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BINOMIAL TRANSFORMS OF THE BALANCING AND LUCAS-BALANCING POLYNOMIALS

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Abstract. In this study, we define the binomial transforms of balancing and Lucas-balancing polynomials. Also, the generating functions, Binet formulas, and summations of these transforms are found by recurrence relations. Furthermore, we establish the relations between these transforms by deriving new formulas. Finally, we obtain the Catalan and Cassini formulas for these transforms.

1. Introduction and Preliminaries

The study of number sequences have been a source of attraction to mathematicians since ancient times. Since then, many of them focused their interest on the study of the fascinating triangular numbers. In 1999, Behera and Panda [1] introduced the notion of balancing numbers $(B_n)_{n\in\mathbb{N}}$ as solutions to a certain Diophantine equation. The recurence relation of this number is $B_{n+1} = 6B_n - B_{n-1}$ for $n \geqslant 1$, where $B_0 = 0,\ B_1 = 1$. A study on the Lucas-balancing numbers $C_n = \sqrt{8B_n^2 + 1}$ was published in 2009 by Panda [11]. The recurrence relation of this number is $C_{n+1} = 6C_n - C_{n-1}$ for $n \geqslant 1$, where $C_0 = 1,\ C_1 = 3$. In addition, the author in [6] studied the binomial sums of the balancing and Lucas-balancing numbers.

Generalizations of balancing numbers can be obtained in various ways (see [7, 8, 10, 12, 14]). A natural extension is to consider for $x \in \mathbb{C}$ sequences of balancing and Lucas-balancing polynomials $(B_n(x))_{n\in\mathbb{N}}$ and $(C_n(x))_{n\in\mathbb{N}}$, respectively.

Balancing and Lucas-balancing polynomials are defined by the recurrence relations

$$B_{n+1}(x) = 6xB_n(x) - B_{n-1}(x), \text{ where } \quad B_0(x) = 0,\ B_1(x) = 1, \tag{1.1}$$

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and

$$C_{n+1}(x) = 6xC_n(x) - C_{n-1}(x), \text{ where } \quad C_0(x) = 1, \ C_1(x) = 3x, \tag{1.2}$$

respectively, in [8, 12].

In addition, some matrix-based transforms can be introduced for a given sequence. The binomial transform is one such transform and there are also other transforms such as the rising and falling binomial transforms (see [2, 4, 5, 13]). Furthermore, there is an interesting study on watermarking and the binomial transform. For example, in [9], the binomial transform-based fragile image watermarking technique has been proposed for color image authentication.

Now we give some preliminaries related our study. Given an integer sequence $X = \{x_0, x_1, x_2, \ldots\}$, the binomial transform B of the sequence X, $B(X) = \{b_n\}$, is given by

$$b_n = \sum_{i=0}^{n} \binom{n}{i} x_i. \tag{1.3}$$

Also, in [3], the author studied the following properties of the binomial transform

$$\sum_{i=0}^{n} \binom{n}{i} i x_i = n(b_n - b_{n-1}) \tag{1.4}$$

and

$$\sum_{i=1}^{n} \binom{n}{i} \frac{x_i}{i} = \sum_{j=1}^{n} \frac{b_j}{j}. \tag{1.5}$$

Motivated by [2, 12], the objective of this paper is to apply the binomial transforms to the balancing $B_n(x)$ and Lucas-balancing polynomials $C_n(x)$. Also, the generating functions of these transforms are found by recurrence relations. Finally, we describe the Catalan and Cassini formulas and the relations between these transforms by deriving new formulas.

2. Binomial transforms of balancing and Lucas-balancing polynomials

In this section, we will mainly focus on binomial transforms of balancing and Lucas-balancing polynomials to get some important results. In fact, as a middle step, we will also present the recurrence relations, generating functions, Binet formulas, and summations. Later, we obtain relations between these transforms by deriving new formulas.

Definition 2.1. *Let $B_n(x)$ and $C_n(x)$ be the balancing and Lucas-balancing polynomials, respectively. The binomial transforms of these polynomials can be expressed as follows:*

i) *the binomial transform of the balancing polynomial is*

$$b_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i(x),$$

ii) *the binomial transform of the Lucas-balancing polynomial is*

$$c_n(x) = \sum_{i=0}^{n} \binom{n}{i} C_i(x).$$

Before starting the results, it is useful to say $\binom{n}{i} = 0$ for $i > n$. We note that, from Definition 2.1 and equations (1.1) and (1.2), for $n \geqslant 0$, we obtain

$$b_n(x) = \begin{cases} (6x+2)^{\frac{n}{2}} B_{\frac{n}{2}}(x), & n \text{ is even} \\ (6x+2)^{\frac{n-1}{2}} (B_{\frac{n+1}{2}}(x) + B_{\frac{n-1}{2}}(x)), & n \text{ is odd} \end{cases} \tag{2.1}$$

and

$$c_n(x) = \begin{cases} (6x+2)^{\frac{n}{2}} C_{\frac{n}{2}}(x), & n \text{ is even} \\ (6x+2)^{\frac{n-1}{2}} (C_{\frac{n+1}{2}}(x) + C_{\frac{n-1}{2}}(x)), & n \text{ is odd.} \end{cases} \tag{2.2}$$

The following lemma will be key to the proof of the next theorem.

Lemma 2.2. *For $n \geqslant 0$, the following equalities hold:*

i) $b_{n+1}(x) - b_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_{i+1}(x)$,
ii) $c_{n+1}(x) - c_n(x) = \sum_{i=0}^{n} \binom{n}{i} C_{i+1}(x)$.

Proof. We will only prove *i)* since the proof of *ii)* is analogous.

By using Definition 2.1 and the well known binomial equality

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1} \text{ for } 1 \leqslant i \leqslant n, \tag{2.3}$$

we obtain

$$\begin{aligned} b_{n+1}(x) &= \sum_{i=1}^{n+1} \binom{n+1}{i} B_i(x) + B_0(x) \\ &= \sum_{i=0}^{n} \binom{n}{i} B_i(x) + \sum_{i=1}^{n+1} \binom{n}{i-1} B_i(x) \\ &= \sum_{i=0}^{n} \binom{n}{i} \left(B_i(x) + B_{i+1}(x) \right), \end{aligned}$$

which is desired result. ☐

Theorem 2.3. *For $n > 0$,*

i) *Recurrence relation of sequences $\{b_n(x)\}$ is*

$$b_{n+1}(x) = (6x+2)\left(b_n(x) - b_{n-1}(x)\right), \tag{2.4}$$

with initial conditions $b_0(x) = 0$, $b_1(x) = 1$.

ii) *Recurrence relation of sequences $\{c_n(x)\}$ is*

$$c_{n+1}(x) = (6x+2)\left(c_n(x) - c_{n-1}(x)\right), \tag{2.5}$$

with initial conditions $c_0(x) = 1$, $c_1(x) = 3x+1$.

Proof. Similar to the proof of the previous theorem, only the first case $i)$ will be proved. We omit the other cases since the proofs are similar.

By considering Definition 2.1, we obtain

$$\begin{aligned} b_{n+1}(x) &= \sum_{i=0}^{n+1} \binom{n+1}{i} B_i(x) \\ &= \sum_{i=0}^{n} \binom{n+1}{i+1} B_{i+1}(x) + B_0(x). \end{aligned}$$

By taking into account equation (2.3), we get

$$b_{n+1}(x) = \sum_{i=0}^{n-1} \binom{n}{i+1} B_{i+1}(x) + \sum_{i=0}^{n} \binom{n}{i} B_{i+1}(x) + B_0(x).$$

By considering recurrence relations of balancing polynomials and the equality $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, we obtain

$$
\begin{aligned}
b_{n+1}(x) &= \sum_{i=0}^{n-2}\binom{n-1}{i+1}B_{i+1}(x) + \sum_{i=0}^{n-1}\binom{n-1}{i}B_{i+1}(x) + 6x\sum_{i=0}^{n}\binom{n}{i}B_i(x) \\
&\quad - \sum_{i=0}^{n}\binom{n}{i}B_{i-1}(x) + B_0(x) \\
&= \sum_{i=0}^{n-1}\binom{n-1}{i}B_i(x) + \sum_{i=0}^{n-1}\binom{n-1}{i}\left(6xB_i(x) - B_{i-1}(x)\right) \\
&\quad + 6x\sum_{i=0}^{n}\binom{n}{i}B_i(x) - \sum_{i=0}^{n}\binom{n}{i}B_{i-1}(x) \\
&= 6xb_n(x) + (6x+1)\,b_{n-1}(x) - \sum_{i=0}^{n-1}\binom{n-1}{i}B_{i-1}(x) \\
&\quad - \sum_{i=0}^{n}\binom{n}{i}B_{i-1}(x) \\
&= 6xb_n(x) + (6x+1)\,b_{n-1}(x) - 2\sum_{i=0}^{n-1}\binom{n-1}{i}B_{i-1}(x) \\
&\quad - \sum_{i=0}^{n-1}\binom{n-1}{i}B_i(x) \\
&= 6xb_n(x) + 6xb_{n-1}(x) - 2\sum_{i=0}^{n-1}\binom{n-1}{i}\left(6xB_i(x) - B_{i+1}(x)\right).
\end{aligned}
$$

From Lemma 2.2, we have

$$b_{n+1}(x) = 6xb_n(x) - 6xb_{n-1}(x) + 2\left(b_n(x) - b_{n-1}(x)\right).$$

which completes the proof in this case. □

The generating functions for balancing and Lucas-balancing polynomials play a vital role to finding many important identities for these polynomials. In the following theorem, we develop the generating functions for the binomial transforms of balancing and Lucas-balancing polynomials.

Theorem 2.4. *The generating functions of the binomial transforms for* $\{b_n(x)\}$ *and* $\{c_n(x)\}$ *are*

i) $\sum\limits_{n=0}^{\infty} b_n(x)s^n = \frac{s}{1-(6x+2)s+(6x+2)s^2}$,

ii) $\sum\limits_{n=0}^{\infty} c_n(x)s^n = \frac{1-(1+3x)s}{1-(6x+2)s+(6x+2)s^2}$,

respectively.

Proof. We omit the balancing case since the proof is similar.

Assume that $c(x,s)$ is the generating function of the binomial transform for $\{C_n(x)\}$. Then we have

$$c\left(x,s\right)=\sum_{n=0}^{\infty}c_n(x)s^n.$$

From Theorem 2.3, we obtain

$$\begin{aligned}c\left(x,s\right)&=c_0(x)+c_1(x)s+(6x+2)\sum_{n=2}^{\infty}\left(c_{n-1}(x)-c_{n-2}(x)\right)s^n\\&=c_0(x)+\left(c_1(x)-(6x+2)c_0(x)\right)s\\&\quad+(6xs+2s)c\left(x,s\right)-(6x+2)s^2c\left(x,s\right).\end{aligned}$$

Now rearrangement the equation implies that

$$c\left(x,s\right)=\frac{c_0(x)+\left(c_1(x)-(6x+2)c_0(x)\right)s}{1-(6x+2)\,s+(6x+2)\,s^2},$$

which is equal to $\sum_{n=0}^{\infty}c_n(x)s^n$ in the theorem. □

Furthermore, we note that $b\left(x,s\right)$ and $c\left(x,s\right)$ may be obtained from the generating functions of the balancing and Lucas-balancing polynomials in [8],

$$f\left(x,s\right)=\frac{s}{1-6xs+s^2}$$

and

$$g\left(x,s\right)=\frac{1-3xs}{1-6xs+s^2}.$$

It is seen by using the following result proved by Prodinger [13]:

$$b\left(x,s\right)=\frac{1}{1-s}f\left(x,\frac{s}{1-s}\right)$$

and

$$c\left(x,s\right)=\frac{1}{1-s}g\left(x,\frac{s}{1-s}\right).$$

Lemma 2.5. *For $n\geqslant 0$, we have $\alpha^n=\alpha b_n(x)-(6x+2)b_{n-1}(x)$ and $\beta^n=\beta b_n(x)-(6x+2)b_{n-1}(x)$, where α and β are the roots of the characteristic equation $\lambda^2-(6x+2)\lambda+(6x+2)=0$.*

Proof. These can be established easily using the principle of mathematical induction. □

To derive new identities, we now present an explicit formula for $\{b_n(x)\}$ and $\{c_n(x)\}$ for $n\geqslant 0$.

Theorem 2.6. *The Binet formulas of sequences $\{b_n(x)\}$ and $\{c_n(x)\}$ are*

i) $b_n(x)=\frac{\alpha^n-\beta^n}{\alpha-\beta}$,
ii) $c_n(x)=\frac{\alpha^n+\beta^n}{2}$,

where α and β are as in the Lemma 2.5.

Proof. i) From Theorem 2.4, we have

$$\sum_{n=0}^{\infty} b_n(x)s^n = \frac{s}{1-(6x+2)\,s+(6x+2)\,s^2}.$$

It is easily seen that

$$\begin{aligned} b\,(x,s) &= \frac{\frac{1}{2\sqrt{9x^2-1}}}{1-\left(3x+1+\sqrt{9x^2-1}\right)s} - \frac{\frac{1}{2\sqrt{9x^2-1}}}{1-\left(3x+1-\sqrt{9x^2-1}\right)s} \\ &= \frac{1}{\alpha-\beta}\left[\sum_{n=0}^{\infty}\alpha^n s^n - \sum_{n=0}^{\infty}\beta^n s^n\right]. \end{aligned}$$

Thus, by the equality of generating function, we get $b_n(x) = (\alpha^n - \beta^n)/(\alpha - \beta)$.

ii) The proof of the binomial transform of Lucas-balancing polynomials can be seen by taking Theorem 2.4, ii) into account similar to the proof of i). □

Now, we give the sums of binomial transforms for balancing and Lucas-balancing polynomials.

Theorem 2.7. *Sums of sequences $\{b_n(x)\}$ and $\{c_n(x)\}$ are*

i) $\sum\limits_{i=0}^{n-1} b_i(x) = (6x+2)b_{n-2}(x) + 1,$

ii) $\sum\limits_{i=0}^{n-1} c_i(x) = (6x+2)c_{n-2}(x) - 3x.$

Proof. *i*) By considering equation (2.4), we have

$$\begin{aligned} b_2(x) &= (6x+2)b_1(x) - (6x+2)b_0(x) \\ b_3(x) &= (6x+2)b_2(x) - (6x+2)b_1(x) \\ b_4(x) &= (6x+2)b_3(x) - (6x+2)b_2(x) \\ &\vdots \\ b_{n-1}(x) &= (6x+2)b_{n-2}(x) - (6x+2)b_{n-3}(x). \end{aligned}$$

Adding these equations, we get

$$\sum_{i=0}^{n-1} b_i(x) = (6x+2)b_{n-2}(x) + 1.$$

ii) Similar to *i*), by considering equation (2.5), we have

$$\begin{aligned}
c_2(x) &= (6x+2)c_1(x) - (6x+2)c_0(x) \\
c_3(x) &= (6x+2)c_2(x) - (6x+2)c_1(x) \\
c_4(x) &= (6x+2)c_3(x) - (6x+2)c_2(x) \\
&\vdots \\
c_{n-1}(x) &= (6x+2)c_{n-2}(x) - (6x+2)c_{n-3}(x).
\end{aligned}$$

Adding these equations, we get

$$\sum_{i=0}^{n-1} c_i(x) = (6x+2)c_{n-2}(x) - 3x.$$

□

Now, we give the sums of the first n of binomial transforms for balancing and Lucas-balancing polynomials with odd and even subscripts.

Proposition 2.8. *Sums of sequences* $\{b_n(x)\}$ *and* $\{c_n(x)\}$ *with odd and even subscripts are*

i) $\sum\limits_{i=0}^{n-1} b_{2i+1}(x) = \frac{1}{12x+5}\left[(6x+2)^2 b_{2n-1}(x) - b_{2n+1}(x) + 6x + 3\right],$

ii) $\sum\limits_{i=0}^{n-1} b_{2i}(x) = \frac{1}{12x+5}\left[(6x+2)^2 b_{2n-2}(x) - b_{2n}(x) + 6x + 2\right],$

iii) $\sum\limits_{i=0}^{n-1} c_{2i+1}(x) = \frac{1}{12x+5}\left[(6x+2)^2 c_{2n-1}(x) - c_{2n+1}(x) - 18x^2 - 9x - 1\right],$

iv) $\sum\limits_{i=0}^{n-1} c_{2i}(x) = \frac{1}{12x+5}\left[(6x+2)^2 c_{2n-2}(x) - c_{2n}(x) - 18x^2 - 6x + 1\right].$

Proof. The proof can be easily established using Theorem 2.6. □

Now, we give the combinatorial equalities of the binomial transforms for balancing and Lucas-balancing polynomials.

Theorem 2.9. *We have*

i) $\sum\limits_{i=0}^{n} \binom{n}{i}(-1)^{n-i} b_i(x) = \frac{1}{(6x+2)^n} b_{2n}(x),$

ii) $\sum\limits_{i=0}^{n} \binom{n}{i}(-1)^{n-i} c_i(x) = \frac{1}{(6x+2)^n} c_{2n}(x),$

iii) $\sum\limits_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1}(3x+1)^{n-2i+1}(9x^2-1)^i = b_n(x),$

iv) $\sum\limits_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i}(3x+1)^{n-2i}(9x^2-1)^i = c_n(x),$

where $n \geqslant 0$.

Proof. i) From Theorem 2.6, we have

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}b_i(x) &= \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\left(\frac{\alpha^i-\beta^i}{\alpha-\beta}\right)\\ &= \frac{1}{\alpha-\beta}\left[(\alpha-1)^n-(\beta-1)^n\right]\\ &= \frac{1}{\alpha-\beta}\left[\left(\frac{\alpha^2}{6x+2}\right)^n-\left(\frac{\beta^2}{6x+2}\right)^n\right]\\ &= \frac{1}{(6x+2)^n}b_{2n}(x).\end{aligned}$$

The proof of ii) it similar to the proof of *i*). The proofs of *iii*) and *iv*) can be easily seen by the method of induction for the binomial transforms of balancing and Lucas-balancing polynomials. □

In the following Theorem, we present the relationship between these binomial transforms.

Theorem 2.10. *For any integer $n \geqslant 0$, we have*

i) $b_{n+1}(x)-(6x+2)b_{n-1}(x)=2c_n(x)$,
ii) $c_{n+1}(x)-(6x+2)c_{n-1}(x)=2(9x^2-1)b_n(x)$,
iii) $2b_n(x)c_n(x)=b_{2n}(x)$,
iv) $b_n(x)b_m(x)=\frac{1}{2(9x^2-1)}\left[c_{n+m}(x)-(6x+2)^m c_{n-m}(x)\right]$,
v) $c_n(x)c_m(x)=\frac{1}{2}\left[c_{n+m}(x)+(6x+2)^m c_{n-m}(x)\right]$.

Proof. i) From Lemma 2.5, we have

$$\alpha^n=\alpha b_n(x)-(6x+2)b_{n-1}(x)$$

$$\beta^n=\beta b_n(x)-(6x+2)b_{n-1}(x).$$

Adding these equations, we get

$$\alpha^n+\beta^n=(6x+2)b_n(x)-2(6x+2)b_{n-1}(x).$$

By using part i) of Theorem 2.3, we acquire

$$2c_n(x)=b_{n+1}(x)-(6x+2)b_{n-1}(x).$$

iii) From Theorem 2.6, we have

$$\begin{aligned}b_n(x)c_n(x) &= \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)\left(\frac{\alpha^n+\beta^n}{2}\right)\\ &= \frac{b_{2n}(x)}{2}.\end{aligned}$$

The proofs of parts ii), iv), and v) are similar to the proof of iii). □

We extend the definitions of $b_n(x)$ and $c_n(x)$ to negative subscripts in the following proposition.

Proposition 2.11. *For any integer $n \geqslant 1$, we have*

i) $b_{-n}(x) = -\frac{1}{(6x+2)^n} b_n(x)$,
ii) $c_{-n}(x) = \frac{1}{(6x+2)^n} c_n(x)$.

Using the Binet formulas, we give the Catalan formulas of these transforms, as the next proposition shows.

Proposition 2.12. *For integers n, r with $r < n$, we have*

i) $b_{n-r}(x)b_{n+r}(x) - b_n^2(x) = -(6x+2)^{n-r} b_r^2(x)$,
ii) $c_{n-r}(x)c_{n+r}(x) - c_n^2(x) = (6x+2)^{n-r}(9x^2-1)b_r^2(x)$.

Remark: For $r = 1$ in Proposition 2.12,

$$b_{n-1}(x)b_{n+1}(x) - b_n^2(x) = -(6x+2)^{n-1}$$

and

$$c_{n-1}(x)c_{n+1}(x) - c_n^2(x) = (6x+2)^{n-1}(9x^2-1),$$

which we call the Cassini formulas for the binomial transforms of balancing and Lucas-balancing polynomials.

Proposition 2.14. *For any integer $n > 0$, we have*

i): $n\left(b_n(x) - b_{n-1}(x)\right) = \sum_{i=0}^{n} \binom{n}{i} i B_i(x)$,
ii): $n\left(c_n(x) - c_{n-1}(x)\right) = \sum_{i=0}^{n} \binom{n}{i} i C_i(x)$,
iii): $\sum_{j=1}^{n} \frac{b_j(x)}{j} = \sum_{i=1}^{n} \binom{n}{i} \frac{B_i(x)}{i}$,
iv): $\sum_{j=1}^{n} \frac{c_j(x)}{j} = \sum_{i=1}^{n} \binom{n}{i} \frac{C_i(x)}{i} + \sum_{j=1}^{n} \frac{1}{j}$.

Proof. We will give the proofs of *ii*) and *iii*), the proof of others are similar.
ii) From Definition 2.1, we obtain

$$\begin{aligned} n\left(c_n(x) - c_{n-1}(x)\right) &= n\left(\sum_{i=0}^{n} \binom{n}{i} C_i(x) - \sum_{i=0}^{n} \binom{n-1}{i} C_i(x)\right) \\ &= n\sum_{i=0}^{n} \binom{n}{i} C_i(x) - n\sum_{i=0}^{n} \frac{n-i}{n} \binom{n}{i} C_i(x) \\ &= \sum_{i=0}^{n} \binom{n}{i} i C_i(x). \end{aligned}$$

iii) From Definition 2.1, we get

$$\begin{aligned}
\sum_{j=1}^{n}\frac{b_j(x)}{j} &= \sum_{j=1}^{n}\frac{1}{j}\left(\sum_{i=0}^{j}\binom{j}{i}B_i(x)\right)\\
&= \sum_{j=1}^{n}\frac{1}{j}\left(\binom{j}{0}B_0(x)+\binom{j}{1}B_1(x)\right.\\
&\qquad \left.+\binom{j}{2}B_2(x)+\cdots+\binom{j}{j}B_j(x)\right)\\
&= \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)B_0(x)+\left(1+\frac{2}{2}+\frac{3}{3}+\cdots+\frac{n}{n}\right)B_1(x)\\
&\qquad +\left(\frac{1}{2}+1+\frac{3}{2}+\cdots+\frac{n-1}{2}\right)B_2(x)+\cdots+\frac{1}{n}B_n(x).
\end{aligned}$$

By considering $B_0(x)=0$ and the properties of summation, we can write

$$\begin{aligned}
\sum_{j=1}^{n}\frac{b_j(x)}{j} &= \binom{n}{1}\frac{B_1(x)}{1}+\binom{n}{2}\frac{B_2(x)}{2}+\cdots+\binom{n}{n}\frac{B_n(x)}{n}\\
&= \sum_{i=1}^{n}\binom{n}{i}\frac{B_i(x)}{i}.
\end{aligned}$$

□

3. Conclusion

In this paper, we first define the binomial transforms of balancing polynomials and give some properties of this new sequence. Thus, we obtain a new generalization for binomial transforms that have a similar recurrence relations to the literature. By taking into account these transforms and its properties, properties of the binomial transforms of balancing and Lucas-balancing numbers can also be obtained. That is,

- If we replace $x=1$ in $b_n(x)$, we obtain the binomial transforms for balancing numbers (the binomial transform of OEIS A001109).
- If we replace $x=1$ in $c_n(x)$, we obtain the binomial transforms for Lucas-balancing numbers (OEIS A084130).
- If we replace $x=k$ in $b_n(x)$, we obtain the binomial transforms for k-balancing numbers.
- If we replace $x=k$ in $c_n(x)$, we obtain the binomial transforms for Lucas k-balancing numbers.

We obtained the generating functions, Binet formulas, summations, and relationships for the binomial transforms of the well-known sequences in the literature.

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