n$ and the function f with the criterion $f(v_0) = 0$ and $f(v_i) \equiv f(e_{0i}) + 1 \pmod{n}$ when $1 \leq i \leq n-1$ and $1 \leq f(e_{0i}) \leq n-1$ is a TDC of $T(\mathbb{T})$ with n colors, we have $\chi_d^{tt}(\mathbb{T}) = \chi_d^t(T(\mathbb{T})) = n = \Delta(\mathbb{T}) + 1$. Finally let $diam(\mathbb{T}) = 3$. Then $\mathbb{T}$ is a tree which is obtained by joining the central vertex v_{p+1} of tree $K_{1,p}$ with the central vertex v_{p+2} of tree $K_{1,q}$, where $V(K_{1,p}) = \{v_i \mid 1 \leq i \leq p+1\}$, $V(K_{1,q}) = \{v_i \mid p+2 \leq i \leq p+q+2\}$, $p \geq q$ and $p+q = n-2$. Hence $\Delta(\mathbb{T}) = p+1$ and $V(T(\mathbb{T})) = V(\mathbb{T}) \cup \mathcal{E}$ where $\mathcal{E} = \{e_{i(p+1)} \mid 1 \leq i \leq p\} \cup \{e_{(p+2)(p+2+i)} \mid 1 \leq i \leq q\} \cup \{e_{(p+1)(p+2)}\}$. Let $f = (V_1, \ldots, V_\ell)$ be a TDC of $T(\mathbb{T})$. Since the subgraph of $T(\mathbb{T})$ induced by $A = \{e_{i(p+1)} \mid 1 \leq i \leq p\} \cup \{v_{p+1}, e_{(p+1)(p+2)}\}$ is a complete graph of order $p+2$, we conclude that $p+2$ colors, say 1, 2, …, $p+2$, are needed to color the vertices in A, and so $\ell \geq p+2$. Also, by this fact that $v_{p+3} \succ_t V_k$ implies $V_k = \{e_{(p+2)(p+3)}\}$ or $V_k = \{v_{p+2}\}$, and since $v_{p+3} \not\succ_t V_s$ where $1 \leq s \leq p+2$, we conclude that a new color is needed to color the vertex in V_k, and so $\ell \geq p+3$. Without loss of generality, we may assume that $f(e_{i(p+1)}) = i$ when $1 \leq i \leq p$, $f(v_{p+1}) = p+1$ and $f(e_{(p+1)(p+2)}) = p+2$. Since also the subgraph of $T(\mathbb{T})$ induced by $A' = \{e_{(p+2)(p+2+i)} \mid 1 \leq i \leq q\} \cup \{v_{p+2}, e_{(p+1)(p+2)}\}$ is a complete graph of order $q+2$, we may assume that $f(v_{p+2}) = p+3$ and $f(e_{(p+2+i)(p+2)}) = i$ when $1 \leq i \leq q$. Now by assigning color 1 to all vertices $v_2, \ldots, v_p, v_{p+4}, \ldots, v_{p+q+2}$, and color 2 to the vertices v_1 and v_{p+3}, we obtain a TDC of $T(\mathbb{T})$ with $p+3$ colors, which implies $\chi_d^{tt}(\mathbb{T}) = \chi_d^t(T(\mathbb{T})) = p+3 = \Delta(\mathbb{T})+2$.

Now by assumption $diam(\mathbb{T}) = r \geq 4$ let $P_r : v_1, v_2, \ldots, v_r, v_{r+1}$ be a longest path of length r in $\mathbb{T}$. Then $deg_{\mathbb{T}}(v_i) = \Delta(\mathbb{T})$ for some $v_i \in V(\mathbb{T}) \setminus \{v_1, v_{r+1}\}$ (because in otherwise we have a cycle in the tree). Let H_{v_i} be the subgraph of $T(\mathbb{T})$ induced by $\{v_i\} \cup \{e_{ij} \mid v_j \in N_{\mathbb{T}}(v_i)\}$ and let $f = (V_1, \ldots, V_\ell)$ be a min-TDC of $T(\mathbb{T})$. Then, since $deg_{\mathbb{T}}(v_1) = deg_{\mathbb{T}}(v_{r+1}) = 1$ and $r \geq 4$, $v_1 \succ_t V_k$ and $v_{r+1} \succ_t V_m$ for some k and m, imply $k \neq m$ and $V_k = \{w\}$ and $V_m = \{w'\}$ where $w \in \{v_2, e_{12}\}$ and $w' \in \{v_r, e_{r(r+1)}\}$. Also there exist a color class V_p other than V_k and V_m such that $w' \succ_t V_p$. Since H_{v_i} is a complete graph of order $1 + deg_{\mathbb{T}}(v_i)$, we have to assign $\Delta(\mathbb{T})+1$ colors to the vertices of H_{v_i}. Since $v_i = v_2$ implies that the $\Delta(\mathbb{T})+1$ colors which are assigned to the vertices of H_{v_i} are different of the colors of the vertices of $V_m \cup V_p$, and simliarly $v_i = v_r$ implies that the $\Delta(\mathbb{T})+1$ colors which are assigned to the vertices of H_{v_i} are different of the colors of the vertices of $V_k \cup V_p$, we have $\ell \geq \Delta(\mathbb{T})+3$, as desired. So we assume $v_i \neq v_2, v_r$. In this case, similarly, the $\Delta(\mathbb{T})+1$ colors which are assigned to the vertices of H_{v_i} are different of the colors of the vertices of $V_k \cup V_m$, and so $\ell \geq \Delta(\mathbb{T})+3$, as desired. $\square$

Corollary 3.3. *The Behzad's conjecture is true for any tree with diameter at most three.*

3.2. **A Nordhaus-Gaddum-like relation for trees.** Finding a Nordhaus-Gaddum-like relation for any parameter in graph theory is one of a tradition work which is started after the following theorem by Nordhaus and Gaddum in 1956 [19].

Theorem 3.4 (Nordhaus and Gaddum, [19])**.** *For any graph G of order n, $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n+1$.*

Here, we will find some Nordhaus-Gaddum-like relations for the total dominator total chromatic number of a tree. For this aim we will find some bounds for the total dominator chromatic number of the complement of a tree. First two lemmas.

Lemma 3.5 (Clark and Holton, [4])**.** *For any complete graph K_n of order at least 2,*

$$\chi'(K_n) = \begin{cases} n-1 & \textit{if } n \textit{ is even}, \\ n & \textit{if } n \textit{ is odd}. \end{cases}$$

Lemma 3.6. *For any tree $\mathbb{T}$ of order $n \geq 3$,*

$$\chi'(\overline{\mathbb{T}}) = \begin{cases} n-2 & \textit{if } \mathbb{T} \textit{ is } P_4 \textit{ or is nonstar or is star with odd } n, \\ n-1 & \textit{if } \mathbb{T} \textit{ is star with even } n. \end{cases}$$

Proof. If $\mathbb{T}$ is the star $K_{1,n-1}$, then $\overline{\mathbb{T}}$ up to isomorphism is the disjoint union of K_{n-1} and K_1, and so

$$\chi'(\overline{\mathbb{T}}) = \begin{cases} n-2 & \text{if } n \text{ is odd}, \\ n-1 & \text{if } n \text{ is even}, \end{cases}$$

by Lemma 3.5. If $\mathbb{T} \cong P_4$, then $\overline{\mathbb{T}} \cong P_4$, and obviously $\chi'(P_4) = 2 = n-2$. Therefore, we assume $\mathbb{T} = (V, E)$ is a nonstar tree of order $n \geq 5$, which implies $\alpha(\mathbb{T}) = \omega(\overline{\mathbb{T}}) \leq n-2$ (recall that $\omega(G)$ is the *clique number* of a graph G, which is the number of vertices in a maximum clique of G). By assumption $V = \{v_1, v_2, \ldots, v_n\}$, we have $V(L(\overline{\mathbb{T}})) = \{e_{ij} \mid v_i v_j \notin E\}$. Since for any leaf v_i in $\mathbb{T}$ the subgraph of $L(\overline{\mathbb{T}})$ induced by $\{e_{ij} \mid v_j \notin N_{\mathbb{T}}(v_i)\}$ is a complete graph of order $n-2$, we have $\chi(L(\overline{\mathbb{T}})) \geq n-2$. On the other hand, Lemma 3.5 and this fact that every m-clique K_m in $\overline{\mathbb{T}}$ with the vertx set $\{v_i \mid i \in I\}$, for some index set I, makes m cliques in $L(\overline{\mathbb{T}})$ with the vertex sets $E_i = \{e_{ij} \mid j \in I - \{i\}\}$ of order $m-1$ such that $E_i \cap E_j = \{e_{ij}\}$ for each $i \neq j$, give us this possibility that we color the vertices of $L(K_m)$ by at most m colors. By a permutation on the used colors in each clique in $\overline{\mathbb{T}}$, if needed, we can color the vertices of $L(\overline{\mathbb{T}})$ by at most $n-2$ colors, that is, $\chi(L(\overline{\mathbb{T}})) \leq n-2$, which implies $\chi'(\overline{\mathbb{T}}) = \chi(L(\overline{\mathbb{T}})) = n-2$ by considering the previous inequality. □

Theorem 3.7. *For any nonstar tree $\mathbb{T}$ of order $n \geq 5$ with ℓ leaves,*

$$\ell + n - 2 \leq \chi_d^{tt}(\overline{\mathbb{T}}) \leq 2n - 4,$$

and this bounds are same for any tree with diameter three.

Proof. Let $\mathbb{T} = (V, E)$ be a nonstar tree $\mathbb{T}$ of order $n \geq 5$ with ℓ leaves which implies $\alpha(\mathbb{T}) = \omega(\overline{\mathbb{T}}) \leq n-2$. By assumption $V = \{v_1, v_2, \ldots, v_n\}$, we have $V(T(\overline{\mathbb{T}})) = V \cup \mathcal{E}$ where $\mathcal{E} = \{e_{ij} \mid v_i v_j \notin E\}$ and $E(T(\overline{\mathbb{T}})) = E(\overline{\mathbb{T}}) \cup \{e_{ij} v_k \mid e_{ij} \in \mathcal{E} \text{ and } k \notin \{i, j\}\} \cup \{e_{ij} e_{i'j'} \mid e_{ij}, e_{i'j'} \in \mathcal{E} \text{ and } \{i, j\} \cap \{i', j'\} \neq \emptyset\}$. For some index set I, let $\mathbb{L} = \{v_i \mid i \in I\}$ be the set of all leaves of $\mathbb{T}$, and let f be a proper vertex coloring of $T(\overline{\mathbb{T}})$. By Lemma 3.6, we have $|f(V(L(\overline{\mathbb{T}})))| = |f(\mathcal{E})| \geq n-2$. On the other hand, since the induced subgraph $\overline{\mathbb{T}}[\mathbb{L}]$ is a complete graph of order ℓ and also for any $v_i \in \mathbb{L}$, since $deg_{\overline{\mathbb{T}}}(v_i) = n-2$, each of the induced subgraphs $H_{v_i} = \overline{\mathbb{T}}[\{v_i\} \cup \{e_{ij} \mid v_j \in N_{\overline{\mathbb{T}}}(v_i)\}]$ of $\overline{\mathbb{T}}$ is a complete graph of order $n-1$ such that $V(H_{v_i}) \cap V(\overline{\mathbb{T}}[\mathbb{L}]) = \{v_i\}$ and $V(H_{v_i}) - \{v_i\} \subset \mathcal{E}$, we conclude that $f(\mathbb{L}) \cap f(\mathcal{E}) = \emptyset$, and so $\chi_d^{tt}(\overline{\mathbb{T}}) = \chi_d^t(T(\overline{\mathbb{T}})) \geq \chi(T(\overline{\mathbb{T}})) \geq n + \ell - 2$.

For the upper bound, first $\Delta(\mathbb{T}) \leq n-2$ implies $diam(\mathbb{T}) \geq 3$, so there exist at least two nonleaf vertices, say v_1 and v_2, such that v_1 is adjacent to v_2 in $\mathbb{T}$ and so $deg_{\overline{\mathbb{T}}}(v_1) \leq deg_{\overline{\mathbb{T}}}(v_2) \leq n-3$. We will give a TDC f in $T(\overline{\mathbb{T}})$ with $2n-4$ color classes. Since $\chi(L(\overline{\mathbb{T}})) = n-2$, by Lemma 3.6, we can assign $n-2$ colors to the vertices in $V(L(\overline{\mathbb{T}}))=\mathcal{E}$. For $i=1,2$, since $N_{T(\overline{\mathbb{T}})}(v_i)\cap\mathcal{E}=\{e_{ij}\mid e_{ij}\in\mathcal{E}\}$, we define $f(v_i)=a_i$ for $i=1,2$ where $a_i\neq f(e_{ij})$ for some $1\leq a_i\leq n-2$. Finally we assign $n-2$ new colors to the $n-2$ vertices of $V-\{v_1,v_2\}$. We claim that f is a TDC of $T(\overline{\mathbb{T}})$. For this aim, we have to show that for any vertex $w\in T(\overline{\mathbb{T}})=V\cup\mathcal{E}$ there exists a color class V_p such that $w\succ_t V_p$. Since $\mathbb{T}$ has always two leaves, say v_k and v_q, and so $deg_{\overline{\mathbb{T}}}(v_k)=deg_{\overline{\mathbb{T}}}(v_q)=n-2$, we have $v_i\succ_t V_p$ for any $v_i\in V$ when $V_p=\{v_k\}$ or $\{v_q\}$. Also $e_{ij}\succ_t V_p$ where $V_p=\{v_i\}$ or $\{v_j\}$ for any $e_{ij}\in\mathcal{E}$ because $e_{12}\notin\mathcal{E}$ and so $e_{ij}\neq e_{12}$.

This bounds are same for any tree with diameter three. Because every tree $\mathbb{T}$ with diameter three is in fact a tree which is obtained by joining the central vertices of two star trees $K_{1,p}$ and $K_{1,q}$ in which $p+q=n-2$. □

By Theorems 3.1 and 3.7 we have the following theorem.

Theorem 3.8. *For any nonstar tree $\mathbb{T}$ of order $n\geq 5$ with ℓ leaves,*

$$n+\ell+3\leq\chi_d^{tt}(\mathbb{T})+\chi_d^{tt}(\overline{\mathbb{T}})\leq\left\lfloor\frac{8n}{3}\right\rfloor+\Delta(\mathbb{T})-3.$$

By Theorems 3.2 and 3.7, we see that while the lower bound in Theorem 3.8 is tight for any tree $\mathbb{T}$ of order $n=5$ with diameter three, but $\chi_d^{tt}(\mathbb{T})+\chi_d^{tt}(\overline{\mathbb{T}})\leq 3n-4<\lfloor\frac{8n}{3}\rfloor+\Delta(\mathbb{T})-3$ when $diam(\mathbb{T})=3$. So we ask the following question.

Question. *Is the upper bound in Theorem 3.8 tight for any tree with order greater than or equal to 5 and diameter greater than or equal to 4?*

4. Graphs which their total dominator total chromatic numbers are equale to their orders

One of the usual questions in graph theory is the following question.

Question. *Let $\mathcal{P}$ be a property defind on a set $\mathcal{S}$ of graphs. Is there any graph in $\mathcal{S}$ of order n with $\mathcal{P}=k$?*

Next two propositions give positive answer to this question when $\mathcal{P}$ is the total dominator total chromatic number of the double star trees and the corona graphs $G\circ P_1$ and $G\circ P_2$. We recall that the *double star tree* $S_{1,n,n}$ is a subdivition graph of $K_{1,n}$ by replacing every edge by a path with lengh 2, and the *m-corona* graph $G\circ P_m$ of a graph G is the graph obtained from G by adding a path of order m to each vertex of G. First three lemmas.

Lemma 4.1. *For any $n\geq 1$, $\gamma_{tm}(S_{1,n,n})=n+1$.*

Proof. Let $S_{1,n,n}$ be a double star with vertex set $V=\{v_i\mid 0\leq i\leq 2n\}$ and edge set $E=\{v_0v_i,v_iv_{n+i}\mid 1\leq i\leq n\}$. Let S be a TDS of $T(S_{1,n,n})$, the total of $S_{1,n,n}$, which its vertex set is

$$V(T(S_{1,n,n}))=V\cup\{e_{0i},e_{i(n+i)}\mid 1\leq i\leq n\}.$$

Since $N_{T(S_{1,n,n})}(v_{n+i}) = \{v_i, e_{i(n+i)}\}$ for $1 \leq i \leq n$ and $N_{T(S_{1,n,n})}(v_{n+i}) \cap S \neq \emptyset$, we have $\{w_1, w_2, \ldots, w_n\} \subseteq S$ where $w_i \in \{v_i, e_{i(n+i)}\}$ for $1 \leq i \leq n$. On the other hand, since $N_{T(S_{1,n,n})}(w_i) \cap S \neq \emptyset$ for $1 \leq i \leq n$, we have $|S| \geq n+1$. Now since the set $\{v_i \mid 0 \leq i \leq n\}$ is a TDS of $T(S_{1,n,n})$, we have $\gamma_{tm}(S_{1,n,n}) = \gamma_t(T(S_{1,n,n})) = n+1$. □

Lemma 4.2. *For any connected graph G of order $n \geq 2$ and any $1 \leq m \leq 2$, $\gamma_{tm}(G \circ P_m) = mn$.*

Proof. Let $G = (V, E)$ be a connected graph of order $n \geq 2$ when $V = \{v_i \mid 1 \leq i \leq n\}$.

Case 1: $m = 1$.

Then $V(G \circ P_1) = V \cup \{v_{n+i} \mid 1 \leq i \leq n\}$ and

$$E(G \circ P_1) = \{v_i v_{n+i} \mid 1 \leq i \leq n\} \cup E.$$

Let S be a min-TDS of $T(G \circ P_1)$, the total of $G \circ P_1$, in which

$$V(T(G \circ P_1)) = \{v_i \mid 1 \leq i \leq 2n\} \cup \{e_{ij} \mid v_i v_j \in E\} \cup \{e_{i(n+i)} \mid 1 \leq i \leq n\}$$

and

$$\begin{aligned} E(T(G \circ P_1)) &= E(G \circ P_1) \\ &\cup \{e_{i(n+i)} v_i, e_{i(n+i)} v_{n+i} \mid 1 \leq i \leq n\} \\ &\cup \{e_{i(n+i)} e_{ik} \mid 1 \leq i \leq n,\ v_i v_k \in E\}. \end{aligned}$$

Since $N_{T(G \circ P_1)}(v_{n+i}) = \{v_i, e_{i(n+i)}\}$ and $N_{T(G \circ P_1)}(v_{n+i}) \cap S \neq \emptyset$ for each $1 \leq i \leq n$, we have $\{w_1, w_2, \ldots, w_n\} \subseteq S$ where $w_i \in \{v_i, e_{i(n+i)}\}$, which implies $|S| \geq n$. Now since $\{v_i \mid 1 \leq i \leq n\}$ is a TDS of $T(G \circ P_1)$, we have $\gamma_{tm}(G \circ P_1) = \gamma_t(T(G \circ P_1)) = n$.

Case 2: $m = 2$.

Then $V(G \circ P_2) = \{v_i \mid 1 \leq i \leq 3n\}$ and

$$E(G \circ P_2) = \{e_{i(n+i)}, e_{(n+i)(2n+i)} \mid 1 \leq i \leq n\} \cup E.$$

Let S be a min-TDS of $T(G \circ P_2)$, the total of $G \circ P_2$, in which

$$\begin{aligned} V(T(G \circ P_2)) &= \{v_i \mid 1 \leq i \leq 3n\} \cup \{e_{ij} \mid v_i v_j \in E\} \\ &\cup \{e_{i(n+i)}, e_{(n+i)(2n+i)} \mid 1 \leq i \leq n\} \end{aligned}$$

and

$$\begin{aligned} E(T(G \circ P_2)) &= E(G \circ P_2) \\ &\cup \{e_{i(n+i)} v_i, e_{i(n+i)} v_{n+i} \mid 1 \leq i \leq n\} \\ &\cup \{e_{i(n+i)} e_{ik} \mid 1 \leq i \leq n,\ v_i v_k \in E\} \\ &\cup \{e_{(n+i)(2n+i)} v_{(n+i)}, e_{(n+i)(2n+i)} v_{(2n+i)}, \\ &\quad e_{(n+i)(2n+i)} e_{i(n+i)} \mid 1 \leq i \leq n\}. \end{aligned}$$

Since for each $1 \leq i \leq n$, $N_{T(G \circ P_2)}(v_{2n+i}) = \{v_{n+i}, e_{(n+i)(2n+i)}\}$ and $N_{T(G \circ P_2)}(v_{2n+i}) \cap S \neq \emptyset$, we have $\{w_1, w_2, \ldots, w_n\} \subseteq S$ where $w_i \in \{v_{n+i}, e_{(n+i)(2n+i)}\}$. Since also every w_i must be dominated by an element $w'_i \in N_{T(G \circ P_2)}(w_i) \cap S$, and all of the elements w_i and w'_i are distinct, we conclude that S includes the set $\{w_i, w'_i \mid 1 \leq i \leq n\}$ of cardinality $2n$, and so $\gamma_t(T(G\circ P_2)) \geq 2n$. On the other hand, since $\{v_i, v_{n+i} \mid 1 \leq i \leq n\}$ is a TDS of $T(G\circ P_2)$, we have $\gamma_{tm}(G\circ P_2) = \gamma_t(T(G\circ P_2)) \leq 2n$, which completes our proof. □

Lemma 4.3. *For any connected graph G with no isolated vertex,*

$$\chi_d^{tt}(G\circ P_m) \leq n(m+1)$$

when $m = 1, 2$.

Proof. Let $G = (V, E)$ be a connected graph with no isolated vertex of order $n \geq 2$. We continue our proof in the following two cases by using the notations in the proof of Lemma 4.2.

Case 1: $m = 1$.

Since V is a min-TDS of $T(G\circ P_1))$ (by Lemma 4.2), Theorem 1.6 implies $\chi_d^t(T(G\circ P_1)) \leq n+\chi(T(G\circ P_1)-V)$. By showing $\chi(H) \leq n$ our proof will be completed in which $H = T(G\circ P_1) - V$. Since $\chi(L(G)) = \chi'(G) \leq n$ (by Theorem 1.7), we have $|N_{T(G\circ P_1)}(e_{i(n+i)}) \cap \mathcal{E}| = deg_G(v_i) \leq n-1$ for each $1 \leq i \leq n$ in which $\mathcal{E} = \{e_{ij} \mid v_iv_j \in E\} \cup \{e_{i(n+i)} \mid 1 \leq i \leq n\}$. Thus each of the subgraphs induced by $\mathcal{E}_i = \{e_{ij} \mid 1 \leq j \leq n \text{ and } j \neq i\} \cup \{e_{i(n+i)}\}$ is a complete graph of order at most n, and so each of them can be colored by a set X_i of colors which has cardinality at most n. If need, we can color each of them in this way that the common vertices in different induced subgraphs have same colors, and so $|X_1 \cup \cdots \cup X_n| \leq n$. Now since $N_H(v_{n+i}) = \{e_{i(n+i)}\}$, we can assign a color from $X_1 \cup \cdots \cup X_n$ to the vertices in $\{v_{n+i} \mid 1 \leq i \leq n\}$, which implies $\chi(H) \leq n$, as desired.

Case 2: $m = 2$.

Since $S = \{v_i, v_{n+i} \mid 1 \leq i \leq n\}$ is a min-TDS of $T(G\circ P_2))$ (by Lemma 4.2), Theorem 1.6 implies $\chi_d^t(T(G\circ P_2)) \leq 2n+\chi(T(G\circ P_2)-S)$. Since, similar to the case $m = 1$, it can be shown that $\chi(T(G\circ P_2)-S) \leq n$, our proof is completed. □

Proposition 4.4. *For any integer $n \geq 1$, $\chi_d^{tt}(S_{1,n,n}) = 2n+1$.*

Proof. Let $S_{1,n,n}$ be a double star with vertex set $V = \{v_i \mid 0 \leq i \leq 2n\}$ and edge set $E = \{v_0v_i, v_iv_{n+i} \mid 1 \leq i \leq n\}$. Let $f = (V_1, V_2, \ldots, V_\ell)$ be a TDC of $T(S_{1,n,n})$, the total of $S_{1,n,n}$, which its vertex set is $V \cup \{e_{0i}, e_{i(n+i)} \mid 1 \leq i \leq n\}$. Since the subgraph of $T(S_{1,n,n})$ induced by $\{e_{0i} \mid 1 \leq i \leq n\} \cup \{v_0\}$ is isomorphic to a complete graph of order $n+1$, we have $\chi_d^t(T(S_{1,n,n})) \geq n+1$, and so we may assume $e_{0i} \in V_i$ for each $1 \leq i \leq n$ and $v_0 \in V_{n+1}$. Since for each $1 \leq i \leq n$, $v_{n+i} \succ_t V_k$ implies $V_k = \{e_{i(n+i)}\}$ or $\{v_i\}$ and $V_k \cap (V_1 \cup \cdots \cup V_n \cup V_{n+1}) = \emptyset$, we have $\chi_d^t(T(S_{1,n,n})) \geq 2n+1$. On the other hand, since $S' = \{v_i \mid 0 \leq i \leq n\}$ is a min-TDS of $T(S_{1,n,n})$ by Lemma 4.1, Theorem 1.6 implies $\chi_d^t(T(S_{1,n,n})) \leq 2n+1$, and so $\chi_d^{tt}(S_{1,n,n}) = \chi_d^t(T(S_{1,n,n})) = 2n+1$. Figure 4 shows the min-TDC $(\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{e_{01}, e_{36}\}, \{e_{02}, v_4, v_5, v_6\}, \{e_{03}, e_{14}, v_{25}\})$ of $T(S_{1,3,3})$ for an example. $\square$

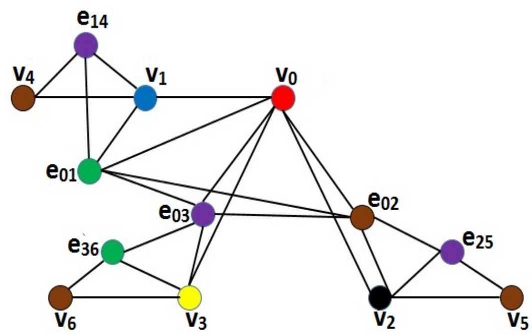

FIGURE 4. A min-TDC of $T(S_{1,3,3})$.

Proposition 4.5. *For any integers $n \geq 2$ and $1 \leq m \leq 2$, $\chi_d^{tt}(K_n \circ P_m) = n(m+1)$.*

Proof. Let $K_n = (V, E)$ be a complete graph of order $n \geq 2$ with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and let $f = (V_1, \ldots, V_\ell)$ be an arbitrary TDC of $T(K_n \circ P_m)$. By Lemma 4.3 and using the notations in its proof, it is sufficient to prove $\ell \geq n(m+1)$.

CASE 1: $m = 1$.

Since for each $1 \leq i \leq n$, $v_{n+i} \succ_t V_k$ implies $V_k \subset \{v_i, e_{i(n+i)}\}$, we may assume $V_k = \{w_i\}$ and $\{v_i, e_{i(n+i)}\} - \{w_i\} = \{w'_i\}$. Since we have to assign one color to each vertex w_i for $1 \leq i \leq n$, and we need n new colors to assign to the vertices of the complete subgraph induced by $\mathcal{E}_i = \{e_{ij} \mid 1 \leq j \leq n,\ j \neq i\} \cup \{w'_i\}$, for $1 \leq i \leq n$, we have $\ell \geq 2n$, as desired.

CASE 2: $m = 2$.

Since $v_{2n+i} \succ_t V_{k_{2n+i}}$ implies $V_{k_{2n+i}} \subset \{v_{n+i}, e_{(n+i)(2n+i)}\}$ for each $1 \leq i \leq n$, we have

$$\ell \geq |\{f(w_i) \mid V_{k_{2n+i}} = \{w_i\},\ 1 \leq i \leq n\}| = n.$$

Also since $w_i \succ_t V_{k_{w_i}}$ implies

$$V_{k_{w_i}} \subset W_i = \{v_i, v_{n+i}, v_{2n+i}, t_{i(n+i)}, t_{(n+i)(2n+i)}\} - \{f(w_i)\}$$

for each $1 \leq i \leq n$, and $W_i \cap W_j = \emptyset$ when $i \neq j$, we have $\ell \geq 2n$. Finally since at least one of the complete graphs of order n induced by $\{v_i\} \cup \{t_{ij} \mid 1 \leq i < j \leq n\}$ or by $\{t_{i(n+i)}\} \cup \{t_{ij} \mid 1 \leq i < j \leq n\}$ has no vertex in-common with $V_{k_{2n+i}} \cup V_{k_{w_i}}$, we need n new colors, which implies $\ell \geq 3n$, as desired.

$\square$

5. Problems

In this introductory paper on total dominator total coloring of a graph, we present some bounds on the parameter and some fundamental properties of the parameter and determine the total dominator total coloring of special classes of graphs. We close with a list of open problems.

Problem 5.1. *Study the total dominator total chromatic number on various graph products, including, among others, the Cartesian product, lexicographic product, direct product.*

Problem 5.2. *Study the the total dominator total chromatic number in certain classes of graphs, including, among others, chordal graphs, split graphs, block graphs, proper interval graphs, Cayley graphs, Mycieleskian graphs, and Kneser graphs.*

Problem 5.3. *Find a family of connected graphs G satisfy*

- $\chi_d^{tt}(G) = \chi_d^t(L(G)) + \chi_d^t(G)$, *or*
- $\chi_d^{tt}(G) = \chi(T(G) - (S_1 \cup S_2)) + \gamma_t(G) + \gamma_t(L(G))$ *where S_1 is a min-TDS of G and S_2 is a min-TDS of $L(G)$.*

Problem 5.4. *Characterize trees $\mathbb{T}$ of order $n \geq 5$ satisfy*

$$\chi_d^{tt}(\mathbb{T}) = \left\lfloor \frac{2n}{3} \right\rfloor + \Delta(\mathbb{T}) + 1.$$

Problem 5.5. *Whether for any connected graph G of order $n \geq 3$,*

$$\chi_d^{tt}(G) \leq \left\lceil \frac{5n}{3} \right\rceil ?$$

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