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SYMMETRIC ASSOCIATION SCHEMES ARISING FROM ABSTRACT REGULAR POLYTOPES

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Abstract. This article investigates the question of when every double coset of a string C-group G relative to its vertex stabilizer subgroup H is represented by an involution. We show that this is the case for every finite string Coxeter group except in the $\{5, 3, 3\}$ case of type H_4, and for the infinite Coxeter groups of Schläfli type $\{4, 4\}$ and $\{3, 6\}$. From this it is immediate that, for every string C-group of these types, the double coset algebra $\mathbb{C}[G/\!\!/H]$ is commutative and all of its characters are realizable over $\mathbb{R}$. In particular, the abstract regular polytopes with these automorphism groups have a polyhedral realization cone.

1. Introduction

The motivation for this article stems from Problem 23 of Schulte and Ivič-Weiss's *Problems on Polytopes* [13], which asks if there are irreducible characters of string C-groups (i.e. automorphism groups of abstract regular polytopes) that have real Schur index 2. This problem originates in the work of McMullen and Monson [8] which adapted some formulas in an earlier work of McMullen to account for the possibility that irreducible constituents of the character afforded by the geometric simplex realization of the abstract regular polytope might not be realizable over the real numbers. It was later observed that irreducible characters with imaginary fields of character values can occur as constituents of the simplex realization of the polytope (see [6] and [7]). However, the existence of irreducible characters of string C-groups that are real-valued but not realizable over the field of real numbers (i.e. of quaternion type), has remained open, and, should they exist, it is of particular interest if they could occur as constituents of the simplex realization of the corresponding polytope.

A finite *string C-group* G of Schläfli type $\{m_1, \ldots, m_r\}$ and of rank $r+1$ is a finite group $G = \langle S \rangle$ whose generating set $S = \{s_0, s_1, \ldots, s_r\}$ consists of

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$r+1$ involutions of order 2 and relations $(s_{i-1}s_i)^{m_i}=1$ for $i=1,\dots,r$ and $(s_j s_i)^2=1$ for $0\le j\le i-2\le r-2$, plus additional relations R that must guarantee finiteness and respect the *intersection condition*: $\langle I\cap J\rangle=\langle I\rangle\cap\langle J\rangle$ for all subsets $I,J\subseteq S$. We will restrict ourselves here to the case of "connected" string C-groups by assuming $m_i\ge 3$ for all $i=1,\dots,r$. To indicate the generic string C-group G with Schläfli type $\{m_1,\dots,m_r\}$ we will write $G=[m_1,\dots,m_r]$. The vertex stablizer of $G=[m_1,\dots,m_r]=\langle s_0,s_1,\dots,s_r\rangle$ is the subgroup $H=\langle s_1,\dots,s_r\rangle$. Note that H depends on the orientation of the Schläfli type of G, since the reversed type is associated with the dual polytope. The geometric structures of the realizations of the corresponding polytope are encoded in its *realization cone*, which is built up from pure realizations, one for each real irreducible character of G that occurs in the *simplex realization* of the polytope, which is obtained from the action of G on the vertices of the polytope, with the trivial representation removed by convention. As the vertices correspond to the cosets of H in G, the character associated to the simplex realization is $(1_H)^G-1_G$ (see, for example, [6]).

The starting point for this article is the observation that $(1_H)^G$ is precisely the character of G that determines the character theory of the double coset algebra $\mathbb{C}[G/\!\!/H]$ via the character theory of Hecke algebras as explained in [2, §11D]. Furthermore, this algebra is precisely the adjacency algebra of the Schurian association scheme $(G/H,G/\!\!/H)$, and $(1_H)^G$ can be identified with its standard character. Ladisch showed the essential Wythoff dimension of the pure realization afforded by a real irreducible character $\tilde{\chi}\neq 1_G$ is its multiplicity $m_{\tilde{\chi}}$ in $(1_H)^G$ [7, Lemma 3.3], and from this we can assert that the formulas for the number of vertices, diagonal classes, and Wythoff space dimension in [8] (see [9, 5B14, 5B17]) match well-known formulas for the order, number of $*$-classes of relations, and rank for these association schemes. Another consequence of [7, Lemma 3.3] and [9, 5B18] is that the realization cone of the polytope will be polyhedral if and only if $m_{\tilde{\chi}}=1$ for every real irreducible character $\tilde{\chi}$ of G occurring in $(1_H)^G-1_G$. In Section 2 we explain how this is equivalent to the property that every simple component of $\mathbb{R}[G/\!\!/H]$ has trivial matrix degree.

It is well-known that symmetric association schemes are always commutative. For double coset algebras this corresponds to having $HgH=(HgH)^{-1}=Hg^{-1}H$ for every double coset $HgH\in G/\!\!/H$. Since symmetric basis elements in finite association schemes always have real character values, it follows that every irreducible constituent of the simplex realization for the polytopes corresponding to these groups will be realizable over $\mathbb{R}$. An even stronger case of commutativity is the case where $G/\!\!/H$ is *involutive*; i.e. every nontrivial double coset HgH contains an element of order 2. It should be clear that

$$\begin{aligned} G/\!\!/H \text{ is involutive } &\implies \mathbb{C}[G/\!\!/H] \text{ is symmetric}\\ &\implies \mathbb{C}[G/\!\!/H] \text{ is commutative.}\end{aligned}$$

(Although all of the reverse implications do not hold for Schurian association schemes in general, we find it interesting we have not found examples where either fails for finite string C-groups relative to their vertex stabilizer subgroups — in every string C-group G where we have found $G/\!\!/H$ to be not involutive, we have found the corresponding $\mathbb{C}[G/\!\!/H]$ to be not commutative.) The main results of this article show that, in the case G is a string C-group and H is its vertex stabilizer subgroup, every double coset HgH is involutive when

(i) G is any finite Coxeter group except the one of Schläfli type $\{5, 3, 3\}$; and
(ii) G is any finite string C-group of Schläfli type $\{4, 4\}$ or $\{3, 6\}$.

A consequence is that, for the polytopes corresponding to string C-groups of these types, every irreducible constituent of the simplex realization will be realizable over $\mathbb{R}$, and the realization cone will be polyhedral. That the realization cone in the case of $\{5, 3, 3\}$ is not polyhedral was observed by Ladisch [7, Example 3.6]. In this case $\mathbb{C}[G/\!\!/H]$ has simple components of dimensions 4 and 9. That irreducible constituents of the simplex realization were realizable over $\mathbb{R}$ and that the realization cone was polyhedral was already noted for the polytopes of type $\{3, 6\}$ in [10] and [1], and for those of type $\{4, 4\}$ in [11]. Monson and Weiss also noted the self-inverse property for double cosets for type $\{4, 4\}$ ([11, p. 461]). We are able to treat all of these cases with a uniform approach - we show every double coset is involutive by showing that every nontrivial double coset is either

(i) represented by an element that is a *palindromic* word in the generators $s_0, s_1, \ldots, s_r$ (the same sequence of generators forwards or backwards); or
(ii) represented by an element of the form pqp^{-1}, where q is a product of commuting involutions.

In our treatment of the finite Coxeter group cases in Section 3, we also make an effort to identify the association schemes that occur. The infinite Coxeter groups are investigated in Section 4, and we also give some evidence that these two infinite cases are likely the only ones of rank 3 for which the involutive property holds.

2. A condition for the realization cone to be polyhedral

The results of [7, §3] clarify the structure of the realization cone of a finite-dimensional polytope. Ladisch's results confirm the equivalence of conditions (b) and (c) of [9, 5B18], that the realization cone of a regular polytope with finite automorphism group G and vertex stabilizer H will be polyhedral if and only if $\tilde{\chi}$ has multiplicity one in $(1_H)^G$ for every real irreducible constituent $\tilde{\chi} \neq 1_G$ of $(1_H)^G$. By [7, Lemma 3.3], this multiplicity is equal to the essential Wythoff dimension of the pure realization whose irreducible representation affords χ, and it is immediate from [7, Theorem

3.5] that the realization cone is polyhedral if and only if all of these multiplicities are 1. Our goal of this section is to interpret this condition in terms of the structure of the real adjacency algebra $\mathbb{R}[G/\!\!/H]$.

We begin with an overview of the character theory of double coset algebras. When H is a subgroup of a finite group G, 1_H^G represents the standard character of the association scheme $(G/H, G/\!\!/H)$ [4]. The correspondence between irreducible characters of the double coset algebra and the constituents of 1_H^G in $\mathrm{Irr}(G)$ is explained by the character theory of Hecke algebras (see [2] and [4]). The irreducible characters $\hat{\chi}$ of $\mathbb{C}[G/\!\!/H]$ are restrictions of the irreducible constituents χ of $(1_H)^G$ to the double coset algebra. As the identity element of the double coset algebra is $e_H := \frac{1}{|H|}\sum_{h\in H} h$, this restriction may not preserve degrees: the degree of $\hat{\chi}$ is the value $\chi(e_H)$, which is at most $\chi(1)$, and will be 0 when χ is not a constituent of $(1_H)^G$. When $((1_H)^G, \chi) \neq 0$, the degree of $\hat{\chi}$ will be equal to $((1_H)^G, \chi)$, and $\chi(1) := m_{\hat{\chi}}$ is the multiplicity of $\hat{\chi}$ in its standard representation. In particular, $\mathbb{C}[G/\!\!/H]$ is commutative if and only if $(1_H)^G$ is multiplicity free as a complex character.

The simple components of $\mathbb{R}G$, and in turn $\mathbb{R}[G/\!\!/H]$, are matrix rings over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The division algebra that occurs determines the real, complex, or quaternionic type of the real irreducible character corresponding to a given simple component. From Ladisch's results it follows that the matrix degree corresponding to a real irreducible character $\tilde{\chi}$ of G with positive multiplicity in $(1_H)^G - 1_G$ coincides with the essential Wythoff dimension for the space of pure realizations corresponding to $\tilde{\chi}$.

Commutativity of $\mathbb{C}[G/\!\!/H]$ is equivalent to the commutativity of $\mathbb{R}[G/\!\!/H]$, and the latter semisimple algebra will be commutative precisely when all of its simple components are isomorphic to $\mathbb{R}$ or $\mathbb{C}$. The adjacency algebras of symmetric association schemes are even more restricted. If $\chi \in \mathrm{Irr}(\mathbb{C}S)$ for an association scheme (X, S), it always holds that $\chi(b^*) = \overline{\chi(b)}$ for all $b \in \mathbb{C}S$ [3, Proposition 4.5]. This means that every irreducible complex character of a symmetric association scheme will be real-valued. Since the adjacency algebras of symmetric association schemes are automatically commutative, this is equivalent to the property that every simple component of $\mathbb{R}[G/\!\!/H]$ is isomorphic to $\mathbb{R}$.

This tells us the answer to Monson's question about the existence of quaternionic constituents of the simplex realization of the polytope is clearly no when the association scheme is commutative or symmetric. Condition (a) and (d) of [9, 5B18] are, on the other hand, easily seen to be equivalent to the property that our Schurian association scheme $(G/H, G/\!\!/H)$ is symmetric, which turns out to be stronger than the multiplicity condition if some of the real irreducible characters occurring in $(1_H)^G$ happen to be of complex or quaternionic type.

Proposition 2.1. *The realization cone of a polytope with automorphism group G and vertex stabilizer H is finite-dimensional and polyhedral if and* *only if* $\mathbb{R}[G/\!\!/H]$ *is finite-dimensional and every simple component of* $\mathbb{R}[G/\!\!/H]$ *is isomorphic to either* $\mathbb{R}$, $\mathbb{C}$, *or* $\mathbb{H}$.

Proof. The realization cone will be finite-dimensional if and only if H has finite index in G, which is equivalent to the double coset algebra being finite-dimensional. When the realization cone is polyhedral, every nontrivial real irreducible constituent $\tilde{\chi}$ of $(1_H)^G$ has multiplicity 1 in $(1_H)^G$. In cases (i) or (ii) above, this implies $((1_H)^G, \chi)$ will also be 1 for each complex irreducible constituent χ of $\tilde{\chi}$. This implies the simple component of $\mathbb{C}[G/\!\!/H]$ corresponding to these characters is commutative: in the first case it will be isomorphic to $\mathbb{R}$ and in the second case it will be isomorphic to $\mathbb{C}$. If $\tilde{\chi} = 2\chi$ for a quaternionic real-valued complex irreducible character χ, then $\tilde{\chi}$ has multiplicity 1 in $(1_H)^G$ when $(1_H)^G, \chi) = 2$. As there is a unique simple component of $\mathbb{R}[G/\!\!/H]$ corresponding to $\tilde{\chi}$ it must have dimension 4, and so being of quaternionic type it will be isomorphic to $\mathbb{H}$.

Conversely, if every simple component of $\mathbb{R}[G/\!\!/H]$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, then each simple component will correspond to an irreducible constituent $\tilde{\chi}$ of $(1_H)^G$ of multiplicity 1. It follows then from [7, Theorem 3.5] that the pure part of the realization cone corresponding to each nontrivial real irreducible constituent of $(1_H)^G$ has dimension 1. Therefore the realization cone is polyhedral. □

We have yet to encounter examples of string C-groups G and vertex stabilizers H where $\mathbb{R}[G/\!\!/H]$ is commutative but $(G/H, G/\!\!/H)$ is not symmetric. We remark that real-valued irreducible characters of quaternionic type are not yet known to occur in simplex realizations of polytopes, it would be even more difficult to show they could occur with the smallest possible positive multiplicity.

3. The structure constants of Schurian association schemes

Suppose H is a subgroup of a finite group G with $[G : H] = n$. Let

$$G/\!\!/H = \{H, Hg_1H, \dots, Hg_dH\}$$

be the partition of G into H-H-double cosets. Each double coset is the union of the distinct left (or distinct right) cosets of H that it contains. For subsets D of G write D^+ for the sum of the elements of D in the complex group algebra $\mathbb{C}G$. We call D^+ the characteristic function of D. When we multiply characteristic functions for two double cosets, we get a nonnegative linear combination of them. The next lemma gives a structure constant formula for Schurian association schemes that is computationally more efficient than the general formula for structure constants of quotient association schemes [15, Theorem 4.1.3]. For another treatment of structure constants for commutative double coset algebras, see [14].

Lemma 3.1. *Suppose* H *is a subgroup of a finite group* G. *Let* Hg_iH *and* Hg_jH *be a pair of* H-H-*double cosets in* G, *and suppose* $Hg_jH =$

$h_1 g_j H \cup \cdots \cup h_e g_j H$ is a partition of Hg_jH into distinct left cosets. Then

$$(Hg_iH)^+(Hg_jH)^+ = \sum_{Hg_kH \in G/\!\!/H} \frac{|Hg_iH||H|n_{ijk}}{|Hg_kH|}(Hg_kH)^+,$$

where $n_{ijk} = \#\{h_u : g_i h_u g_j \in Hg_kH\}$. This number n_{ijk} is independent of the choice of double coset representatives.

Proof. Let $Hg_iH = Hg_ib_1 \cup \cdots \cup Hg_ib_f$ and $Hg_jH = Hg_ja_1 \cup \cdots \cup Hg_ja_e$ be partitions of Hg_iH and Hg_jH into distinct right cosets. We have

$$\begin{aligned}
(Hg_iH)^+ \cdot (Hg_jH)^+ &= \textstyle\sum_{v=1}^{f} \sum_{u=1}^{e} (Hg_ib_v)^+(Hg_ja_v)^+ \\
&= \textstyle\sum_{v=1}^{f} \sum_{u=1}^{e} (H^+g_ib_v)(H^+g_ja_u) \\
&= f \textstyle\sum_{u=1}^{e} H^+g_iH^+g_ja_u \\
&= f(H^+g_i) \textstyle\sum_{u=1}^{e} H^+g_ja_u \\
&= f(H^+g_i)(Hg_jH)^+.
\end{aligned}$$

Now let $Hg_jH = h_1g_jH \cup \cdots \cup h_eg_jH$ be a partition of Hg_jH into distinct left cosets. For each double coset Hg_kH, let $n_{ijk} = \#\{h_u : g_ih_ug_j \in Hg_kH\}$. To see that this does not depend on the choice of double coset representatives, note that n_{ijk} is the number of distinct left cosets h_ug_jH in the partition of Hg_jH which are sent into Hg_kH on left multiplication by g_i. Replacing g_i by g_ix for $x \in H$ only permutes the left cosets h_ug_jH to xh_ug_jH. Since these give the same collection of left cosets h_ug_jH in a different order, the same number will be mapped into Hg_kH under left multiplication by g_i. Then

$$\begin{aligned}
(Hg_iH)^+ \cdot (Hg_jH)^+ &= f(H^+g_i)(Hg_jH)^+ \\
&= f \textstyle\sum_{u=1}^{e} H^+g_ih_ug_jH^+ \\
&= f \textstyle\sum_{Hg_kH} n_{ijk}H^+g_kH^+.
\end{aligned}$$

Finally, by counting the number of elements in the support of $H^+g_kH^+$ we see that $H^+g_kH^+ = \frac{|H|^2}{|Hg_kH|}(Hg_kH)^+$. Therefore,

$$\begin{aligned}
(Hg_iH)^+ \cdot (Hg_jH)^+ &= f \textstyle\sum_{Hg_kH} n_{ijk}H^+g_kH^+ \\
&= \sum_{Hg_kH} \frac{|Hg_iH|n_{ijk}|H|^2}{|H||Hg_kH|}(Hg_kH)^+ \\
&= \sum_{Hg_kH} \frac{|Hg_iH||H|n_{ijk}}{|Hg_kH|}(Hg_kH)^+.
\end{aligned}$$

This proves the lemma. □

Borrowing some notation from association schemes, we will write $Hg_{k^*}H = Hg^{-1}H$, and identify the double coset H with Hg_0H. Considering $G/\!\!/H$ as the quotient association scheme of G modulo H, the standard basis of $\mathbb{C}[G/\!\!/H]$ will be the set of normalized characteristic functions $\{\frac{1}{|H|}(Hg_iH)^+ : Hg_iH \in G/\!\!/H\}$ [4, Proposition 3.4]. The *valency* of each standard basis element $b_i := \frac{1}{|H|}(Hg_iH)^+$ is the number of left cosets of H contained in Hg_iH.

By Lemma 3.1, the structure constants of $\mathbb{C}[G/\!\!/H]$ in the standard basis are given by

$$\lambda_{ijk} = \frac{|Hg_iH|n_{ijk}}{|Hg_kH|}$$

where n_{ijk} is the number of distinct left cosets $h_u g_j H$ in a left coset partition of Hg_jH such that $Hg_ih_ug_jH = Hg_kH$.

4. The involutive double coset property for finite string Coxeter groups

In the case $G = \langle s_0, s_1, \ldots, s_r\rangle$ is a finite string C-group of rank $r+1$ and $H = \langle s_1, \ldots, s_r\rangle$ is its vertex stabilizer subgroup, it is clear that for every nontrivial double coset $HgH \in G/\!\!/H$,

(i) HgH can be represented by a word g that starts and ends with s_0, and
(ii) if a reduced word presentation of g contains k s_0's, then k is the least positive integer for which $(HgH)^+$ appears with nonzero coefficient in the decomposition of $((Hs_0H)^+)^k$.

The latter makes it possible to inductively generate all double coset representatives that require $k+1$ s_0's from those that contain k s_0's. All of the new double coset representatives will be the old ones appended on the left by *left prefixes*: left-reduced words of the form $s_0s_1...-$ that contain only one s_0. (The reviewer has remarked this observation reminds one of the circuit criterion for polytopes, see [9, 2F3, 2F4].)

Our goal for this section is to determine all cases among finite string Coxeter groups G where all double cosets relative to the vertex stabilizer subgroup H are involutive. Where possible we identify the finite association scheme $G/\!\!/H$ in terms of its identification in the database of small association schemes, its intersection array, or as an extension of smaller schemes. Where convenient in what follows, we will write $a := s_0$, $b := s_1$, $c := s_2$, etc.

Example 4.1. *Type I.* Let $G = [m] = \langle a, b\rangle$ be the finite dihedral group of order $2m$ in its Coxeter presentation in terms of two noncommuting reflections. The vertex stabilizer subgroup $H = \langle b\rangle$ has order 2, so $[G : H] = m$. Suppose $|G/\!\!/H| = h$. Since the only possible left prefix is $ab-$ and $(ab)^m = 1$, the list of distinct double cosets is

$$H, HaH, H(ab)aH, \ldots, H(ab)^{h-2}aH,$$

where $\frac{m+1}{2} \le h \le \frac{m}{2}+1$. This is because the first time $k \ge \frac{m}{2}-1$, we have

$$H(ab)^k aH = H(ba)^{m-k}aH = Hb(ab)^{m-k-2}abaaH = H(ab)^{m-k-2}aH,$$

so the double coset is represented by a word with fewer a's. It is clear that each of these double coset representatives is palindromic, so $G/\!\!/H$ is involutive. Furthermore, the valency of $H(ab)^kaH$ will be 2 unless m is even and $h = \frac{m}{2}+1$, in which case the valency of $H(ab)^{h-2}aH$ will be 1.

Note that this association scheme agrees with the one generated by an m-polygon (i.e. an undirected m-cycle). This is precisely what we obtain when we consider the underlying graph of the 1-skeleton of this polytope.

Example 4.2. *Type A.* Let $G = [3, 3, 3, \ldots, 3] = \langle s_0, s_1, \ldots, s_r\rangle$ be the string Coxeter group of type A and rank $r+1$, and let H be the corresponding vertex stabilizer subgroup. Since $s_{i-1}s_is_{i-1} = s_is_{i-1}s_i$ for $i = 1, \ldots, r$, the list of possible left prefixes consists only of the increasing strings

$$s_0s_1-, s_0s_1s_2-, s_0s_1s_2s_3-, \ldots, s_0s_1 \cdots s_r - .$$

But when any of these is appended on the left to s_0 to try to produce a double coset representative with two occurrences of s_0 in its reduced word, we get

$$H(s_0s_1 \cdots s_k)s_0H = Hs_0s_1s_0H = Hs_1s_0s_1H = Hs_0H.$$

This means H and Hs_0H are the only two double cosets in $G /\!/ H$. Since $G \simeq S_{r+2}$, we have that $[G : H] = r+2$, and so the valency of Hs_0H is $r+1$. This is the association scheme K_{r+2}, the association scheme corresponding to the complete graph on $r + 2$ vertices. This again agrees with the $r + 2$-simplex graph that underlies the 1-skeleton of this polytope. That this has a unique pure realization was noted in [9, §5B]. This is also immediate from the fact that $|G /\!/ H| = 2$.

Remark: The double coset algebra we considered for type A is the adjacency algebra for association scheme generated by the Johnson graph $J(n, 1)$. In general the double coset algebras of $G /\!/ H_k$ where $H_k = \langle S \setminus \{s_{k-1}\}\rangle$ for $k - 1 = 0, 1, \ldots, r$ agree with the adjacency algebras for the association schemes generated by the Johnson graphs $J(n, k)$. $J(n, k)$ is also the underlying graph that would be associated with our double coset $H_ks_{k-1}H_k$ in the standard representation of the association scheme. It was earlier observed that the Johnson graph $J(n, k)$ is the graph associated with the vertices and edges of the hypersimplex polytope $\Delta_{n,k}$ (see [12]).

Example 4.4. *Type B,* $m_1 = 3$. Let $G = [3, \ldots, 3, 4] = \langle s_0, s_1, \ldots, s_r\rangle$, and let $H = \langle s_1, \ldots, s_r\rangle$. We have that $[G : H] = 2r + 2$, and the list of left prefixes is

$$s_0s_1-, s_0s_1s_2-, \ldots, s_0s_1 \cdots s_{r-1}s_r-, s_0s_1 \cdots s_{r-1}s_rs_{r-1}-,$$
$$\ldots, s_0s_1 \cdots s_{r-1}s_rs_{r-1} \cdots s_1 - .$$

Using this we can show, for all ranks, there are only three distinct double cosets: H, Hs_0H, and $Hs_0s_1 \cdots s_{r-1}s_rs_{r-1} \cdots s_1s_0H$, with valencies 1, $2r$, and 1, respectively. Since these double coset representatives are palindromic, $G /\!/ H$ is involutive. These association schemes are the wreath products $C_2 \wr K_{r+1}$. This Schläfli type is associated with the $(r+2)$-cross polytopes (a.k.a. $(r+2)$-orthoplexes). That they have two pure realizations was noted in [9, §5B].

We will leave the dual case of type B, $m_1 = 4$ to the end, as the association schemes in that case have unbounded dimension.

Example 4.5. *Type F_4.* Let $G = [3,4,3] = \langle a,b,c,d \rangle$, and let $H = \langle b,c,d \rangle$. Then $[G:H] = 24$. The list of possible left prefixes is

$$ab-, abc-, abcb-, abcd-, abcdb-, abcdbc-, \text{ and } abcdbcb- .$$

(Recall that these need only be left-reduced.) By appending these to a we get two new double coset representatives with two a's: $H\,abcb\,aH$ and $H\,abcdbcb\,aH$. The first of these representatives is palindromic, the second is involutory since it is equal to $(abc)db(cba)$. When we append each left prefix to these two representatives to try to make one with three a's, we find we can cancel an a except in the case of $H(abcdbcb)(abcdbcb)aH$. This representative is also reduced and palindromic. The valencies of H, HaH, $HabcbaH$, $HabcdbcbaH$ and $HabcdbcbabcdbcbaH$ are 1, 8, 6, 8, and 1, respectively, so we know these are all of the double cosets in $G/\!\!/H$. Therefore, $G/\!\!/H$ is involutive.

Remark: The association scheme we generated is denoted by `as24no42` in the classification of small association schemes [5]. While this association scheme is neither P- nor Q-polynomial, its adjacency algebra is isomorphic to the polynomial algebra generated by the matrix

$$b_1 = \begin{bmatrix} 0 & 8 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 0 & 4 & 0 & 4 & 0 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 8 & 0 \end{bmatrix}$$

associated with our double coset HaH, and this matrix can be used to write the other elements of the table algebra basis as polynomials in b_1. Note that this matrix realization of b_1 depends on our ordering of the double cosets. The graph underlying the 1-skeleton of the regular polytope of type F_4 is the 8-regular graph associated with the 24-cell, and this corresponds to our double coset HaH. It is interesting to note that the other basis element of degree 8 does not generate $\mathbb{C}[G/\!\!/H]$ in this case, in fact b_1 is the only standard basis element that generates the algebra.

Example 4.7. *Type H_3, $m_1 = 3$.* Let $G = [3,5] = \langle a,b,c \rangle$. There are 12 left cosets of $H = \langle b,c \rangle$ in G. The list of possible left prefixes is

$$ab-, abc-, abcb-, \text{ and } abcbc- .$$

Working inductively from HaH, we find just one double coset $H\,abcb\,aH$ represented by a reduced palindromic word with two a's, and one more double coset $Habcbc\,abcb\,aH$ represented by a palindromic word with three a's. The valencies of H, HaH, $HabcbaH$, and $HabcbcabcbaH$ are 1, 5, 5, and 1, respectively. So $G/\!\!/H$ is involutive. This association scheme appears as `as12no9` in [5], it is the distance regular graph with intersection array $[5,4,1;1,4,5]$. The polytope of type $\{3,5\}$ is the icosahedron, and its underlying graph is exactly this distance-regular graph. That it has three pure realizations was noted in [9, §5B].)

Example 4.8. *Type* H_3*,* $m_1 = 5$. Let $G = [5,3] = \langle a,b,c\rangle$ and $H = \langle b,c\rangle$. This time we find 20 left cosets of H in G, and the list of left prefixes is just $ab-$ and $abc-$. Working from HaH, we can produce one new double coset $HabaH$ with two a's, and one $HabcabaH$ with three a's. Appending $ab-$ after an $abc-$ will result in $Hab\,abc\,ab\ldots aH = Hbababa\,c\,ab\ldots aH = Hababcb\ldots aH$, so two a's will be cancelled. So any additional double coset representatives result from the prefix $abc-$. We get one $H(abc)(abc)abaH$ with four a's, one $H(abc)(abc)(abc)abaH$ with five a's but that is all, because the respective valencies of the six double cosets H, HaH, $HabaH$, $Hab\,ca\,baH$, $HabcabcabaH$, and $HabcabcabcabaH$ are 1, 3, 6, 6, 3, and 1. The first three are palindromic, the fourth is involutory. To see that the last two are involutory as well, note that

$$HabcabcabaH = H\,abac\,b\,cabaH$$

is represented by a palindrome, and

$$HabcabcabcabaH = H\,abacb\,ca\,bcaba\,H.$$

(This association scheme is P-polynomial with intersection array

$$[3,2,1,1,1;1,1,1,2,3].$$

It appears as `as20no35` in [5].

On the other hand, the regular polytope of type $\{5,3\}$ is that of the dodecahedron, and its underlying graph coincides with the distance-regular graph associated with this intersection array. That there are 5 pure realizations in this case was noted in [9, §5B].)

Example 4.9. *Type* H_4*,* $m_1 = 3$. Let $G = [3,3,5] = \langle a,b,c,d\rangle$ and $H = \langle b,c,d\rangle$. This time $[G:H] = 120$, and the list of possible left prefixes is:

$$ab-, abc-, abcd-, abcdc-, abcdcb-, abcdcd-, abcdcdb-, abcdcdbc-,$$
$$abcdcdbcd-, abcdcdbcdc-, \text{ and } abcdcdbcdcb- .$$

Starting with HaH, we can inductively produce nine double cosets:

$$\begin{aligned} G/\!\!/H \quad &= \{H, HaH, H\,abcdcb\,aH, H\,abcdcdbcdcb\,aH,\\ &H\,abcdcbdcd\,abcdcb\,aH,\\ &H\,abcdcbdcdc\,abcdcdbcdcb\,aH,\\ &H\,abcdcbdcdc\,abcdcbdcd\,abcdcb\,aH,\\ &H\,abcdcdbcdc\,abcdcbdc\,abcdcdbc\,abcdcb\,aH,\\ &H\,abcdcdbcb\,abcdcdbcdcb\,abcdcdbcd\,abcdcb\,aH\}. \end{aligned}$$

Their valencies are 1, 12, 20, 12, 30, 12, 20, 12, and 1, respectively. We have verified that each of these double cosets is involutive. This association scheme is not P-polynomial, but it is polynomial - the left regular matrix associated with the standard basis element $b_1 = \frac{1}{|H|}(HaH)^+$ with respect to our ordering of basis elements is

$$\begin{bmatrix} 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 4 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \end{bmatrix}$$

and the rational adjacency algebra of the association scheme is isomorphic to the ring of matrix polynomials $\mathbb{Q}[b_1]$. The basis element b_6 corresponding to the sixth double coset listed above also generates $\mathbb{C}[G/\!\!/H]$ in this case. The underlying graph of the 1-skeleton the regular polytope of type $\{3,3,5\}$ is the 12-regular graph with 120 vertices associated with the 600-cell. Ladisch used the character multiplicities to deduce that the realization cone will be polyhedral in this case [7].

Example 4.10. *Type H_4, $m_1 = 5$.* This is the only finite Coxeter group for which $G/\!\!/H$ produces a noncommutative association scheme. When $G = [5,3,3] = \langle a,b,c,d\rangle$ and $H = \langle b,c,d\rangle$, we find that $[G:H] = 600$, and $|G/\!\!/H| = 45$. $G/\!\!/H$ has 9 nonself-inverse pairs of double cosets. $(1_H)^G$ is not multiplicity-free — it has fifteen irreducible constituents of multiplicity 1, three of multiplicity 2, and one of multiplicity 3. (One of the irreducible characters of multiplicity 2 occurring here is the restriction of the unique noncuspidal irreducible character of G of degree 48. This character is unique among characters of finite Coxeter groups, as it has local Schur index 2 at precisely the primes 2 and 3, it has rational Schur index 2 and real Schur index 1.) Ladisch used the same character theoretic approach to show the realization cone in this case is the unique nonpolyhedral one among the finite Coxeter groups [7, Example 3.6].

The underlying graph of the 1-skeleton of the regular polytope of type $\{5,3,3\}$ is the 4-regular graph with 600 vertices associated with the 120-cell. This graph corresponds to the HaH double coset in our association scheme. Being noncommutative, $\mathbb{C}[G/\!\!/H]$ is not generated as an algebra by the single basis element $(HaH)^+$. In fact the unital subalgebra of $\mathbb{C}[G/\!\!/H]$ generated by $(HaH)^+$ has dimension 27. Our computer calculations show $\mathbb{C}[G/\!\!/H]$ is generated by the pair of double cosets HaH and $HabcabadcbaH$.

Example 4.11. *Type B, $m_1 = 4$.* Let $G = [4,3,\dots,3] = \langle s_0, s_1, \dots, s_r\rangle$, and let $H = \langle s_1,\dots,s_r\rangle$. The list of left prefixes is

$$s_0s_1-, s_0s_1s_2-, \dots, s_0s_1\cdots s_r - .$$

Working from Hs_0H, $Hs_0s_1s_0H$ gives a new coset, and all of the others give the same one. After that $Hs_0s_1s_2s_0s_1s_0H$ is a new one with three s_0's, and again all of the other prefixes do not give new double cosets. We claim that the double cosets with k s_0's are represented by

$$H(s_0 s_1 \cdots s_{k-1})(s_0 s_1 \cdots s_{k-2}) \cdots (s_0 s_1) s_0 H.$$

This representative is reduced. If the left prefix is $s_0 s_1 \cdot s_\ell$ with $\ell > k-1$, we can move the $s_k \cdots s_\ell$ on the left past everything to the right and reduce to the given form. If the left prefix is $s_0 s_1 \cdots s_\ell-$ with $\ell < k-1$, then the first two prefixes reduce to

$$\begin{aligned}
&(s_0 s_1 \cdots s_\ell)(s_0 s_1 \cdots s_{\ell-1} s_\ell \cdots s_{k-2}) s_0 s_1 - \\
&= (s_0 s_1 \cdots s_{\ell-2}) s_0 s_1 \cdots s_{\ell-1} s_{\ell-2} s_{\ell-1} s_\ell \cdots s_{k-2} s_0 s_1 - \\
&= (s_0 s_1 \cdots s_{\ell-3}) s_0 s_1 \cdots s_{\ell-2} s_{\ell-1} s_{\ell-2} s_{\ell-1} \cdots s_{k-2} s_0 s_1 - \\
&\vdots \\
&= s_0 s_1 s_0 (s_2 s_1 s_2) \cdots s_{k-2} s_0 s_1 - \\
&= s_0 s_1 s_0 s_1 s_2 s_1 s_3 \cdots s_{k-2} s_0 s_1 - \\
&= s_1 s_0 s_1 s_0 s_2 s_1 s_3 \cdots s_{k-2} s_0 s_1 - \\
&= s_1 s_0 s_1 s_2 s_3 \cdots s_{k-2} s_0 s_1 s_0 s_1 - \\
&= s_1 s_0 s_1 s_2 s_3 \cdots s_{k-2} s_1 s_0 s_1 s_0 -,
\end{aligned}$$

and the s_0 on the right will cancel with the next s_0 on the right. So there is a unique double coset whose reduced representative contains k s_0's, for $k = 1, \ldots, r$. So $|G /\!\!/ H| = r + 1$. A similar calculation shows that each of these double cosets has the form pqp^{-1} where q is a product of commuting involutions, so $G /\!\!/ H$ is involutive.

The underlying graph of the 1-skeleton the regular polytope of type $\{4, 3, \ldots, 3\}$ and rank $(r+1)$ is the $(r+1)$-dimensional hypercube. Again for this type, our association schemes are generated by the basis elements corresponding to these graphs, as $[G : H] = 2^r$, $|G /\!\!/ H| = r + 1$, and the elements have valencies $\binom{r}{k}$ for $k = 0, \ldots, r$. These association schemes are P- and Q-polynomial and have intersection array $[r, r-1, \ldots, 1; 1, 2, \ldots, r]$.

Question. *If an abstract regular polytope with automorphism group G and vertex stabilizer H has a polyhedral realization cone, will the underlying graph of its 1-skeleton generate $(G/H, G /\!\!/ H)$ as a polynomial association scheme?*

5. The involutive double coset property for infinite Coxeter groups

If G is an infinite string Coxeter group of type $\{m_1, \ldots, m_r\}$ with vertex stabilizer H which has the property that every double coset HgH is involutive (or self-inverse), then this property will hold in every finite homomorphic image. This is of interest to our problem because this will imply the double coset algebra of the resulting Schurian association scheme is involutive (or symmetric) and hence it will be commutative and all of its irreducible characters will be realizable over $\mathbb{R}$.

Given our characterization of the finite Coxeter groups for which $G/\!\!/H$ is involutive, it makes sense to check the involutive property for the infinite Coxeter groups obtained by increasing one of the parameters in the type by one. The $\{5,3\}$ to $\{5,3,3\}$ example shows adding a 3 at the end may not preserve the involutive double cosets property. So the list of infinite Coxeter groups of rank 3 or 4 that we would like to check are the ones of type $\{4,4\}$, $\{6,3\}$ and $\{3,6\}$, $\{4,4,3\}$ and $\{3,4,4\}$, $\{4,3,4\}$, $\{3,5,3\}$, and possibly $\{3,3,6\}$, $\{4,3,5\}$, and $\{3,4,5\}$. It would also make sense to check the rank 5 Coxeter group of types $\{3,4,3,3\}$ and $\{3,3,4,3\}$ obtained by adding one to a middle parameter of the rank 5 type A Coxeter group, but we have been unable to settle these rank 5 cases.

Some of these cases can be eliminated quickly by finding noninvolutive double cosets:

Proposition 5.1. *The following double cosets of $G/\!\!/H$ are noninvolutive when G is of the type given and H is the vertex stabilizer subgroup:*

(i) $H\,abcd\,abcb\,aH$ *if G has type* $\{4,4,3\}$;
(ii) $H\,abcdcb\,abcdbcb\,aH$ *if G has type* $\{3,4,4\}$; *and*
(iii) $H\,abcbcd\,abcb\,aH$ *if G has type* $\{3,5,3\}$;

Another sieve technique one can use to eliminate cases is to find finite string C-groups G of a given type for which the character $(1_H)^G$ induced from the vertex stabilizer H is not multiplicity-free. We will write $\{m_1,m_2\}_\ell$ for the rank 2 string C-group of type $\{m_1,m_2\}$ with the additional string order relation $(abc)^\ell = 1$.

Proposition 5.2. *$G/\!\!/H$ is noncommutative for the following finite string C-groups G when H is the vertex stabilizer subgroup:*

(i) *the string C-group* $\{6,3\}_6$;
(ii) *the string C-group* $\{4,5\}_6$;
(iii) *the string C-group* $\{5,4\}_8$; *and*
(iv) *the string C-group* $\{3,7\}_8$.

By (i), we can eliminate type $\{6,3\}$ from our list. Once we show the groups of type $\{4,4\}$ and $\{3,6\}$ have involutive double cosets with respect to the vertex stabilizer, parts (ii), (iii), and (iv) tell us we should not expect to find more of these with rank 3.

Theorem 5.3. *Let G be a finite string C-group of type $\{4,4\}$, and let H be its vertex stabilizer subgroup. Then every double coset of $G/\!\!/H$ is involutive.*

Proof. It suffices to show the involutive property holds for every double coset of $H = \langle b,c\rangle$ in the infinite Coxeter group $G = \langle a,b,c\rangle$, since this property will be inherited by every finite homomorphic image of G.

To do this, we show nontrivial double cosets of H in G fall into three infinite families, then show that each of these infinite families is represented by an element of order 2. Let n be the number of a's occurring in a reduced double coset representative as a word in the defining generators a, b, and c.

We claim that every nontrivial double coset of H in G lies in one of these three infinite families:

F_1: $H(abcb)^{n-1}(a)H$;
F_2: $H(abc)^{n-2}(aba)H$; and
F_3: $H(abcb)^{n-k-2}(abc)^k(aba)H$, where k is odd and $< n-2$.

Since G is of type $\{4,4\}$, the possible left prefixes in this case are: $ab-$, $abc-$, and $abcb-$. Starting from HaH, we can get two new double cosets $HabaH$ and $HabcbaH$ whose representatives require two a's. Appending $abcb-$ on the left preserves the families F_1 and F_3. Appending $abc-$ on the left preserves the familiy F_2. Appending $abcb-$ to a member of family F_2 with an odd number of a's will result in a member of F_3. Since

$$\begin{aligned}(abc)(abc)b &= (aba)(cbcb)\\ &= (aba)(bcbc)\\ &= (abab)(cbc)\\ &= (baba)(cbc)\\ &= b(abc)(abc),\end{aligned}$$

we have that b commutes with $(abc)^2$. This implies that when we append $abcb-$ to a member of family F_2 that has an even number of a's, the result will be equivlaent to the member of F_2 with one additional a. If we append $abc-$ on the left to a double coset in family F_1 with n a's, we get

$$\begin{aligned}H\,(abc)\,(abcb)^{n-1}\,aH &= H\,(abc)\,(abcb)\,(abcb)^{n-2}\,a\,H\\ &= H\,(abc)^2\,(abcb)^{n-2}\,aH\\ &= H\,(abcb)\,(abc)^2\,(abcb)^{n-3}\,aH\\ &= H\,(abcb)^2\,(abc)^2\,(abcb)^{n-4}\,aH\\ &\vdots\\ &= H\,(abcb)^{n-2}\,(abc)^2\,aH\\ &= H\,(abcb)^{n-2}\,(abc)\,(aba)H,\end{aligned}$$

which is a member of F_3. If we append $abc-$ to a double coset in family F_3 with n a's, a similar calculation shows

$$H\,(abc)\,(abcb)^{n-k-2}\,(abc)^k\,aH = H\,(abcb)^{n-k-3}\,(abc)^{k+2}\,(aba)H,$$

so the result will be in F_3.

We leave it to the reader to show that appending $ab-$ preserves these families and does not produce a new double coset that is not already obtained by appending $abc-$ or $abcb-$.

Finally, we need to show every double coset in these three families is involutive. It is easy to see that the given representatives of members of family F_1 are palindromic. For family F_2, we have

$$H(abc)(aba)H = H(ab)ca(ba)H$$

and

$$H(abc)(abc)(aba)H = H(abac)b(caba)H.$$

When $m > 2$,

$$H(abc)^m(aba)H = H(abc)(abc)^{m-2}(aba)(cba)H$$

so by induction we can conclude $H(abc)^m(aba)H$ has the form $Hpqp^{-1}H$ with

$$q = \begin{cases} ca & m \text{ odd} \\ b & m \text{ even} \end{cases}.$$

For family F_3, we have four base cases:

$$\begin{aligned}
H(abcb)(abc)(aba)H &= H(abc)baba(cba)H, \\
H(abcb)^2(abc)(aba)H &= H(abc)(babcbaba)(cba)H \\
&= H(abc)(bab)ca(bab)(cba)H, \\
H(abcb)(abc)^3(aba)H &= H(abc)(baba)(cbc)(aba)(cba)H \\
&= H(abc)(aba)bcbc(aba)(cba)H, \text{ and} \\
H(abcb)^2(abc)^3(aba)H &= H(abcb)(abc)^3(abc)b(aba)(cba)H \\
&= H(abcb)(abc)^3(abc)(aba)(bcba)H \\
&= H(abcb)(aba)(cbca)b(acbc)(aba)(bcba)H.
\end{aligned}$$

If k is odd and > 4, then for all $m \geq 1$, we can use the fact that b commutes with $(abc)^2$ to show

$$H(abcb)^m(abc)^k(aba)H = H(abc)^2(abcb)^m(abc)^{k-4}(aba)(cba)^2H.$$

Also for all $m \geq 1$, we have that

$$\begin{aligned}
H(abcb)^m(abc)(aba)H &= H(abcb)(abcb)^{m-2}(abc)(baba)(cba)H \\
&= H(abcb)(abcb)^{m-2}(abc)(aba)(bcba)H,
\end{aligned}$$

and

$$\begin{aligned}
H(abcb)^m(abc)^3(aba)H &= H(abcb)(abcb)^{m-2}(abc)^3baba(cba)H \\
&= H(abcb)(abcb)^{m-2}(abc)^3(aba)(bcba)H.
\end{aligned}$$

These identities allow us to always reduce the problem to one of the base cases. So each double coset $H(abcb)^{n-k-2}(abc)^k(aba)H$ in the family F_3 is equal to one of the form $Hpqp^{-1}H$ with

$$q = \begin{cases} abab & n \text{ even}, k \equiv 1 \mod 4 \\ ca & n \text{ even}, k \equiv 1 \mod 4 \\ bcbc & n \text{ odd}, k \equiv 3 \mod 4 \\ b & n \text{ odd}, k \equiv 3 \mod 4 \end{cases}.$$

Therefore, every double coset of $H = \langle b, c\rangle$ in $G = \langle a, b, c\rangle$ is involutive when G is the infinite Coxeter group of type $\{4, 4\}$. The same will be true for any finite string C-group of type $\{4, 4\}$. This completes the proof of the theorem. $\square$

Theorem 5.4. *Let G be a finite string C-group of type $\{3,6\}$, and let H be its vertex stabilizer subgroup. Then every double coset of $G/\!/H$ is involutive.*

Proof. We use the same approach as in the previous theorem. This time the possible left prefixes are $abc-$, $abcb-$, $abcbc-$, and $abcbcb-$. We get two double cosets with two a's: $HabcbaH$ and $HabcbcbaH$. After that appending $abc-$ or $abcb-$ will not increase the number of a's, since $Habc\,abc- = Haba\,cbc- = Hbab\,cbc- = Habcbc-$ and $Habcb\,abcb- = Habcabacb- = Habacbcab- = Habcbcab-$. Furthermore, b commutes with $(abcbc)^2$, since

$$\begin{aligned} b(abcbc)(abcbc) &= (bab)cbcabcbc = ab(ac)b(ca)bcbc \\ &= abc(aba)cbcbc = abc(bab)cbcbc \\ &= abcba(bcbcbc) = abcba(cbcbcb) \\ &= (abcbc)(abcbc)b. \end{aligned}$$

As in the previous theorem, this identity allows us to again sort the double cosets into three infinite families:

F_1: $H(abcbcb)^{n-1}aH$;
F_2: $H(abcbc)^{n-2}(abcba)H$; and
F_3: $H(abcbcb)^{n-k-2}(abcbc)^k(abcba)H$, where k is odd and $< n-2$.

Finally, we claim all of these double cosets are involutive. The given representatives of F_1 are palindromic, so we need to show all members of F_2 and F_3 are equivalent to an $Hpqp^{-1}H$ with q of order 2. For family F_2, the identity

$$H(abcbc)^m(abcba)H = H(abcbc)(abcbc)^{m-2}(abcba)(cbcba)H$$

for all $m > 2$ allows us to reduce to one of two base cases, depending on whether m is even or odd: $m = 0$: $H(abcba)H$, which is pallidromic, and $m = 1$: $H(abcbc)(abcba)H = H(abcb)(ca)(bcba)H$.

For the family F_3, when $k > 4$ and $m \geq 2$ we can use the fact that b commutes with $(abcbc)^2$ to establish the identity

$$\begin{aligned} &H(abcbcb)^m(abcbc)^k(abcba)H \\ &= H(abcbc)^2(abcbcb)^m(abcbc)^{k-4}(abcba)(cbcba)^2H, \end{aligned}$$

so we can assume $k = 1$ or 3. Now consider, for $m \geq 2$,

$$\begin{aligned} &H(abcbcb)^m(abcbc)(abcba)H \\ &= H(abcbcb)^{m-1}(abcbc)babcbc(abcba)H \\ &= H(abcbcb)^{m-1}(abcbc)abacba(cbcba)H \\ &= H(abcbcb)^{m-1}(abcbc)abcbab(cbcba)H \\ &= H(abcbcb)(abcbcb)^{m-2}(abcbc)(abcba)(bcbcba)H. \end{aligned}$$

Also, for $m \geq 2$,

$$\begin{aligned} &H(abcbcb)^m(abcbc)^3(abcba)H \\ &= H(abcbcb)^{m-1}(abcbc)^3b(abcbc)(abcba)H \\ &= H(abcbcb)^{m-1}(abcbc)^3abacba(cbcba)H \\ &= H(abcbcb)(abcbcb)^{m-2}(abcbc)^3(abcba)(bcbcba)H, \end{aligned}$$

so the problem is reduced to one of four base cases:

$$\begin{aligned} H(abcbc)(abcba)H &= H(abcb)ca(bcba)H, \\ H(abcbc)^3(abcba)H &= H(abcba)(cbcb)ca(bcbc)(abcba)H, \\ H(abcbcb)(abcbc)(abcba)H &= H(abcbc)(ab)ac(ba)(cbcba)H, \text{ and} \\ H(abcbcb)(abcbc)^3(abcba)H &= H(abcbc)^3babcba(cbcba)H \\ &= H(abcbcabc)bcabcbcbab(cbacbcba)H \\ &= H(abcbcabcba)cbcbcb(abcbacbcba)H. \end{aligned}$$

Therefore, all of the double cosets in $G/\!\!/H$ are involutive. The same will be true of any homomorphic image of G. This completes the proof that any string C-group of type $\{3,6\}$ will have involutive double cosets. □

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