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REFINING OVERPARTITIONS BY PROPERTIES OF NONOVERLINED PARTS

A. M. ALANAZI, B. M. ALENAZI, W. J. KEITH, AND A. O. MUNAGI

Abstract. We study new classes of overpartitions of numbers based on the properties of nonoverlined parts. Several combinatorial identities are established by means of generating functions and bijective proofs. We show that our enumeration function satisfies a pair of infinite Ramanujan-type congruences modulo 3. Lastly, by conditioning on the overlined parts of overpartitions, we give a seemingly new identity between the number of overpartitions and a certain class of ordinary partition functions. A bijective proof for this theorem also includes a partial answer to a previous request for a bijection on partitions doubly restricted by divisibility and frequency.

1. Introduction

An overpartition of a positive integer n is a partition of n, where the first occurrence of each part-size may be overlined. Overpartitions generalize ordinary partitions. We denote the number of overpartitions of n by $\overline{p}(n)$, with $\overline{p}(0) = 1$. For example, $\overline{p}(3) = 8$ enumerates the following overpartitions:

$$(3),\ (\overline{3}),\ (2,1),\ (\overline{2},1),\ (2,\overline{1}),\ (\overline{2},\overline{1}),\ (1,1,1),\ (\overline{1},1,1).$$

The three overpartitions with no overlined parts are the ordinary partitions of 3.

Given a positive integer ℓ, a partition λ is called ℓ-regular if no part of λ is divisible by ℓ.

Munagi and Sellers [6] studied combinatorial and arithmetic properties of overpartitions when the overlined parts are ℓ-regular. Alanazi and Munagi [1] later investigated certain combinatorial identities satisfied by ℓ-regular overpartitions. Some other authors have also investigated the arithmetic properties of related overpartition functions (see for example [2, 7, 8, 9, 10]).

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Let $\overline{R_\ell^*}(n)$ denote the number of overpartitions of n where nonoverlined parts are ℓ-regular, and denote by $A_\ell(n)$ the number of overpartitions of n in which the overlined parts are ℓ-regular.

The generating functions are given by

$$\sum_{n=0}^{\infty} \overline{R_\ell^*}(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{n\ell})(1+q^n)}{(1-q^n)}, \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} A_\ell(n) q^n = \prod_{n=1}^{\infty} \frac{(1+q^n)}{(1-q^n)(1+q^{n\ell})}. \tag{1.2}$$

Expectedly, there is a sort of symmetry between these two enumeration functions, as the following assertion shows.

Theorem 1.1. *We have*

$$\begin{cases} A_\ell(n) = \overline{R_\ell^*}(n) & \textit{if} \quad n < 2\ell, \\ A_\ell(n) > \overline{R_\ell^*}(n) & \textit{if} \quad n \geq 2\ell. \end{cases}$$

In section 2 we give two proofs of Theorem 1.1, and consider several identities between $\overline{R_\ell^*}(n)$ and different restricted partition functions. In section 3 we establish some related congruences modulo 3 using the Ramanujan theta functions. In section 4 we discuss the parity of $A_\ell(n)$, and prove a special identity that it satisfies (Theorem 4.2) which seems to be new. In that section, we also give a second, bijective proof, which includes a partial answer to a question posed in [5].

2. Combinatorial Identities for $\overline{R_\ell^*}(n)$

We will use the following special notation throughout this paper: if $G(n)$ is an enumeration function, the corresponding enumerated set will be denoted by $G[n]$, where G represents any letter symbol.

Proof of Theorem 1.1. We give two proofs.
First proof. If we multiply the right-hand-side of Equation (1.2) by

$$\prod_{m=1}^{\infty} (1-q^{2\ell m}),$$

we obtain

$$\prod_{m=1}^{\infty} \frac{(1+q^m)(1-q^{2\ell m})}{(1-q^n)(1+q^{m\ell})} = \prod_{m=1}^{\infty} \frac{(1+q^{\ell m})(1+q^m)(1-q^{\ell m})}{(1-q^m)(1+q^{m\ell})}. \tag{2.1}$$

The combinatorial interpretation of the coefficient of q^n in (2.1) is:
"the number of overpartitions of n in which overlined parts are ℓ-regular, nonoverlined parts that are multiples of ℓ are distinct, and other nonoverlined parts are unrestricted."

It follows immediately from the pentagonal number theorem expansion,

$$\prod_{m=1}^{\infty}(1-q^{2\ell m}) = 1 - q^{2\ell} - q^{4\ell} + q^{10\ell} + q^{14\ell} - q^{24\ell} - \cdots$$

and the monotonicity of the coefficients of $\sum_{n=0}^{\infty} A_\ell(n)q^n$ (since appending a nonoverlined 1 is an injection from $A_\ell(n)$ to $A_\ell(n+1)$), that

$$\prod_{m=1}^{\infty}\frac{(1+q^m)}{(1-q^m)(1+q^{m\ell})} > \prod_{m=1}^{\infty}\frac{(1+q^m)(1-q^{2\ell m})}{(1-q^m)(1+q^{m\ell})} = \prod_{m=1}^{\infty}\frac{(1-q^{m\ell})(1+q^m)}{(1-q^m)}$$

which is equivalent to

$$\prod_{m=1}^{\infty}\left(1 - q^{m\ell} + q^{2m\ell} - q^{3m\ell} + \cdots\right) > \prod_{m=1}^{\infty}(1-q^{m\ell}).$$

That is, $\sum_{n=0}^{\infty} A_\ell(n)q^n > \sum_{n=0}^{\infty} \overline{R_\ell^*}(n)q^n$. Here we write $f(q) > g(q)$ if the coefficient of q^n in $f(q)$ is greater than the coefficient of q^n in $g(q)$ for all $n > 0$. In particular, this is the case when $n \geq 2\ell$.

It is clear that when $n < 2\ell$, the interpretation of (2.1) is identical with that of $\sum_{n=0}^{\infty} \overline{R_\ell^*}(n)q^n$.

Second proof. We consider two cases.

Case 1: $n < 2\ell$.

Let $\lambda \in \overline{R_\ell^*}[n]$ and define the map $\overline{R_\ell^*}[n] \to A_\ell[n]$. If $\ell \notin \lambda$, then λ is fixed. But if $\ell \in \lambda$, then ℓ is overlined and so maps to a nonoverlined $\ell \in \beta \in A_\ell[n]$. Note that $\overline{\ell}$ cannot occur more than once in λ, and ℓ cannot occur more than once in β because $n < 2\ell$. Hence, we have a one-to-one correspondence, proving $A_\ell(n) = \overline{R_\ell^*}(n)$.

Case 2: $n \geq 2\ell$.

We apply the same map and obtain an injection. Since $n \geq 2\ell$, we can find some $\gamma \in A_\ell[n]$ in which ℓ occurs more than once. Then γ has no preimage in $\overline{R_\ell^*}[n]$ since at most one copy of ℓ may be overlined in any member of $\overline{R_\ell^*}[n]$. Thus we have a strict injection when $n \geq 2\ell$. Hence, $A_\ell(n) > \overline{R_\ell^*}(n)$.

This completes the proof. □

Theorem 2.1. *Let $B_4(2n)$ be the number of partitions of $2n$ in which no part is divisible by 4, odd parts occur with even multiplicity and even parts are unrestricted. Then*

$$B_4(2n) = \overline{R_2^*}(n).$$

Proof. The generating function for $B_4(2n)$ is

$$\sum_{n=0}^{\infty} B_4(2n)q^{2n} = \prod_{n=1}^{\infty} \frac{(1+q^{2n}+q^{4n}+\cdots)(1+q^{2(2n-1)}+q^{4(2n-1)}+\cdots)}{(1+q^{4n}+q^{8n}+q^{12n}+\cdots)}$$

(since an odd part is not divisible by 4)

$$= \prod_{n=1}^{\infty} \frac{(1-q^{4n})}{(1-q^{2n})(1-q^{2(2n-1)})}.$$

Thus on replacing q^2 by q we obtain

$$\sum_{n=0}^{\infty} B_4(2n)q^{n} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})}{(1-q^{n})(1-q^{2n-1})}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1+q^{n})}{(1-q^{n})} = \sum_{n=0}^{\infty} \overline{R_2^*}(n)q^n.$$

The theorem follows by comparing the coefficients of q^n in the extremes of the foregoing equations.

We now give a bijective proof. We will start from a partition counted by $B_4(2n)$ and obtain the corresponding overpartition counted by $\overline{R_2^*}(n)$, under the map $\beta_2 : B_4[2n] \to \overline{R_2^*}[n]$.

Let $\lambda = (c_1^{u_1}, c_2^{u_2}, \ldots) \in B_4[2n], c_1 > c_2 > \cdots$. If $c_j = 2m$, then from the definition of $B_4(2n)$, $2 \nmid m$. In order to assign the $c_j^{u_j} \in \lambda$, we first obtain the 2-adic expansion:

$$u_j = m_0 + 2m_1 + \cdots + 2^r m_r, m_i \in \{0,1\}.$$

Thus $c_j^{u_j} = c_j^{m_0}, c_j^{2m_1}, c_j^{2^2 m_2}, \ldots, c_j^{2^r m_r}$. Note that if $\lambda_i = c_j^{2^i m_i}$, we use the convention $\beta_2(\lambda) = \bigcup_{\lambda_i \in \lambda}(\beta_2(\lambda_i))$. Setting $k = 2^i m_i$ (note that k is 1, 2 or a multiple of 4), we get,

$$\beta_2 : c_j^k = (2m)^k \mapsto \begin{cases} m & \text{if } k = 1, \\ \overline{2m} & \text{if } k = 2, \\ \overline{km} & \text{if } k \equiv 0 \pmod 4. \end{cases}$$

For all other c_j (these are odd so k is even) we define a second map f by: $\lambda \mapsto f(\lambda) = \bigcup_{c \in \lambda} f_c(c^k)$, where

$$f_c(c^k) = \begin{cases} \overline{c} & \text{if } k = 2, \\ c^2 & \text{if } k = 4, \end{cases}$$

and if $4 < k \equiv r \pmod 4$, $r \in \{0, 2\}$, the image is a sequence of parts:

$$f_c(c^k) = f_c(c^r), c^{\frac{k-r}{2}}.$$

The inverse map β_2^{-1} will be applied to $\overline{c}$, where c is a multiple of 4, so c can be written as $c = 4^r s$ where $4 \nmid s$. Now we define β_2^{-1} as follows:

$$\beta_2^{-1}(\overline{4^r s}) = \begin{cases} (2s)^{4^r} & \text{if } s \equiv 1 \pmod 2, \\ (s)^{2(4^r)} & \text{if } s \equiv 0 \pmod 2. \end{cases}$$

The inverse map f^{-1} is analogously given by:

$$f_c^{-1}(\overline{c}) = c^2, \quad f_c^{-1}(c^k) = \begin{cases} 2c & \text{if } k = 1, \\ c^4 & \text{if } k = 2, \end{cases}$$

and if $2 < k \equiv r \pmod 2$, $1 \le r \le 2$, then

$$f_c^{-1}(c^k) = f_c^{-1}(c^r), c^{2(k-r)}.$$

This bijection is illustrated in Table 1 with the case $n = 36$. □

$B_4[72]$	$\to$	2-adic exp of even parts	$\to$	$\overline{R_2^*}[36]$
(2^{36})	$\to$	$(2^{32}, 2^4)$	$\to$	$(\overline{32}, \overline{4})$
(6^{12})	$\to$	$(6^8, 6^4)$	$\to$	$(\overline{24}, \overline{12})$
$(10^7, 2)$	$\to$	$(10^4, 10^3, 2)$	$\to$	$(\overline{20}, \overline{10}, 5, 1)$
(1^{72})	$\to$	(1^{72})	$\to$	(1^{36})
$(2^{30}, 6^2)$	$\to$	$(2^{16}, 2^8, 2^4, 2^2, 6^2)$	$\to$	$(\overline{16}, \overline{8}, \overline{4}, \overline{2}, \overline{6})$
$(2^{32}, 3^2, 1^2)$	$\to$	$(2^{32}, 3^2, 1^2)$	$\to$	$(\overline{32}, \overline{3}, \overline{1})$
$(2^{20}, 1^{32})$	$\to$	$(2^{16}, 2^4, 1^{32})$	$\to$	$(\overline{16}, \overline{4}, 1^{16})$
$(6^4, 3^6, 2^{10}, 1^{10})$	$\to$	$(6^4, 3^6, 2^8, 2^2, 1^{10})$	$\to$	$(\overline{12}, 3^2, \overline{3}, \overline{8}, \overline{2}, 1^4, \overline{1})$
$(14^5, 2)$	$\to$	$(14^4, 14, 2)$	$\to$	$(\overline{28}, 7, 1)$

TABLE 1. The correspondence between $B_4[72]$ and $\overline{R_2^*}[36]$.

The following is an extension of Theorem 2.1 (which is the case $\ell = 2$).

Theorem 2.2. *Let $B_{2\ell}(2n)$ be the number of 2ℓ-regular partitions of $2n$ in which odd parts occur with even multiplicity and even parts are unrestricted. Then*

$$B_{2\ell}(2n) = \overline{R_\ell^*}(n).$$

Proof. This is analogous to the proof of the previous theorem. The generating function for $B_{2\ell}(2n)$ is

$$\begin{aligned} \sum_{n=0}^{\infty} B_{2\ell}(2n) q^{2n} &= \prod_{n=1}^{\infty} \frac{(1 + q^{2n} + q^{4n} + \cdots)(1 + q^{2(2n-1)} + q^{4(2n-1)} + \cdots)}{(1 + q^{2\ell n} + q^{4\ell n} + q^{6\ell n} + \cdots)} \\ &= \prod_{n=1}^{\infty} \frac{(1 - q^{2\ell n})}{(1 - q^{2n})(1 - q^{2(2n-1)})}. \end{aligned}$$

But on replacing q^2 by q we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} B_{2\ell}(2n) q^n &= \prod_{n=1}^{\infty} \frac{(1 - q^{\ell n})}{(1 - q^n)(1 - q^{2n-1})} \\ &= \prod_{n=1}^{\infty} \frac{(1 - q^{\ell n})(1 + q^n)}{(1 - q^n)} = \sum_{n=0}^{\infty} \overline{R_\ell^*}(n) q^n. \end{aligned} \tag{2.2}$$

Hence the theorem follows.

Secondly, we give a bijective proof with $\beta_\ell : B_{2\ell}[2n] \to \overline{R_\ell^*}[n]$.
Let

$$\lambda = (c_1^{u_1}, c_2^{u_2}, \dots) \in B_{2\ell}[2n], c_1 > c_2 > \cdots .$$

Note that

$$c_j = \ell m \implies 2 \nmid m.$$

We first obtain the 2-adic expansion

$$u_j = m_0 + 2m_1 + \cdots + 2^r m_r,\ m_i \in \{0,1\}.$$

Then each $c_j^{u_j} \in \lambda$ is equivalent to $c_j^{u_j} = c^{2m_1}, c^{2^2 m_2}, \dots, c^{2^r m_r}$, where $c = c_j$. Thus for each j we set $k = 2^i m_i$ and obtain

$$\beta_\ell : c_j^k = (\ell m)^k, \mapsto \begin{cases} \frac{\ell m}{2} & \text{if } k = 1, \\ \overline{\ell m} & \text{if } k = 2, \\ \overline{\frac{k\ell m}{2}} & \text{if } k \equiv 0 \pmod 4. \end{cases}$$

(Note that when ℓ is odd, the case $k = 1$ does not apply).

Then for all other c_j, we apply the map f, in extended form, as follows:

$$f_c(c^k) = \begin{cases} \frac{c}{2} & \text{if } k = 1, \\ \overline{c} & \text{if } k = 2, \\ \frac{c}{2}, \overline{c} & \text{if } k = 3, \\ c^2 & \text{if } k = 4, \end{cases}$$

and if $4 < k \equiv r \pmod 4$, $r \in \{0, 2\}$, the image is a sequence of parts:

$$f_c(c^k) = f_c(c^r), c^{\frac{k-r}{2}}.$$

Note that the cases $k = 1, 3$ refer to even parts only since odd parts occur with even multiplicities.

See Table 2 (second and third columns) for an illustration with $n = 25$ and $\ell = 4$. □

Theorem 2.3. *Let $V_\ell(2n)$ denote the number of partitions of $2n$ in which even parts appear at most $\ell - 1$ times and odd parts occur with even multiplicities. Then*

$$V_\ell(2n) = \overline{R_\ell^*}(n).$$

Proof. The generating function for $V_\ell(2n)$ is

$$\begin{aligned} \sum_{n=0}^{\infty} V_\ell(2n) q^{2n} &= \prod_{n=1}^{\infty} (1 + q^{2n} + \cdots + q^{(\ell-1)2n})(1 + q^{2(2n-1)} + q^{4(2n-1)} + \cdots) \\ &= \prod_{n=1}^{\infty} \frac{(1 - q^{2\ell n})}{(1 - q^{2n})(1 - q^{2(2n-1)})}. \end{aligned}$$

$V_4(50)$	$\xrightarrow{\alpha}$	$B_8(50)$	$\xrightarrow{f,\beta_\ell}$	$\overline{R_4^*}(25)$
$(48,2)$	$\to$	$(12^4,2)$	$\to$	$(\overline{24},1)$
$(32,12,4,2)$	$\to$	$(2^{16},12,4,2)$	$\to$	$(\overline{16},6,2,1)$
$(24^2,2)$	$\to$	$(6^8,2)$	$\to$	$(6^4,1)$
$(24,16,8,1^2)$	$\to$	$(6^4,4^4,2^4,1^2)$	$\to$	$(6^2,\overline{8},2^2,\overline{1})$
$(16^3,2)$	$\to$	$(4^{12},2)$	$\to$	$(\overline{16},\overline{8},1)$
$(16^2,12,4,2)$	$\to$	$(4^8,12,4,2)$	$\to$	$(\overline{16},6,2,1)$
$(16,12^2,8,1^2)$	$\to$	$(12^2,4^4,2^4,1^2)$	$\to$	$(\overline{12},\overline{8},2^2,\overline{1})$
$(12^3,8,6)$	$\to$	$(12^3,2^4,6)$	$\to$	$(\overline{12},6,2^2,\overline{1})$
$(8^3,7^2,4^3)$	$\to$	$(2^{12},7^2,4^3)$	$\to$	$(2^6,\overline{7},\overline{4},2)$
$(8,4^3,3^6,2^3,1^6)$	$\to$	$(4^3,3^6,2^7,1^6)$	$\to$	$(\overline{4},\overline{3},3^2,\overline{2},2^3,\overline{1},1^2)$

TABLE 2. Bijections of Theorems 2.2 and 2.3, $n = 25, \ell = 4$.

Thus replacing q^2 by q gives (with Equation (2.2)),

$$\sum_{n=0}^{\infty} V_\ell(2n)q^{2n} = \prod_{n=1}^{\infty} \frac{(1-q^{\ell n})}{(1-q^n)(1-q^{2n-1})} = \sum_{n=0}^{\infty} \overline{R_\ell^*}(n)q^n,$$

as required.

For a combinatorial proof, we employ a new bijection, α, to compose with f and β_ℓ:

$$V_\ell[2n] \xrightarrow{\alpha} B_{2\ell}[2n] \xrightarrow{f,\beta_\ell} \overline{R_\ell^*}[n].$$

If $\lambda = (c_1, c_2, \ldots) \in B_\ell[2n]$, then each $c_i = c$ can be expressed uniquely in the form $c = (2\ell)^r m$ with $r \geq 0$ such that $2\ell \nmid m$.
Define $\alpha : V_\ell[2n] \to A_{2\ell}[2n]$ by setting $\alpha(\lambda) = \bigcup_{c\in\lambda} \alpha_c(c)$, with

$$\alpha_c(c) = \alpha_c((2\ell)^r m), \mapsto \begin{cases} c & \text{if } r = 0, \\ 2m^{(2^{r-1}\ell^r)} & \text{if } \ell \nmid m \text{ and } r > 0, \\ (\frac{2m}{\ell})^{(2^{r-1}\ell^{r+1})} & \text{if } \ell \mid m \text{ and } r > 0. \end{cases}$$

This correspondence is illustrated for $n = 25$ and $\ell = 4$ in Table 2 (first and third columns). $\square$

Theorem 2.4. *Denote by $H_\ell(2n)$ the number of partitions of $2n$ where parts which are multiples of 2ℓ occur exactly twice, each part $\equiv \ell \pmod{2\ell}$ appears at most three times and odd parts occur with even multiplicities. Then*

$$H_\ell(2n) = \overline{R_\ell^*}(n).$$

Proof. The generating function proof runs as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} H_\ell(2n)q^{2n} &= \prod_{n=1}^{\infty} \frac{(1-q^{\ell n})(1+q^{4\ell n})(1+q^{(2\ell n-\ell)}+q^{2(2\ell n-\ell)}+q^{3(2\ell n-\ell)})}{(1-q^{2n})(1-q^{2(2n-1)})} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^{\ell n})(1-q^{8\ell n})(1-q^{4(2\ell n-\ell)})}{(1-q^{4\ell n})(1-q^{(2\ell n-\ell)})(1-q^{2n})(1-q^{2(2n-1)})} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^{\ell n})(1-q^{4\ell n})}{(1-q^{4\ell n})(1-q^{\ell(2n-1)})(1-q^{2n})(1-q^{2(2n-1)})} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^{\ell 2n})(1-q^{\ell(2n-1)})}{(1-q^{\ell(2n-1)})(1-q^{2n})(1-q^{2(2n-1)})} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^{\ell 2n})}{(1-q^{2n})(1-q^{2(2n-1)})}.\end{aligned}$$

Lastly, the proof follows from by replacing q^2 with q, that is,

$$\sum_{n=0}^{\infty} H_\ell(2n)q^{n} = \prod_{n=1}^{\infty} \frac{(1-q^{\ell n})(1+q^{n})}{(1-q^{n})} = \sum_{n=0}^{\infty} \overline{R_\ell^*}(n)q^n.$$

For a bijection, we will apply the map $f : H_\ell[2n] \to R_\ell^*[n]$, as the reader may verify. $\square$

The proof of the following identity is similar to the foregoing ones.

Theorem 2.5. *Let $Q_\ell(n)$ denote the number of partitions of n in which parts divisible by ℓ are distinct and parts not divisible by ℓ occur with multiplicity $r\ell$, $r \geq 0$. Then*

$$Q_\ell(\ell n) = \overline{R_\ell^*}(n). \tag{2.3}$$

3. Congruence Properties

We denote by $p(n \mid S)$ the number of partitions of n that satisfy property S.

Theorem 3.1. $\overline{R_\ell^*}(n) \equiv 1 \pmod 2$ *if and only if*

$$\ell \mid n \text{ and } p\left(\frac{n}{\ell} \mid \text{distinct parts}\right) \text{ is odd.}$$

Proof. Firstly, using generating functions we have

$$\begin{aligned}
\sum_{n\geq 0}\overline{R_\ell^*}(n)q^n &= \prod_{n\geq 1}\frac{(1-q^{\ell n})(1+q^n)}{(1-q^n)}\times\prod_{n\geq 1}\frac{(1-q^{\ell n})(1-q^n)}{(1-q^{\ell n})(1-q^n)}\\
&= \prod_{n\geq 1}\frac{(1-q^{\ell n})^2(1-q^{2n})}{(1-q^n)^2(1-q^{\ell n})}\\
&\equiv \prod_{n\geq 1}\frac{(1-q^{2\ell n})(1-q^{2n})}{(1-q^{2n})(1-q^{\ell n})} \pmod 2\\
&= \prod_{n\geq 1}\frac{(1-q^{2\ell n})}{(1-q^{\ell n})} \pmod 2\\
&= \prod_{n\geq 1}(1+q^{\ell n}) \pmod 2\,.
\end{aligned}$$

Secondly, for a bijective proof we first assume that $\overline{R_\ell^*}(n)\equiv 1 \pmod 2$. Construct a partition $\lambda=(c_1^{u_1},c_2^{u_2},\ldots,c_r^{u_r})$ by overlining the first occurrence of each part-size such that multiples of ℓ are overlined. So there are a power of 2 such overpartitions λ unless all the parts are multiples of ℓ. From the definition of $\overline{R_\ell^*}(n)$ the overlined parts are distinct and $\ell\mid n$. Their number is the same as the number $p(n/\ell\mid \text{distinct parts})$ of ordinary partitions of n/ℓ into distinct part sizes. Therefore, we need to consider only the case of λ when all the c_i are multiples of ℓ such that $\ell\mid n$. Since $\overline{R_\ell^*}(n)$ is odd and $\ell\mid n$, it follows that $p(n/\ell\mid \text{distinct parts})$ must be odd.

Conversely, if $\ell\mid m$ and $p(n/\ell\mid\text{distinct parts})\equiv 1 \pmod 2$, the result is immediate from the previous argument. □

There is a nice expression in terms of the Ramanujan theta functions [3]:

$$\sum_{n=0}^{\infty}\overline{R_\ell^*}(n)q^n=\frac{f(-q^{2\ell})}{\varphi(-q)},$$

where f (different from the previous map notation) is defined by

$$f(-q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}=\prod_{n=1}^{\infty}(1-q^n),$$

and

$$\varphi(q)=\sum_{n=-\infty}^{\infty}q^{n^2}=\prod_{n=1}^{\infty}(1+q^{2n-1})^2(1-q^{2n}). \tag{3.1}$$

Consider the following functions, where $f_a^b=\prod_{n=1}^{\infty}(1-q^{an})^b$,

$$D(q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}=\frac{f_1^2}{f_2}=\varphi(-q)$$

and

$$Y(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n} = \frac{f_1 f_6^2}{f_2 f_3}.$$

Hirschhorn and Sellers [4] showed that

$$D(q) = D(q^9) - 2qY(q^3),$$

$$D(q)D(\omega q)D(\omega^2 q) = \frac{D(q^3)^4}{D(q^9)}, \text{ where } \omega = e^{2\pi i/3}.$$

Now we can prove the following result.

Theorem 3.2. *For all* $n \geq 0$,

$$\overline{R_3^*}(9n+4) \equiv 0 \pmod 3 \quad \textit{and} \quad \overline{R_3^*}(9n+7) \equiv 0 \pmod 3.$$

Proof. Using the f_a^b notation we have

$$\begin{aligned}
\sum_{n\geq 0} \overline{R_3^*}(n)q^n &= \prod_{n\geq 1} \frac{(1-q^{3n})(1+q^n)}{(1-q^n)} \times \prod_{n\geq 1} \frac{(1-q^n)}{(1-q^n)} \\
&= \prod_{n\geq 1} \frac{(1-q^{3n})(1-q^{2n})}{(1-q^n)^2} = \frac{f_3 f_2}{f_1^2} = \frac{f_3}{D(q)} \\
&= \frac{f_3}{D(q)} \frac{D(\omega q)}{D(\omega q)} \frac{D(\omega^2 q)}{D(\omega^2 q)} \\
&= \frac{f_3 D(q^9)}{D(q^3)^4} (D(q^9) - 2\omega q Y(q^3))(D(q^9) - 2\omega^2 q Y(q^3)) \\
&= \frac{f_3 D(q^9)}{D(q^3)^4} (D(q^9)^2 + 2qD(q^9)Y(q^3) + 4q^2 Y(q^3)^2).
\end{aligned}$$

Next, we can 3-dissect $\overline{R_3^*}$ to get

$$\begin{aligned}
\sum_{n\geq 0} \overline{R_3^*}(3n+1)q^n &= \frac{f_1 D(q^3)}{D(q)^4} (2D(q^3)Y(q)) \\
&= 2f_1 \frac{f_3^4}{f_6^2} \frac{f_1 f_6^2}{f_2 f_3} \frac{f_2^4}{f_1^8} \\
&= \frac{2f_2^3 f_3^3}{f_1^6} \\
&\equiv \frac{2f_2^3 f_3^3}{f_3^2} \pmod 3 \\
&= 2f_2^3 f_3 \\
&\equiv 2f_6 f_3 \pmod 3.
\end{aligned}$$

It is clear that the series $2f_6 f_3$ cannot contain terms of the form q^{3n+1} nor q^{3n+2}. Therefore, for all $n \geq 0$,

$$\overline{R_3^*}(3(3n+1)+1) = \overline{R_3^*}(9n+4) \equiv 0 \pmod 3,$$

and

$$\overline{R_3^*}(3(3n+2)+1) = \overline{R_3^*}(9n+7) \equiv 0 \pmod{3}.$$

□

Theorem 3.3. *For all $n \geq 0$ and all $j \geq 3$, $\overline{R_{3^j}^*}(27n+18) \equiv 0 \pmod{3}$.*

Proof. From recent work of Alanazi et al. [2], we define the function $R_\ell(n)$ to be the number of overpartitions of n in which no part is divisible by ℓ. Then for all $n \geq 0$ and all $j \geq 3$, $R_{3^j}(27n+18) \equiv 0 \pmod{3}$ where

$$\sum_{n\geq 0} R_{3^j}(n)q^n = \prod_{n\geq 1} \frac{(1-q^{3^j n})}{(1-q^n)} \prod_{n\geq 1} \frac{(1+q^n)}{(1+q^{3^j n})}. \tag{3.2}$$

Next, note that

$$\sum_{n\geq 0} \overline{R_{3^j}^*}(n)q^n = \prod_{n\geq 1} \frac{(1-q^{3^j n})}{(1-q^n)} \prod_{n\geq 1}(1+q^n). \tag{3.3}$$

It is then clear from (3.2) and (3.3) that

$$\sum_{n\geq 0} \overline{R_{3^j}^*}(n)q^n = \left(\prod_{n\geq 1}(1+q^{3^j n})\right) \sum_{n\geq 0} R_{3^j}(n)q^n.$$

Lastly, since $\prod_{n\geq 1}(1+q^{3^j n})$ is a function of q^{27}, the theorem follows. □

4. A New Identity for $A_\ell(n)$

We give the counterpart of Theorem 3.1 for the dual enumeration function $A_\ell(n)$, i.e., the number of overpartitions of n in which the overlined parts are ℓ-regular, followed by a special identity with a class of ordinary partition functions.

Theorem 4.1. *$A_\ell(n) \equiv 1 \pmod{2}$ if and only if*

$$\ell \mid n \text{ and } p\left(\frac{n}{\ell}\right) \equiv 1 \pmod{2}.$$

The proof of this theorem is analogous to that of Theorem 3.1, and so is omitted.

Lastly, we prove the following identity.

Theorem 4.2. *Let $W_\ell(2n)$ denote the number of partitions of $2n$ in which odd parts occur with multiplicity $2, 4, \ldots,$ or $2(\ell-1)$ and even parts are unrestricted. Then we have*

$$W_\ell(2n) = A_\ell(n).$$

Proof. We give two proofs—a generating function proof and a bijective proof. The generating function for $W_\ell(2n)$ is

$$\begin{aligned}\sum_{n=0}^{\infty} W_\ell(2n)q^{2n} &= \prod_{n=1}^{\infty}(1+q^{1\cdot 2n}+q^{2\cdot 2n}+\cdots)\\ &\qquad \times (1+q^{2(2n-1)}+\cdots+q^{2(\ell-1)(2n-1)})\\ &= \prod_{n=1}^{\infty}\frac{(1-q^{2\ell(2n-1)})}{(1-q^{2n})(1-q^{2(2n-1)})}.\end{aligned}$$

On the other hand, if we replace q^2 by q we obtain,

$$\begin{aligned}\sum_{n=0}^{\infty} W_\ell(2n)q^{2n} &= \prod_{n=1}^{\infty}\frac{(1-q^{\ell(2n-1)})}{(1-q^{n})(1-q^{2n-1})}\\ &= \prod_{n=1}^{\infty}\frac{(1-q^{\ell(2n-1)})}{(1-q^{n})(1-q^{2n-1})}\times\frac{(1-q^{n})(1-q^{2\ell n})}{(1-q^{n})(1-q^{2\ell n})}\\ &= \prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{\ell n})}{(1-q^{n})^2(1-q^{2\ell n})}\\ &= \prod_{n=1}^{\infty}\frac{(1+q^{n})}{(1-q^{n})(1+q^{\ell n})}\times\frac{(1-q^{n})(1-q^{\ell n})}{(1-q^{n})(1-q^{\ell n})}\\ &= \sum_{n=0}^{\infty} A_\ell(n)q^n.\end{aligned}$$

Hence the proof. □

4.1. **Bijective Proof of Theorem 4.2.** Although the generating function argument is straightforward, an interesting element of a second, bijective proof of Theorem 4.2 will be a bijection for a general class of partitions referred to in [5], partially answering a request there for a generalization of a more restricted bijection, which we will detail at the appropriate point in the proof.

Given an element of $W_\ell(2n)$, our strategy is as follows:

- Construct an arbitrary partition α of some weight $|\alpha| \le n$.
- Construct a partition into distinct parts, none of which are divisible by ℓ, of weight $n-|\alpha|$.
- Overline the distinct parts from the latter partition and combine the two to obtain an overpartition in $A_\ell(n)$.

Proof. Let $\omega \in W_\ell(2n)$, $\omega = ({\omega_1}^{m_1}, {\omega_2}^{m_2}, \dots)$. Let the indices $\{e_1, e_2, \dots, e_t\}$ demarcate the even part sizes, and $\{o_1, o_2, \dots, o_s\}$ the odd part sizes. The odd part sizes appear an even number of times but at most $2(\ell-1)$.

From the even parts of ω, construct the intermediate partition

$$\alpha = \left(\left(\frac{\omega_{e_1}}{2}\right)^{m_{e_1}}, \left(\frac{\omega_{e_2}}{2}\right)^{m_{e_2}}, \dots, \left(\frac{\omega_{e_t}}{2}\right)^{m_{e_t}}\right).$$

Observe that α can be an arbitrary partition.

Of the odd parts $({\omega_{o_1}}^{m_{o_1}}, \dots, {\omega_{o_s}}^{m_{o_s}})$, reduce the frequencies by half to obtain a partition

$$\beta = (\omega_{o_1}^{(m_{o_1}/2)}, \dots, \omega_{o_s}^{(m_{o_s}/2)}).$$

This will be a partition into odd parts appearing less than ℓ times, and clearly any such partition can arise.

Our goal is now a bijection between partitions of any N into odd parts appearing less than ℓ times, and partitions of N into distinct parts not divisible by ℓ.

Such partitions were considered in [5]. It is easy to see that in general, the number of partitions of integers into parts not divisible by s, each part size appearing less than t times, has generating function symmetric in s and t, namely

$$\prod_{n=1}^{\infty} \frac{(1-q^{sn})(1-q^{tn})}{(1-q^n)(1-q^{stn})}.$$

This is also the generating function of the number of partitions of integers into parts not divisible by t, each part size appearing less than s times, and equality of the sets is immediate but there remains the question of a bijection between the two sets.

In [5] it was shown that, in the case where s and t were coprime, these two classes could be mapped to each other by a double use of *Glaisher's bijections*, which individually are a standard set of maps in combinatorial partition theory. They are bijections ϕ_t between partitions with parts not divisible by t and those with parts appearing fewer than t times. We describe these maps below.

Let λ be a partition into parts not divisible by t. Suppose that part size m appears k times, with

$$k = a_{m,0}t^0 + a_{m,1}t^1 + a_{m,2}t^2 + \dots,$$

with $0 \leq a_{m,i} < t$. That is, the frequency of appearance of m is written in base-t notation

$$k = (\cdots a_{m,2}a_{m,1}a_{m,0})_t.$$

Then $\mu = \phi_t(\lambda)$ is the partition that contains $a_{m,i}$ parts of each size $t^i m$, and this is a partition into parts appearing fewer than t times. To reverse the map, ${\phi_t}^{-1}(\mu)$ will be the partition in which each part of size $t^i m$, with m not divisible by t, becomes t^i appearances of part size m.

For the present purpose we require the case $t = 2$, $s = \ell$. The aforementioned result in the case when s and t are coprime is Lemma 4.3, which we append at the end of this section. The lemma establishes the required map when ℓ is odd; we now complete the proof by exhibiting the necessary map when ℓ is even.

Suppose $\ell = 2^r k$, k odd.

We consider β. Suppose part size $\omega_{o_i} = g_i$ to save repeated indices. Write

$$\frac{m_{o_i}}{2} = a_i + C_i k, \quad 0 \le a_i < k, \quad 0 \le C_i \le 2^r - 1.$$

The partition

$$\gamma = ({g_1}^{a_1}, {g_2}^{a_2}, \dots)$$

is a partition into odd parts appearing less than k times. We defined k to be odd, so by the Lemma, the partition $\delta = \phi_2 {\phi_k}^{-1}(\gamma)$ is a partition into parts not divisible by k, appearing less than twice, i.e. parts are distinct.

For each g_i in β for which $C_i \neq 0$, write

$$C_i = c_{i,0}2^0 + c_{i,1}2^1 + c_{i,2}2^2 + \dots, \quad c_{i,j} \in \{0, 1\}.$$

Using this expression, insert into δ parts of size $2^j k g_i$ whenever $c_{i,j} = 1$.

Since $\ell = 2^r k$ and $C_i < 2^r$, with k and g_i odd, we have that $2^j k g_i$ is never divisible by 2^r and so ℓ never divides any of the parts added to δ in this step, which are all multiples of k.

The map is easily reversible (simply separate the multiples of k and the nonmultiples) and one-to-one in both directions, and hence a bijection.

We conclude our desired bijection from $W_\ell(2n)$ to $A_\ell(n)$ by overlining the parts of δ. We then take α and insert the distinct overlined parts. By construction, only parts not divisible by ℓ will be overlined, and hence the resulting overpartition is in $A_\ell(n)$. Reversing the map is a simple matter of separating the overlined parts and reversing the individual maps above, and hence we have the desired bijection. □

Example. Let $\ell = 12 = 2^2 3$. Let $\omega = (24, 11^6, 9^2, 8, 6^2, 2, 1^{22}) \in W_{12}(152)$.

The even parts yield

$$\alpha = (12, 4, 3^2, 1).$$

The odd parts after halving frequencies yield

$$\beta = (11^3, 9, 1^{11}).$$

Taking the residue modulo 3 of the frequencies, we have the partition

$$\gamma = (9, 1^2).$$

We then construct

$$\delta = \phi_2 {\phi_3}^{-1}((9, 1^2)) = \phi_2((1^{11})) = (8, 2, 1).$$

The remaining odd parts in β are $(11^3, 1^9)$. They are mapped thus:

$$(11^3) = (11^{3\cdot(2^0)}) \longrightarrow (3 \cdot 11 \cdot 2^0) = (33),$$
$$(1^9) = (1^{3\cdot(2^0+2^1)}) \longrightarrow (3 \cdot 1 \cdot 2^1, 3 \cdot 1 \cdot 2^0) = (6, 3).$$

Inserting the distinct parts $(33, 8, 6, 3, 2, 1)$ into α and overlining sizes where an insertion occurred, we obtain $(\overline{33}, 12, \overline{8}, \overline{6}, 4, \overline{3}^3, \overline{2}, \overline{1}^2) \in A_{12}(76)$.

For completeness, we close with a restatement of the previously published Lemma required in the proof above.

Lemma 4.3. *Let s and t be coprime. The map $\phi_t {\phi_s}^{-1}$ is a bijection that maps the set of partitions of N with parts not divisible by t, appearing fewer than s times, to the set of partitions of N with parts not divisible by s, appearing fewer than t times.*

Example. Let $\lambda = (25^2, 7^4, 5, 1, 1)$ be a partition of 84 into parts not divisible by 2, appearing fewer than 5 times. Then

$$\rho = {\phi_5}^{-1}(\lambda) = (7^4, 1^{57})$$

and ρ is a partition into parts divisible by neither 2 nor 5. Finally

$$\phi_2(\rho) = (32, 28, 16, 8, 1)$$

and as desired this is a partition into distinct parts not divisible by 5.

Proof. Consider a partition λ of N into parts not divisible by t, appearing fewer than s times, with s and t coprime. Then applying ${\phi_s}^{-1}$ produces parts of size m from parts of size $s^i m$. This only divides sizes by powers of s and thus parts which were not divisible by t remain not divisible by t.

Hence ${\phi_s}^{-1}(\lambda)$ is a partition of N into parts simultaneously not divisible by either t or s. Applying ϕ_s to any such partition produces a partition into parts appearing less than s times, and since part sizes can only be multiplied by powers of s, coprime to t, the resulting parts are also not divisible by t. Hence the maps are onto in both directions.

Applying instead ϕ_t, we obtain a partition into parts appearing fewer than t times. Again, ϕ_t multiplies part sizes by powers of t, coprime to s, and hence no part is produced divisible by s. The map is onto as above, and so we have the required bijection between the two sets. □

Remark. The proof above required coprimality. An intermediate set—partitions into parts divisible by neither s nor t—is equal in size to the other two when s and t are coprime, but such a set does not appear when $\gcd(s, t) > 1$. It is an open conjecture of Bridget Tenner, mentioned in [5], that for s and t not coprime, some map $(\phi_t {\phi_s}^{-1})^k$ exhibits the required bijection. If this conjecture could not be established, another bijection exhibiting the equality was requested. We believe that a bijection along the lines of Theorem 4.2 can be constructed for general s and t, but Glaisher's bijections are an important set of maps in combinatorial partition theory, and it would remain interesting to show Tenner's conjecture independently.

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Department of Mathematics, University of Tabuk, P.O. Box 741, Tabuk 71491
E-mail address: am.alenezi@ut.edu.sa

Department of Mathematics, University of Tabuk, P.O. Box 741, Tabuk 71491
E-mail address: bashayer14140@gmail.com

Department of Mathematical Sciences, Michigan Technological University
E-mail address: wjkeith@mtu.edu

School of Mathematics, University of the Witwatersrand, P.O. Wits 2050, Johannesburg
E-mail address: augustine.munagi@wits.ac.za