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DEGREE CONDITIONS OF NEARLY INDUCED MATCHING EXTENDABLE GRAPHS

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Abstract. A graph G is induced matching extendable (shortly, IM-extendable) if every induced matching of G is included in a perfect matching of G. The IM-extendable graph was first introduced by Yuan in [11]. A graph G is nearly IM-extendable if $G \vee K_1$ is IM-extendable. We show in this paper that: (1) Let G be a graph with $2n-1$ vertices, where $n \geq 2$. If for each pair of nonadjacent vertices u and v in G, $d(u) + d(v) \geq 2\lceil 4n/3 \rceil - 3$, then G is nearly IM-extendable. (2) Let G be a claw-free graph with $2n-1$ vertices, where $n \geq 2$. If for each pair of nonadjacent vertices u and v in G, $d(u) + d(v) \geq 2n-1$, then G is nearly IM-extendable. Minimum degree conditions of nearly IM-extendable graphs and nearly IM-extendable claw-free graphs are also obtained in this paper. It is also shown that all these results are best possible.

1. Introduction

Graphs considered in this paper are finite, simple, and undirected. For a graph G, $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. The edge joining two vertices x and y is written as xy. The number of edges in G incident with a vertex u is called the degree of u, and denoted by $d(u)$. The minimum degree of G is denoted by $\delta(G)$. For $X \subseteq V(G)$, the neighbour set $N_G(X)$ of X is defined by

$$N_G(X) = \{y \in V(G) \setminus X : \text{ there is } x \in X \text{ such that } xy \in E(G)\}.$$

The neighbour set of vertex u in $V(G)$ is denoted by $N(u)$. For $W \subseteq V(G)$, set

$$E(W) = E(G[W]) = \{uv \in E(G) : u, v \in W\}.$$

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For $M \subseteq E(G)$, set

$$V(M) = \{x \in V(G) : \text{ there is } y \in V(G) \text{ such that } xy \in M\}.$$

When no confusion can occur, a single-element set $\{a\}$ will be denoted by a for short. A component H of G is odd (even) if $|V(H)|$ is odd (even). The number of odd components of G is denoted by $o(G)$. Let G_1 and G_2 be two disjoint graphs. We denote their union by $G_1 \cup G_2$. The join $G_1 \vee G_2$ is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2. A graph G is called claw-free, if it does not contain $K_{1,3}$ as an induced subgraph.

We call $M \subseteq E(G)$ a matching [5] of G, if $V(e) \cap V(f) = \emptyset$ for every two distinct edges $e, f \in M$. A matching M of G is perfect [5], if $V(M) = V(G)$. We say a matching M of G is an induced matching [2, 3] if $E(V(M)) = M$. A connected graph G is said to be n-extendable [6], if $|V(G)| \geq 2n + 2$, G has a perfect matching, and for every matching M of G with $|M| = n$, there is a perfect matching M^* of G, such that $M \subseteq M^*$. A graph G is said to be induced matching extendable [11] (shortly, IM-extendable), if every induced matching of G is included in a perfect matching of G. A graph G is said to be nearly IM-extendable, if $G \vee K_1$ is IM-extendable.

Research on IM-extendable graphs can be found, for example, in [4], [7], [9]–[15]. For notation and terminology not defined here we refer to [1].

In this paper, we will give some degree sum conditions and minimum degree conditions of nearly IM-extendable graphs. We also show that these results are best possible. The following four Lemmas will be very useful in our proofs.

Lemma 1.1 ([6]). *Let G be a graph with p vertices, where p is even, and let n be an integer with $1 \leq n < p/2$. Suppose that for each pair of nonadjacent vertices u and v in G, $d(u) + d(v) \geq p + 2n - 1$. Then G is n-extendable.*

Lemma 1.2 (Tutte's Theorem [8]). *A graph G has a perfect matching if and only if for every vertex subset $S \subset V(G)$, $o(G - S) \leq |S|$.*

Lemma 1.3 ([5]). *If G is a connected claw-free graph with even number of vertices, then G has a perfect matching.*

Lemma 1.4 ([1]). *Suppose that G is a graph with minimum degree at least $\frac{1}{2}(|V(G)| + k - 2)$, where $1 \leq k \leq |V(G)| - 1$, then G is k-connected.*

2. Main Results

Theorem 2.1. *Let G be a graph with $2n - 1$ vertices, where $n \geq 2$. If for each pair of nonadjacent vertices u and v in G, $d(u) + d(v) \geq 2\lceil 4n/3 \rceil - 3$, then G is nearly IM-extendable, and the result is best possible.*

Proof. When $n = 2$, G is a graph with 3 vertices. We can easily check that G is nearly IM-extendable if and only if G is C_3. Hence we suppose $n \geq 3$ in the sequel. Let $F = G \vee K_1$, where $K_1 = s$ and $|V(F)| = 2n$. We need only show that F is IM-extendable.

Because s is adjacent to each vertex in $V(G)$, we have

$$d_F(u) + d_F(v) \geq 2\left\lceil \frac{4n}{3} \right\rceil - 3 + 2 = 2\left\lceil \frac{4n}{3} \right\rceil - 1 \tag{2.1}$$

for each pair of nonadjacent vertices u and v in F, where $d_F(w)$ is the degree of vertex w in F. Let M be an induced matching in F. The following Claim holds.

Claim 1. $|M| \leq \lceil n/3 \rceil$.

Suppose to the contrary that $|M| \geq \lceil n/3 \rceil + 1$. Because $n \geq 3$, $|M| \geq 2$. There must be two nonadjacent vertices x and y in $V(M)$. By (2.1),

$$d_F(x) + d_F(y) \geq 2\left\lceil \frac{4n}{3} \right\rceil - 1.$$

It follows that there exists $t \in \{x, y\}$ such that

$$d_F(t) \geq \left\lceil \frac{4n}{3} \right\rceil.$$

Because $d_F(t) \leq |V(F)| - |V(M)| + 1$, we have

$$\left\lceil \frac{4n}{3} \right\rceil \leq 2n - 2(\left\lceil \frac{n}{3} \right\rceil + 1) + 1 = 2\left\lfloor \frac{2n}{3} \right\rfloor - 1 \leq \frac{4n}{3} - 1,$$

a contradiction. This completes the proof of Claim 1.

Hence we must have $|M| \leq \lceil n/3 \rceil$. By Lemma 1.1, F is $\lceil n/3 \rceil$-extendable. So F is IM-extendable and G is nearly IM-extendable.

Now we show that the result is best possible. We construct a graph G with $2n-1$ vertices and degree sum $2\lceil 4n/3 \rceil - 4$ which is not nearly IM-extendable as follows.

Let $V(G) = A_1 \cup A_2 \cup A_3$ and $A_i \cap A_j = \emptyset$, where $1 \leq i < j \leq 3$.

Case 1. $2n \equiv 0 \pmod 3$. *Let* $2n - 1 = 3m - 1$.

In this case, $2\lceil 4n/3 \rceil - 4 = 4m - 4$ and m is even. Let $|A_1| = m$, $|A_2| = m + 1$, $|A_3| = m - 2$. Let $E(A_1)$ be a matching of size $m/2$,

$$E(A_i, A_j) = \{xy | x \in A_i, y \in A_j\}, \text{ for } 1 \leq i < j \leq 3,$$

$$E(A_2) = E(A_3) = \emptyset.$$

The edge set of G is defined by

$$E(G) = \left(\bigcup_{i=1}^{3} E(A_i)\right) \cup \left(\bigcup_{1 \leq i < j \leq 3} E(A_i, A_j)\right).$$

Clearly, the minimum degree sum of the constructed graph G is $4m - 4 = 2\lceil 4n/3 \rceil - 4$. Let $F = G \vee K_1$, then $E(A_1)$ is an induced matching of F. Let $H = F - A_1$, we have $o(H - A_3 \cup K_1) = |A_2| > |A_3| + 1$. By Lemma 1.2, H has no perfect matching. So F is not IM-extendable and G is not nearly IM-extendable.

Case 2. $2n \equiv 1 \pmod 3$. *Let* $2n-1=3m$.

In this case, $2\lceil 4n/3\rceil - 4 = 4m-2$ and m is odd. Let $|A_1| = |A_2| = m+1$, $|A_3| = m-2$. Let $E(A_1)$ be a matching of size $(m+1)/2$; and the remaining part of $E(G)$ is the same as Case 1. Clearly, the minimum degree sum of G is $4m-2$. It is easy to check that G is not nearly IM-extendable by a similar argument as above.

Case 3. $2n \equiv 2 \pmod 3$. *Let* $2n-1=3m+1$.

In this case, $2\lceil 4n/3\rceil - 4 = 4m$ and m is even. Let $|A_1| = m+2$, $|A_2| = m+1$, $|A_3| = m-2$. Let $E(G)$ be the same as before except that $E(A_1)$ is a matching of size $(m+2)/2$. Clearly, the minimum degree sum of G is $4m$, and G is not nearly IM-extendable.

We now summarize the above three cases as follows.

Let $V(G) = A_1 \cup A_2 \cup A_3$, $|V(G)| = 2n-1 = 3m+k-1$, and $A_i \cap A_j = \emptyset$, where $1 \le i < j \le 3$, $k = 0, 1, 2$.

Let $|A_1| = m+k$, $|A_2| = m+1$, $|A_3| = m-2$. Let $E(A_1)$ be a matching of size $|A_1|/2$,

$$E(A_i, A_j) = \{xy | x \in A_i, y \in A_j\}, \text{ for } 1 \le i < j \le 3.$$

The edge set of G is defined by

$$E(G) = E(A_1) \cup \Big(\bigcup_{1\le i<j\le 3} E(A_i, A_j)\Big).$$

Then G is a graph with $2n-1$ vertices and degree sum $2\lceil 4n/3\rceil - 4$ which is not nearly IM-extendable, and so the result is best possible. □

The following Corollary can be obtained from Theorem 2.1 directly. We can use the graphs constructed in the proof of Theorem 2.1 to show that the result in Corollary 2.2 is best possible.

Corollary 2.2. $\lceil 4n/3\rceil - 1$ *is the minimum integer* δ *such that every graph* G *with minimum degree at least* δ *is nearly IM-extendable, where* $3 \le |V(G)| = 2n-1$.

Theorem 2.3. *Let* G *be a claw-free graph with* $2n-1$ *vertices, where* $n \ge 2$. *If for each pair of nonadjacent vertices* u *and* v *in* G, $d(u)+d(v) \ge 2n-1$, *then* G *is nearly IM-extendable, and the result is best possible.*

Proof. Let $F = G \vee K_1$, where $K_1 = s$ and $|V(F)| = 2n$. Since s is adjacent to each vertex in $V(G)$, for each pair of nonadjacent vertices u and v in F,

$$d_F(u) + d_F(v) \ge 2n+1. \tag{2.2}$$

By Lemma 1.1, F is 1-extendable. We need only show that F is IM-extendable.

Let M be an induced matching of F. If $s \in V(M)$, then $|M| = 1$. So $F - V(M)$ has a perfect matching. If $s \notin V(M)$, we choose an arbitrary vertex $t \in F - V(M) - s$. If $o(F - V(M) - \{s,t\}) = 0$, by Lemma 1.3,

$F-V(M)-\{s,t\}$ has a perfect matching. Otherwise, $o(F-V(M)-\{s,t\}) \geq 2$. Let F_1 and F_2 be two odd components of $F-V(M)-\{s,t\}$. Choose $x \in V(F_1)$ and $y \in V(F_2)$ arbitrarily. By (2.2),

$$d_F(x) + d_F(y) \geq 2n+1. \tag{2.3}$$

Because G is claw-free, we have

$$|N(x) \cap V(M)| \leq 4 \text{ and } |N(y) \cap V(M)| \leq 4.$$

It follows that

$$d_F(x) \leq |V(F_1)| - 1 + 4 + 2 = |V(F_1)| + 5 \tag{2.4}$$

and

$$d_F(y) \leq |V(F_2)| + 5. \tag{2.5}$$

Because

$$|V(F_1)| + |V(F_2)| + |V(M)| + 2 \leq |V(F)|, \tag{2.6}$$

we have

$$d_F(x) + d_F(y) \leq |V(F)| - |V(M)| - 2 + 10 = |V(F)| - |V(M)| + 8,$$

and so

$$2n+1 \leq 2n+8-|V(M)|.$$

This implies that $|M| \leq 3$. If $|M| = 1$, because F is 1-extendable, $F-V(M)$ has a perfect matching. We distinguish the following two cases.

Case 1. $|M| = 3$. *Then* $n > 4$.

By (2.3), (2.4), and (2.5), we have $|V(F_1)| + |V(F_2)| \geq 2n-8$. By (2.6), we have $|V(F_1)| + |V(F_2)| \leq 2n-8$. So $|V(F_1)| + |V(F_2)| = 2n-8$. This implies that $F-V(M)-\{s,t\} = F_1 \cup F_2$. It follows that there exists $i \in \{1,2\}$ such that $|V(F_i)| \geq n-4$. Without loss of generality, suppose that $|V(F_2)| \geq n-4$. Then $|V(F_1)| \leq n-4$.

Claim 1.1. $N(t) \cap (V(F_1) \cup V(F_2)) \neq \emptyset$.

Otherwise, because $|N(t) \cap V(M)| \leq 4$, we have $d(t) \leq 4$ and $d(x) \leq n-1$. So, $d(t)+d(x) \leq n+3 < 2n-1$, a contradiction to the fact that the minimum degree sum of G is at least $2n-1$. Hence $N(t) \cap (V(F_1) \cup V(F_2)) \neq \emptyset$.

Without loss of generality, suppose that $N(t) \cap V(F_2) \neq \emptyset$. By Lemma 1.3, $G[V(F_2) \cup \{t\}]$ has a perfect matching M_2. Let w be a vertex of F_1 such that w is not a cut vertex of F_1 (in fact, w can be any vertex of F_1, which we will show in the sequel). So $F_1 - w$ is connected. By Lemma 1.3, $F_1 - w$ has a perfect matching M_1. Hence $M_1 \cup \{sw\}$ is a perfect matching of $F_1 \vee s$. So $F-V(M)$ has a perfect matching $M_1 \cup M_2 \cup \{sw\}$.

Remark. *For more details and for further research, we give some characterizations of* F_1 *and* F_2 *as follows, which also show that* w *can be any vertex of* F_1 *(or* F_2*, if* $N(t) \cap V(F_2) = \emptyset$*).*

We have

Claim 1.2. *F_1 is a complete graph.*

Suppose that there are two vertices $u, v \in V(F_1)$ such that $uv \notin E(F_1)$. Then $d(u) + d(v) \leq n - 1 + n - 1 = 2n - 2 < 2n - 1$, a contradiction. Hence Claim 1.2 holds.

Claim 1.3. *If $|V(F_1)| \geq 5$, F_2 is 3-connected.*

Because $2n - 1 \leq d(x) + d(y) \leq n + d(y)$, we have $d(y) \geq n - 1$. That is $\delta(F_2) \geq n - 6$. Because $|V(F_1)| + |V(F_2)| = 2n - 8$ and $|V(F_1)| \geq 5$, we have $|V(F_2)| \leq 2n - 13$. By Lemma 1.4, F_2 is 3-connected.

Claim 1.4. *If $|V(F_1)| < 5$ and $N(t) \cap V(F_2) = \emptyset$, F_2 is a complete graph.*

Suppose to the contrary that there are two vertices $u, v \in V(F_2)$ such that $uv \notin E(F_2)$. If $F_1 = K_1$, we have $|V(F_2)| = 2n - 9$. Then $d(u) + d(t) \leq 2n - 11 + 4 + 5 = 2n - 2 < 2n - 1$, a contradiction. If $|V(F_1)| = 3$, we have $|V(F_2)| = 2n - 11$. Then $d(u) + d(t) \leq 2n - 13 + 4 + 7 = 2n - 2 < 2n - 1$, a contradiction again. Hence Claim 1.4 holds.

From the above four claims, we know that if $N(t) \cap V(F_2) \neq \emptyset$, F_1 is a complete graph; if $N(t) \cap V(F_2) = \emptyset$, either F_2 is a complete graph or F_2 is 3-connected.

Case 2. $|M| = 2$. *Then $n > 3$.*

By (2.3), (2.4), and (2.5), we have $|V(F_1)| + |V(F_2)| \geq 2n - 8$. By (2.6), we have $|V(F_1)| + |V(F_2)| \leq 2n - 6$. So $|V(F_1)| + |V(F_2)| = 2n - 6$ or $|V(F_1)| + |V(F_2)| = 2n - 8$. When $n \leq 7$, we can check that $F - V(M)$ has a perfect matching. Now we suppose $n > 7$ in the sequel.

When $n > 7$, we can obtain that $|V(F_1)| + |V(F_2)| = 2n - 6$. This implies $F - V(M) - \{s, t\} = F_1 \cup F_2$. It follows that there exists $i \in \{1, 2\}$ such that $|V(F_i)| \geq n - 3$. Without loss of generality, suppose $|V(F_2)| \geq n - 3$. Then $|V(F_1)| \leq n - 3$. We have

Claim 2.1. *$N(t) \cap (V(F_1) \cup V(F_2)) \neq \emptyset$.*

Otherwise, $d(t) \leq 4$ and $d(x) \leq n$. So, $d(t) + d(x) \leq n + 4 < 2n - 1$, a contradiction to the fact that the minimum degree sum is at least $2n - 1$. Hence $N(t) \cap (V(F_1) \cup V(F_2)) \neq \emptyset$.

From a similar discussion of Claim 1.1, we can prove that $F - V(M)$ has a perfect matching. We also give some characterizations of F_1 and F_2 as follows.

Claim 2.2. *If $N(t) \cap V(F_2) \neq \emptyset$, F_1 is a complete graph.*

Suppose that there are two vertices $u, v \in V(F_1)$ such that $uv \notin E(F_1)$. If $N(t) \cap V(F_1) = \emptyset$, then $d(u) + d(v) \leq n - 1 + n - 1 = 2n - 2 < 2n - 1$, a contradiction. If $N(t) \cap V(F_1) \neq \emptyset$, because $N(t) \cap V(F_2) \neq \emptyset$ and G is a claw-free graph, we know that $|N(t) \cap \{u, v\}| \leq 1$. Assume that $v \notin N(t)$. So $d(u) + d(v) \leq |V(F_1)| - 2 + 5 + |V(F_1)| - 2 + 4 = 2|V(F_1)| + 5 \leq 2n - 1$. Since $d(u) + d(v) \geq 2n - 1$, we have $d(u) + d(v) = 2n - 1$. This implies that

$|N(u) \cap V(M)| = 4$ and $|N(v) \cap V(M)| = 4$. So $N(F_2) \cap V(M) = \emptyset$. But now $d(v)+d(y) \leq |V(F_1)|-2+4+|V(F_2)|-1+1 = |V(F_1)|+|V(F_2)|+2 = 2n-4 < 2n-1$, a contradiction. Hence F_1 is a complete graph and Claim 2.2 holds.

Claim 2.3. *If $|V(F_1)| \geq 9$, F_2 is 3-connected.*

Because $2n-1 \leq d(x)+d(y) \leq n+1+d(y)$, we have $d(y) \geq n-2$. That is $\delta(F_2) \geq n-7$. Because $|V(F_1)|+|V(F_2)| = 2n-6$ and $|V(F_1)| \geq 9$, we have $|V(F_2)| \leq 2n-15$. By Lemma 1.4, F_2 is 3-connected.

Claim 2.4. *If $|V(F_1)| < 9$ and $N(t) \cap V(F_2) = \emptyset$, F_2 is 3-connected.*

If $F_1 = K_1$, since $|V(F_1)|+|V(F_2)| = 2n-6$, we have $|V(F_2)| = 2n-7$. Because $d(t) \leq 5$, $d(y) \geq 2n-6$. So $\delta(F_2) \geq 2n-10$. Since $n > 7$, by Lemma 1.4, F_2 is 5-connected. The cases when $|V(F_1)| = 3, 5$, and 7 can be proved similarly.

From these four claims, we can deduce that if $N(t) \cap V(F_2) \neq \emptyset$, F_1 is a complete graph; if $N(t) \cap V(F_2) = \emptyset$, F_2 is 3-connected.

In both Case 1 and Case 2, we have proved that $F - V(M)$ has a perfect matching. According to the above analysis, we know that F is IM-extendable, and so G is nearly IM-extendable.

The result is best possible. For if $G = K_1 \vee (K_x \cup K_{2n-2-x})$, where $K_1 = t$ and x is an odd number such that $1 \leq x < 2n-2$, we can easily check that G is a claw-free graph with minimum degree sum $2n-2$. Let $F = G \vee s$ and $M = st$. Since $F - V(M)$ has two odd components, F is not IM-extendable, and so G is not nearly IM-extendable.

□

Theorem 2.4. *$2\lfloor n/2 \rfloor$ is the minimum integer δ such that every claw-free graph G with minimum degree at least δ is nearly IM-extendable, where $3 \leq |V(G)| = 2n-1$.*

Proof. Let $F = G \vee K_1$, where $K_1 = s$ and $|V(F)| = 2n$. Since s is adjacent to each vertex in $V(G)$, $\delta(F) \geq 2\lfloor n/2 \rfloor + 1$. By Lemma 1.1, when n is even, F is 1-extendable.

Now we show that F is IM-extendable. Let M be an induced matching of F. For each vertex $x \in V(F) - V(M) - s$, since G is claw-free, $|N(x) \cap V(M)| \leq 4$.

Claim 1. *If $s \in V(M)$, $F - V(M)$ has a perfect matching.*

If $s \in V(M)$, then $|M| = 1$. If n is even, since F is 1-extendable, $F - V(M)$ has a perfect matching. If n is odd, we show that $o(F - V(M)) = 0$, and so $F - V(M)$ has a perfect matching. Suppose to the contrary that $o(F - V(M)) \geq 2$. Then there must exist an odd component F_1 of $F - V(M)$ such that $|V(F_1)| \leq n-2$. For any vertex $v \in V(F_1)$, we have $d(v) \leq n-2 < 2\lfloor n/2 \rfloor$, a contradiction. Hence Claim 1 holds.

When $s \notin V(M)$, we choose an arbitrary vertex $t \in F - V(M) - s$. If $o(F - V(M) - \{s,t\}) = 0$, by Lemma 1.3, $F - V(M) - \{s,t\}$ has a perfect matching. Otherwise, $o(F - V(M) - \{s,t\}) \geq 2$. Let F_1 and F_2 be two odd components of $F - V(M) - \{s,t\}$. We distinguish the following two cases.

Case 1. *n is even.*

In this case, $\delta = n$. It is obvious that $F - V(M)$ has a perfect matching, which can be obtained from Theorem 2.3 directly.

Case 2. *n is odd. Let $\delta = n - 1 = 2m$.*

Subcase 2.1. $|M| = 1$.

We can deduce that $|V(F_i)| - 1 + 2 + 2 \geq \delta(F) \geq 2m+1$ and so $|V(F_i)| \geq 2m - 1$, $i = 1, 2$. Because

$$|V(F)| \geq |V(F_1)| + |V(F_2)| + |V(M)| + 2, \tag{2.7}$$

we have

$$4m + 2 \geq 2m - 1 + 2m - 1 + 2 + 2 = 4m + 2.$$

This implies that $|V(F_1)| = |V(F_2)| = 2m - 1$.

When $m = 1$, we have $|V(F_1)| = |V(F_2)| = 1$. Suppose that $F_1 = u$ and $F_2 = v$. Since $\delta = 2m = 2$ and G is claw-free, we know that $N(t) \cap \{u, v\} \neq \emptyset$. Without loss of generality, suppose that $tu \in E(G)$. Then $\{tu\} \cup \{sv\}$ is a perfect matching of $F - V(M)$.

When $m \geq 2$, since $d(t) \geq \delta = 2m$, it is easy to check that $N(t) \cap (V(F_1) \cup V(F_2)) \neq \emptyset$. Without loss of generality, suppose that $N(t) \cap V(F_1) \neq \emptyset$. By Lemma 1.3, $G[V(F_1) \cup \{t\}]$ has a perfect matching M_1. Let w be a vertex of F_2 such that w is not a cut vertex of F_2. So $F_2 - w$ is connected. By Lemma 1.3, $F_2 - w$ has a perfect matching M_2. Hence $M_2 \cup \{sw\}$ is a perfect matching of $F_2 \vee s$. So $F - V(M)$ has a perfect matching $M_1 \cup M_2 \cup \{sw\}$.

Subcase 2.2. $|M| \geq 2$.

Because for each vertex $x \in V(F) - V(M) - s$, $|N(x) \cap V(M)| \leq 4$, we can deduce that $|V(F_i)| - 1 + 4 + 2 \geq \delta(F) \geq 2m + 1$ and so $|V(F_i)| \geq 2m - 3$, $i = 1, 2$. By (2.7),

$$4m + 2 \geq 2m - 3 + 2m - 3 + 4 + 2 = 4m.$$

We have $|M| \leq 3$ and $|V(F - V(M \cup F_1 \cup F_2) - \{s,t\})| \leq 2$. Without loss of generality, suppose $|V(F_1)| \leq |V(F_2)|$. Since $\delta = 2m$ and G is claw-free, this implies the following two possible cases:

(a) $|V(F_1)| = 2m - 3$, $|V(F_2)| = 2m - 1$, $|M| = 2$.
(b) $|V(F_1)| = |V(F_2)| = 2m - 3$, $|M| = 3$.

When $m = 2$ or $m = 3$, because G is claw-free and $d(t) \geq 2m$, we can check that $N(t) \cap (V(F_1) \cup V(F_2)) \neq \emptyset$. When $m \geq 4$, since $d(t) \geq 2m \geq 8$, obviously we have $N(t) \cap (V(F_1) \cup V(F_2)) \neq \emptyset$. From a similar discussion of Subcase 2.1, we can prove that $F - V(M)$ has a perfect matching.

According to the above analysis, we know that F is IM-extendable, and so G is nearly IM-extendable.

The above result is best possible. For if

$$G = \begin{cases} K_1 \vee 2K_{n-1} & n \text{ is even,} \\ K_1 \vee (K_n \cup K_{n-2}) & n \text{ is odd,} \end{cases}$$

where $K_1 = t$, we can easily check that G is a claw-free graph with $\delta(G) = 2\lfloor n/2 \rfloor - 1$. Let $F = G \vee s$ and $M = st$. Since $F - V(M)$ has two odd components, F is not IM-extendable, and so G is not nearly IM-extendable. □

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