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$K_n(\lambda)$ IS FULLY $\{P_5, C_6\}$-DECOMPOSABLE

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ABSTRACT. Let P_{k+1} denote a path of length k, C_ℓ denote a cycle of length ℓ, and $K_n(\lambda)$ denote the complete multigraph on n vertices in which every edge is taken λ times. In this paper, we have obtained the necessary conditions for a $\{P_{k+1}, C_\ell\}$-decomposition of $K_n(\lambda)$ and proved that the necessary conditions are also sufficient when $k = 4$ and $\ell = 6$.

1. INTRODUCTION

All graphs considered here are finite and undirected with no loops. For the standard graph-theoretic terminology the reader is referred to [2]. A simple graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. A complete graph on n vertices is denoted by K_n. If more than one edge joining two vertices are allowed, the resulting object is called a *multigraph*. Let $K_n(\lambda)$ denotes the *complete multigraph* on n vertices and in which every edge is taken λ times. A *complete bipartite graph* is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$. In $K_{m,n}(\lambda)$, we label the vertices in the partite set X as $\{x_1, x_2, \ldots, x_m\}$ and Y as $\{x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. A *cycle* is a closed trail with no repeated vertex other than the first and last vertex. A cycle with ℓ edges is denoted by C_ℓ. A *path* is an open trail with no repeated vertex. A path with k edges is denoted by P_{k+1}. The complete bipartite graph $K_{1,m}$ is called a *star* and is denoted by S_m. For $m \geq 3$, the vertex of degree m in S_m is called the *center* and any vertex of degree 1 in S_m is called an *end vertex*.

Let G be a graph and G_1 be a subgraph of G. Then $G \backslash G_1$ is obtained from G by deleting the edges of G_1. Let G_1 and G_2 be subgraphs of G. The *union* $G_1 \cup G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. We say that G_1 and G_2 are *edge-disjoint* if they have no edge in common. If G_1 and G_2 are edge-disjoint, we denote their union by $G_1 + G_2$. A *decomposition* of a graph G is a collection of

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edge-disjoint subgraphs $G_1, G_2, \ldots, G_n$ of G such that every edge of G is in exactly one G_i. Here it is said that G is *decomposed* or *decomposable* into $G_1, G_2, \ldots, G_n$. If G has a decomposition into p_1 copies of $G_1, \ldots, p_n$ copies of G_n, then we say that G has a $\{p_1G_1, \ldots, p_nG_n\}$-decomposition. If such a decomposition exists for all values of $p_1, \ldots, p_n$ satisfying trivial necessary conditions, then we say that G has a $\{G_1, \ldots, G_n\}_{\{p_1,\ldots,p_n\}}$-decomposition or G is fully $\{G_1, \ldots, G_n\}$-decomposable. We say that G is decomposed into P_5 and C_6 if each $G_i \simeq P_5$ or C_6.

In [6], Priyadharsini and Muthusamy gave necessary and sufficient conditions for the existence of $\{pG_1, qG_2\}$-decomposition of $K_n(\lambda)$, when

$$(G_1, G_2) \in \{(P_n, S_{1,n-1}), (C_n, S_{1,n-1}), (P_n, C_n)\}.$$

In [9], Shyu gave the necessary conditions for a $\{pP_{k+1}, qC_\ell\}$-decomposition of K_n and proved that K_n is fully $\{P_{k+1}, C_k\}$-decomposable, when k is even, n is odd, $n \geq 5k + 1$ and settled the case $k = 4$ completely. In [10], Shyu proved that K_n is fully $\{P_4, C_3\}$-decomposable. In [5], Jeevadoss and Muthusamy proved that K_n is fully $\{P_{k+1}, C_k\}$-decomposable, when k is even and n is odd with $n > 4k$. In [4], Ilayaraja and Muthusamy proved that K_n is fully $\{P_4, C_4\}$-decomposable. In [7], Sarvate and Zhang obtained necessary and sufficient conditions for the existence of a $\{pP_3, qK_3\}$-decomposition of $K_n(\lambda)$, when $p = q$. In [8], Shyu gave the necessary conditions for a $\{pC_k, qP_{k+1}, rS_k\}$-decomposition of K_n and proved that K_n is fully $\{C_4, P_5, S_4\}$-decomposable, when n is odd. In this paper we prove that $K_n(\lambda)$ is fully $\{P_5, C_6\}$-decomposable.

2. Preliminaries

For convenience we denote $V(K_n(\lambda)) = \{x_1, x_2, \ldots, x_n\}$. The notation $(x_1, x_2, \ldots, x_\ell)$ denotes a cycle with vertices $x_1, x_2, \ldots, x_\ell$ and edges $x_1x_2, x_2x_3, \ldots, x_{\ell-1}x_\ell, x_\ell x_1$, and $(x_1x_2 \ldots x_{k+1})$ is a path with vertices $x_1, x_2, \ldots, x_{k+1}$ and edges $x_1x_2, x_2x_3, \ldots, x_kx_{k+1}$.

We recall here some results on P_{k+1} and C_ℓ-decompositions that are useful for our proofs.

Theorem 2.1 (Bryant, et al. [1]). *Let λ, n and ℓ be integers with $n, \ell \geq 3$ and $\lambda \geq 1$. There exists a decomposition of $K_n(\lambda)$ into C_ℓ if and only if $\ell \leq n$, $\lambda(n-1)$ is even and ℓ divides $\lambda\binom{n}{2}$. There exists a decomposition of $K_n(\lambda)$ into C_ℓ and a perfect matching if and only if $\ell \leq n$, $\lambda(n-1)$ is odd and ℓ divides $\lambda\binom{n}{2} - \frac{n}{2}$.*

Theorem 2.2 (Tarsi [11]). *Necessary and sufficient conditions for the existence of a P_{k+1}-decomposition of $K_n(\lambda)$ are $\lambda\binom{n}{2} \equiv 0 \pmod k$ and $n \geq k+1$.*

Theorem 2.3 (Lee [3]). *For positive integers λ, m, n and ℓ with $\lambda m \equiv \lambda n \equiv \ell \equiv 0 \pmod 2$ and $\min\{m, n\} \geq \frac{\ell}{2} \geq 2$, the multigraph $K_{m,n}(\lambda)$ is C_ℓ-decomposable if one of the following conditions holds:*

(i) λ is odd and ℓ divides mn,

(ii) λ *is even and* ℓ *divides* $2mn$,
(iii) λ *is even and* λm *or* λn *is divisible by* ℓ.

Theorem 2.4 (Truszczynski [12]). *Let* k *be a positive integer and let* m *and* n *be positive even integers such that* $m \geq n$. $K_{m,n}(\lambda)$ *has a* P_{k+1}*-decomposition if and only if* $m \geq \lceil \frac{k+1}{2} \rceil, n \geq \lceil \frac{k}{2} \rceil$ *and* $\lambda mn \equiv 0 \pmod{k}$.

Lemma 2.5 (Shyu [9]). *Let* k *and* n *be positive integers such that* $k \geq 3$ *and* $n \geq 2$. *Suppose that for* $i \in \{1, 2, \ldots, n\}$, C_i *denotes the cycle* $(x_{(i,1)}, x_{(i,2)}, \ldots, x_{(i,k)})$ *of length* k. *If* $x_{(1,1)} = x_{(2,1)} = \cdots = x_{(n,1)}$, $x_{(i+1,2)} \notin \{x_{(i,1)}, x_{(i,2)}, \ldots, x_{(i,k)}\}$ *for* $i \in \{1, 2, \ldots, n-1\}$, *and* $x_{(1,2)} \notin \{x_{(n,1)}, x_{(n,2)}, \ldots, x_{(n,k)}\}$, *then* $\bigcup_{i=1}^{n} C_i$ *can be decomposed into* n *paths of length* k.

Theorem 2.6 (Shyu [9]). *Let* n, ℓ *and* k *be positive integers such that* n *is odd and* $n \geq \max\{\ell, k+1\}$. *If* K_n *can be decomposed into* p *copies of* P_{k+1} *and* q *copies of* C_ℓ *for nonnegative integers* p *and* q, *then* $pk + q\ell = e(K_n)$ *and* $p \neq 1$.

Theorem 2.7 (Shyu [9]). *Let* n, ℓ *and* k *be positive integers such that* n *is even and* $n \geq \max\{\ell, k+1\}$. *If* K_n *can be decomposed into* p *copies of* P_{k+1} *and* q *copies of* C_ℓ *for nonnegative integers* p *and* q, *then* $pk + q\ell = e(K_n)$ *and* $p \geq \frac{n}{2}$.

In [9], Shyu gave the necessary conditions for a $\{pP_{k+1}, qC_\ell\}$-decomposition of K_n and proved that K_n is fully $\{P_{k+1}, C_k\}$-decomposable, when k is even, n is odd, $n \geq 5k+1$ and settled the case $k = 4$ completely.

In the following theorems, we discuss the necessary conditions for a $\{pP_{k+1}, qC_\ell\}$-decomposition of $K_n(\lambda)$, when $\lambda \geq 1$.

Theorem 2.8. *Let* λ, n, k *and* ℓ *be positive integers such that* n *is odd or* n *and* λ *are both even and* $n \geq \max\{\ell, k+1\}$. *If* $K_n(\lambda)$ *can be decomposed into* p *copies of* P_{k+1} *and* q *copies of* C_ℓ *for nonnegative integers* p *and* q, *then* $pk + q\ell = \lambda\binom{n}{2}$ *and* $p \neq 1$.

Proof. Condition $pk + q\ell = \lambda\binom{n}{2}$ is trivial. On the contrary, suppose that $p = 1$. Let P denote the only path of length k in the decomposition. It follows that the starting and end vertices of P have odd degree $\lambda(n-1)-1$ in $K_n(\lambda)\backslash P$. Therefore, $K_n(\lambda)\backslash P$ can not be decomposed into cycles. We obtained a contradiction. □

Theorem 2.9. *Let* λ, n, k *and* ℓ *be positive integers such that* λ *is odd,* n *is even and* $n \geq \max\{\ell, k+1\}$. *If* $K_n(\lambda)$ *can be decomposed into* p *copies of* P_{k+1} *and* q *copies of* C_ℓ *for nonnegative integers* p *and* q, *then* $pk + q\ell = \lambda\binom{n}{2}$ *and* $p \geq \frac{n}{2}$.

Proof. Condition $pk + q\ell = \lambda\binom{n}{2}$ is trivial. Let D be an arbitrary decomposition of $K_n(\lambda)$ into p copies of P_{k+1} and q copies of C_l; let $P^{(1)}, P^{(2)}, \ldots, P^{(p)}$ denote those p copies of P_{k+1} in D. By assumption,

$$K_n(\lambda)\backslash(P^{(1)} \cup P^{(2)} \cup \ldots \cup P^{(p)})$$

has a C_ℓ-decomposition. It follows that each vertex of

$$K_n(\lambda)\backslash(P^{(1)} \cup P^{(2)} \cup \ldots \cup P^{(p)})$$

has even degree. Since λ is odd and n is even, each vertex of $K_n(\lambda)$ must be an end vertex of at least one $P^{(i)}(1 \le i \le p)$. It implies that $2p \ge n$. □

We prove that the above necessary conditions are sufficient for a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_n(\lambda)$ in Theorem 3.5.

3. Main Result

In this section, we discuss a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_n(\lambda)$, when $\lambda \ge 1$. Since $K_n(\lambda)$ cannot be decomposed into P_5 and C_6 when $n \le 5$, we discuss the decompositions for $n \ge 6$.

Remark 3.1: The necessary conditions for the existence of a $\{P_5, C_6\}_{\{p,q\}}$-decomposition in $K_n(\lambda)$ are satisfied when $n \equiv 0, 1 \pmod 4$ if $\lambda \ge 1$ and $n \equiv 2, 3 \pmod 4$ if λ is even, i.e., there does not exist nonnegative integers p and q satisfying $4p + 6q = \lambda\binom{n}{2}$ when $n \equiv 2, 3 \pmod 4$ if λ is odd.

In the following lemma, we discuss a $\{P_5, C_6\}_{\{p,q\}}$-decompositions of $K_{4,6}$, which we use further to decompose $K_n(\lambda)$ into $\{pP_5, qC_6\}$.

Lemma 3.2. *If p and q are nonnegative integers such that $4p+6q = 24$, then $K_{4,6}$ is fully $\{P_5, C_6\}$-decomposable.*

Proof. $(p, q) \in \{(0, 4), (3, 2), (6, 0)\}$. By Theorem 2.3, $K_{4,6}$ is $\{0P_5, 4C_6\}$-decomposable. The graph $K_{4,6}$ can be decomposed into

$$3P_5 : (x_1x_6x_4x_5x_3), (x_3x_{10}x_1x_8x_4), (x_4x_7x_2x_9x_1)$$

and

$$2C_6 : (x_1, x_5, x_2, x_6, x_3, x_7), (x_2, x_8, x_3, x_9, x_4, x_{10}).$$

By Theorem 2.4, $K_{4,6}$ is $\{6P_5, 0C_6\}$-decomposable. Therefore $K_{4,6}$ is fully $\{P_5, C_6\}$-decomposable. □

Remark 3.3: The graph $K_{4,4}(3)$ can be decomposed into

$$\begin{aligned} 8C_6 : \quad & (x_1, x_5, x_2, x_6, x_3, x_7), (x_1, x_6, x_4, x_8, x_3, x_5), (x_1, x_8, x_2, x_7, x_4, x_5), \\ & (x_2, x_5, x_4, x_6, x_3, x_8), (x_1, x_6, x_2, x_5, x_3, x_7), (x_1, x_6, x_2, x_7, x_4, x_8), \\ & (x_1, x_7, x_3, x_6, x_4, x_8), (x_2, x_8, x_3, x_5, x_4, x_7). \end{aligned}$$

Thus $K_{4,4}(3)$ is $\{0P_5, 8C_6\}$-decomposable.

Based on Lemma 2.5, we have the following remark.

Remark 3.4: Let $C_6^1 = (x_1, x_2, x_3, x_4, x_5, x_6)$ and $C_6^2 = (y_1, y_2, y_3, y_4, y_5, y_6)$. If $x_1 = y_1$, $x_6 \notin \{y_3, y_4\}$, $x_2 \notin V(C_6^2)$ and $y_2 \notin V(C_6^1)$, then $C_6^1 \cup C_6^2$ can be decomposed into 3 copies of

$$P_5 : (x_2x_3x_4x_5x_6), (x_6x_1y_2y_3y_4), (y_4y_5y_6y_1x_2).$$

We now prove our main result.

Theorem 3.5. *For any nonnegative integers p and q and any integer $n \ge 6$, there exists a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_n(\lambda)$ if and only if $4p + 6q = \lambda\binom{n}{2}$, (i) $p \ge \frac{n}{2}$, if λ is odd and n is even, (ii) $p \ne 1$ otherwise.*

Proof. The necessary part follows from Theorem 2.8 and 2.9. From remark 3.1, we have $n \equiv 0, 1 \pmod 4$ if $\lambda \geq 1$ and $n \equiv 2, 3 \pmod 4$ if λ is even. First we prove the result for $6 \leq n \leq 17$, next we generalise it for any $n > 17$ by applying mathematical induction. As we discuss $\{pP_5, qC_6\}$-decompositions of $K_n(\lambda)$ for all possible choices of p and q, we have the following cases:

CASE 1: $n = 6$.

If $\lambda = 2$, then $(p, q) \in \{(0, 5), (3, 3), (6, 1)\}$. By Theorem 2.1, $K_6(2)$ is $\{0P_5, 5C_6\}$-decomposable. The graph $K_6(2)$ can be decomposed into

$$3P_5 : (x_1x_6x_3x_2x_4), (x_6x_3x_4x_5x_1), (x_6x_5x_1x_2x_4)$$

and

$$3C_6 : C^1 = C^2 = (x_1, x_3, x_5, x_2, x_6, x_4), C^3 = (x_1, x_2, x_3, x_4, x_5, x_6).$$

The above $3P_5$ in $K_6(2)$ along with $(x_1x_3x_5x_2x_6), (x_1x_6x_4x_3x_2)$ and $(x_6x_5x_4x_1x_2)$ (Since $C^1 \cup C^3$ can be decomposed into $3P_5$), and a C^2, we get the required $\{6P_5, 1C_6\}$-decomposition.

If $\lambda = 4$, then $(p, q) \in \{(0, 10), (3, 8), (6, 6), (9, 4), (12, 2), (15, 0)\}$. We write

$$\begin{aligned} K_6(4) = K_6(2) + K_6(2) &= \{(0, 5), (3, 3), (6, 1)\} + \{(0, 5), (3, 3), (6, 1)\} \\ &= \{(0, 10), (3, 8), (6, 6), (9, 4), (12, 2)\}. \end{aligned}$$

By Theorem 2.2, $K_6(4)$ is $\{15P_5, 0C_6\}$-decomposable.

If $\lambda \geq 6$, then the proof is divided into the following cases.

$\lambda \equiv 0 \pmod 4$. We write $K_6(\lambda) = \frac{\lambda}{4}K_6(4)$.

$\lambda \equiv 2 \pmod 4$. We write $K_6(\lambda) = K_6(\lambda - 2) + K_6(2) = \frac{\lambda-2}{4}K_6(4) + K_6(2)$. Therefore $K_6(\lambda)$ is fully $\{P_5, C_6\}$-decomposable.

CASE 2: $n = 7$.

If $\lambda = 2$, the graph $K_7(2)$ can be decomposed into

$$\begin{aligned} 7C_6 : &C^1 = (x_1, x_2, x_4, x_7, x_6, x_5), C^2 = (x_1, x_3, x_7, x_4, x_5, x_6), \\ &C^3 = (x_2, x_4, x_1, x_7, x_6, x_3), C^4 = (x_2, x_5, x_7, x_1, x_3, x_6), \\ &C^5 = (x_4, x_6, x_2, x_7, x_3, x_5), C^6 = (x_4, x_1, x_2, x_7, x_5, x_3), \\ &C^7 = (x_1, x_5, x_2, x_3, x_4, x_6). \end{aligned}$$

By applying remark 3.4 to $C^1 \cup C^2$, $C^3 \cup C^4$, $C^5 \cup C^6$, we get all the possible decompositions.

If $\lambda = 4$, then $(p, q) \in \{(0, 14), (3, 12), (6, 10), \ldots, (21, 0)\}$ (we see that the values of p increases by 3 and the values of q decreases by 2). By Theorem 2.2, $K_7(4)$ is $\{21P_5, 0C_6\}$-decomposable. By taking $K_7(4) = 2K_7(2)$, we get all the above possible decompositions.

If $\lambda \geq 6$, then the proof is divided into the following cases.

$\lambda \equiv 0 \pmod 4$. We write $K_7(\lambda) = \frac{\lambda}{4}K_7(4)$.

$\lambda \equiv 2 \pmod 4$. We write $K_7(\lambda) = K_7(\lambda - 2) + K_7(2) = \frac{\lambda-2}{4}K_7(4) + K_7(2)$.

CASE 3: $n = 8$.

If $\lambda = 1$, then $(p, q) \in \{(4, 2), (7, 0)\}$. By Theorem 2.2, K_8 is $\{7P_5, 0C_6\}$-decomposable. The graph K_8 can be decomposed into

$$4P_5 : (x_3x_1x_8x_2x_6), (x_4x_2x_7x_1x_5), (x_8x_6x_4x_5x_2), (x_1x_6x_3x_5x_7)$$

and

$$2C_6 : (x_1, x_2, x_3, x_8, x_7, x_4), (x_3, x_4, x_8, x_5, x_6, x_7).$$

If $\lambda = 2$, then $(p, q) \in \{(2, 8), (5, 6), (8, 4), \ldots, (14, 0)\}$. The graph $K_8(2)$ can be decomposed into

$$2P_5 : (x_1x_7x_5x_6x_3), (x_1x_5x_8x_4x_3)$$

and

$$\begin{aligned} 8C_6 : &C^1 = (x_1, x_7, x_4, x_2, x_3, x_6), C^2 = (x_1, x_3, x_8, x_5, x_4, x_2), \\ &C^3 = (x_1, x_3, x_8, x_4, x_7, x_5), C^4 = (x_1, x_2, x_3, x_4, x_5, x_6), \\ &C^5 = C^6 = (x_8, x_6, x_7, x_3, x_5, x_2), C^7 = C^8 = (x_1, x_8, x_7, x_2, x_6, x_4). \end{aligned}$$

The above $2P_5$ in $K_8(2)$ along with $(x_7x_1x_3x_2x_4), (x_7x_4x_5x_8x_3)$ and $(x_4x_2x_1x_6x_3)$ (Since $C^1 \cup C^2$ can be decomposed into $3P_5$), and C^3, C^4, C^5, C^6, C^7, C^8, we get the required $\{5P_5, 6C_6\}$-decomposition. By taking $K_8(2) = 2K_8$, we get all the other possible decompositions.

If $\lambda = 3$, then $(p, q) \in \{(6, 10), (9, 8), (12, 6), \ldots, (21, 0)\}$. By taking $K_8(3) = K_8(2) + K_8$, we get all the above possible decompositions.

If $\lambda = 4$, then $(p, q) \in \{(4, 16), (7, 14), (10, 12), \ldots, (28, 0)\}$. By taking $K_8(4) = 2K_8(2)$, we get all the above possible decompositions.

If $\lambda = 5$, then $(p, q) \in \{(5, 20), (8, 18), (11, 16), \ldots, (35, 0)\}$. The graph $K_8(5)$ can be decomposed into

$$\begin{aligned} 5P_5 : &(x_8x_5x_4x_6x_2), (x_5x_8x_4x_3x_6), (x_2x_4x_3x_7x_1), \\ &(x_4x_2x_3x_1x_7), (x_3x_7x_1x_6x_2) \end{aligned}$$

and

$$\begin{aligned} 20C_6 : &(x_3, x_4, x_5, x_8, x_2, x_6), (x_8, x_5, x_6, x_2, x_3, x_4), \\ &(x_1, x_7, x_2, x_6, x_4, x_3), (x_8, x_5, x_6, x_1, x_7, x_2), \\ &3 \text{ copies of } (x_8, x_2, x_7, x_3, x_6, x_4), (x_1, x_3, x_2, x_4, x_5, x_6), \\ &5 \text{ copies of } (x_8, x_3, x_5, x_1, x_4, x_7), (x_8, x_6, x_7, x_5, x_2, x_1). \end{aligned}$$

By taking $K_8(5) = K_8(3) + K_8(2)$, we get all the other possible decompositions.

If $\lambda = 6$, then $(p, q) \in \{(0, 28), (3, 26), (6, 24), \ldots, (42, 0)\}$. By Theorem 2.1, $K_8(6)$ is $\{0P_5, 28C_6\}$-decomposable. The graph $K_8(6)$ can be decomposed into

$$3P_5 : (x_4x_6x_8x_7x_3), (x_8x_2x_7x_6x_4), (x_8x_2x_4x_7x_3)$$

and

$$\begin{aligned} 26C_6 : &(x_1, x_5, x_6, x_4, x_7, x_3), (x_1, x_5, x_7, x_3, x_8, x_2), \\ &(x_8, x_2, x_5, x_1, x_6, x_4), (x_1, x_5, x_4, x_6, x_3, x_7), \\ &(x_1, x_5, x_3, x_4, x_2, x_8), (x_6, x_4, x_1, x_5, x_8, x_2), \\ &(x_1, x_2, x_3, x_7, x_8, x_6), 5 \text{ copies of } (x_2, x_6, x_7, x_5, x_8, x_3), \\ &(x_1, x_3, x_6, x_5, x_4, x_8), (x_1, x_4, x_3, x_5, x_2, x_7), \\ &4 \text{ copies of } (x_1, x_2, x_4, x_7, x_8, x_6). \end{aligned}$$

By taking $K_8(6) = K_8(4) + K_8(2)$, we get all the other possible decompositions.

If $\lambda \geq 7$, then the proof is divided into the following cases.

$\lambda \equiv 0 \pmod 6$. We write $K_8(\lambda) = \frac{\lambda}{6}K_8(6)$.

$\lambda \equiv 1 \pmod 6$. We write $K_8(\lambda) = K_8(\lambda - 1) + K_8 = \frac{\lambda-1}{6}K_8(6) + K_8$.

$\lambda \equiv 2 \pmod 6$. We write $K_8(\lambda) = K_8(\lambda - 2) + K_8(2) = \frac{\lambda-2}{6}K_8(6) + K_8(2)$.

$\lambda \equiv 3 \pmod 6$. We write $K_8(\lambda) = K_8(\lambda - 3) + K_8(3) = \frac{\lambda-3}{6}K_8(6) + K_8(3)$.

$\lambda \equiv 4 \pmod 6$. We write $K_8(\lambda) = K_8(\lambda-4) + K_8(4) = \frac{\lambda-4}{6}K_8(6) + K_8(4)$.
$\lambda \equiv 5 \pmod 6$. We write $K_8(\lambda) = K_8(\lambda-5) + K_8(5) = \frac{\lambda-5}{6}K_8(6) + K_8(5)$.
Case 4: $n = 9$.
The graph K_9 can be decomposed into

$$\begin{aligned} 6C_6 : &C^1 = (x_1, x_9, x_4, x_7, x_8, x_5), C^2 = (x_1, x_2, x_3, x_4, x_6, x_8), \\ &C^3 = (x_9, x_2, x_6, x_3, x_7, x_5), C^4 = (x_9, x_8, x_3, x_1, x_7, x_6), \\ &C^5 = (x_5, x_6, x_1, x_4, x_8, x_2), C^6 = (x_5, x_3, x_9, x_7, x_2, x_4). \end{aligned}$$

By applying remark 3.4 to $C^1 \cup C^2$, $C^3 \cup C^4$, $C^5 \cup C^6$, we get all the possible decompositions. If $\lambda \geq 2$, by taking $K_9(\lambda) = \lambda K_9$, we get all the possible decompositions.
Case 5: $n = 10$.

If $\lambda = 2$, then $(p, q) \in \{(0, 15), (3, 13), (6, 11), \ldots, (21, 1)\}$. By Theorem 2.1, $K_{10}(2)$ is $\{0P_5, 15C_6\}$-decomposable. We write $K_{10}(2) = K_{10}(2)\backslash K_6(2) + K_6(2)$. The graph $K_{10}(2)\backslash K_6(2)$ can be decomposed into

$$3P_5 : (x_8x_7x_5x_{10}x_9), (x_9x_{10}x_2x_7x_5), (x_8x_7x_2x_{10}x_5)$$

and

$$\begin{aligned} 8C_6 : &C^1 = (x_{10}, x_4, x_8, x_9, x_7, x_6), C^2 = (x_{10}, x_1, x_9, x_2, x_8, x_3), \\ &C^3 = (x_{10}, x_4, x_8, x_9, x_7, x_6), C^4 = (x_{10}, x_1, x_9, x_2, x_8, x_3), \\ &C^5 = (x_7, x_4, x_9, x_5, x_8, x_{10}), C^6 = (x_7, x_1, x_8, x_6, x_9, x_3), \\ &C^7 = (x_7, x_{10}, x_8, x_5, x_9, x_4), C^8 = (x_7, x_3, x_9, x_6, x_8, x_1). \end{aligned}$$

By applying remark 3.4 to $C^1 \cup C^2$, $C^3 \cup C^4$, $C^5 \cup C^6$, $C^7 \cup C^8$ we get the decompositions $(p, q) \in \{(6, 6), (9, 4), (12, 2), (15, 0)\}$ in $K_{10}(2)\backslash K_6(2)$. By combining these copies of P_5 and C_6 along with the copies of P_5 and C_6 in $K_6(2)$, we get all the above possible decompositions.

If $\lambda = 4$, then $(p, q) \in \{(0, 30), (3, 28), (6, 26), \ldots, (45, 0)\}$. By Theorem 2.2, $K_{10}(4)$ is $\{45P_5, 0C_6\}$-decomposable. By taking $K_{10}(4) = 2K_{10}(2)$, we get all the other possible decompositions.

If $\lambda \geq 6$, then the proof is divided into the following cases.
$\lambda \equiv 0 \pmod 4$. We write $K_{10}(\lambda) = \frac{\lambda}{4}K_{10}(4)$.
$\lambda \equiv 2 \pmod 4$. We write $K_{10}(\lambda) = K_{10}(\lambda-2) + K_{10}(2) = \frac{\lambda-2}{4}K_{10}(4) + K_{10}(2)$.
Case 6: $n = 11$.

If $\lambda = 2$, then $(p, q) \in \{(2, 17), (5, 15), (8, 13), \ldots, (26, 1)\}$. We write $K_{11}(2) = K_{11}(2)\backslash K_7(2) + K_7(2)$. The graph $K_{11}(2)\backslash K_7(2)$ can be decomposed into

$$2P_5 : (x_{11}x_2x_{10}x_9x_8), (x_8x_6x_{10}x_1x_{11})$$

and

$$\begin{aligned} 10C_6 : &C^1 = (x_{10}, x_3, x_9, x_2, x_8, x_6), C^2 = (x_{10}, x_9, x_8, x_3, x_{11}, x_7), \\ &C^3 = (x_{11}, x_1, x_{10}, x_2, x_9, x_3), C^4 = (x_8, x_3, x_{10}, x_7, x_{11}, x_2), \\ &C^5 = C^6 = (x_8, x_5, x_{11}, x_6, x_9, x_4), C^7 = C^8 = (x_{10}, x_4, x_{11}, x_9, x_1, x_8), \\ &C^9 = C^{10} = (x_{11}, x_{10}, x_5, x_9, x_7, x_8). \end{aligned}$$

The above $2P_5$ in $K_{11}(2)\backslash K_7(2)$ along with

$$(x_{10}x_5x_9x_7x_8),(x_8x_{11}x_{10}x_3x_9),(x_{10}x_6x_8x_2x_9),$$
$$(x_8x_3x_{10}x_1x_{11}),(x_{10}x_2x_9x_3x_{11}),(x_{10}x_7x_{11}x_2x_8),$$
$$(x_{11}x_5x_8x_4x_9),(x_{10}x_8x_1x_9x_{11}),(x_9x_6x_{11}x_4x_{10}),$$
$$(x_{10}x_4x_{11}x_5x_8),(x_{10}x_8x_1x_9x_{11}),(x_{11}x_6x_9x_4x_8),$$
$$(x_{11}x_7x_{10}x_9x_8),(x_8x_{11}x_{10}x_5x_9) \text{ and } (x_{11}x_3x_8x_7x_9)$$

(Since $C^1\cup C^9, C^3\cup C^4, C^5\cup C^7, C^6\cup C^8, C^2\cup C^{10}$ can be decomposed into $15P_5$), we get a $\{17P_5, 0C_6\}$-decomposition in $K_{11}(2)\backslash K_7(2)$. The above $2P_5$, $3P_5$ obtained from $C^1\cup C^9$ in $K_{11}(2)\backslash K_7(2)$ and $C^2, C^3, C^4, C^5, C^6, C^7, C^8$, C^{10} we get a $\{5P_5, 8C_6\}$-decomposition in $K_{11}(2)\backslash K_7(2)$. By combining these copies of P_5 and C_6 along with the copies of P_5 and C_6 in $K_7(2)$, we get all the above possible decompositions.

If $\lambda = 4$, then $(p,q) \in \{(4,34),(7,32),(10,30),\ldots,(55,0)\}$. By Theorem 2.2, $K_{11}(4)$ is $\{55P_5, 0C_6\}$-decomposable. By taking $K_{11}(4) = 2K_{11}(2)$, we get all the above possible decompositions.

If $\lambda = 6$, then $(p,q) \in \{(0,53),(3,53),(6,51),\ldots,(81,1)\}$. By Theorem 2.3, $K_{5,5}(6)$ is $\{0P_5, 25C_6\}$-decomposable. By taking $K_{11}(6) = 2K_6(6) + K_{5,5}(6)$, we get the decomposition $(p,q) = (3,53)$. By taking $K_{11}(6) = K_{11}(4) + K_{11}(2)$, we get all the other possible decompositions.

If $\lambda = 8$, then $(p,q) \in \{(2,72),(5,70),(8,68),\ldots,(110,0)\}$. By Theorem 2.2, $K_{11}(8)$ is $\{110P_5, 0C_6\}$-decomposable. By taking $K_{11}(8) = K_{11}(6) + K_{11}(2)$, we get all the above possible decompositions.

If $\lambda = 10$, then $(p,q) \in \{(4,89),(7,87),(10,85),\ldots,(136,1)\}$. By taking $K_{11}(10){=}K_{11}(6) + K_{11}(4)$, we get all the above possible decompositions.

If $\lambda = 12$, then $(p,q) \in \{(0,110),(3,108),(6,106),\ldots,(165,0)\}$. By Theorem 2.2, $K_{11}(12)$ is $\{165P_5, 0C_6\}$-decomposable. By taking $K_{11}(12) = 2K_{11}(6)$, we get all the above possible decompositions.

If $\lambda \geq 14$, then the proof is divided into the following cases.

$\lambda \equiv 0 \pmod{12}$. We write $K_{11}(\lambda) = \frac{\lambda}{12}K_{11}(12)$.

$\lambda \equiv 2 \pmod{12}$. We write $K_{11}(\lambda) = K_{11}(\lambda-2) + K_{11}(2) = \frac{\lambda-2}{12}K_{11}(12) + K_{11}(2)$.

$\lambda \equiv 4 \pmod{12}$. We write $K_{11}(\lambda) = K_{11}(\lambda-4) + K_{11}(4) = \frac{\lambda-4}{12}K_{11}(12) + K_{11}(4)$.

$\lambda \equiv 6 \pmod{12}$. We write $K_{11}(\lambda) = K_{11}(\lambda-6) + K_{11}(6) = \frac{\lambda-6}{12}K_{11}(12) + K_{11}(6)$.

$\lambda \equiv 8 \pmod{12}$. We write $K_{11}(\lambda) = K_{11}(\lambda-8) + K_{11}(8) = \frac{\lambda-8}{12}K_{11}(12) + K_{11}(8)$.

$\lambda \equiv 10 \pmod{12}$. We write $K_{11}(\lambda) = K_{11}(\lambda-10) + K_{11}(10) = \frac{\lambda-10}{12}K_{11}(12) + K_{11}(10)$.

Case 7: $n = 12$.

If $\lambda = 1$, then $(p,q) \in \{(6,7),(9,5),(12,3),(15,1)\}$. We write $K_{12} = (K_{12}\backslash K_9) + K_9$. The graph $K_{12}\backslash K_9$ can be decomposed into

$$6P_5 : (x_6x_{10}x_5x_{11}x_3),(x_{12}x_{10}x_4x_{11}x_8),$$
$$(x_5x_{12}x_6x_{11}x_9),(x_4x_{12}x_7x_{11}x_{10}),$$
$$(x_2x_{12}x_9x_{10}x_1),(x_{11}x_{12}x_8x_{10}x_7)$$

and a

$$C_6 : (x_{12},x_1,x_{11},x_2,x_{10},x_3).$$

By combining these copies of P_5 and C_6 along with the copies of P_5 and C_6 in K_9, we get all the above possible decompositions.

If $\lambda = 2$, then $(p, q) \in \{(0, 22), (3, 20), (6, 18), \dots, (33, 0)\}$. By Theorem 2.2, $K_{12}(2)$ is $\{33P_5, 0C_6\}$-decomposable. By Theorems 2.3 and 2.4 $K_{6,6}$ is $\{\{0P_5, 6C_6\}, \{9P_5, 0C_6\}\}$-decomposable. By taking $K_{12}(2) = 2K_6(2) + 2K_{6,6}$, we get all the above possible decompositions.

If $\lambda \geq 3$, then the proof is divided into the following cases.
$\lambda \equiv 0 \pmod 2$. We write $K_{12}(\lambda) = \frac{\lambda}{2} K_{12}(2)$.
$\lambda \equiv 1 \pmod 2$. We write $K_{12}(\lambda) = K_{12}(\lambda - 1) + K_{12} = \frac{\lambda-1}{2} K_{12}(2) + K_{12}$.
Case 8: $n = 13$.

If $\lambda = 1$, then $(p, q) \in \{(0, 13), (3, 11), (6, 9), \dots, (18, 1)\}$. We write $K_{13} = (K_{13} \backslash K_9) + K_9$. The graph $K_{13} \backslash K_9$ can be decomposed into

$$\begin{aligned} 7C_6 : & C^1 = (x_{13}, x_7, x_{10}, x_8, x_{11}, x_3), C^2 = (x_{13}, x_4, x_{10}, x_5, x_{12}, x_6), \\ & C^3 = (x_{11}, x_7, x_{12}, x_8, x_{13}, x_9), C^4 = (x_{11}, x_5, x_{13}, x_2, x_{10}, x_{12}), \\ & C^5 = (x_{11}, x_4, x_{12}, x_3, x_{10}, x_6), C^6 = (x_{11}, x_1, x_{10}, x_9, x_{12}, x_{13}), \\ & C^7 = (x_{13}, x_1, x_{12}, x_2, x_{11}, x_{10}). \end{aligned}$$

By applying remark 3.4 to $C^1 \cup C^2$, $C^3 \cup C^4$, $C^5 \cup C^6$, we get the decompositions $(p, q) \in \{(0, 7), (3, 5), (6, 3), (9, 1)\}$ in $K_{13} \backslash K_9$. By combining these copies of P_5 and C_6 along with the copies of P_5 and C_6 in K_9, we get all the above possible decompositions.

If $\lambda = 2$, then $(p, q) \in \{(0, 26), (3, 24), (6, 22), \dots, (39, 0)\}$. By Theorem 2.2, $K_{13}(2)$ is $\{39P_5, 0C_6\}$-decomposable. By taking $K_{13}(2) = 2K_{13}$, we get all the above possible decompositions.

If $\lambda \geq 3$, then the proof is divided into the following cases.
$\lambda \equiv 0 \pmod 2$. We write $K_{13}(\lambda) = \frac{\lambda}{2} K_{13}(2)$.
$\lambda \equiv 1 \pmod 2$. We write $K_{13}(\lambda) = K_{13}(\lambda - 1) + K_{13} = \frac{\lambda-1}{2} K_{13}(2) + K_{13}$.
Case 9: $n = 14$.

By taking $K_{14}(\lambda) = K_8(\lambda) + K_6(\lambda) + 2\lambda K_{4,6}$, we get all the possible decompositions.
Case 10: $n = 15$.

By taking $K_{15}(\lambda) = K_9(\lambda) + K_7(\lambda) + 2\lambda K_{4,6}$, we get all the possible decompositions.
Case 11: $n = 16$.

If $\lambda = 1$, then $(p, q) \in \{(9, 14), (12, 12), (15, 10), \dots, (30, 0)\}$. We write $K_{16} = (K_{16} \backslash K_{13}) + K_{13}$. The graph $K_{16} \backslash K_{13}$ can be decomposed into

$$\begin{aligned} 9P_5 : & (x_{11} x_{14} x_2 x_{15} x_{13}), (x_{15} x_1 x_{16} x_5 x_{14}), (x_4 x_{16} x_{12} x_{14} x_6), \\ & (x_{11} x_{16} x_9 x_{14} x_7), (x_5 x_{15} x_7 x_{16} x_{10}), (x_1 x_{14} x_{16} x_{15} x_{12}), \\ & (x_{11} x_{15} x_{14} x_{13} x_{16}), (x_2 x_{16} x_8 x_{15} x_9), (x_3 x_{15} x_{10} x_{14} x_8) \end{aligned}$$

and a

$$C_6 : (x_{16}, x_3, x_{14}, x_4, x_{15}, x_6).$$

By Theorem 2.1, K_{13} is $\{0P_5, 13C_6\}$-decomposable. We have,
$K_{16} = (K_{16} \backslash K_{13}) + K_{13} = \{(9, 1)\} + \{(0, 13)\} = \{(9, 14)\}$. The graph $K_{2,8}$ is $\{4P_5, 0C_6\}$-decomposable. By taking $K_{16} = 2K_8 + 2K_{6,4} + K_{2,8}$, we get all the other possible decompositions.

If $\lambda = 2$, then $(p, q) \in \{(0, 40), (3, 38), (6, 36), \ldots, (60, 0)\}$. By Theorem 2.2, $K_{16}(2)$ is $\{60P_5, 0C_6\}$-decomposable. By Theorems 2.3 and 2.4, $K_{6,6}(2)$ is $\{\{0P_5, 12C_6\}, \{18P_5, 0C_6\}\}$-decomposable. By taking $K_{16}(2) = K_{10}(2) + K_6(2) + K_{6,6}(2) + 2K_{4,6}$, we get all the above possible decompositions.

If $\lambda \geq 3$, then the proof is divided into the following cases.

$\lambda \equiv 0 \pmod 2$. We write $K_{16}(\lambda) = \frac{\lambda}{2}K_{16}(2)$.

$\lambda \equiv 1 \pmod 2$. We write $K_{16}(\lambda) = K_{16}(\lambda - 1) + K_{16} = \frac{\lambda-1}{2}K_{16}(2) + K_{16}$.

Case 12: $n = 17$.

If $\lambda = 1$, then $(p, q) \in \{(4, 20), (7, 18), (10, 16), \ldots, (34, 0)\}$. The graph $K_{2,8}$ is $\{4P_5, 0C_6\}$-decomposable. By taking $K_{17} = 2K_9 + 2K_{6,4} + K_{2,8}$, we get all the above possible decompositions.

If $\lambda = 2$, then $(p, q) \in \{(2, 44), (5, 42), (8, 40), \ldots, (68, 0)\}$. By Theorem 2.2, $K_{17}(2)$ is $\{68P_5, 0C_6\}$-decomposable. By Theorems 2.3 and 2.4, $K_{6,6}(2)$ is $\{\{0P_5, 12C_6\}, \{18P_5, 0C_6\}\}$-decomposable. By taking $K_{17}(2) = K_{11}(2) + K_7(2) + K_{6,6}(2) + 2K_{4,6}$, we get all the above possible decompositions.

If $\lambda = 3$, then $(p, q) \in \{(0, 68), (3, 66), (6, 64), \ldots, (102, 0)\}$. By Theorem 2.1, $K_{17}(3)$ is $\{0P_5, 68C_6\}$-decomposable. By taking $K_{17}(3) = 2K_9(3) + 4K_{4,4}(3)$, we get the decomposition when $(p, q) = (3, 66)$ and by taking $K_{17}(3) = K_{17}(2) + K_{17}$, we get all the other possible decompositions.

If $\lambda \geq 4$, then the proof is divided into the following cases.

$\lambda \equiv 0 \pmod 3$. We write $K_{17}(\lambda) = \frac{\lambda}{3}K_{17}(3)$.

$\lambda \equiv 1 \pmod 3$. We write $K_{17}(\lambda) = K_{17}(\lambda - 1) + K_{17} = \frac{\lambda-1}{3}K_{17}(3) + K_{17}$.

$\lambda \equiv 2 \pmod 3$. We write $K_{17}(\lambda) = K_{17}(\lambda - 2) + K_{17}(2) = \frac{\lambda-2}{3}K_{17}(3) + K_{17}(2)$.

Now we prove the result for $n > 17$. We apply mathematical induction on n and split the proof into four cases as follows:

$n \equiv 0 \pmod 4$. Let $n = 4r$, where $r \geq 2$. If $2 \leq r \leq 4$, the result follows from Cases 3, 7 and 11. Now for some $t > 4$ we assume that there exists a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_{4r}(\lambda)$ for all r where $2 \leq r < t$. Next, we write

$$\begin{aligned} K_{4t}(\lambda) &= K_{4(t-3)}(\lambda) + K_{12}(\lambda) + K_{4(t-3),12}(\lambda) \\ &= K_{4(t-3)}(\lambda) + K_{12}(\lambda) + (t-3)K_{4,12}(\lambda) \\ &= K_{4(t-3)}(\lambda) + K_{12}(\lambda) + (2t-6)\lambda K_{4,6}. \end{aligned}$$

By the induction hypothesis, there exists a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_{4(t-3)}(\lambda)$, and by case 7 and by Lemma 3.2 there exist $\{P_5, C_6\}_{\{p,q\}}$-decompositions of $K_{12}(\lambda)$ and $K_{4,6}$, respectively. Therefore a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_{4t}(\lambda)$ exists.

$n \equiv 1 \pmod 4$. Let $n = 4r + 1$, where $r \geq 2$. If $2 \leq r \leq 4$, the result follows from cases 4, 8 and 12. Now for some $t > 4$ we assume that there exists a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_{4r+1}(\lambda)$ for all r where $2 \leq r < t$. Next, we write

$$\begin{aligned} K_{4t+1}(\lambda) &= K_{4(t-3)+1}(\lambda) + K_{13}(\lambda) + K_{4(t-3),12}(\lambda) \\ &= K_{4(t-3)+1}(\lambda) + K_{13}(\lambda) + (t-3)K_{4,12}(\lambda) \\ &= K_{4(t-3)+1}(\lambda) + K_{13}(\lambda) + (2t-6)\lambda K_{4,6}. \end{aligned}$$

By the induction hypothesis, there exists a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_{4(t-3)+1}(\lambda)$, and by case 8 and by Lemma 3.2 there exist $\{P_5, C_6\}_{\{p,q\}}$-decompositions of $K_{13}(\lambda)$ and $K_{4,6}$, respectively. Therefore a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_{4t+1}(\lambda)$ exists.

$n \equiv 2 \pmod 4$. Let $n = 4r + 2$, where $r \geq 1$. If $1 \leq r \leq 3$, the result follows from cases 1, 5 and 9. Now for some $t > 3$ we assume that there exists a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_{4r+2}(\lambda)$ for all r where $1 \leq r < t$. Next, we write

$$\begin{aligned} K_{4t+2}(\lambda) &= K_{4(t-1)}(\lambda) + K_6(\lambda) + K_{4(t-1),6}(\lambda) \\ &= K_{4(t-1)}(\lambda) + K_6(\lambda) + (t-1)\lambda K_{4,6}. \end{aligned}$$

By the induction hypothesis, there exists a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_{4(t-1)}(\lambda)$, and by case 1 and by Lemma 3.2 there exist $\{P_5, C_6\}_{\{p,q\}}$-decompositions of $K_6(\lambda)$ and $K_{4,6}$, respectively. Therefore a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_{4t+2}(\lambda)$ exists.

$n \equiv 3 \pmod 4$. Let $n = 4r + 3$, where $r \geq 1$. If $1 \leq r \leq 3$, the result follows from cases 2, 6 and 10. Now for some $t > 3$ we assume that there exists a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_{4r+3}(\lambda)$ for all r where $1 \leq r < t$. Next, we write

$$\begin{aligned} K_{4t+3}(\lambda) &= K_{4(t-1)+1}(\lambda) + K_7(\lambda) + K_{4(t-1),6}(\lambda) \\ &= K_{4(t-1)+1}(\lambda) + K_7(\lambda) + (t-1)\lambda K_{4,6}. \end{aligned}$$

By the induction hypothesis, there exists a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_{4(t-1)+1}(\lambda)$, and by case 2 and by Lemma 3.2 there exist $\{P_5, C_6\}_{\{p,q\}}$-decompositions of $K_7(\lambda)$ and $K_{4,6}$, respectively. Therefore a $\{P_5, C_6\}_{\{p,q\}}$-decomposition of $K_{4t+3}(\lambda)$ exists, and the result follows by mathematical induction. □

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