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TWO FAMILIES OF STRONGLY WALK REGULAR GRAPHS FROM THREE-WEIGHT CODES OVER $\mathbb{Z}_4$

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Abstract. A necessary condition for a $\mathbb{Z}_4$-code to be a three-weight code for the Lee weight is given. Two special constructions of three-weight codes over $\mathbb{Z}_4$ are derived. The coset graphs of their duals are shown to be strongly 3-walk-regular, a generalization of strongly regular graphs.

1. Introduction

Strongly walk-regular graphs (SWRG) were introduced in [2] as a generalization of strongly regular graphs. Recently, a simple numerical condition bearing on the homogeneous weights of three-weight codes over rings was introduced to check if the coset graph of their dual codes is SWRG [5, 7]. In parallel, in a series of papers [6, 9, 10, 11, 12, 13, 14, 15, 16], Shi *et al.* constructed and studied one-weight codes, two-weight and three-weight codes over various finite rings. The general construction method is based on trace codes, see [4, Chapt. 18] for a survey. Especially, Shi *et al.* considered the construction of one-Lee weight and two-Lee weight projective codes over $\mathbb{Z}_4$ in [10]. Later, the authors also determined the linearity of these codes completely in [14].

Inspired by the above works, we revisit $\mathbb{Z}_4$ codes. The alphabet $\mathbb{Z}_4$ has been an important and popular example of ring alphabet ever since the paper [3], where, already the coset graph of the Preparata code is used to construct a distance regular graph of diameter three.

We give two special constructions of projective three-weight codes over $\mathbb{Z}_4$, with explicit weight distributions. Using the weight information, the spectrum of the coset graph of the dual codes are determined. Using a spectral condition of [2], these graphs are shown to be 3-SWRG.

This work is organized as follows. In Section 2, we recall some background and notations about linear codes over $\mathbb{Z}_4$, and their coset graphs. In Section 3, some useful conditions for a linear code to have three-Lee weight over $\mathbb{Z}_4$ are given. The structures of three-Lee weight projective linear codes are discussed in Section 4. Moreover, we also

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give some examples to illustrate the results. In section 5, we discuss the linearity of the Gray images of these codes. Section 6 concludes the article.

2. Definitions and Notation

2.1. $\mathbb{Z}_4$ **codes.** In this section, we first recall the codes over $\mathbb{Z}_4$ in [5]. Let $\mathbb{Z}_4 = \{0, 1, 2, 3\}$, and $\mathbb{Z}_4^n = \{(x_1, x_2, \dots, x_n) | x_i \in \mathbb{Z}_4, 1 \le i \le n\}$. A linear code C over $\mathbb{Z}_4$ of length n is a $\mathbb{Z}_4$-submodule of $\mathbb{Z}_4^n$. The Lee weights of $0, 1, 2, 3 \in \mathbb{Z}_4$ are $0, 1, 2, 1$ respectively. For any $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{Z}_4^n$, we define

$$W_L(\mathbf{x}) = \sum_{i=1}^{n} W_L(x_i).$$

Each element $x \in \mathbb{Z}_4$ has a 2-adic expansion $x = \alpha(x) + 2\beta(x)$, where $\alpha(x), \beta(x) \in \mathbb{F}_2$, the Gray map from $\mathbb{Z}_4$ to $\mathbb{F}_2^2$ is given by $\Phi(x) = (\beta(x), \alpha(x) + \beta(x))$. Define $\alpha(\mathbf{x}) = (\alpha(x_1), \dots, \alpha(x_n))$, where $\alpha(0) = \alpha(2) = 0$ and $\alpha(1) = \alpha(3) = 1$. This map can be extended to $\mathbb{Z}_4^n$ naturally. Φ is a weight-preserving map from ($\mathbb{Z}_4^n$, Lee weight) to ($\mathbb{F}_2^{2n}$, Hamming weight).

Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ be two elements of $\mathbb{Z}_4^n$, the inner product of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{Z}_4^n$ is defined by $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$, and the componentwise multiplication $*$ of $\mathbf{x}$ and $\mathbf{y}$ is $\mathbf{x} * \mathbf{y} = (x_1y_1, x_2y_2, \dots, x_ny_n)$, where the operation is performed in $\mathbb{Z}_4$. The dual code of C is defined as $C^\perp = \{\mathbf{x} \in \mathbb{Z}_4^n | \mathbf{x} \cdot \mathbf{y} = 0, \forall\ \mathbf{y} \in C\}$. A Lee weight projective code C of length n over $\mathbb{Z}_4$ is a linear code such that the minimum Lee weight of its dual code is at least three.

It is well known that a nonzero linear code C over $\mathbb{Z}_4$ has a generator matrix which after a suitable permutation of the coordinates can be written in the form (see [3])

$$G_{k_1,k_2} = \begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2D \end{pmatrix}, \tag{2.1}$$

where I_{k_1} and I_{k_2} denote the $k_1 \times k_1$ and $k_2 \times k_2$ identity matrices, respectively, A and D are $\mathbb{Z}_2$-matrices, B is a $\mathbb{Z}_4$-matrix, and $|C| = 4^{k_1}2^{k_2}$.

2.2. **Graphs.** An *eigenvalue* of a graph Γ (i.e., an eigenvalue of its adjacency matrix) is called a *restricted eigenvalue* if there is a corresponding eigenvector which is not a multiple of the all-one vector $\mathbf{1}$. Note that for an η-regular connected graph, the restricted eigenvalues are simply the eigenvalues different from η.

Definition 2.1. *Let T be a finite abelian group and $S \subseteq T$ a subset satisfying $S = -S$ and $0_T \notin S$. The corresponding* Cayley graph $\mathrm{C}(T, S)$ *has vertex set equal to T; two vertices $g, h \in T$ are adjacent in $\mathrm{C}(T, S)$ iff $g - h \in S$.*

For a precise definition of a Cayley graph attached to a code in a canonical way we refer to [8]. Now, we recall the relation between the weight distribution of a linear code over $\mathbb{Z}_4$ and the eigenvalues of the syndrome graph of its dual code. This extension of Lemma 3.4 in [1] was derived in Theorem 3.4 in [8], thus its proof is omitted here.

Theorem 2.2. *Suppose that C is a regular, projective linear code over $\mathbb{Z}_4$ with Lee weights w_i and corresponding weight distribution $A_i = |\{x \in C; W_L(x) = w_i\}|$. Then the eigenvalues of $\Gamma(C^\perp)$ are $2n - 2w_i$ with multiplicity A_i.*

3. Some preliminaries

We motivate the constructions of the next section by two necessary conditions for a $\mathbb{Z}_4$-code to be a three-weight code.

Lemma 3.1. *Let C be a linear code over $\mathbb{Z}_4$ with type $4^{k_1}2^{k_2}$ of length n. There is no three-Lee weight projective linear code when $k_1 = 1$.*

Proof. Suppose there is a three-Lee weight projective linear code when $k_1 = 1$. Let $\mathbf{r_0}, \mathbf{r_1}, \ldots, \mathbf{r_{k_2}}$ be the rows of generator matrix G_{1,k_2}. Let $\mathbf{r_0} = (r_{00}, r_{01}, \ldots, r_{0,n-1})$. We claim that r_{0j} can only take 1 or 3 for $0 \leq j \leq n-1$. Otherwise, if $r_{0j} = 0$ or 2, there is $(0, \ldots, 2, \ldots, 0)$ in its dual code $C^{\perp}$, this is a contradiction since C is projective. Thus $\omega_1 = w_L(\mathbf{r_0}) = n$, $\omega_2 = w_L(\mathbf{2r_0}) = 2n$. We assume that $\omega_3 = 2n_2$, where n_2 is the number of 2 in the row $\mathbf{r_i}$ for $1 \leq i \leq k_2$, thus $w_L(\mathbf{2r_0} + \mathbf{r_i}) = 2n - 2n_2$ it is easy to check that $2n - 2n_2$ can't equal to ω_1, ω_2 and ω_3. This completes the proof. □

Lemma 3.2. *Let C be a linear code over $\mathbb{Z}_4$ with type $4^{k_1}2^{k_2}$ of length n, where $k_1 \geq 1$ and $k_2 \geq 1$. Let $\mathbf{r_1}, \ldots, \mathbf{r_{k_1}}, \mathbf{r_{k_1+1}}, \ldots, \mathbf{r_{k_1+k_2}}$ be the rows of the generator matrix G_{k_1,k_2}. If $\mathbf{r} = (r_{00}, r_{01}, \ldots, r_{0,n-1})$ is a linear combination of the first k_1 rows and r_{0j} can only take 1 or 3 for $0 \leq j \leq n-1$, then there is no three-Lee weight projective linear code.*

Here are two fundamental lemmas.

Lemma 3.3. *Let C be an N-Lee weight linear code of length n with nonzero Lee weights $\omega_1, \omega_2, \ldots, \omega_N$ over $\mathbb{Z}_4$ and G be its generator matrix. If C' is generated by*

$$G' = \begin{pmatrix} G & G & G & G \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{pmatrix},$$

where $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length n, $i \in \mathbb{Z}_4$, then C' is an $(N+1)$-Lee weight linear code of length $n' = 4n$ withh nonzero Lee weights $\omega'_1 = 4\omega_1, \omega'_2 = 4\omega_2, \ldots, \omega'_N = 4\omega_N$ and $\omega'_{N+1} = 4n$. Moreover, $A'_{\omega_1} = A_{\omega_1}$, $A'_{\omega_2} = A_{\omega_2}$, $\ldots$, $A'_{\omega_N} = A_{\omega_N}$, and $A'_{\omega_{N+1}} = 3|C|$, where A'_{ω_i} is the number of the codewords of weight ω'_i in C', $1 \leq i \leq N+1$, and A_{ω_j} is the number of the codewords of weight ω_j in C, $1 \leq j \leq N$.

Proof. It is easily seen that $n' = 4n$, $\omega'_i = 4\omega_i$ and $A'_{\omega_i} = A_{\omega_i}$, $1 \leq i \leq N$. Without loss of generality, let $\mathbf{c}' = (\mathbf{c}\ \mathbf{c}\ \mathbf{c}\ \mathbf{c})$ and $\mathbf{r}_0 = (\mathbf{0}\ \mathbf{1}\ \mathbf{2}\ \mathbf{3})$, where $\mathbf{c} \in C$ and $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length n, $i \in \mathbb{Z}_4$. Thus $W_L(\mathbf{r}_0) = W_L(2\mathbf{r}_0) = W_L(3\mathbf{r}_0) = 4n$. Denote the number of i in $\mathbf{c}$ by n_i, $i \in \mathbb{Z}_4$.

(1) If the order of $\mathbf{c}$ is 2, then $n = n_0 + n_2$, and

$$\begin{aligned} W_L(\mathbf{r}_0 + \mathbf{c}') &= W_L(\mathbf{c}) + W_L(\mathbf{c} + \mathbf{1}) + W_L(\mathbf{c} + \mathbf{2}) + W_L(\mathbf{c} + \mathbf{3}) \\ &= 2n_2 + n + 2(n - n_2) + n \\ &= 4n. \end{aligned}$$

(2) If the order of $\mathbf{c}$ is 4, then $n = n_0 + n_1 + n_2 + n_3$, and

$$\begin{aligned} W_L(\mathbf{r}_0 + \mathbf{c}') &= W_L(\mathbf{c}) + W_L(\mathbf{c}+\mathbf{1}) + W_L(\mathbf{c}+\mathbf{2}) + W_L(\mathbf{c}+\mathbf{3}) \\ &= (n_1 + 2n_2 + n_3) + (n_0 + 2n_1 + n_2) \\ &\quad + (2n_0 + n_1 + n_3) + (n_0 + n_2 + 2n_3) \\ &= 4(n_1 + n_3 + n_2 + n_0) \\ &= 4n. \end{aligned}$$

Similarly, we can also prove $W_L(2\mathbf{r}_0 + \mathbf{c}') = W_L(3\mathbf{r}_0 + \mathbf{c}') = 4n = \omega'_{N+1}$. Therefore, $A'_{\omega_{N+1}} = 3(|C| - 1) + 3 = 3|C|$. □

Remark. Obviously, if $\omega_i = n$ for some i, then C' is still an N-Lee weight code and $A'_{\omega_i} = A_{\omega_i} + 3|C|$.

Similar to the discussion of Lemma 3.3, we have the following lemma.

Lemma 3.4. *Let C be an N-Lee weight linear code of length n with nonzero Lee weights $\omega_1, \omega_2, \ldots, \omega_N$ over $\mathbb{Z}_4$ and G be its generator matrix. If C' is generated by*

$$G' = \begin{pmatrix} G & G \\ \mathbf{0} & \mathbf{2} \end{pmatrix},$$

where $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length n, $i \in 2\mathbb{Z}_4$, then C' is an $(N+1)$-Lee weight linear code of length $n' = 2n$ with nonzero Lee weights $\omega'_1 = 2\omega_1, \omega'_2 = 2\omega_2, \ldots, \omega'_N = 2\omega_N$ and $\omega'_{N+1} = 2n$. Moreover, $A'_{\omega_1} = A_{\omega_1}$, $A'_{\omega_2} = A_{\omega_2}$, $\ldots$, $A'_{\omega_N} = A_{\omega_N}$, and $A'_{\omega_{N+1}} = |C|$, where A'_{ω_i} is the number of the codewords of weight ω'_i in C', $1 \le i \le N+1$, and A_{ω_j} is the number of the codewords of weight ω_j in C, $1 \le j \le N$.

Remark: Obviously, if $\omega_i = n$ for some i, then C' is still an N-Lee weight code and $A'_{\omega_i} = A_{\omega_i} + |C|$.

Let C be an N-Lee weight linear code of length n with type $4^s 2^t$ over $\mathbb{Z}_4$ and G be its generator matrix. Assume the nonzero Lee weights of C are $\omega_1, \omega_2, \ldots$, and ω_N. If $N = 2$, $\omega_1 \ne n$, and $\omega_2 \ne n$, or $N = 3$, $\omega_i = n$, for some $i \in \{1, 2, 3\}$, then we can construct a three-Lee weight code with type $4^{k_1} 2^{k_2}$, where $k_1 \ge s + 1$, and $k_2 \ge t + 1$.

4. Constructions

In this section, we will give the constructions of three-weight codes with type $4^{k_1} 2^{k_2}$ over $\mathbb{Z}_4$.

Proposition 4.1. *Let C be a linear code over $\mathbb{Z}_4$ with type $4^2 2^0$ of length 3 with the following generator matrix:*

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}.$$

It is easy to check that C is a two-Lee weight linear code with nonzero Lee weights $\omega_1 = 2$, and $\omega_2 = 4$. Moreover, $A_{\omega_1} = 6$, and $A_{\omega_2} = 9$.

We can generalize Propsition 4.1 to the general case by Lemma 3.3 and Lemma 3.4.

Theorem 4.2. *Let C_1 be the linear code over $\mathbb{Z}_4$ with type $4^2 2^{k_2}$ of length $n = 3\cdot 2^{k_2}$ with generator matrix*

$$G_{k_2+2,n} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{1} \\ G_{0,k_2} & G_{0,k_2} & G_{0,k_2} \end{pmatrix}, \tag{4.1}$$

where $\mathbf{i}$ is the row vector $(\underbrace{i, i, \ldots, i}_{2^{k_2}})$, $i = 1$ or 2 and $(c_{3,j}, \ldots, c_{k_2+2,j})^\top, c_{i,j} = 0$ or $2, 3 \le i \le k_2 + 2, 1 \le j \le 2^{k_2}$ runs over all distinct column vectors of G_{0,k_2}. Then C_1 is a three-Lee weight code with nonzero weights $\omega_1 = 2^{k_2+1}, \omega_2 = 3\cdot 2^{k_2}, \omega_3 = 2^{k_2+2}$, of respective frequencies $A_{\omega_1} = 6,\ A_{\omega_2} = 2^{k_2+4} - 16,\ A_{\omega_3} = 9$.

Proof. Let $\mathbf{r_{01}}, \mathbf{r_{02}}, \mathbf{r_1}, \ldots, \mathbf{r_{k_2}}$ be the rows of generator matrix $G_{k_2+2,n}$, where

$$\mathbf{r_{01}} = (\underbrace{1, 1, \ldots, 1}_{2^{k_2}}, \underbrace{1, 1, \ldots, 1}_{2^{k_2}}, \underbrace{2, 2, \ldots, 2}_{2^{k_2}}), \mathbf{r_{02}} = (\underbrace{2, 2, \ldots, 2}_{2^{k_2}}, \underbrace{1, 1, \ldots, 1}_{2^{k_2}}, \underbrace{1, 1, \ldots, 1}_{2^{k_2}}).$$

It is easy to check that $W_L(p_{01}\mathbf{r_{01}} + p_{02}\mathbf{r_{02}}) = 2^{k_2+1}$or 2^{k_2+2} unless $p_{01} = p_{02} = 0$ for any $p_{0i} \in \mathbb{Z}_4,\ 1 \le i \le 2$. Moreover, we can also prove that $W_L(p_{01}\mathbf{r_{01}} + p_{02}\mathbf{r_{02}} + p_1\mathbf{r_1} + \cdots + p_{k_2}\mathbf{r_{k_2}}) = 3\cdot 2^{k_2}$ unless $p_1 = \cdots = p_{k_2} = 0$ for any $p_{0i} \in \mathbb{Z}_4,\ 1 \le i \le 2,\ p_j \in \mathbb{F}_2,\ 1 \le j \le k_2$. Hence C_1 contains 9 codewords with weight 2^{k_2+2}, 6 codewords of weight 2^{k_2+1} and $2^{k_2+4} - 16$ codewords with weight $3 \cdot 2^{k_2}$. □

Example 1. If $k_1 = 2$ and $k_2 = 2$, then $n = 12$ and $\omega_1 = 8$, $\omega_2 = 12$, $\omega_3 = 16$, according to Theorem 4.2, there exists a three-Lee weight code with the generator matrix:

$$G_{2,2} = \begin{pmatrix} 1&1&1&1&1&1&1&1&2&2&2&2 \\ 2&2&2&2&1&1&1&1&1&1&1&1 \\ 0&0&2&2&0&0&2&2&0&0&2&2 \\ 0&2&0&2&0&2&0&2&0&2&0&2 \end{pmatrix}.$$

Corollary 4.3. *The coset graph of $C_1^\perp$ is a 3-SWRG.*

Proof. From [2] we know that a graph with three restricted eigenvalues is 3-SWRG iff they add up to zero. Translating in terms of weights of C_1 by Theorem 2.2 we see that it is equivalent to check that $\omega_1 + \omega_2 + \omega_3 = 3n$. This is easy to check by the explicit formulas of Theorem 4.2. □

In the above theorem, if we assume

$$G_{2,k_2} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{1} \\ G_{0,k_2} & G_{0,k_2} & G_{0,k_2} \end{pmatrix} = (G_1^2, G_2^2, G_3^2), \tag{4.2}$$

then we have the following result.

Theorem 4.4. *Let C_2 be the linear code over $\mathbb{Z}_4$ with type $4^{k_1}2^{k_2}$ of length $n = 3\cdot 2^{2k_1+k_2-4}$ with a generator matrix G_{k_1,k_2}, defined inductively as follows. G_{2,k_2} is as above. Assume*

$G_{m,k_2} = (G_1^m, G_2^m, G_3^m)$, *and if* $\mathbf{i}$ *is the row vector* $(i, i, \ldots, i)$, $i \in \mathbb{Z}_4$, $m \geq 2$ *define* G_{m+1,k_2} *as*

(4.3)

$$\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ G_1^m & G_1^m & G_1^m & G_1^m & G_2^m & G_2^m & G_2^m & G_2^m & G_3^m & G_3^m & G_3^m & G_3^m \end{pmatrix}.$$

Then C_2 *is a three-Lee weight code with nonzero weights* $\omega_1 = 2^{2k_1+k_2-3}$, $\omega_2 = 3 \cdot 2^{2k_1+k_2-4}$, $\omega_3 = 2^{2k_1+k_2-2}$, *of respective frequencies* $A_{\omega_1} = 6$, $A_{\omega_2} = 2^{2k_1+k_2} - 16$, *and* $A_{\omega_3} = 9$, *where* $k_1 \geq 2$ *and* $k_2 \geq 1$.

Proof. If $k_1 = 2$, the result is proved by Theorem 2. Assume the result is valid for any $k_1 = m$. Namely, if $G_{m,k_2} = (G_1^m, G_2^m, G_3^m)$ is the generator matrix of a three-Lee weight code with nonzero weights $\omega_1 = 2^{2m+k_2-3}$, $\omega_2 = 3 \cdot 2^{2m+k_2-4}$ and $\omega_3 = 2^{2m+k_2-2}$. Now we consider the case when $k_1 = m+1$, the generator matrix of the linear code C is (4), it is easily seen that C is a three-Lee weight code with nonzero weight $\overline{\omega}_1 = 2^{2m+k_2-1} = 4\omega_1$, $\overline{\omega}_2 = 3 \times 2^{2m+k_2-2} = 4\omega_2$, $\overline{\omega}_3 = 2^{2m+k_2} = 4\omega_3$. By the induction hypothesis, C is a three-Lee weight code over $\mathbb{Z}_4$. □

Example 2. If $k_1 = 3, k_2 = 1$, then $n = 24$, $\omega_1 = 16$, $\omega_2 = 24$, $\omega_3 = 32$, according to Theorem 4.3, there is a three-Lee weight code with the generator matrix $G_{3,1}$:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \end{pmatrix},$$

where

$$G_{2,1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 \end{pmatrix}.$$

The analogue of Corollary 4.3 is the following result, the proof of which is omitted.

Corollary 4.5. *The coset graph of* $C_2^{\perp}$ *is a* 3*-SWRG.*

Here we give another method to construct three-Lee weight codes.

Proposition 4.6. *Consider the code* C *generated by the following generator matrix:*

$$G_0 = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 1 & 1 & 2 \end{pmatrix}.$$

It is easy to check that C *is a three-Lee weight linear code with nonzero Lee weights* $\omega_1 = 4$, $\omega_2 = 8$, *and* $\omega_3 = 6$. *Moreover,* $A_{\omega_1} = 18$, $A_{\omega_2} = 21$, *and* $A_{\omega_3} = 24$.

Theorem 4.7. *Let* C_3 *be a linear code over* $\mathbb{Z}_4$ *with type* $4^{k_1}2^{k_2}$ *of length* n. *If the columns of the generator matrix* $G_{(k_1+k_2)\times n}$ *are all distinct nonzero vectors*

$$(c_1, c_2, \ldots, c_{k_1}, c_{k_1+1}, \ldots, c_{k_1+k_2})^T,$$

where $(c_1, c_2, \ldots, c_{k_1})^T$ *is one column of* G_0 *defined above,* $c_i \in \mathbb{Z}_4, 4 \leq i \leq k_1$ *when* $k_1 \geq 4$, *and* c_j = *0 or 2,* $k_1 + 1 \leq j \leq k_1 + k_2$, *then* C_3 *is a three-Lee weight code of length* $n = 3 \times 2^{2k_1+k_2-5}$ *with nonzero Lee weights* $\omega_1 = 2^{2k_1+k_2-4}$, $\omega_2 = 2^{2k_1+k_2-3}$ *and* $\omega_3 = 3 \times 2^{2k_1+k_2-5}$. *Moreover,* $A_{\omega_1} = 18$, $A_{\omega_2} = 21$, *and* $A_{\omega_3} = 4^{k_1}2^{k_2} - 40$, *respectively.*

Proof. Denote the generator matrix of C_3 by G_{k_1,k_2}.

If $k_1 = 3$, $k_2 = 0$, then $G_{3,0} = G_0$, and the result is valid from the statement of Proposition 4.6. Assume the result is valid for any $k_2 = m$ and we denote the nonzero weights by ω_1', ω_2' and ω_3'. Since

$$G_{3,m+1} = \begin{pmatrix} G_{3,m} & G_{3,m} \\ \mathbf{0} & \mathbf{2} \end{pmatrix},$$

where $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length $3 \times 2^{m+1}$, $i \in 2\mathbb{Z}_4$, according to Lemma 3.4 and the remark of Lemma 3.4, C_3 is a three-Lee weight code with nonzero Lee weights $\omega_1 = 2\omega_1'$, $\omega_2 = 2\omega_2'$ and $\omega_3 = 2\omega_3'$, By induction hypothesis, the results are valid.

It is easy to check that

$$G_{k_1+1,k_2} = \begin{pmatrix} G_{k_1,k_2} & G_{k_1,k_2} & G_{k_1,k_2} & G_{k_1,k_2} \\ \mathbf{0}' & \mathbf{1}' & \mathbf{2}' & \mathbf{3}' \end{pmatrix},$$

where $\mathbf{i}'$ is the row vector $(i, i, \ldots, i)$ of length $3 \times 2^{2k_1+k_2-5}$, $i \in \mathbb{Z}_4$. Similar to the proof of (1), we can prove C_3 is a three-Lee weight code by Lemma 3.3 and the following remark of Lemma 3.3. □

Example 3. If $k_1 = 3, k_2 = 1$, then $w_1 = 8$, $w_2 = 16$, and $w_3 = 12$, then, according to Theorem 4.7, there is a three-Lee weight code with the generator matrix:

$$G_1 = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.4}$$

Like Corollary 2, we have the following graphical consequence, whose proof is omitted.

Corollary 4.8. *The coset graph of $C_3^\perp$ is 3-SWRG.*

Here we prove the codes we get are projective.

Theorem 4.9. *The codes C_2 and C_3 obtained in Theorem 4.3 and Theorem 4.4 are projective, respectively.*

Proof. Take C_3 for example. Let $C_3^\perp$ be the dual code of C_3, G be the generator matrix of C_3, and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be the columns of G, where n is the length of C_3. It's easy to check that every column of G contains 1 and any two columns of G are not multiple of each other by ± 1. If $\mathbf{c} \in C_3^\perp$ and $\mathbf{c} \neq \mathbf{0}$, then we only need to prove $W_L(\mathbf{c}) \geq 3$. Obviously, $W_L(\mathbf{c})$ cannot be 1 because all columns are nonzero. Assume $W_L(\mathbf{c}) = 2$. Since every column of G contains at least a 1, $\mathbf{c}$ cannot be written as $\mathbf{c} = (0, \ldots, 0, 2, 0, \ldots, 0)$. Then we have 2 cases:

Case 1 Without loss of generality, let $\mathbf{c} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$, where $c_i = c_j = 1, i < j$. Then $\mathbf{x}_i + \mathbf{x}_j = \mathbf{0}$. Thus, $\mathbf{x}_j = 3\mathbf{x}_i$, $\mathbf{x}_i$ and $\mathbf{x}_j$ are in proportion by the unit 3, contradiction.

Case 2 Without loss of generality, let $\mathbf{c} = (0, \ldots, 0, 1, 0, \ldots, 0, 3, 0, \ldots, 0)$, where $c_i = 1, c_j = 3, i < j$. Then $\mathbf{x}_i + \mathbf{x}_j = \mathbf{0}$. Thus, $\mathbf{x}_j = \mathbf{x}_i$, contradiction.

Therefore, C_3 is projective. Similarly, we can prove C_2 is projective. □

5. Linearity of Gray Images

For our purpose, we first require the following classic lemma of [3, Th.4].

Lemma 5.1. *If C is a linear code over $\mathbb{Z}_4$, with generators $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}$, then $\Phi(C)$ is linear if and only if $2\alpha(\mathbf{x_i}) * \alpha(\mathbf{x_j})$ in C for all i, j, satisfying $1 \le i \le j \le m$.*

We can now state and prove the following results.

Theorem 5.2. *The Gray image $\phi(C_2)$ of C_2 is nonlinear.*

Proof. It is enough to prove the result is valid for $k_1 = 2$. Let $\mathbf{x_1}$ and $\mathbf{x_2}$ be two row vectors of $G_{k_2+2,n}$. Clearly, if either $\mathbf{x_1}$ or $\mathbf{x_2}$ is from the last k_2 rows, it is easy to check that $2\alpha(\mathbf{x_1}) * \alpha(\mathbf{x_2}) = 0 \in C_2$. When

$$\mathbf{x_1} = (\underbrace{1,1,\ldots,1}_{2^{k_2}}, \underbrace{1,1,\ldots,1}_{2^{k_2}}, \underbrace{2,2,\ldots,2}_{2^{k_2}}), \mathbf{x_2} = (\underbrace{2,2,\ldots,2}_{2^{k_2}}, \underbrace{1,1,\ldots,1}_{2^{k_2}}, \underbrace{1,1,\ldots,1}_{2^{k_2}}),$$

we have

$$2\alpha(\mathbf{x_1}) * \alpha(\mathbf{x_2}) = (\underbrace{0,0,\ldots,0}_{2^{k_2}}, \underbrace{2,2,\ldots,2}_{2^{k_2}}, \underbrace{0,0,\ldots,0}_{2^{k_2}}).$$

The weight of $2\alpha(\mathbf{x_1}) * \alpha(\mathbf{x_2})$ is ω_1. According to Theorem 4.2, $A_{\omega_1} = 6$. However, we can easily check that $2\alpha(\mathbf{x_1}) * \alpha(\mathbf{x_2})$ is not in C_2, because it is not the linear combination of the first two rows. Therefore C_2 is nonlinear. □

Example 4. Consider the code defined in Example 1. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ denote the first and second rows of $G_{2,2}$, respectively. Then we can get

$$2\alpha(\mathbf{x}_1) * \alpha(\mathbf{x}_2) = 2\alpha(111111112222) \ * \ \alpha(222211111111) = (000022220000).$$

However, we can easily check that $2\alpha(\mathbf{x_1}) * \alpha(\mathbf{x_2})$ is not in C_2, because it is not the linear combination of the first two rows. Therefore C_2 is nonlinear.

Similar to Theorem 5.2, we have the following theorem.

Theorem 5.3. *The Gray image $\phi(C_3)$ of C_3 is nonlinear.*

Proof. It is easy to check that the first three rows of G_{k_1,k_2} can be written as:

$$\underbrace{G_0 \ G_0 \ \cdots \ G_0}_{4^{k_1-3}2^{k_2}},$$

which is permutation-equivalent to the matrix

$$G_0' = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{2} \end{pmatrix}, \tag{5.1}$$

where $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length $4^{k_1-3}2^{k_2}$, $i \in \mathbb{Z}_4$. Let $\mathbf{r}_1$ and $\mathbf{r}_3$ be the first and third rows of G_0', respectively, then $W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_3)) = 2 \times 4^{k_1-3}2^{k_2} = 2^{2k_1+k_2-5}$. By Theorem 4.7, C_3 is a three-Lee weight code with nonzero Lee weights $\omega_1 = 2^{2k_1+k_2-4}$, $\omega_2 = 2^{2k_1+k_2-3}$ and $\omega_3 = 3 \times 2^{2k_1+k_2-5}$. Thus, $2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_3) \notin C_3$. By Lemma 5.1, $\phi(C_3)$ is nonlinear. □

Example 5. Consider the code defined in Example 3. Let $\mathbf{r}_1$ and $\mathbf{r}_3$ denote the first and third rows of G_1 in (4.4), respectively. Then the Lee weight of

$$2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_3) = 2\alpha(123000123000) \ * \ \alpha(112112112112) = (200000200000)$$

is 4, which can not be contained in C_3 since C_3 is a three-Lee weight code with $\omega_1 = 8$, $\omega_2 = 16$, and $\omega_3 = 12$.

6. Conclusion and open problems

In this paper, we have constructed two infinite families of three-weight projective codes over $\mathbb{Z}_4$. From there strongly walk-regular graphs were built. There are two main directions of inquiry from that point on. One is to look at other families of three-weight $\mathbb{Z}_4$-codes. The other is to extend this research to other rings. In particular, chain rings are the natural candidates to replace $\mathbb{Z}_4$ as an alphabet for our three-weight codes.

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