Contributions to Discrete Mathematics

Volume 17, Number 1, Pages 70–108
ISSN 1715-0868

SOME RELATIONAL STRUCTURES WITH POLYNOMIAL GROWTH AND THEIR ASSOCIATED ALGEBRAS II. FINITE GENERATION.

MAURICE POUZET AND NICOLAS M. THIÉRY

Abstract. The *profile* of a relational structure R is the function φ_R which counts for every nonnegative integer n the number, possibly infinite, $\varphi_R(n)$ of substructures of R induced on the n-element subsets, isomorphic substructures being identified. If φ_R takes only finite values, this is the Hilbert function of a graded algebra associated with R, the *age algebra* $\mathbb{K}.\mathcal{A}(R)$ introduced by P. J. Cameron. In a previous paper, we studied the relationship between the properties of a relational structure R and those of its age algebra, particularly when R admits a finite monomorphic decomposition. This setting still encompasses well-studied graded commutative algebras like invariant rings of finite permutation groups or the rings of quasisymmetric polynomials.

The main theorem of this paper characterizes combinatorially when the age algebra is finitely generated in this setting. For tournaments, this boils down to the profile being bounded. We further investigate how far the well known algebraic properties of invariant rings and quasisymmetric polynomials extend to age algebras; notably, we explore the Cohen-Macaulay property in the special case of invariants of permutation groupoids. Finally, we exhibit sufficient conditions on the relational structure that make naturally the age algebra into a Hopf algebra.

For a homogeneous structure with a profile bounded by a polynomial, Cameron conjectured in the early eighties that the profile is asymptotically polynomial; Macpherson further conjectured that the age algebra is finitely generated. This was proven recently by Falque and the second author. The combined results support the conjecture that—assuming finite kernel—profiles bounded by a polynomial are asymptotically polynomial, and give hope for a complete characterization of when the age algebra is finitely generated.

Received by the editors December 5, 2019, and in revised form October 1, 2021.

2010 *Mathematics Subject Classification.* 05E15, 03E20.

Key words and phrases. Relational structure, profile, polynomial growth, kernel, age, age algebra, graded commutative algebra, Hilbert function, invariants of a permutation group, quasi-symmetric polynomials.

This work was done under the auspices of the Intas programme Universal algebra and Lattice theory, and supported by CMCU Franco-Tunisien "Outils mathématiques pour l'informatique". Some of the results (invariants of permutation groupoids) were announced in an extended abstract for FPSAC'05 [47]. An early short version of this paper (without proofs) was presented at the ROGICS'08 conference, May 12-17, Mahdia (Tunisia) [48]. The authors would like to express their gratitude to the organizers of the latter, Professors Y. Boudabbous and N. Zaguia.

1. Introduction

This paper is about some connections between commutative algebra and relational structures. For general background on commutative algebras, we refer for example to [15, 11] and for relational structures to [19].

In [8, section 2] Cameron defined the *orbit algebra* of a permutation group G acting on an infinite set E; by design, the Hilbert function $h_{\mathbb{K}.\mathcal{A}(G)}$ of this graded commutative algebra coincides with the *orbital profile* of G, namely the function that counts, for every non-negative integer n, the number $\varphi_G(n)$ of orbits of G acting on the finite subsets of size n of E (see [15, 1.9] for Hilbert functions and graded commutative algebras). The main motivation was to study properties of orbital profiles, and in particular a phenomenon of jumps in the possible growth rates.

Similar phenomenon had been observed in the more general context of relational structures (permutation groups being in correspondence with homogeneous relational structures). There, the profile of a relational structure R on E counts, for every integer n, the number $\varphi_R(n)$ of substructures of R induced on the n-element subsets of E, isomorphic substructures being identified. In [10], Cameron proposed to generalize the approach, defining the age algebra of a relational structure. Familiar algebras like invariant rings of finite permutation groups, algebras of quasisymmetric polynomials [25] or the shuffle algebra over a finite alphabet can be realized as such age algebras.

This paper investigates relationships between combinatorial properties of a relational structure R and algebraic properties of its age algebra $\mathbb{K}.\mathcal{A}(R)$. Specifically, we consider the following conditions:

Conditions 1.1.

- (BP) *the profile is bounded above by a polynomial;*
- (QP) *the profile is eventually a quasipolynomial; equivalently its generating series is of the form:*

$$\frac{P(Z)}{(1-Z^{n_1})(1-Z^{n_2})\cdots(1-Z^{n_k})},$$

where $n_1 \leq \cdots \leq n_k$ and $P(Z) \in \mathbb{Z}[Z]$;

- (QPP) *same as (QP) with $P \in \mathbb{N}[Z]$;*
- (AP) *the profile is asymptotically equivalent to a polynomial;*
- (FG) *the age algebra is finitely generated;*
- (CM) *the age algebra is Cohen-Macaulay (for the definition of Cohen-Macaulay graded commutative algebras, see e.g.* [12, §2.5] *or* [15, § 18.2]*).*

We also consider the following condition:

Condition.

(H) the age algebra is a graded Hopf algebra; in particular it is free (for background on Hopf algebras and their relations to combinatorics, see e.g. [26]*).*

What is known? We start with the obvious or well know relations between those conditions:

- (QP) $\implies$ (BP), (AP) $\implies$ (BP);
- (QP) $\implies$ (AP), using that the profile is non decreasing (Pouzet, [18, ex. 8 p. 113] for relational structures; Cameron, [7, Theorem 2.2] for permutation groups);
- The two conditions of (QP) are equivalent as a straightforward consequence of [53, Proposition 4.4.1];
- (FG) $\implies$ (QP) by a general property of graded commutative algebras (see e.g. [11, Chapter 9, §2]);
- (CM) $\implies$ (QPP);
- (CM) $\implies$ (FG).

For the examples mentioned earlier, all of Conditions 1.1 are equivalent. This is not an isolated phenomenon. Let us consider the case of a permutation group, or equivalently of a homogeneous relational structure. Cameron conjectured (BP) $\implies$ (AP) (see [9, p. 69]) and Macpherson asked whether (BP) $\implies$ (AP) [36, p. 286]. Falque and the second author recently provided a positive answer; in fact, all of Conditions 1.1 are equivalent [17], [16]. This derives from a complete classification of permutation groups with profile bounded by a polynomial in terms of finite permutation groups with decorated block systems.

In a previous paper [49], we made the following conjecture:

Conjecture. *Let R be a relational structure with finite kernel. Then, (BP) $\implies$ (AP), that is the profile φ_R is eventually a quasipolynomial whenever φ_R is bounded by some polynomial.*

We then introduced the notion of monomorphic decomposition (whose nature resembles that of block systems for permutation groups), restricted ourselves to the setting of the relational structure admitting a finite monomorphic decomposition and proved our conjecture there.

Content of this paper. Keeping the same setting but moving to a more algebraic perspective, this paper investigates the conditions (FG), (CM), and (H). This setting encompasses invariant rings of finite permutation groups and rings of quasisymmetric polynomials for which conditions (FG) and (CM) hold; the latter fact is a theorem of Garsia and Wallach [22]. For other examples, (FG) and (CM) fail. Our main result is a combinatorial characterization of when (FG) holds (Theorem 3.5).

In section 2, we briefly review relational structures, their orbit algebras and monomorphic decompositions. We refer to [49] for a detailed approach.

In addition, we mention there is a relationship between the order properties of an age and properties of the ideals of the age algebra.

section 3 is devoted to our main theorem (Theorem 3.5). We start by giving the key ideas on an example, and proceed with the general proof. With the help of [4], we show that the age algebra of a tournament is finitely generated if and only if the profile of the tournament is bounded (Theorem 3.7). Indeed, if an age algebra is finitely generated, the profile is bounded above by a polynomial and according to [4], tournaments with profile bounded by a polynomial have a finite monomorphic decomposition (meaning simply that these tournaments are lexicographical sums of acyclic tournaments indexed by a finite tournament) and our characterization applies.

In section 4, we further restrict the setting to *invariant rings of permutation groupoids*, defined as age algebras of some appropriate relational structures (section 4). This setting provides a very tight generalization of invariant rings of permutation groups which still includes quasisymmetric polynomials. We analyze in detail which properties of invariant rings of permutation groups carry over—or not—to permutation groupoids (cf. Propositions 4.17 and 4.9, and Theorems 3.5 and 4.16). To this end we use, in particular, techniques from [21].

Finally, in section 5, we give some sufficient conditions on the relational structure to endow the age algebra with a further structure of (coassociative) Hopf algebra, and recover several classical Hopf algebras. The age algebra is then a free algebra, which imposes a very rigid form for the Hilbert series and thus for the profile.

General comments and perspectives. For orbit algebras, our main theorem does not bring new insight; indeed, by Theorem 2.18, an orbit algebra whose homogeneous structure admits a finite monomorphic decomposition is isomorphic to the invariant ring of a finite permutation group (or straightforward quotient thereof); the latter is well known to be finitely generated.

There are other classes of structures for which polynomially bounded profile amounts to the existence of a finite monomorphic decomposition (e.g. permutations [37] and ordered graphs [2, 1]) and for which we may use our characterization.

Problem 1.3. *For relational structures admitting a finite monomorphic decomposition, characterize combinatorially when the age algebra is Cohen-Macaulay. This would provide an alternate proof of Garsia and Wallach's theorem for quasisymmetric functions* [22]*.*

Problem 1.4. *For general relational structures, characterize combinatorially when the age algebra is finitely generated. The remaining open case is when the minimal monomorphic decomposition has infinitely many blocs, of which a finite number are infinite. Examples A.9 and A.10 of* [49] *show that, in this case and even just for graphs, the age algebra can be finitely generated, or not. The results of* [17], [16] *for permutation groups suggest* *that an approach may be to exhibit a variant of the notion of monomorphic decomposition that would better mimic block systems.*

Problem 1.5. *Under the hypothesis of Conjecture 1, it can be shown that there is a* uniformly prehomogeneous *relational structure M of age $\mathcal{A}(M) = \mathcal{A}(R)$. Indeed, the age is well quasi ordered by embeddability, this even by addition of finitely many constants (see the last chapter of* [42] *and* [43]*); hence the test of* [40] *applies.*

Consider the automorphism group $G = \mathrm{Aut}(M)$ of M. Beware that, even though the profile of R (and thus of M) is polynomial, the profile of G need not be polynomial (see Example A.9). When the profile of G is indeed bounded by a polynomial, can the results of [17, 16] *be exploited to control the profile of M? Maybe to show, at least, that it is asymptotically polynomial?*

Ultimately the notion of age algebra may not be quite right, and should be adapted to ensure that all of Conditions 1.1 are equivalent:

Problem 1.6. *Devise some* sensible *alternative graded algebra structure on $\mathbb{K}.\mathcal{A}(R)$ which is finitely generated whenever the profile is bounded above by a polynomial.*

Problem 1.7. *Devise some* sensible *alternative graded algebra structure on $\mathbb{K}.\mathcal{A}(R)$ which is Cohen-Macaulay whenever the Hilbert series has the appropriate form (by Proposition 4 of* [3] *such an algebra always exists).*

Let R be a relational structure on a set E. It induces an equivalence relation on the finite subsets of E by setting $A \sim_R B$ whenever the structures induced by R on A and B respectively are isomorphic. This equivalence relation is *hereditary*:

$$|A| = |B|$$

and

$$|\{X \subset A : X \sim_R C\}| = |\{X \subset B : X \sim_R C\}|$$

whenever A, B, and C are finite subsets of E such that $A \sim_R B$ (hereditary equivalence relations were introduced in [45]; see also [6]. The definition of age algebra extends straightforwardly to hereditary equivalence relations.

Problem 1.8. *Generalize the results of this paper to hereditary equivalence relations and the corresponding algebras.*

2. On the profile and age algebra of a relational structure

2.1. Relational structures and their monomorphic decompositions.

A *n-ary relation on a set E* is any subset ρ of E^n. It is identified with the predicate $(e_1, \dots, e_n) \mapsto (e_1, \dots, e_n) \in \rho$. A *relational structure* is a pair $R := (E, (\rho_i)_{i \in I})$ made of a set E and a family of m_i-ary relations ρ_i on E. We denote by $R_{\restriction A}$, that we call *restriction* of R to A, the substructure induced by R on a subset A of E. We consider these substructures up to isomorphism. If needed, we consider *isomorphic types*, objects associated to relational structures in such a way that the types $\tau(S_1)$ and $\tau(S_2)$ of two relational structures are equal if and only if the two relational structures are isomorphic. In our case, we may identify the isomorphic type $\tau(R_{\restriction A})$ of the substructure of R induced on a finite subset A of E to its *orbit* $\tau(A) := \{A' \subseteq E : R_{\restriction_{A'}}$ is isomorphic to $R_{\restriction_A}\}$.

Let R be a relational structure on a set E. A subset B of E is a *monomorphic part* of R if for every integer n and every pair A, A' of n-element subsets of E the induced structures on A and A' are isomorphic whenever $A \setminus B = A' \setminus B$. A partition of E into monomorphic parts is a *monomorphic decomposition* of R.

A straightforward yet crucial property of such partitions is given in Fact 2.1:

Let $(E_x)_{x\in X}$ be a set partition of E. Set $X_\infty := \{x \in X : |E_x| = \infty\}$; for a finite subset A of E, set $d_x(A) := |A \cap E_x|$, and denote by $\mathbf{d}(A) := (d_x(A))_{x\in X}$ the statistics of intersection sizes.

Fact 2.1. The partition $(E_x)_{x\in X}$ is a monomorphic decomposition of R if and only if the induced structures on two finite subsets A and A' of E are isomorphic whenever $\mathbf{d}(A) = \mathbf{d}(A')$.

The proof is left as an exercise for the reader.

As stated in [49, Proposition 2.12], each monomorphic part of E is included in a maximal one with respect to inclusion and these maximal parts form a monomorphic decomposition of R. Hence, every monomorphic decomposition is finer than the decomposition into maximal parts. We call this partition the *minimal monomorphic decomposition*. The *monomorphic dimension* of R is the number of infinite parts in its minimal monomorphic decomposition.

Revisiting these notions, Oudrar and Pouzet (see [38, p. 168, § 7.2.5], [39, 46]) define this partition in a direct way as follows: Say that two elements x and y of E are *equivalent* and set $x \simeq_R y$ if for every finite subset F of $E \setminus \{x, y\}$, the restrictions of R to $\{x\} \cup F$ and $\{y\} \cup F$ are isomorphic.

Lemma 2.2. *The relation $\simeq_R$ is an equivalence relation. Furthermore, the equivalence classes of $\simeq_R$ are the maximal parts of R, hence they form the minimal monomorphic decomposition.*

Proof. See Lemma 7.48 and Lemma 7.49 in Section 7.2.5 of [38] and also the proof of Proposition 3 page 292 of [46]. □

As an immediate consequence, we obtain the following result (occurring as a part of Corollary 2.13 of [49]).

Corollary 2.3. *Every automorphism of R induces a permutation of the maximal parts of R.*

2.2. **Profile and age algebra of a relational structure.** The *age* of a relational structure $R := (E, (\rho_i)_{i\in I})$ is the set $\mathcal{A} := \mathcal{A}(R)$ of finite substructures of R, isomorphic substructures being identified. This set was introduced by Fraïssé (see [19]). The *kernel* of R is the set $\operatorname{kernel}(R) := \{x \in E : \mathcal{A}(R - \{x\}) \neq \mathcal{A}(R)\}$. We say that $\mathcal{A}(R)$ is *inexhaustible* and R is *age-inexhaustible* if two members of $\mathcal{A}(R)$ can be embedded in a third with disjoint images. This condition amounts to say that the kernel of R is empty (see 10.6.2 in [19]).

The *profile* of R is the function φ_R which counts for every integer n the number $\varphi_R(n)$ of substructures of R induced on the n-element subsets, isomorphic substructures being identified. Clearly, this function only depends upon the age of R. We recall that the profile of an infinite relational structure is nondecreasing (Exercise 8 p. 113 by the first author in [18]; see also [41]. Furthermore, provided that some mild conditions hold, namely either the signature $\mu := (m_i)_{i \in I}$ is bounded or $\operatorname{kernel}(R)$, the kernel of R, is finite, there are jumps in the behavior of the profile: the growth of φ_R is either polynomial or as fast as every polynomial [42, 43] and [2, 1, 34] for independent developments on this theme.

Nota Bene: Example A.10 illustrates some peculiarities arising when the profile is infinite. **In the sequel, and for the sake of simplicity of exposition, we always make the assumption that $\varphi_R(n)$ is finite for all** n. This assumption holds as soon as I is finite.

Let $\mathbb{K}$ be a field of characteristic 0. Cameron associates a graded algebra $\mathbb{K}.\mathcal{A}(R)$ to each relational structure R [10]. This algebra $\mathbb{K}.\mathcal{A}(R)$, called the *age algebra of* R depends only upon the age of R. Its main feature is that its Hilbert function coincides with the profile of R as long as it takes only finite values.

The graded algebra $\mathbb{K}.\mathcal{A}(R)$ is the direct sum $\bigoplus_{n<\omega} \mathbb{K}.\mathcal{A}(R)_n$, where $\mathbb{K}.\mathcal{A}(R)_n$ is the set of $\mathbb{K}$-linear combinations of elements of $\mathcal{A}(R)_n$, the set of substructures of R induced on the n-element subsets of E considered up to isomorphy.

Multiplication is induced from disjoint unions of sets. Namely write $A_1 \uplus A_2 = A$ if A is the disjoint union of A_1 and A_2. Then, the product of $\tau_1 \in \mathcal{A}(R)_n$, and $\tau_2 \in \mathcal{A}(R)_m$ is defined by

$$\tau_1.\tau_2 := \sum_{\tau \in \mathcal{A}(R)_{n+m}} c^{\tau}_{\tau_1,\tau_2} \tau \tag{2.1}$$

where,

$$c^{\tau}_{\tau_1,\tau_2} := |\{(A_1, A_2) : A_1 \uplus A_2 = A, \tau(A_1) = \tau_1, \tau(A_2) = \tau_2 \text{ and } \tau(A) = \tau\}|. \tag{2.2}$$

We recall two results:

Let e be the sum of isomorphic types of the one-element restrictions of R (we can identify it to $\sum_{a \in E} \{a\}$). Let U be the graded free algebra $\mathbb{K}[e] = \bigoplus_{n=0}^{\infty} \mathbb{K}e^n$.

Theorem 2.4 ([10]). *If R is infinite then e is not a zero divisor; namely for any $u \in \mathbb{K}.\mathcal{A}(R)$, $eu = 0$ if and only if $u = 0$.*

This result implies that φ_R is nondecreasing. Indeed, the image of a basis of the vector space $\mathbb{K}.\mathcal{A}(R)_n$ under multiplication by e is a linearly independent subset of $\mathbb{K}.\mathcal{A}(R)_{n+1}$.

We recall the following result:

Theorem 2.5 ([44]). *If R is age-inexhaustible then the age algebra $\mathbb{K}.\mathcal{A}(R)$ has no nonzero divisor.*

In the sequel, we give some general properties relating relational structures and algebras.

2.3. **Operations on relational structures and age algebras.** In this section and in section 2.6, a useful technical device is to embed the age algebra $\mathbb{K}.\mathcal{A}(R)$ of a relational structure R in a larger algebra, the set algebra, whose definition we recall now (see e.g. section 1.2 of [49]).

Let $\mathbb{K}$ be a field of characteristic 0, and E be a set. For $n \geq 0$, denote by $[E]^n$ the set of the subsets of E of size n, and let $\mathbb{K}^{[E]^n}$ be the vector space of maps $f : [E]^n \to \mathbb{K}$. The *set algebra* is the graded connected commutative algebra $\mathbb{K}^{[E]^{<\omega}} := \bigoplus_n \mathbb{K}^{[E]^n}$, where the product of $f : [E]^m \to \mathbb{K}$ and $g : [E]^n \to \mathbb{K}$ is defined as $fg : [E]^{m+n} \to \mathbb{K}$ such that:

$$(fg)(A) := \sum_{(A_1,A_2)\,:\,A=A_1\uplus A_2} f(A_1)g(A_2). \tag{2.3}$$

Identifying a set S with its characteristic function χ_S, elements of the set algebra can be viewed as (possibly infinite but of bounded degree) linear combination of sets; the unit is the empty set, and the product of two sets is their disjoint union, or 0 if their intersection is nontrivial.

The desired embedding of $\mathbb{K}.\mathcal{A}(R)$ into $\mathbb{K}^{[E]^{<\omega}}$ is obtained by mapping an isomorphism type τ to its orbit sum $\sum_{A\,:\,\tau(A)=\tau} A$.

Remark: The original definition of the age algebra, which we used above, requires the assumption that the profile is finite. See e.g. Example A.10 for what can go wrong otherwise. The set algebra offers a natural frame to formulate an equivalent definition that extends beyond this assumption and further to hereditary equivalence relations (this is for example the presentation adopted in [44]).

Namely, consider a relational structure R on a set E with profile not necessarily finite, or more generally a hereditary equivalence $\equiv$ on the finite subsets of E (see [45] for the definition of hereditary equivalence). Say that a map $f : [E]^n \to \mathbb{K}$ is *invariant* if it is constant on each equivalence class on $[E]^n$ induced by R (or $\equiv$) (e.g. $f(A) = f(A')$ whenever the restrictions $R_{\restriction A}$ and $R_{\restriction A'}$ are isomorphic). Consider the space $\mathbb{K}.\mathcal{A} = \bigoplus_n \mathbb{K}.\mathcal{A}_n$, where $\mathbb{K}.\mathcal{A}_n$ is the set of all invariant maps for a given n. Observe that $\mathbb{K}.\mathcal{A}$ is a subalgebra of the set algebra, and call it the age algebra.

Proposition 2.7.

(i) *Let $R := (E, (\rho_i)_{i\in I})$ be a relational structure, E' be a subset of E and $R' := R_{\restriction E'}$ be the structure induced by R on E'. Then, $\mathbb{K}.\mathcal{A}(R')$ is a quotient of $\mathbb{K}.\mathcal{A}(R)$.*

(ii) *Let $R := (E, (\rho_i)_{i\in I})$ be a relational structure, I' be a subset of I, and $R' := (E, (\rho_i)_{i\in I'})$. Then, $\mathbb{K}.\mathcal{A}(R')$ is a subalgebra of $\mathbb{K}.\mathcal{A}(R)$.*

(iii) *Let $R := (E, (\rho_i)_{i\in I})$ and $R' := (E', (\rho'_i)_{i\in I'})$ be two relational structures on disjoint sets, and define their* direct sum *as the relational structure $R \oplus R' := (E \cup E', (\rho_i)_{i\in I}, (\rho'_i)_{i\in I'})$. If needed, add an appropriate unary relation to ensure that singletons of E and E' are not isomorphic. Then, $\mathbb{K}.\mathcal{A}(R \oplus R')$ is isomorphic to the tensor product $\mathbb{K}.\mathcal{A}(R) \otimes \mathbb{K}.\mathcal{A}(R')$. Furthermore, if $(E_i)_i$ and $(E'_j)_j$ are (minimal) monomorphic decompositions of respectively R and R', then $(E_i)_i \cup (E'_j)_j$ form a (minimal) monomorphic decomposition of $R \oplus R'$.*

Proof. (i) At the level of the set algebra, the vector space spanned by the sets which are not subsets of E' is an ideal; so the linear map ϕ which kills those sets is an algebra morphism. Furthermore looking at the image of basis elements shows that $\mathbb{K}.\mathcal{A}(R')$ is the image by ϕ of $\mathbb{K}.\mathcal{A}(R)$.

(ii) Each isomorphism class for R splits into one or more isomorphism class(es) for R'; hence each basis element for $\mathbb{K}.\mathcal{A}(R)$ is accordingly the sum of one or more basis element(s) for $\mathbb{K}.\mathcal{A}(R')$.

(iii) Identify each subset A of $E \cup E'$ with the element $(A \cap E) \otimes (A \cap E')$ of the tensor product and check that a basis element for $\mathbb{K}.\mathcal{A}(R \otimes R')$ is the tensor product of a basis element for $\mathbb{K}.\mathcal{A}(R)$ by a basis element of $\mathbb{K}.\mathcal{A}(R')$. The resulting Hilbert series is the product of the Hilbert series for R and R'. □

2.3.1. *Wreath products.* Let R be a relational structure. A *local isomorphism* of R is any isomorphism from a restriction of R to another restriction. We denote by $\mathrm{Loc}(R)$ the set of local isomorphisms of R. Endowed with the partial composition product, $\mathrm{Loc}(R)$ becomes a monoid (see e.g. section 4.1). The wreath product of two such monoids $\mathrm{Loc}(R)$ and $\mathrm{Loc}(R')$ can be defined in the same vein as for permutation groups: $\mathrm{Loc}(R)$ acts independently within each E_x while $\mathrm{Loc}(R')$ acts globally on the $(E_x)_{x\in E'}$.

We define the wreath product $R \wr R'$ of two relational structure R and R' on the product $E \times E'$ of their domain in such a way that, for every local isomorphism f' of R' and each family $(f_{x'})_{x'\in dom f'}$ of local isomorphisms of E, the map $\bigcup_{x'\in dom(f')} \overline{f_{x'}}$ (where $\overline{f_{x'}}(x, x') := (f_{x'}(x), f'(x'))$ for all $x \in dom(f_{x'})$) is a local isomorphism of $R \wr R'$. Namely, the *wreath product* of R and R' is the relational structure $R \wr R' := (E \times E', (\overline{\rho}_i)_{i\in I}, (\overline{\rho}'_i)_{i\in I'})$, where

$$\overline{\rho}_i\left((e_1, e'_1), \dots, (e_k, e'_k)\right) \text{ if and only if } \rho_i(e_1, \dots, e_k) \text{ and } e'_1 = \cdots = e'_k,$$

and

$$\overline{\rho}'_i\left((e_1, e'_1), \dots, (e_k, e'_k)\right) \text{ if and only if } \rho'_i(e'_1, \dots, e'_k).$$

Remark: **Beware of that, for the sake of consistency with the notation for groups and monoids, we changed convention: $R \wr R'$ in this paper was denoted $R' \wr R$ in [49].**

Proposition 2.9. *Let R and R' be relational structures, and $(E_i)_{i \in I}$ be a monomorphic decompositions of R. Then $(E_i \times \{x\})_{i \in I, x \in E'}$ forms a monomorphic decomposition of $R \wr R'$. In particular, if R is monomorphic, then the $E_x := E \times \{x\}$ for $x \in E'$ form a monomorphic decomposition of $R \wr R'$.*

Proof. Direct consequence of Proposition 2.7 (ii) and (iii). □

Remark: $\mathrm{Loc}(R \wr R')$ includes the wreath product $\mathrm{Loc}(R) \wr \mathrm{Loc}(R')$. This inclusion may be strict: take for example the wreath product of a 2-antichain R with itself; then $R \wr R$ is a 4-antichain. Any local bijection is in $\mathrm{Loc}(R \wr R)$ whereas only these preserving the two defining 2-blocks are in $\mathrm{Loc}(R) \wr \mathrm{Loc}(R)$.

2.4. **Initial segments of an age and ideals of a ring.** We recall that a *final segment* of a poset P is a subset $F \subseteq P$ with the property that if x is in F and $x \leq y$ then $y \in F$. *Initial segment* are defined dually. An *ideal* I of P is a nonempty initial segment which is up-directed: for any $x, y \in I$ there exists $z \in I$ such that $x, y \leq z$.

In this section, we order types of finite structures by *embeddability*: if τ_1, τ_2 are relational structures, we set $\tau_1 \leq \tau_2$ if τ_1 is the type of a structure induced on some substructure of type τ_2. The age $\mathcal{A}(R)$ of a relational structure is an ideal. Reciprocally, it is well known that any countable ideal I of finite types is the age of a countable structure [19, §10.2.1 p. 278].

Final segments play for posets the same role as ideals for rings. We describe briefly the correspondence between initial segments of an age and ideals of the age algebra.

Theorem 2.11. *Let $\mathcal{A} := \mathcal{A}(R)$ be the age of a relational structure R and $\mathbb{K}.\mathcal{A}$ be its age algebra. Recall that we assume that R has finite profile. If $\mathcal{A}'$ is an initial segment of $\mathcal{A}$ then:*

(i) the vector subspace $J := \mathbb{K}.(\mathcal{A} \setminus \mathcal{A}')$ spanned by $\mathcal{A} \setminus \mathcal{A}'$ is an ideal of $\mathbb{K}.\mathcal{A}$. Moreover, the quotient of $\mathbb{K}.\mathcal{A}$ by J is a ring isomorphic to the ring $\mathbb{K}.\mathcal{A}'$.
(ii) if J is irreducible then $\mathcal{A}'$ is a subage of $\mathcal{A}$;
(iii) J is a prime ideal if and only if $\mathcal{A}'$ is an inexhaustible age.

Proof. (*i*) Since J is a subspace of $\mathbb{K}.\mathcal{A}$, it suffices to show that:

$$\tau_1 \in \mathcal{A} \setminus \mathcal{A}' \text{ and } \tau_2 \in \mathcal{A} \text{ implies } \tau_1.\tau_2 \in J. \tag{2.4}$$

With the notations of the product in the age algebra (see equation 1, page 6) $\tau_1.\tau_2$ is a linear combination of some elements τ's of $\mathcal{A}$. For those τ's with nonzero coefficients, we have $\tau_1 \leq \tau \in \mathcal{A}$. Since $\tau_1 \in \mathcal{A} \setminus \mathcal{A}'$, we have $\tau \in \mathcal{A} \setminus \mathcal{A}' \subseteq J$. Hence $\tau_1.\tau_2 \in J$, as required.

(ii) For convenience, we consider here complements of initial segments, that is *final segments*. Let $F(\mathcal{A})$ be the set of final segments of $\mathcal{A}$ and $\mathrm{Id}(\mathbb{K}.\mathcal{A})$ be the set of ideals of the algebra $\mathbb{K}.\mathcal{A}$, these sets being ordered by inclusion. Let $\varphi : F(\mathcal{A}) \to \mathrm{Id}(\mathbb{K}.\mathcal{A})$ defined by setting $\varphi(F) := \mathbb{K}.F$ if $F \neq \emptyset$ and $\varphi(F) := \{0\}$ otherwise. As shown in (i) this map is well-defined. It preserves arbitrary joins and meets that is:

$$\mathbb{K}.\bigcup_{i\in I} F_i = \sum_{i\in I} \mathbb{K}.F_i; \tag{2.5}$$

$$\mathbb{K}.\bigcap_{i\in I} F_i = \bigcap_{i\in I} \mathbb{K}.F_i. \tag{2.6}$$

The first equality is obvious. For the second equality, we have trivially $\mathbb{K}.\bigcap_{i\in I} F_i \subseteq \bigcap_{i\in I} \mathbb{K}.F_i$. For the converse, let $P \in \bigcap_{i\in I} \mathbb{K}.F_i$. We have $P = \sum_{j\in I_i} P_{ij}$ with $P_{ij} \in \mathbb{K}.F_i$ for each $i \in I$. By definition, members of $\mathcal{A}$ are linearly independent, thus the decomposition of P into a linear sum of members of $\mathcal{A}$ is unique. It follows that P_{ij} is independent of I_i thus belongs to $\bigcap_{i\in I} F_i$, hence $P \in \mathbb{K}.\bigcap_{i\in I} F_i$. This completes the proof that the second equality holds.

The map φ restricted to $F(\mathcal{A}) \setminus \{\emptyset\}$ is one-to-one. The inverse image of a meet-irreducible element of $\mathrm{Id}(\mathbb{K}.\mathcal{A})$—that is an irreducible ideal of the algebra $\mathbb{K}.\mathcal{A}$—is a meet-irreducible element of $F(\mathcal{A})$. As it is well-known, for an arbitrary poset $\mathcal{P}$, the meet-irreducible members of $F(\mathcal{P})$ are exactly the complements of ideals of $\mathcal{P}$; in the case $\mathcal{P} = \mathcal{A}$ the ideals are the subages of $\mathcal{A}$. This completes the proof of (ii).

(iii) Suppose that $\mathcal{A}'$ is not an inexhaustible age. Then there are $S'_1, S'_2 \in \mathcal{A}'$ such that no $R \in \mathcal{A}'$ extends S'_1, S'_2 on the disjoint union of their domains. But then the product $S'_1.S'_2$ is in $\mathbb{K}.(\mathcal{A} \setminus \mathcal{A}')$, proving that $\mathbb{K}(\mathcal{A} \setminus \mathcal{A}')$ is not prime. Conversely, let $P, Q \in \mathbb{K}.\mathcal{A}$ such that $PQ \in J := \mathbb{K}.(\mathcal{A}\setminus\mathcal{A}')$. We may write $P = P_1 + P_2$, $Q = Q_1 + Q_2$ with $P_1, Q_1 \in J$, $P_2, Q_2 \in \mathbb{K}.(A')$. We have $P_2Q_2 \in J$. Since $\mathbb{K}.\mathcal{A}'$ is isomorphic to the quotient $\mathbb{K}.\mathcal{A}/J$, the product P_2Q_2 in $\mathbb{K}.\mathcal{A}'$ is 0. But since $\mathcal{A}'$ is inexhaustible, $\mathbb{K}.\mathcal{A}'$ has no nonzero divisor by Theorem 2.5, hence P or Q belongs to J, proving that J is prime. □

Problem 2.12. *Does the converse of* (ii) *hold. That is, is* $J := \mathbb{K}.(\mathcal{A} \setminus \mathcal{A}')$ *an irreducible ideal of* $\mathbb{K}.\mathcal{A}$ *whenever* $\mathcal{A}'$ *is a subage of* $\mathcal{A}$*?*

Ages are special cases of *hereditary classes of relational structures* (a class is *hereditary* if any relational structure which embeds in some member of the class is itself in the class; see [19, 38]). The definition of age algebra carries over straightforwardly to such classes and the correspondence between initial segments and ideals would be best further explored in this context.

2.5. **Well-quasi-ordering of an age.** The age algebra of a relational structure being graded and connected, it is finitely generated if and only if it is a Noetherian ring (see [15, §1.4] for the definition of Noetherian). There are posets which play a role as important in the theory of ordered sets as noetherian rings in the theory of rings. These posets, studied first by Higman [28], are called *well-quasi-ordered*, in brief *wqo*. Namely, a poset $\mathcal{P}$ is *wqo* if the set $F(\mathcal{P})$ of final segments of $\mathcal{P}$ is Noetherian with respect to the inclusion order.

Consider an age $\mathcal{A}(R)$ ordered by embeddability. By (i) of Theorem 2.11, $F(\mathcal{A}(R))$ embeds into the collection of ideals of $\mathbb{K}.\mathcal{A}(R)$. Consequently:

Proposition 2.13. *If the age algebra $\mathbb{K}.\mathcal{A}(R)$ is finitely generated then the age of R is well-quasi-ordered by embeddability.*

The reciprocal does not hold. Indeed, the age of a relational structure is well-quasi-ordered as soon as the age has polynomial growth and finite kernel [42, 43] whereas we have seen that the age algebra is not necessarily finitely generated in this case. In fact, well-quasi-ordering of the age does not even imply that the profile is bounded above by a polynomial: indeed, any age with nonpolynomially bounded profile contains a well-quasi-ordered age with the same property (use Lemma 4.1 of [49] and the remark above it).

Problem 2.14. *Is the profile of a relational structure R bounded by some exponential whenever the age R is well-quasi-ordered by embeddability?*

2.6. **The age algebra as a subring of a polynomial ring.**

Theorem 2.15. *If R has a monomorphic decomposition into finitely many infinite parts $(E_x)_{x\in X}$, then the age algebra $\mathbb{K}.\mathcal{A}(R)$ is isomorphic to a subalgebra $\mathbb{K}[X]^R$ of $\mathbb{K}[X]$.*

This may be generalized:

- *If some parts are finite, $\mathbb{K}.\mathcal{A}(R)$ is isomorphic to a subalgebra of the quotient ring $\mathbb{K}[X]/(x^{|E_x|+1} : |E_x| < \infty)$*
- *If there are infinitely many parts, $\mathbb{K}.\mathcal{A}(R)$ is isomorphic to a subalgebra of $\mathbb{K}[[X]]/(x^{|E_x|+1} : |E_x| < \infty)$ made of series of bounded degree.*

Proof. Let $(E_x)_{x\in X}$ be a finite set partition of E into infinite subsets. We first use it to define an embedding of $\mathbb{K}[X]$ into the set algebra $\mathbb{K}^{[E]^{<\omega}}$. For an exponent vector $\mathbf{d} := (d_x)_{x\in X} \in \mathbb{N}^X$, define the monomial $X^{\mathbf{d}} := \prod_{x\in X} x^{d_x}$ and set $\mathbf{d}! := \prod_{x\in X} d_x!$, where, as usual, $d!$ denotes the factorial of d. Set furthermore $O(\mathbf{d}) := \{A \subseteq E : \mathbf{d}(A) := \mathbf{d}\}$ and let $\chi_{O(\mathbf{d})}$ be the characteristic map of $O(\mathbf{d}) \in \mathbb{K}^{[E]^{<\omega}}$, which is best interpreted as the (possibly infinite) sum $\sum_{\mathbf{d}(A)=\mathbf{d}} A$. Let $\phi : \mathbb{K}[X] \hookrightarrow \mathbb{K}^{[E]^{<\omega}}$ be defined by setting $\phi(X^{\mathbf{d}}) := \mathbf{d}!\chi_{O(\mathbf{d})}$.

Claim. ϕ is a one-to-one morphism of algebras.

Indeed, we have

$$\chi_{O(\mathbf{d})}\chi_{O(\mathbf{d}')} = \frac{(\mathbf{d}+\mathbf{d}')!}{\mathbf{d}!\mathbf{d}'!}\chi_{O(\mathbf{d}+\mathbf{d}')}.$$

For an isomorphic type τ in the age of R, viewed as a subset of E, set $\mathbf{d}(\tau) := \{\mathbf{d}(A) : A \in \tau\}$ and $\mu(\tau) := \sum_{\mathbf{d}\in\mathbf{d}(\tau)} \frac{1}{\mathbf{d}!} X^{\mathbf{d}}$. Note that any monomial $X^{\mathbf{d}}$ in $\mathbb{K}[X]$ appears in exactly one polynomial $\mu(\tau)$; in particular, the later are linearly independent. Finally, let $\mathbb{K}[X]^R$ be the subset of $\mathbb{K}[X]$ made of finite linear combinations of polynomials of the form $\mu(\tau)$.

Claim. The map ϕ induces an isomorphism from $\mathbb{K}[X]^R$ onto $\mathbb{K}.\mathcal{A}(R)$. Indeed, we have:

$$\phi(\mu(\tau)) = \sum_{\mathbf{d}\in\mathbf{d}(\tau)} \phi(\frac{1}{\mathbf{d}!} X^{\mathbf{d}}) = \sum_{\mathbf{d}\in\mathbf{d}(\tau)} \chi_{O(\mathbf{d})} = \chi_\tau.$$

The generalizations are straightforward. □

In the sequel, when the monomorphic decomposition of a relational structure is clear from the context, we often use polynomials as a convenient and compact way of writing elements of its age algebra.

At this stage, a natural question is to characterize the subalgebras of $\mathbb{K}[X]$ that can be constructed as age algebras of relational structures with a finite monomorphic decomposition. As suggested in the introduction, the context can be generalized to algebras of hereditary equivalence relations. This makes the answer simpler: in the sequel, we obtain a one-to-one correspondence with subalgebras of $\mathbb{K}[X]$ defined by "grouping monomials together".

Let $\sim$ be an equivalence relation on $E^{<\omega}$, with equivalence classes $(\tau_i)_{i\in I}$. Consider the subspace $\mathbb{K}.\mathcal{A}(\sim)$ of $\mathbb{K}^{[E]^{<\omega}}$ spanned by the χ_{τ_i}, where τ_i ranges through the finite equivalence classes. For simplicity, we assume that there are finitely many equivalence classes for each size of subset.

Lemma 2.16. *$\mathbb{K}.\mathcal{A}(\sim)$ is a subalgebra of $\mathbb{K}^{[E]^{<\omega}}$ if and only if $\sim$ is hereditary.*

Proof. The if part is the same as for age algebras. For the reciprocal, consider a set C, its equivalence class τ, and define $e_d \in \mathbb{K}.\mathcal{A}(\sim)$ as $\sum \chi_{\tau_i}$ where i ranges through the finite equivalence classes of sets of size d. Equivalently, e_d is the (infinite) sum of all subsets of E of size d.

$$\chi_\tau = \sum_{A\,:\,|A|=|C|+d} c_C^A\, A,$$

where $c_C^A = |\{X \subset A : X \sim C\}|$. One recognizes the coefficients appearing in the hereditary condition, and it follows that $c_C^A = c_C^B$ whenever $A \sim B$. □

Proposition 2.17. *Let A be a graded subalgebra of $\mathbb{K}[X]$ that admits a basis $(B_i)_{i\in I}$ such that each monomial of $\mathbb{K}[X]$ appears in exactly one B_i. Then, there exist a hereditary equivalence relation $\sim$ whose algebra $\mathbb{K}.\mathcal{A}(\sim)$ is isomorphic to A.*

Proof. Write S_i the support of B_i. By construction, $(S_i)_{i\in I}$ forms a partition of the monomials of $\mathbb{K}[X]$. Consider the induced equivalence relation of $E^{<\omega}$, with equivalence classes given by $\tau_i := \mathbf{d}^{-1}(S_i)$ for $i \in I$. Let $\mathbb{K}.\mathcal{A}(\sim)$ be the subspace of $\mathbb{K}^{[E]^{<\omega}}$ spanned by the χ_{τ_i}.

We now prove that the basis elements B_i of the former are mapped one to one by ϕ to the basis elements χ_{τ_i} of the latter.

Up to rescaling the variables in X once for all, we may assume without loss of generality that the basis elements of degree 1 are sums of variables. In particular, A contains $e = x_1 + \cdots + x_n$ and therefore also

$$e^d = \sum_{\mathbf{d} \,:\, |\mathbf{d}|=d} \frac{x^{\mathbf{d}}}{\mathbf{d}!}.$$

Up to rescaling the B_i's, we may therefore assume without loss of generality that each B_i is of the form $\sum_{\mathbf{d}\in S_i} x^{\mathbf{d}/\mathbf{d}!}$, where S_i is the support of B_i. Therefore, $\phi(B_i) = \chi_{\tau_i}$, as desired.

It follows that $\mathbb{K}.\mathcal{A}(\sim)$ is indeed an algebra, and therefore, using Lemma 2.16, that $\sim$ is hereditary. □

2.7. **Case of orbital algebras.** As pointed out by Cameron [9], invariant rings of finite permutation groups are special cases of orbital algebras. They are also algebras of relational structures admitting a finite monomorphic decomposition (see Example A.16 of [49]). The converse holds, in the sense that, possibly up to some straightforward quotienting, invariant rings of finite permutation groups are exactly the orbital algebras of groups admitting a finite monomorphic decomposition.

Let us state this more precisely. Let G be a permutation group on a set E. Choose any relational structure R encoding the orbits of G. Note that the definition of a monomorphic decomposition depends only on the isomorphism relation between finite subsets of E, and thus is independent of the chosen relational structure. We can thus forget about the relational structure, and all the concepts of minimal monomorphic decomposition, monomorphic dimension, etc, are well defined for G itself.

Theorem 2.18. *Let G be a permutation group on a set E, and assume that the minimal monomorphic decomposition $(E_x)_{x\in X}$ of E is finite.*

Then G induces a finite subgroup $\tilde{G}$ of the symmetric group $\mathfrak{S}_X$ on X. If the components are all infinite, then the orbital algebra is isomorphic to the invariant ring $\mathbb{K}[X]^{\tilde{G}}$ of $\tilde{G}$. Otherwise, it is isomorphic to the quotient thereof obtained by setting $x^{|E_x|+1} = 0$ for all x such that E_x is finite.

Proof. By Corollary 2.3, every permutation $\sigma \in G$ induces a permutation $\overline{\sigma}$ of the components $(E_x)_{x\in X}$, which we can identify with a permutation of X. Choose any relational structure encoding the orbits of G, and consider the isomorphism Φ from $K[X]^R$ to $\mathbb{K}.\mathcal{A}(R)$ as in the proof of Theorem 2.15. If all components are infinite, it is easy to see that $K[X]^R$ is nothing but the invariant ring $K[X]^{\tilde{G}}$. Otherwise, the same quotienting occurs as in Theorem 2.15. □

3. Finite generation

This section is devoted to our main result: the combinatorial characterization of relational structures admitting a finite monomorphic decomposition whose algebra is finitely generated. We set up the ground with a special case and an example before proceeding to the general case. We conclude with the case of tournaments.

3.1. Finite generation for bounded profiles.

We recall the following result.

Theorem 3.1 (Theorem 1.5 of [49]). *Let R be a relational structure with E infinite. Then, the following properties are equivalent:*

- *(a) The profile of R is bounded.*
- *(b) R is almost-monomorphic.*
- *(b') R has a monomorphic decomposition into finitely many parts, at most one being infinite.*
- *(c) R is almost-chainable.*
- *(d) The Hilbert series is of the following form, with $P(Z) \in \mathbb{N}[Z]$ and $P(1) \neq 0$:*

$$\mathcal{H}_R = \frac{P(Z)}{1-Z}.$$

- *(e) The age algebra is a finite dimensional free-module over the free-algebra $\mathbb{K}[e]$, where $e := \sum_{a \in E} \{a\}$; in particular it is finitely generated and Cohen-Macaulay.*

See [49] for the definition of almost-monomorphy and almost-chainability. Note that, in [49], Theorem 1.5 is stated before the introduction of monomorphic decompositions, and thus does not mention (b'). The equivalence between (b) and (b') follows from Theorem 2.25 of [49].

3.2. Proof of nonfinite generation on a prototypical example.

The age algebra of a relational structure R admitting a finite monomorphic decomposition is not necessarily finitely generated. A prototypical example is this:

Example 3.2. *Let G be the direct sum $K_{(1,\omega)} \oplus \overline{K}_\omega$ of an infinite claw and an infinite independent set. There are two infinite monomorphic parts, E_1 the set of leaves of the wheel and E_2 the independent set, and one finite, E_3, containing the center c of the wheel. Each isomorphism type consists of a wheel and an independent set, so the Hilbert series is*

$$\mathcal{H}_G(Z) = (1 + \frac{Z^2}{1-Z})\frac{1}{1-Z} = \frac{1-Z+Z^2}{(1-Z)^2}.$$

What makes this relational structure special is that the monomorphic decomposition (E_1, E_2, E_3) is minimal, whereas (E_1, E_2) is not *a minimal monomorphic decomposition of the restriction of R to $E_1 \cup E_2$. We now prove that this causes the age algebra $\mathbb{K}.\mathcal{A}(G)$ not to be finitely generated.*

Consider the subalgebra $U := \mathbb{K}[e]$, *where* $e := \sum_{a \in V(G)} \{a\}$. *In each degree* d, *it is spanned by the sum* b_d *of all subsets of size* d *of* E, *since* $e^d = d! b_d$. *Key fact: any element* s *of* $\mathbb{K}.\mathcal{A}(G)$ *can be uniquely written as* $s =: a(s) + b(s)$ *where* $b(s)$ *is in* U, *and all subsets in the support of* $a(s)$ *contain the unique element* c *of* E_3. *Note in particular that* $a(s)a(s') = 0$ *for any* s, s' *homogeneous of positive degree.*

Suppose that S *is a finite generating set of* $\mathbb{K}.\mathcal{A}(G)$*; we may suppose that* S *is made of homogeneous elements of positive degree. By the key observation above,* $\{a(s) : s \in S\}$ *generates* $\mathbb{K}.\mathcal{A}(G)$ *as a* U*-module. It follows that the graded dimension of* $\mathbb{K}.\mathcal{A}(G)$ *is bounded by* $|S|$. *But this graded dimension is the profile of* G *which grows linearly. This gives a contradiction.*

3.3. **Combinatorial characterization.** The previous example suggests that the finite generation of the age algebra is related to the behavior of the minimal monomorphic decomposition with respect to restriction. This is indeed the case, and we get a complete characterization of when the age algebra is finitely generated.

Definition 3.3. *Let* $(E_x)_{x \in X}$ *be the minimal monomorphic decomposition of* R. *Whenever restricting* R *to some union* $\bigcup_{x \in X'} E_x$ *of infinite monomorphic parts,* $(E_x)_{x \in X'}$ *remains a monomorphic decomposition. If it always remains minimal, the decomposition* $(E_x)_{x \in X}$ *is called* hereditary minimal.

Remark: It is sufficient to check the condition on pairs of infinite parts. Namely, a monomorphic decomposition $(E_x)_{x \in X}$ is hereditary minimal if and only if, for any two infinite monomorphic parts E_x and E_y, the monomorphic decomposition (E_x, E_y) of the restriction of R on $E_x \cup E_y$ is minimal.

Theorem 3.5. *Let* R *be a relational structure admitting a finite monomorphic decomposition. Let* $(E_x)_{x \in X}$ *be its minimal monomorphic decomposition, and* X_∞ *be the set of indices of the infinite monomorphic parts. Then, the following propositions are equivalent:*

(a) The monomorphic decomposition is hereditary minimal;
(b) The age algebra $\mathbb{K}.\mathcal{A}(R)$ *is finitely generated;*
(c) For some large enough integer D, *the age algebra* $\mathbb{K}.\mathcal{A}(R)$ *contains a free graded subalgebra isomorphic to* $\mathrm{Sym}(x^D : x \in X_\infty)$, *and is a module of finite type thereupon.*

The implication (c) $\Rightarrow$ (b) is immediate; we prove separately (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a).

For the former implication, the proof is based on the Stanley-Reisner ring approach of [21] to construct generators of the invariant rings of a permutation group as a module over symmetric functions (also dubbed "chain-product trick" by the second author), together with the layer addition lemma used in [49] to prove that the Hilbert series is a rational fraction. For this, we need the tools and notations of section 3 of [49], albeit with a small variant on the chosen total order on the monomials of $\mathbb{K}[X]$; this variant is necessary to handle the fact that hereditary minimality is only about the infinite components E_x of the relational structure.

Say for short that a variable $x \in X$ is *finite* if E_x is finite. Write $\deg_{<\infty}(m)$ for the degree of a monomial $m \in \mathbb{K}[X]$ in the finite variables. To compare two monomials m and m' in $\mathbb{K}[X]$, first compare their degree in the finite variables; e.g. set $m > m'$ if $\deg_{<\infty}(m) < \deg_{<\infty}(m')$; in case of a tie, proceed as in [49] by comparing the shapes of m and m', and breaking ties with the usual lexicographic order. When there is no finite variable, nothing changes. The *leading monomial* $\operatorname{lm}(p)$ of a polynomial p, and the *orbit sum* $o(m)$ of a monomial m are defined as usual. We recall here the less mainstream definition of the *chain support*

The key property of leading monomials of age algebras is reminiscent of Stanley-Reisner rings. To each set $S \subseteq X$, associate the monomial $x_S := \prod_{i\in S} x_i$. By square free factorization, any monomial $m \in \mathbb{K}[X]$ can be written in a unique way as a product $m = x_{S_1}^{e_1} \dots x_{S_r}^{e_r}$ where $\emptyset \subset S_1 \subset \cdots \subset S_r \subset X$ is a chain of nonempty subsets of X, and the e_i are positive. Each S_i is a *layer* of m, and $S_1 \subset \cdots \subset S_r$ is the *chain support* of m.

With this order on monomials, Lemma 3.2 of [49] becomes:

Lemma 3.6. *Let m be a leading monomial, and S be a layer of m with $S \subseteq X_\infty$. Then, mx_S is again a leading monomial.*

Proof. Proceed as in the proof of Lemma 3.2 of [49]. Since we are considering only layers $S \subset X_\infty$, the tweak in the total order on monomials does not interfere. □

Proof of (a) ⇒ (c) of Theorem 3.5. Consider the restriction R' of R on the union E' of its infinite components. Since the monomorphic decomposition is hereditary minimal, the monomorphic decomposition $(E_x)_{x\in X_\infty}$ remains minimal. Take d large enough as in Lemma 2.15 of [49]. By Corollary 2.16, for $i = 1, \dots, k$, the collection $\mathcal{C}_i$ of all subsets of E' of size di and shape $(d, \dots, d)$ is closed under R' orbits. Identifying the elementary symmetric function $e_i(x^d, x \in X_\infty)$ in $\mathbb{K}[X]$ with (a constant factor of) the sum of all the elements of $\mathcal{C}_i$ in the set algebra, we derive that this symmetric function belongs to the age algebra of R'.

Let $\overline{\mathcal{C}_i}$ be the closure of $\mathcal{C}_i$ under orbits in R, and let $\overline{e}_i \in \mathbb{K}.\mathcal{A}(R)$ be the sum of the elements in $\overline{\mathcal{C}_i}$. Consider the graded algebra morphism induced by the canonical projection Π from $\mathbb{K}[X]$ to $\mathbb{K}[X_\infty]$, and note that $\Pi(\overline{e}_i) = e_i$. Thereby, Π induces a morphism from $\mathbb{K}[\overline{e}_i, i = 1, \dots, k]$ to $\mathbb{K}[e_i, i = 1, \dots, k]$. The latter is the ring of symmetric functions $\operatorname{Sym}(x^d, x \in X_\infty)$, hence freely generated by the e_i's. By dimension count, the former is the free graded subalgebra of $\mathbb{K}[R]$ generated by the $\overline{e}_i$'s, isomorphic to $\operatorname{Sym}(x^d, x \in X_\infty)$ through Π, as desired.

Claim 1: Let m be a leading monomial with chain support C. Assume that $m = x_S^d m'$ for some leading monomial m', $S \in C$, and $d \geq 0$. Then,

$$\operatorname{lm}(o(m')\overline{e}_{|S|}) = m.$$

Proof. Take a monomial ab appearing in the product $o(m')\overline{e}_{|S|}$, with a contributed by $o(m')$ and b by $e_{|S|}$.

Since $a \leq m'$, we must have $\deg_{<\infty}(a) \geq \deg_{<\infty}(m')$. If the comparison is strict or if $\deg_{<\infty}(b) > 0$, then $\deg_{<\infty}(ab) > \deg_{<\infty}(m)$ and thus $ab < m$ as desired. Otherwise, $\deg_{<\infty}(a) = \deg_{<\infty}(m')$, and b is of the form $x^d_{S'}$ for some $S' \subseteq X^\infty$ with $|S'| = |S|$. Given that the shape of a is at most that of m', that the shape of $x^d_{S'}$ coincide with that of x^d_S, and that S is in the chain support of m', the shape of ab is at most that of m. If equality does not hold, we are done. Otherwise, the shape of a coincides with that of m', and S' is in the chain support of m', and it follows that $ab \leq_{\text{lex}} m$, and therefore $ab \leq m$, as desired. □

Fix a chain $C := \emptyset \subset S_1 \subset \cdots \subset S_r \subseteq X$, and let lm_C be the set of leading monomials of the age algebra with this chain support. As in Section 3.2 of [49], consider $\mathcal{J} := \mathbb{K}.\text{lm}_C \oplus \mathcal{I}$ of $\mathbb{K}[S_1, \dots, S_l]$; By Lemma 3.6, this is a monomial ideal. Dickson's Lemma states that $\mathcal{J}$ is finitely generated as an ideal, that is as a module over $\mathbb{K}[S_1, \dots, S_l]$. It is in fact also finitely generated as a module over $\mathbb{K}[S_1^d, \dots, S_l^d]$, with a canonical finite set G_C of monomials as generators (see e.g. Lemma 2.3.2 (a) of [55]).

Let $\mathcal{G}$ be the finite collection of the orbitsums whose leading monomial is in G_C for some chain C. We conclude by proving the following claim.

Claim 2: $\mathcal{G}$ generates $\mathbb{K}.\mathcal{A}$ as a module over $\mathbb{K}[\overline{e}_i]$.

Take an orbitsum $o(m)$ in $\mathbb{K}.\mathcal{A}$ with m its leading monomial, and let C be the chain support of m. Assume by induction that, for any leading monomial $m' < m$, the orbitsum $o(m')$ is in the $\mathbb{K}[\overline{e}_i]$-module spanned by $\mathcal{G}$. We prove that this holds for $o(m)$ too.

If m is in G_C, o is in $\mathcal{G}$ and we are done. Otherwise, write m as $m = s^d_S m'$ where $S \subset X_\infty$. Using Claim 1, $o(m)$ is in the $\mathbb{K}[\overline{e}_i]$-module spanned by $o(m')$ together with orbitsums of strictly smaller leading monomials. Therefore, by induction, $o(m)$ is in the $\mathbb{K}[\overline{e}_i]$-module spanned by $\mathcal{G}$ as desired. □

Proof of (b) ⇒ (a) of Theorem 3.5. Assume that the monomorphic decomposition is not hereditary minimal but the age algebra $\mathbb{K}.\mathcal{A} := \mathbb{K}.\mathcal{A}(R)$ is finitely generated.

The proof follows the same path as in Example 3.2: we first reduce the problem to the monomorphic dimension 2 case. Then, we construct a subalgebra $\mathcal{B}$ of $\mathbb{K}.\mathcal{A}$ which is "small" (dimension 1 in each degree) compared to $\mathbb{K}.\mathcal{A}$ (dimension asymptotically equivalent to d in degree d). Using the fact that $\mathbb{K}.\mathcal{A}$ is finitely generated as an algebra we prove that $\mathbb{K}.\mathcal{A}$ is a finitely generated module over $\mathcal{B}$. This is impossible dimension-wise.

Claim 1: we may assume without loss of generality that R is of monomorphic dimension 2. Otherwise, consider two monomorphic parts E_1 and E_2 such that R restricted to $H = E_1 \cup E_2$ is monomorphic. By the minimality of the decomposition, H is not a monomorphic part of R: there exist two finite subsets A and A' of E such that A and A' are not isomorphic yet coincide outside H. Let G be the finite subset $A\backslash H = A'\backslash H$. Then, R restricted to $H \cup G$ is of monomorphic dimension 2. The age algebra of the restriction is a quotient of the age algebra of R and is therefore still finitely generated, as desired.

Keeping the above notations, we now have $E = H \cup G$ where $H = E_1 \cup E_2$. Consider the graded subalgebra $\mathcal{B}$ of $\mathbb{K}.\mathcal{A}$ spanned in each degree d by the sum $e_d(E)$ of all subsets of size d ($\mathcal{B}$ can be alternatively defined as the free graded commutative algebra generated by the sum of all points $e_1(E)$, or the age algebra of the trivial relational structure on E).

Claim 2: $\mathbb{K}.\mathcal{A}$ is a finite-module over $\mathcal{B}$. This yields the desired contradiction because the graded dimension of such a module is bounded by a constant, whereas by Lemma 2.15 of [49], the graded dimension of $\mathbb{K}.\mathcal{A}$ is asymptotically equivalent to d.

It remains to prove Claim 2.

Since the restriction of R to H is monomorphic, any orbitsum of degree d either contains all subsets of size d of H, or none. Therefore, any s in the age algebra decomposes uniquely as $s =: a(s) + b(s)$, where $b(s)$ is in $\mathcal{B}$, and all the subsets in the support of $a(s)$ intersect G nontrivially.

Key fact: any product $a(s_1) \cdots a(s_k)$ of $k > |G|$ homogeneous elements s_i of positive degree is zero ($k > |G|$ subsets of E which intersect G nontrivially cannot be disjoint).

Let S be a finite generating set of the age algebra made of homogeneous elements of positive degree. Then, $\{a(s), s \in S\}$ generates $\mathbb{K}.\mathcal{A}$ as a $\mathcal{B}$-algebra. By the above key fact the finite collection of all products $a(s_1) \cdots a(s_k)$ of $k \leq |G|$ elements of S generates $\mathbb{K}.\mathcal{A}$ as a $\mathcal{B}$-module. □

3.4. **Finite generation for tournaments.** The existence of a very simple tournament (Example A.8 of [49]) whose age algebra is not finitely generated is not an accident; in fact, the age algebra of a tournament is very seldom finitely generated.

Theorem 3.7. *The age algebra of a tournament T is finitely generated if and only if the profile is bounded.*

This is a consequence of Theorem 3.5 thanks to some simple remarks and a structural theorem on the monomorphic parts of a tournament.

Remarks 3.8.

(a) A monomorphic part of size at least 4 is acyclic;
(b) The union of two monomorphic parts of size at least 4 is acyclic;
(c) A minimal monomorphic decomposition of a tournament with at least two infinite parts cannot be hereditary minimal.

Theorem 3.9 (Boudabbous-Pouzet [4])**.** *Let T be an infinite tournament whose profile is bounded above by a polynomial. Then, T is a lexicographical sum $\sum_{i \in D} A_i$ of acyclic tournaments A_i indexed by a finite tournament D.*

Proof of Theorem 3.7. Suppose that φ_T is bounded. Then, by Theorem 3.1, $\mathbb{K}.\mathcal{A}(T)$ is finitely generated. Conversely, suppose that $\mathbb{K}.\mathcal{A}(T)$ is finitely generated. Then, the profile φ_T has polynomial growth, and by Theorem 3.9 the minimal monomorphic decomposition of T is finite. Applying Theorem 3.5, this decomposition is hereditary minimal, and by Remark 3.8 (c), it has a single infinite monomorphic maximal part. Therefore, the profile is bounded. □

Theorem 3.7 was stated as Theorem 3.5 in [43]. The argument provided there expands on the idea given in Example 3.2 but, as stated, is incorrect. The correction is straightforward. Assume that the tournament T is a lexicographical sum $T = \sum_{i\in D} A_i$ of acyclic tournaments—indexed by a finite tournament D—at least two of which, A_j, A_k, are infinite. Then, according to Lemma 9 of [4] T contains a sub-tournament $T' = \sum_{i\in D'} A'_i$ with the same property where D' has at most 5 elements and all A'_i but two, say $A'_{j'}$ and $A'_{k'}$, are singletons. Supposing D' with minimum size (hence between 3 and 5), the union of the $A'_{i'}$'s, for $i' \notin \{j', k'\}$, is equal to kernel(T'), the kernel of T'. As stated in Theorem 3.5, if $\mathbb{K}.\mathcal{A}(T)$ is finitely generated, $\mathbb{K}.\mathcal{A}(T')$ is finitely generated. Then we may repeat the proof given in Example 3.2 with kernel(T') playing the role of $\{c\}$ and obtain a contradiction (the proof of Theorem 3.5 deals only with D of size 3, in which case kernel(T') is a singleton).

4. Invariant rings of permutation groupoids

The common feature of invariant rings of permutation groups and other interesting examples like the rings of quasisymmetric polynomials (Example A.18 of [49]) is that they can be realized as age algebras of a relational structure R of the form $K_\infty \wr (X, (\rho_i)_{i\in I})$, where X is a finite set and K_∞ is an infinite clique (or similar monomorphic relational structure). In this section, we study further such age algebras which we call *invariant rings of permutation groupoids*.

Our motivations are twofold. On the one hand, relate, in this simpler yet rich setting, the properties of the profile to algebraic properties of the invariant ring. In particular, find conditions under which the invariant ring is Cohen-Macaulay. On the other hand, generalize the theory, algorithms, and techniques of invariant rings of permutation groups to a larger class of subrings of $\mathbb{K}[X]$. For background on invariant theory, see e.g. [11, §.7] or [52].

4.1. **Permutation groupoids, inverse monoids and representations.**

In this section, we briefly review the groupoid and inverse monoid structures on the local isomorphisms of a relational structure, and their representations. For details, see e.g. [35, Chapter 4], [54].

Let $(X, (\rho_i)_{i\in I})$ be a finite relational structure and $G := \mathrm{Loc}(R)$ be its collection of local isomorphisms. Recall that it can be endowed with a groupoid structure by composing two local isomorphisms f and g whenever the codomain of the first agrees with the domain $\mathrm{dom}g$ of the second.

The properties of G are as follows:

- G contains all local identities of X;
- G is stable under composition, inverse, and restrictions.

We call a collection of local bijections of a set X satisfying the above properties and endowed with the composition product a *permutation groupoid* on X.

Proposition 4.1. *Any permutation groupoid on a finite set X can be realized as the groupoid of local isomorphisms on some relational structure on X.*

A permutation group of X induces a permutation groupoid on X by considering all the restrictions of its automorphisms.

Alternatively, G can be endowed with an *inverse monoid* structure, which we denote by $\overline{G}$, by taking the partial composition as product; see [60]:

$$fg : \begin{cases} g^{-1}(\mathrm{im}g \cap \mathrm{dom}f) & \hookrightarrow\!\!\!\!\rightarrow f(\mathrm{im}g \cap \mathrm{dom}g) \\ x \mapsto f(g(x)) \end{cases},$$

where $\mathrm{im}g$ and $\mathrm{dom}g$ denote respectively the image and domain of g.

The groupoid and inverse monoid structures are tightly related through their algebras. Define as usual the groupoid algebra $\mathbb{K}.G$ of G as the vector space of formal linear combinations of G, endowed with the product obtained by extending by bilinearity the groupoid product of G.

Define similarly the monoid algebra of G starting from its inverse monoid structure. The latter algebra is isomorphic to $\mathbb{K}.G$, by mapping a partial bijection f to $\overline{f} = \sum_{A \subset \mathrm{dom}f} f_{\restriction A}$ in $\mathbb{K}.G$. The inverse isomorphism can be defined by inclusion-exclusion. In the sequel, we identify both algebras, interpreting $(\overline{f})_{f\in G}$ as an alternative basis of $\mathbb{K}.G$. The map $f \mapsto \overline{f}$ also defines an embedding of $\overline{G}$ in $\mathbb{K}.G$.

Because of this isomorphism, the representations of G and $\overline{G}$ coincide. In particular, they are semisimple.

4.2. **Invariant rings of permutation groupoids.** Take a relational structure R of the form $K_\infty \wr (X, (\rho_i)_{i\in I})$. The running example to keep in mind is that of quasisymmetric functions (Example A.18 of [49]). The monomorphic decomposition $(E_x := K_\infty \times \{x\})_{x\in X}$ of R is minimal, and even hereditary minimal. This makes the age algebra into a subring of $\mathbb{K}[X]$.

We extend the natural action of a permutation of X on $\mathbb{K}[X]$ as follows. Take a monomial $X^{\mathbf{d}}$ and a local bijection f. If the *support* of $X^{\mathbf{d}}$ (that is: $\{x \in X : d_x > 0\}$) coincides with the domain of f, set $f.X^{\mathbf{d}} := \prod_{x, d_x > 0} f(x)^{d_x}$; otherwise set $f.X^{\mathbf{d}} := 0$.

Lemma 4.2. *Let A and B be two finite isomorphic subsets of R. Then, there exists a local isomorphism f of $(X, (\rho_i)_{i\in I})$ such that $X^B = f.X^A$. In particular, the shape of A and B are identical.*

Proof. Let g be a local isomorphism from A to B. Two points of A in the same monomorphic part must be sent by g to the same monomorphic part. Therefore, there exists a local isomorphism f of $(X, (\rho_i)_{i\in I})$, and local isomorphisms $(f_x)_{x\in X}$ of K_∞ such that every point (j, x) of A is mapped to $g(j, x) = (f_x(j), f(x))$. □

This proposition should be interpreted as follows. The monomorphic parts are strings, and the finite subsets of E are sets of beads threaded on those strings and sliding freely on them. Furthermore, some local permutations of the nonempty strings are allowed, corresponding to the local isomorphisms of $(X, (\rho_i)_{i\in I})$. For example, for quasisymmetric functions, one may move the string 1 to the string 3, and the string 2 to the string 4, assuming that all other strings contain no beads.

Corollary 4.3. *The age and therefore the age algebra depend only on the groupoid G of the local isomorphisms of $(X, (\rho_i)_{i\in I})$. Reciprocally, the age algebra characterizes the groupoid.*

By analogy with invariant rings of permutation group, we therefore write it $\mathbb{K}[X]^G$, and call it the *invariant ring of the permutation groupoid G*. If G is induced by a permutation group, then both of their invariant rings coincide.

The basic notions of invariant rings of permutation groups, like *G-isomorphic monomials* and *G-orbits* extend straightforwardly; in particular one may define the *orbit sum* $O(X^{\mathbf{d}})$ of a monomial $X^{\mathbf{d}}$ as the sum of all the monomials in its orbit.

Proposition 4.4. *The orbitsums form a vector space basis of the invariant ring $\mathbb{K}[X]^G$. The latter is a graded connected commutative algebra which contains symmetric functions in X.*

Remark: The action being by local permutation, the isomorphism of two monomials does not depend on the actual values of the exponents, but only on the partition of X induced by them.

Formally: let f be a function from $\mathbb{N}$ to $\mathbb{N}$ such that $f(0) = 0$, and define $f(X^{\mathbf{d}}) := \prod_x x^{f(d_x)}$. Then, if $X^{\mathbf{d}}$ and $X^{\mathbf{d}'}$ are isomorphic then so are $f(X^{\mathbf{d}})$ and $f(X^{\mathbf{d}'})$.

In this setting, Lemma 3.2 of [49] becomes immediate.

Corollary 4.6. *Consider the lexicographic monomial order. If $X^{\mathbf{d}}$ is a leading monomial in $\mathbb{K}[X]^G$, and if $S \subseteq X$ is a layer of $X^{\mathbf{d}}$, then $X^{\mathbf{d}}X_S$ is again a leading monomial.*

Proof. Duplicating a layer in $X^{\mathbf{d}}$ amounts to apply a strictly increasing function f. This function preserves both isomorphism and the monomial order. □

4.3. **Permutation groupoids versus permutation groups.** Here, we discuss briefly how the action of a permutation groupoid on polynomials differs from that of a permutation group.

4.3.1. *Restriction.* The *restriction* $G_{\restriction X'}$ of a permutation groupoid G to a subset X' is the set of all local functions f in G such that $\mathrm{dom} f \subseteq X'$ and $\mathrm{im} f \subseteq X'$, which is again a permutation groupoid. Furthermore, the orbits of monomials in $\mathbb{K}[X']$ are unchanged by this restriction. In particular, the invariant ring of $G_{\restriction X'}$ is simply the quotient of the invariant ring of G obtained by killing all the variables x_i with $i \notin X'$. This simple fact is one of the motivations for considering permutation groupoids instead of just permutation groups (for which the restriction to a subset is not clearly defined). This may indeed give opportunities for induction techniques on the size of the underlying set.

Proposition 4.7. *Any permutation groupoid on a finite set is the restriction of a permutation groupoid induced by a permutation group of some superset. However, this superset may need to be infinite.*

Proof. This is an immediate consequence of the fact that every finite relational structure R embeds into a countable homogeneous structure (structure for which local isomorphisms of finite domain extend to automorphisms) [20]. In several instances, the permutation groupoid may be chosen finite [27]. □

Examples.

(a) *The permutation groupoid on* $\{1,2,3\}$ *generated by the rank* 1 *local bijection* $1 \mapsto 2$ *is the restriction of the permutation group on* $\{1,2,3,4\}$ *generated by the permutation* $(1,2)(3,4)$.

(b) *The local automorphism permutation groupoid of the chain* $a < b$ *is the restriction of the cyclic group* C_3 *on* $\{a,b,c\}$.

(c) *Consider a relational structure* R *such that there exists three elements* a, b, c *and a binary relation* $<$ *which restricts on* $\{a,b,c\}$ *to the chain* $a < b < c$. *Typically,* R *is a chain of length at least* 3 *(giving* $\mathrm{QSym}(X)$ *as invariants) or a poset of height at least* 3. *Then, there exists no relational structure* $\overline{R}$ *on a finite superset where all local isomorphisms extend to global isomorphisms.*

   Consider indeed the local isomorphism $a \mapsto a, c \mapsto b$*, and extend it to a global isomorphism* σ *of* $\overline{R}$. *It is easy to check that* $a < \sigma(b) < b < c$ *is again a chain, which implies that* $\sigma(b) \notin \{a,b,c\}$. *By induction,* $a < \sigma^k(b) < \cdots < \sigma(b) < c$ *is again a chain, which proves that all* $\sigma^k(b)$ *are distinct. Hence* R' *is infinite.*

4.3.2. *Multiplicativity.* As for a permutation group, the groupoid algebra $\mathbb{K}.G$ is semisimple, and the action of G extends to a linear representation of G. However, for a permutation groupoid, the action is not multiplicative on polynomials. Take for example $f := \mathrm{id}_{\{1,2\}}$, $P := x_1$ and $Q := x_2$.

This requires a bit of care in the upcoming generalization of the Reynolds operator and explains why it is not any more a Sym-module morphism.

Multiplicativity can be partially recovered by considering the inverse monoid $\overline{G}$, whose action on polynomials is given by $\overline{f}.X^{\mathbf{d}} := \prod_{x,d_x>0} f(x)^{d_x}$ if $\{x, d_x > 0\} \subseteq \mathrm{dom} f$ and $f.X^{\mathbf{d}} := 0$ otherwise. This action is multiplicative.

4.4. **The Reynolds operator.** The first essential feature of invariant rings is the so-called *Reynolds operator* $\mathcal{R}$, a projector on the invariant ring. The following proposition states that this operator still exists for invariants of permutation groupoids, albeit missing the important property of being a $\mathbb{K}[X]^G$-module morphism. In particular, although $\mathbb{K}[X]^G$ still contains the ring of symmetric polynomials $\mathrm{Sym}(X)$, R is no longer a $\mathrm{Sym}(X)$-module morphism. Recall also that, for a permutation group, $\mathbb{K}[X]^G$ is the isotopic component for the trivial representation of $\mathbb{K}.G$ in $\mathbb{K}[X]$; furthermore, the Reynolds operator is the unique central idempotent of $\mathbb{K}.G$ projecting on the trivial representation. These properties fail for a permutation groupoid; in fact $\mathbb{K}[X]$ is not even stable under the action by $\mathbb{K}.G$.

Proposition 4.9. *There exists an idempotent $\mathcal{R}$ in the groupoid algebra $\mathbb{K}.G$ which projects $\mathbb{K}[X]$ onto the invariant ring $\mathbb{K}[X]^G$:*

$$\mathcal{R} := \sum_{A \subseteq X} \frac{1}{|\{g \in G\,:\, \mathrm{dom} g = A\}|} \sum_{g \in G\,:\,\mathrm{dom} g = A} g.$$

Furthermore, the four following conditions are equivalent:

(i) G is induced by a permutation group.
(ii) $\mathcal{R}$ is a $\mathbb{K}[X]^G$-module morphism;
(iii) $\mathcal{R}$ is a $\mathrm{Sym}(X)$*-module morphism;*
(iv) $\ker \mathcal{R}$ *is a* Sym-*module;*

Proof. By construction, $\mathcal{R}$ is in the groupoid algebra. One easily checks that, up to a scalar factor, the image of a monomial by $\mathcal{R}$ is the orbitsum of that monomial, and the image of an orbitsum is itself. Therefore, it projects onto the invariant ring.

(i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv) are well known or obvious.

(iv) $\Rightarrow$ (i): Assume that $\ker R$ is a Sym-module and that G does not come from a permutation group, and let $f : A \mapsto B$ be a local bijection which does not extend to a permutation of X. Let x_X be the product of the variables, and m be a monomial with support A with all exponents distinct. Then, $f.m$ is in the orbit of m, whereas $x_X f.m$ is not in the orbit of $x_X m$. Therefore, $\mathcal{R}(m - f.m) = 0$ whereas $x_X \mathcal{R}(m - f.m) = \mathcal{R}(x_X m - x_X f.m) \neq 0$, a contradiction. □

4.5. **Fine grading, the chain product, and degree bounds.** Recall that the *degree bound* $\beta(A)$ of a finitely generated graded algebra A is the smallest integer such that A is generated by its elements of degree at most

$\beta(A)$. In [21], Garsia and Stanton construct an associated finely-graded algebra of the invariant ring $\mathbb{K}[X]^G$ of a permutation group G by letting G act on the Stanley-Reisner ring of the boolean lattice. They use it to exhibit $\mathrm{Sym}(X)$-module generators in some special cases and prove, in general, the quadratic degree bound $\beta(\mathbb{K}[X]^G) \leq \binom{|X|}{2}$. A notable feature is that this approach is combinatorial and thus characteristic free!

In this section, we show that this construction generalizes essentially straightforwardly to permutation groupoids, and derive some properties of the invariant ring: degree bounds and finite generation; in the following section we further derive a necessary conditions for being Cohen-Macaulay.

As in [58], we follow the basic approach of realizing the Stanley-Reisner ring as $\mathbb{K}[X]$ endowed with another product $\star$, called the *chain product*. The chain product preserves a finer grading, and many algebraic properties of the invariant ring with respect to the chain product transfer back to the usual product. We refer to e.g. [15, § 5.1] for background on associated graded algebras.

Given a subset S of X, set $x_S := \prod_{i\in S} x_i$. By square-free decomposition, any monomial x^d can be identified uniquely with a *multichain* $S_1 \subseteq \cdots \subseteq S_k$ of nested subsets of X, so that:

$$x^d = x_{S_1} \cdots x_{S_k}.$$

We call each S_k a *layer* of x.

The *fine degree* of the monomial x^d is the integer vector $(r_1, \dots, r_n)$ where each r_i counts the (possibly null) number of repetitions of the layer of size i of x^d. Orbit sums are homogeneous with respect to the fine degree; the invariant ring is therefore graded as a vector space.

One may compute the fine Hilbert series of the invariant ring as follows: take a representative Y for each orbit of subsets of X; consider the group G_Y of local automorphisms of domain Y; use Pólya enumeration to compute the fine generating series for monomials with full support Y. Sum all the results.

The fine grading is not preserved by multiplication; however it still defines a filtration on $\mathbb{K}[X]$. The *chain product* $\star$ of two monomials $x^d := x_{S_1} \cdots x_{S_k}$ and $x^{d'} := x_{S'_1} \cdots x_{S'_k}$ is defined by:

$$x^d \star x^{d'} := \begin{cases} x^d x^{d'} & \text{if } \{S_1, \dots, S_k, S'_1, \dots, S'_k\} \text{ is again} \\ & \text{a multichain of subsets,} \\ 0 & \text{otherwise.} \end{cases}$$

For example, $x_1 \star x_1 = x_1^2$, $x_1 \star x_2 = 0$, $x_1x_3^2 \star x_1x_2x_3^2 = x_1^2x_2x_3^4$, and $x_1x_3^2 \star x_1x_2 = 0$.

The chain product endows $\mathbb{K}[X]$ with a second algebra structure $(\mathbb{K}[X], \star)$, isomorphic to the quotient

$$\mathbb{K}[x_S, S \subseteq X]/\{x_S x_{S'} = 0 : S \not\subseteq S' \text{ and } S' \not\subseteq S\}.$$

$(\mathbb{K}[X], \star)$ is also finely graded, fine degrees being added term-by-term. In fact, $(\mathbb{K}[X], \star)$ is exactly the associated graded algebra of $\mathbb{K}[X]$ with respect to the fine degree filtration. Beware however that $(\mathbb{K}[X], \star)$ is not an integral domain.

The elementary symmetric functions

$$e_d := \sum_{S \subseteq X, |S|=d} x_S$$

are still algebraically independent and generate $(\mathrm{Sym}(X)_n, \star)$. This is no longer true for, say, the symmetric powersums. The following simple fact turns out to be an essential key:

Remark: Consider the chain product of a monomial $x_{S_1} \cdots x_{S_k}$ by the elementary symmetric function e_d. It is the sum of all monomials
$x_{S_1} \cdots x_S \cdots x_{S_k}$, where S is of size k, and fits in the multichain $S_1 \subseteq \cdots \subseteq S \subseteq \cdots \subseteq S_k$. In particular, if $x_{S_1} \cdots x_{S_k}$ contains a layer S of size k, then $x_{S_1} \cdots x_{S_k} \star e_k$ is the unique monomial obtained by replicating this layer.

More generally, $(\mathbb{K}[X]^G, \star)$ is a subring of $(\mathbb{K}[X], \star)$. In particular, $(\mathbb{K}[X]^G, \star)$ is a $\mathrm{Sym}(X)$-module. Furthermore, we may transfer the following algebraic properties from $(\mathbb{K}[X], \star)$ to $\mathbb{K}[X]^G$, as in the case of permutation groups [21].

Proposition 4.11.

(a) *A family F of finely homogeneous invariants of positive degree which generates $(\mathbb{K}[X]^G, \star)$, also generates $\mathbb{K}[X]^G$;*
(b) *$\beta(\mathbb{K}[X]^G, \star) \geq \beta(\mathbb{K}[X]^G)$;*
(c) *A family F of finely homogeneous invariants which generates $(\mathbb{K}[X]^G, \star)$ as a $\mathrm{Sym}(X)$-module also generates $\mathbb{K}[X]^G$ as a $\mathrm{Sym}(X)$-module;*
(d) *If $(\mathbb{K}[X]^G, \star)$ is a free $\mathrm{Sym}(X)$-module, then so is $\mathbb{K}[X]^G$.*

Proof. This is a standard fact about filtrations and associated graded connected algebras. The key of the proof is that, if p and q are finely homogeneous, the maximal finely homogeneous component of pq is exactly $p \star q$. (a) and (c) follow by induction over the fine grading. Then, (b) follows straightaway from (a), and (d) from (c) by a simple Hilbert series argument. □

The converse of (a) and (b) do not hold. In fact, with most permutation groups, the degree bound $\beta(\mathbb{K}[X]^G, \star)$ is much larger than $\beta(\mathbb{K}[X]^G)$. We conjecture that the converse of (c) and (d) hold. However (d) no longer holds in a slightly larger setting which includes the r-quasisymmetric polynomials of F. Hivert [30]; a counterexample is $\mathrm{QSym}^2(X_3)$ which exhibits an obstruction in the fine Hilbert series.

We now generalize the quadratic degree bound of Garsia and Stanton to permutation groupoids.

Theorem 4.12. *Let G be a permutation groupoid acting on X and $n = |X|$. Then, the invariant ring $\mathbb{K}[X]^G$ is a finitely generated algebra and* $\mathrm{Sym}(X)$*-module, in degree at most* $(|X|(|X|+1))/2$*. This degree bound is tight.*

Note that, as usual, when G does not act transitively on the variables, the degree bound can be greatly improved by considering the symmetric polynomials on each transitive component instead.

Proof. The set of orbit sums $o(x_{S_1} \cdots x_{S_k})$, where $S_1 \subsetneq \cdots \subsetneq S_k$ is a chain, generates $(\mathbb{K}[X]^G, \star)$ as a $(\mathrm{Sym}, \star)$-module. This transfers back to $\mathbb{K}[X]^G$ and Sym.

Note that we may need to consider chains with $S_k = X$; hence the degree bound of $(|X|(|X|+1))/2$ instead of $\binom{|X|}{2}$ for permutation groups. For an example where the bound is achieved, consider the group G made of the identity together with all the local bijections of $X = \{1, \dots, n\}$ whose domain is of size at most $|X| - 1$; then, $\mathbb{K}[X]^G$ is freely generated as a Sym-module by 1 and the *staircase* monomials $x_1^{d_1} \cdots x_n^{d_n}$ with $1 \leq d_i \leq i$. □

4.6. **The Cohen-Macaulay property.** Invariant rings of permutation groups are always Cohen-Macaulay and in fact free $\mathrm{Sym}(X)$-modules. The key ingredients are that $\mathbb{K}[X]$ is a free $\mathrm{Sym}(X)$-module and the Reynolds operator a $\mathrm{Sym}(X)$-module morphism. A more involved result is that, for all n, $\mathrm{QSym}(X_n)$ is also a free $\mathrm{Sym}(X_n)$-module [22].

As Example A.19 of [49] shows, this property does not hold for all permutation groupoids G. Still, $\mathbb{K}[X]^G$ and $(\mathbb{K}[X]^G, \star)$ being finitely generated over $\mathrm{Sym}(X)$, they are Cohen-Macaulay if and only if they are free $\mathrm{Sym}(X)$-modules.

Problem 4.13. *Characterize the permutation groupoids G whose invariant ring $\mathbb{K}[X]^G$ (or $(\mathbb{K}[X]^G, \star)$) is Cohen-Macaulay.*

In practice, a first test is to compute the (fine) degrees of tentative free generators of $\mathbb{K}[X]^G$ by dividing the (fine) Hilbert series of $\mathbb{K}[X]^G$ by that of $\mathrm{Sym}(X)$.

The following theorem is a straightforward extension of Theorem 6.1 of [21] applied to the quotient of the Stanley Reiner ring of the boolean lattice by a permutation group.

Theorem 4.14. $(\mathbb{K}[X]^G, \star)$ *is a free* $\mathrm{Sym}(X)$*-module if and only if the incidence matrix between generators and orbits of maximal chains is invertible. In particular, for a set F of finely homogeneous invariants whose fine degrees are given by the Hilbert series of* $\mathbb{K}[X]^G$*, the three following conditions are equivalent: F spans* $\mathbb{K}[X]^G$ *as a* $\mathrm{Sym}(X)$*-module, F is a free* $\mathrm{Sym}(X)$*-family, and F is a* $\mathrm{Sym}(X)$*-basis of* $\mathbb{K}[X]^G$.

This immediately gives us a necessary condition on the number of generators.

Corollary 4.15. *If* $(K[X]^G, \star)$ *is a free* $\mathrm{Sym}(X)$*-module, then it is of rank* $|X|!/|G(X,X)|$*, where* $G(X,X)$ *is the underlying permutation group of* G*.*

As a consequence, we recover that the ring $\mathrm{QSym}(X)$ of quasisymmetric polynomials in X has to be of rank $|X|!$ over $\mathrm{Sym}(X)$.

4.7. **SAGBI bases.** Elimination theory is a traditional approach to making algebra effective. Typical instances include Gauß elimination and echelon forms for subspaces of vector spaces, Gröbner bases for ideals of polynomial rings, strong generating sets for permutation groups, or Knuth-Bendix completions for congruences on words.

SAGBI bases (Subalgebra Analog of Gröbner Bases for Ideals) were introduced in [33, 51] to develop a similar elimination theory for subalgebras of polynomial rings. However, as for congruences on words, not all subalgebras admit a finite SAGBI basis. It remains a long open problem to characterize those subalgebras which admit one.

The following theorem states that, as in the case of permutation groups, invariant rings of permutation groupoids seldom have finite SAGBI bases. For example, $\mathrm{QSym}(X_n)$, represented as a subring of $\mathbb{K}[X]$, has no finite SAGBI basis whenever $n > 1$. In particular, $\mathrm{QSym}(X_2)$ becomes the smallest example of finitely generated algebra which has no finite SAGBI basis (the standard example being the invariant ring of the alternating group A_3). Still, SAGBI bases and SAGBI-Gröbner bases provide a useful device in the computational study of invariant rings of permutation groups [56], and should play the same role with permutation groupoids.

We refer to [33, 51] for all definitions and basic properties of SAGBI bases (admissible term orders, etc.).

Theorem 4.16. *Let* G *be a permutation groupoid, and* $<$ *be any admissible term order on* $\mathbb{K}[X]$*. Then, the invariant ring* $\mathbb{K}[X]^G$ *has a finite SAGBI basis with respect to* $<$ *if, and only if,* G *is induced by a permutation group generated by reflections (that is transpositions).*

The following proof is a close variant on the short proof given by the second author in [57] in the special case of permutation groups. For the sake of readability and completeness, we include it in full here. The key fact is that a submonoid M of $\mathbb{N}^n$ is finitely generated if and only if, the convex cone $C := \mathbb{R}_+ M$ it spans in $\mathbb{R}_+^n$ is finitely generated (that is C is a *polyhedral cone*). For details, see for example [5, Corollary 2.10]. In particular, C must be the intersection of finitely many half spaces, and thus closed for the euclidean topology.

Proof. The 'if' part is easy, a finite SAGBI basis being given by the elementary symmetric polynomials in the variables in each G-transitive components.

Without loss of generality, we may assume $X = \{1, \dots, n\}$ with $x_1 > \dots > x_n$. Let M be the monoid of initial monomials in $\mathbb{K}[X]^G$, seen as a submonoid of $\mathbb{N}^n$, and $C := \mathbb{R}_+ M$ be the convex cone it spans in $\mathbb{R}_+^n$.

At this stage, we cannot give an explicit description of C, but we can construct a convex cone C' which approximates it closely enough for our purposes. By the standard characterization of admissible term orders on $\mathbb{K}[X]$ [59], there exists a family of n linear forms $l = (l_1, \dots, l_n)$ such that $x^d > x^{d'}$ if and only if $l(d) >_{\text{lex}} l(d')$, where we denote by $l(d)$ the n-uple $l_1(d_1, \dots, d_n), \dots, l_n(d_1, \dots, d_n)$. Given two vectors v and v' in $\mathbb{R}_+^n$, we write $v > v'$ if $l(v) >_{\text{lex}} l(v')$. The partial action of G on monomials extends naturally to a partial action on $\mathbb{R}_+^n$: whenever the support of $v = (v_1, \dots, v_n)$ in contained in the domain of a local bijection $f \in G$, $f.v$ is the vector obtained by permuting the nonzero entries of v according to f. Let C' be the subset of all vectors v of $\mathbb{R}_+^n$ such that $v > f.v$ for all $f.v$ in the G-orbit of v. In fact, C' is a convex cone with nonempty interior (it contains the n linearly independent vectors $(1, 0, \dots, 0), (1, 1, 0, \dots, 0), \dots, (1, \dots, 1)$). By construction, M consists of the points of C' with integer coordinates. It follows that $C \subseteq C' \subseteq \overline{C}$, where $\overline{C}$ is the topological closure of C.

Assume now that M is finitely generated. Then, C is a closed convex cone, and C and C' simply coincide.

Assume further that G is not generated by transpositions. Then, there exists $a < b$ such that the transposition (a, b) is not in G, while a is in the G-orbit of b. Choose such a pair $a < b$ with b minimal. We claim that there is no transposition (a', b) in G with $a' < b$. Otherwise, a and a' are in the same G-orbit, and by minimality of b, $(a, a') \in G$; thus, $(a, b) = (a, a')(a', b)(a, a') \in G$. Pick $g \in G$ such that $g.b = a$, and for $t \geq 0$, define the vector in $\mathbb{R}_+^n$:

$$u_t := (nt, (n-1)t, \dots, (n-b+2)t, n-b+1, (n-b)t, \dots, t, 1)\,.$$

Note that $u_1 = (n, \dots, 1)$ is in C, whereas $u_0 = (0, \dots, 0, n-b+1, 0, \dots, 0)$ is not in C because $g.u_0 > u_0$.

Take t such that $0 < t \leq 1$. Then, the vector u_t has no zero coefficients, and in particular its G-orbit coincides with its orbit with respect to the underlying permutation group $G(X, X)$. Furthermore, the entries of u_t are all distinct, except when $t = (n-b)/(n-a')$ for some $a' < b$, in which case the a'-th and b-th entries are equal. Since $(a', b) \notin G$, the orbit of u_t is of size $|G(X, X)|$, and there exists a unique permutation $f_t \in G(X, X)$ such that $f_t.u_t$ is in C.

Let $t_0 = \inf\{t \geq 0, u_t \in C\}$. If $u_{t_0} \notin C$, then u_{t_0} is in the closure of C, but not in C, a contradiction. Otherwise, $u_{t_0} \in C$, and $t_0 > 0$ because $u_0 \notin C$. For any permutation f, $\{f.u_t, t \geq 0\}$ is a half-line; so, C being convex and closed, $I_f := \{t, f.u_t \in C\}$ is a closed interval $[x_f, y_f]$. For example, $I_{\text{id}} = [t_0, 1] \subsetneq [0, 1]$. Since the interval $[0, 1]$ is the union of all the I_f, there exists $f \neq \text{id}$ such that $t_0 \in I_f$. This contradicts the uniqueness of f_{t_0}. □

4.8. **Stability by derivation.** We denote by ∂_i the derivative with respect to the variable x_i, and consider the derivation $D := \sum_{i \in X} \partial_i$ on $\mathbb{K}[X]$.

Proposition 4.17. *Let G be a permutation groupoid. Then, $\mathbb{K}[X]^G$ is stable by the derivation D if and only if G is induced by a permutation group. On the other hand, $\mathbb{K}[X]^G$ is always stable with respect to the action of the* rational Steenrod operators $S_k := \sum_i x_i^{k+1}\partial_i$ *for $k \geq 0$ (see* [31] *for details on the rational Steenrod operators).*

Proof. The if-part is trivial since D commutes with the action of the symmetric group $\mathfrak{S}_X$ on $\mathbb{K}[X]$. Similarly, the rational Steenrod operators always stabilize $\mathbb{K}[X]^G$ because they commute with the action of any local bijection on $\mathbb{K}[X]^G$.

Assume now that $\mathbb{K}[X]^G$ is stable by derivation. Let $f : A \mapsto B$ be a local bijection such that $A \subsetneq X$, and take i in $X \backslash A$. We just need to prove that f extends to a local bijection g in G with domain $A \cup \{i\}$. Applying induction, any local bijection in G will then extend to a permutation, as desired.

Take a monomial m whose support is A and whose exponents are all distinct and at least 2, and consider the derivation $p = D(o(mx_i))$ of the orbitsum of the monomial mx_i in $\mathbb{K}[X]^G$. The monomial m occurs in p; hence, by invariance of p, $f(m)$ also occurs in p, as the derivative of some monomial $g(mx_i)$ in the orbit of mx_i. By the choice of the exponents of m, f and g must coincide on A, while at the same time i belongs to the domain of g. □

Example 4.18. QSym(X_2) *has no graded derivation of degree -1.*

5. Hopf algebra structure

Let R be a relational structure on a set E. In this section, we propose one approach to try to endow its age algebra with a Hopf algebra structure by looking at copies of R within R.

Assume that there exists two disjoint subsets E_1 and E_2 of E such that:

(a) R restricted to $E_i, i = 1, 2$ is isomorphic to R, or at least has the same age as R.
(b) The isomorphism type of a set $A \subseteq E_1 \cup E_2$ is entirely determined by the isomorphism types of $A \cap E_1$ and $A \cap E_2$.

Define the following graded algebra morphism on the set algebra:

$$\Delta_E^{E_1,E_2} : \begin{cases} \mathbb{K}^{[E]^{<\omega}} & \twoheadrightarrow \mathbb{K}^{[E_1]^{<\omega}} \otimes \mathbb{K}^{[E_2]^{<\omega}} \\ A & \mapsto \begin{cases} A \cap E_1 \otimes A \cap E_2 & \text{if } A \subseteq E_1 \cup E_2, \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{5.1}$$

Lemma 5.1. *$\Delta_E^{E_1,E_2}$ induces a coproduct Δ on $\mathbb{K}.\mathcal{A}(R)$, that is a graded algebra morphism from $\mathbb{K}.\mathcal{A}(R)$ to $\mathbb{K}.\mathcal{A}(R) \otimes \mathbb{K}.\mathcal{A}(R)$.*

Proof. All we have to check is that $\Delta_E^{E_1,E_2}$ indeed maps $\mathbb{K}.\mathcal{A}(E)$ to $\mathbb{K}.\mathcal{A}(E_1) \otimes \mathbb{K}.\mathcal{A}(E_2)$. Things are easier in the graded dual which is the quotient of the set algebra by the isomorphism equivalence relation. There, we need to check that the dual product is well defined; that is, given two types $\tau_1 \in \mathcal{A}(E_1)$ and $\tau_2 \in \mathcal{A}(E_2)$, the type of the product $A_1 \cup A_2$ of two representatives A_1 and A_2 of the types τ_1 and τ_2 respectively shall be independent of that choice of representatives. This is exactly condition (b). By condition (a), Δ can then be interpreted as going from $\mathbb{K}.\mathcal{A}(R)$ to $\mathbb{K}.\mathcal{A}(R) \otimes \mathbb{K}.\mathcal{A}(R)$ and is, by construction, an algebra morphism. □

Let in addition ϕ_1, ϕ_2 be isomorphisms from E to E_1, E_2 respectively, and denote by $E_{i,j} := \phi_i(E_j)$ the induced copy of E_j inside E_i. Consider the induced bijection

$$\Phi : \begin{cases} E_{1,1} \times E_{1,2} \times E_2 & \hookrightarrow\!\!\!\!\rightarrow E_1 \times E_{2,1} \times E_{2,2} \\ (a,b,c) & \mapsto (\phi_1^{-1}(a), \phi_1 \circ \phi_2 \circ \phi_1^{-1} \circ \phi_2^{-1}(b), \phi_2(c)) \end{cases}$$

and assume further

(c) Φ is an isomorphism: namely, if A, B, C are three subsets of $E_{1,1}$, $E_{1,2}$, E_2 respectively and $A', B', C' = \Phi(A, B, C)$, then $A \cup B \cup C \subset E$ and $A' \cup B' \cup C' \subset E$ have the same type in $\mathcal{A}(R)$.

Proposition 5.2. *Let R, E_1, E_2, Δ, ϕ_1, ϕ_2, as above satisfying (a) and (b) and (c). Then the coproduct is coassociative, turning $\mathbb{K}.\mathcal{A}(R)$ into a freely generated graded connected commutative Hopf algebra. In particular, its Hilbert series is of the form:*

$$\mathcal{H}_R(Z) = \prod_i \frac{1}{(1 - Z^{d_i})}, \tag{5.2}$$

where the product may be infinite.

Proof. Working in the graded dual as in the proof of Lemma 5.1, we need to check the dual product is associative. Take A, B, C representatives of three orbits; without loss of generality they may be chosen in $E_{1,1}, E_{1,2}, E_2$ respectively. Take $A', B', C' = \Phi(A, B, C)$, and note that A', B', C' are representatives of the same orbit. Condition (c) guarantees exactly that the product $(A \cup B) \cup C$ and $A' \cup (B' \cup C')$ are in the same orbit, as desired.

Since $\mathbb{K}.\mathcal{A}(R)$ is graded connected, most axioms of Hopf algebras (in particular concerning the antipode) are satisfied for free. The age algebra is then a graded commutative Hopf algebra. Using the classical Milnor-Moore theorem, it must be a free commutative algebra. The form of the Hilbert series follows. □

A. More examples of relational structures and age algebras

A.1. Examples with or without a Hopf age algebra structure.

Example A.1. *Consider the interval $(-1, 1)$ of $\mathbb{Q}$ equipped either with just the equality or its natural total order, and some relational structure S on some set F. Construct the relational structure $R := \mathbb{Q} \wr S$ on $E := \mathbb{Q} \times F$ by substituting each rational number by a copy of F. Define $E_1 = (-1, 0) \times F$*

and $E_2 = (0,1) \times F$. *Consider the affine isomorphisms* ϕ_1 *and* ϕ_2 *mapping* E *on* E_1 *and* E_2 *respectively, so that, e.g.,* $E_{2,1} = (0, 1/2)$.

Then, the axioms (a), (b), (c) of Proposition 5.2 are satisfied, and $\mathbb{K}.\mathcal{A}(R)$ *is a Hopf algebra. When* S *is an infinite chain and* $\mathbb{Q}$ *is equipped respectively with its natural antichain or chain structure, one get respectively the Hopf algebras of symmetric and quasisymmetric functions on the monomial basis.*

Example A.9 of [49] *is obtained by taking a finite graph* G *for* S *and the antichain on* $\mathbb{Q}$. *We recover the free algebra generated by connected finite restrictions of* S. *This generalizes immediately for any relational structure* S.

In the examples above, the construction mimics the standard trick of doubling the alphabet to construct Hopf algebras (see e.g. [14, section, 3.2] or [29, 50]) which is our original inspiration. The existence of a coassociative coproduct is a very strong constraint, and it is not clear whether there exist interesting coassociative examples where the construction goes beyond this trick.

Example A.2. *Let* $X := x_1, \ldots, x_n$, *and consider the polynomial rings* $\mathbb{K}[X]$ *realized as the invariant ring of the trivial permutation groupoid* $\mathbb{K}[X] = \mathbb{K}[X]^{\mathrm{id}}$. *One can take as relational structure* $R := X \times \mathbb{Q}$ *where each piece* $\{x_i\} \times \mathbb{Q}$ *is colored differently by a unary predicate. Taking* $E_1 := X \times (-1, 0)$ *and* $E_2 := X \times (0, 1)$, *Proposition 5.2 endows* $\mathbb{K}[X]$ *with its usual Hopf algebra structure where the generators* x_i *are primitive.*

Example A.3. *The age algebra of a direct sum* $R := K_\omega \oplus \cdots \oplus K_\omega$ *of* k *infinite cliques is the algebra of symmetric polynomials on* k *variables (see* [49, Example A.2]*). Since it is a free algebra, it can be endowed with a Hopf algebra structure (for example by making its generators group-like). Yet, we have not found a way to achieve this using Proposition 5.2 on this particular relational structure* R.

Example A.4 (The Planar Shuffle Algebra)**.** *In* [49, section A.4]*, we realized the Planar Shuffle Algebra of Gerritzen* [13, 24, 23] *as an age algebra.*

Consider the infinite tree T *depicted in Figure 1 of* [49, section A.4]. *Recall that the relational structure consists of the infix total order and three ternary relations on the set* E *of leaves of* T. *Choose two nonleaf children* x_1 *and* x_2 *of the root of* T, *and define* E_1 *and* E_2 *respectively as the leaves of the subtrees* T_1 *and* T_2 *of* T *dangling from* x_1 *and* x_2 *respectively. Choose isomorphisms* ϕ_1 *and* ϕ_2 *from* E *to* E_1 *and* E_2 *respectively. Define* $E_{i,j}$ *accordingly.*

Then, conditions (a) and (b) of Proposition 5.2 are satisfied, but not condition (c). Take indeed a, b, c *in* $E_{1,1}, E_{1,2}, E_{2,2}$ *respectively, and define* $a', b', c' := \Phi(a, b, c)$ *in* $E_1, E_{2,1}, E_{2,2}$ *respectively. The set* $\{a, b, c\}$ *has type* $((\circ, \circ), \circ)$ *whereas the set* $\{a', b', c'\}$ *has type* $(\circ, (\circ, \circ))$. *We recover the non-coassociative coproduct of the Planar Shuffle Algebra which splits the children of the root nodes in two consecutive ranges in all possible ways, and reduces* *the two resulting trees. For example:*

$$\Delta(\circ) = \circ \otimes 1 + 1 \otimes \circ,$$
$$\Delta((\circ,\circ)) = (\circ,\circ) \otimes 1 + \circ \otimes \circ + 1 \otimes (\circ,\circ),$$
$$\Delta(((\circ,\circ),\circ)) = ((\circ,\circ),\circ) \otimes 1 + (\circ,\circ) \otimes \circ + 1 \otimes ((\circ,\circ),\circ).$$

Iterating the above, $\circ \otimes \circ \otimes \circ$ *appears in* $(\Delta \otimes \mathrm{id})(\Delta(((\circ,\circ),\circ)))$ *but not in* $(\mathrm{id} \otimes \Delta)(\Delta(((\circ,\circ),\circ)))$*; that's the dual of the aforementioned counter example to (c).*

As far as we know, there currently is no known coassociative coproduct for this algebra, though it's likely to exist.

A.2. **Examples with a finite monomorphic decomposition.**

Example A.5. *This example features another age algebra that is not finitely generated.*

Consider the relational structure $R := (E, \rho)$*, where* $E := \mathbb{N} \times \{1,2,3\}$ *is endowed with the ternary relation*

$$\rho := \{((i,1),(j,2),(k,3)),\ i,j,k \in \mathbb{N}\}.$$

The minimal monomorphic decomposition is given by

$$(E_i := \mathbb{N} \times \{i\})_{i \in X := \{1,2,3\}}.$$

A basis of the age algebra $\mathbb{K}.\mathcal{A} = \mathbb{K}.\mathcal{A}(R)$ *in degree* d *is given by*

$$x^{\mathbf{d}}, \quad \text{for } d_i > 0, d_1 + d_2 + d_3 = d$$

together with

$$\sum_{\substack{\mathbf{d}\ :\ d_1 = 0 \text{ or} \\ d_2 = 0 \text{ or} \\ d_3 = 0, \\ d_1 + d_2 + d_3 = d}} x^{\mathbf{d}}.$$

The profile is given by $\phi_R(d) = \binom{d-1}{2} + 1$ *and the Hilbert series is*

$$\left(\frac{x}{1-x}\right)^3 + \frac{1}{1-x} = \frac{x^3 + x^2 - 2x + 1}{(1-x)^3}.$$

By construction, the restriction of E *on* $E_1 \cup E_2$ *is monomorphic. Therefore, the minimal monomorphic decomposition is not hereditary minimal and the age algebra is not finitely generated.*

Example A.6. *A variation of Example A.19 of* [49] *featuring a finitely generated age algebra with a Hilbert series whose numerator cannot be chosen with nonnegative coefficients. This one has only two monomorphic parts which are both infinite.*

Let $R := (E, \rho)$*, where* $E := \mathbb{N} \times \{0,1\}$*,* $\rho := (\mathbb{N} \times \{0\})^3 \cup (\mathbb{N} \times \{1\})^3$*. Then* R *has two monomorphic parts, namely* $\mathbb{N} \times \{0\}$ *and* $\mathbb{N} \times \{1\}$*. Each type of* n*-element restriction has a representative made of a* $m+k$ *element subset of* $\mathbb{N} \times \{0\}$ *and of a* m*-element subset of* $\mathbb{N} \times \{1\}$ *such that* $n = 2m+k$*; these representatives are nonisomorphic, except if* $n = 2$ *(in the later* *case, all* 2 *-element restrictions are isomorphic, hence we may eliminate the representative corresponding to* $m = 1, k = 0$*). With this observation, a straightforward computation shows that* $\varphi_R(0) = \varphi_R(1) = \varphi_R(2) = 1$ *and* $\varphi_R(n) = \lfloor n/2 \rfloor + 1$ *for* $n \geq 3$*. Hence the generating series*

$$H_{\varphi_R} = \frac{1}{(1-Z)(1-Z^2)} - Z^2 = \frac{1 - Z^2 + Z^3 + Z^4 - Z^5}{(1-Z)(1-Z^2)}.$$

But, then H_{φ_R} *cannot be written as a quotient of the form*

$$\frac{P}{(1-Z)(1-Z^k)}$$

where P *is a polynomial with nonnegative integer coefficients. Suppose indeed that* H_{φ_R} *is of this form. We may assume* k *even (otherwise, multiply* P *and* $(1-Z)(1-Z^k)$ *by* $(1+Z^k)$*. Set* $k' := k/2$*. Multiplying* $1 - Z^2 + Z^3 + Z^4 - Z^5$ *and* $(1-Z)(1-Z^2)$ *by* $1 + Z^2 + \cdots + Z^{2(k'-1)}$*, we get*

$$P = (1 - Z^2 + Z^3 + Z^4 - Z^5)(1 + Z^2 + \cdots + Z^{2(k'-1)}).$$

Hence, the term of largest degree has a negative coefficient, a contradiction.

Example A.7. *Another variation on Example A.19 of* [49]*, with four variables; now the numerator can take either positive or negative coefficients.*

Let $R := (E, (\rho, U_2, U_3))$*, where*

$$\begin{aligned} E :=& \mathbb{N} \times \{0,1,2,3\}, \\ \rho :=& \{((n,i),(m,j)) : i = 0, j \in \{1,2\} \text{ or } i = 1, j = 3\}, \\ U_i :=& \mathbb{N} \times \{i\} \text{ for } i \in \{2,3\}. \end{aligned}$$

Then R *has four monomorphic components, namely* $\mathbb{N} \times \{0\}, \mathbb{N} \times \{1\}, \mathbb{N} \times \{2\}, \mathbb{N} \times \{3\}$*. Let* S *be the induced structure on four elements of the form* $(x_i, i), i \in \{0,1,2,3\}$*. A crucial property is that* S *has only two nontrivial local isomorphisms, namely the map sending* $(x_0, 0)$ *onto* $(x_1, 1)$ *and its inverse. From this follows that the induced substructures on two* n*-element subsets* E *are isomorphic if either they have the same number of elements on each* $\mathbb{N} \times \{i\}$ *or one subset is included into* $\mathbb{N} \times \{0\}$*, the other into* $\mathbb{N} \times \{1\}$*. Hence, the generating series* $\mathcal{H}_{\varphi_R}$ *is*

$$\frac{1}{(1-Z)^4} - \frac{Z}{1-Z} = \frac{1 - Z + 3Z^2 - 3Z^3 + Z^4}{(1-Z)^4}.$$

We may write it

$$\mathcal{H}_{\varphi_R} = \frac{Q_1}{(1-Z)(1-Z^4)(1-Z^5)(1-Z^5)},$$

where

$$\begin{aligned} Q_1 := 1 &+ 2Z + 6Z^2 + 10Z^3 + 14Z^4 + 17Z^5 \\ &+ 18Z^6 + 14Z^7 + 10Z^8 + 6Z^9 + Z^{10}, \end{aligned}$$

as well as

$$\mathcal{H}_{\varphi_R} = \frac{Q}{(1-Z)(1-Z^5)^3},$$

where

$$Q_2 := 1 + 2Z + 6Z^2 + 10Z^3 + 15Z^4 + 18Z^5 + 22Z^6 + 18Z^7 + 15Z^8 + 10Z^9 + 6Z^{10} + Z^{12} + Z^{16}.$$

A.3. **An example with polynomial growth but infinitely many monomorphic parts.**

Examples. *Consider the direct sum* $R' := \overline{K}_\infty \wr K_{1,1} \oplus R$ *of the infinite matching and the relational structure* R *of Example 3.2. The profile has polynomial growth, but* R *has infinitely many monomorphic parts and the age algebra is not finitely generated.*

A.4. **A uniformly prehomegenous example with polynomial growth whose automorphism group has nonpolynomial growth.** A relational structure R is *prehomegeneous* if, for every finite subset F of the domain of R, there is a finite superset $\overline{F}$ such that every local isomorphism defined on F which extends to $\overline{F}$ extends to an automorphism of R. If the cardinality of $\overline{F}$ can be bounded by a function of the cardinality of F, then R is *uniformly prehomogeneous*.

For a homogeneous relational structure R, the age of R (and thus its profile) coincides with that of the automorphism group of R. The following example shows that—unlike what one might have hoped—the profile of a uniformly prehomogeneous structure can be much smaller than that of its automorphism group $\mathrm{Aut}(R)$

Example A.9. *Let* $R := (E, \rho, \rho')$ *be the relation on* $E := \{1,2\} \times \mathbb{Q}$*, where* ρ *is the quaternary relation:*

$$\{((1,j),(2,j),(1,j'),(2,j')), j < j' \in \mathbb{Q}\}.$$

The automorphism group of R *is the permutation group* $G = \mathrm{id}_2 \times \mathrm{Aut}(\mathbb{Q})$ *endowed by the action of* $\mathrm{Aut}(\mathbb{Q})$ *by permutation of the blocks* $\{(1,j),(2,j)\}$. *Orbits under* G *are in bijections with words in the three letters* $\{1\}$,$\{2\}$, $\{1,2\}$. *The profile of* G *is exponential, with Hilbert series*

$$\mathcal{H}_G = \frac{1}{1+2X+X^2}.$$

On the other hand, the profile of R *is linear: indeed, if a subset* A *contains a single point in a given block, then the function defined on that subset by moving this point to any block not touched by* A *while fixing all other points of* S *is a local automorphism; hence, each orbit is determined by a pair* (a,b)*, where* a *gives the number of blocks containing two elements and* b *the number of blocks containing one element (exception:* $(2, n-2)$ *and* $(0,n)$ *give the same orbit).*

Finally, R is uniformly homogeneous: let A be a subset, and B be the union of the blocks containing at least one element of A; Then $|B| \leq 2|A|$, and any local isomorphism with domain A which extends to a local isomorphism with domain B extends to an automorphism of R.

A.5. **An example with nonfinite profile.**

Example A.10. *This example illustrates why some care needs to be taken when defining the age algebra for a relational structure with nonfinite profile.*

Take an infinite set E, with one binary relation for each couple (i, j) of distinct elements of E, which holds just on (i, j). In degree 1, there is a single type; let e_1 be the corresponding element in the age algebra. Then, e_1^2 shall be the sum of all the types of degree 2, of which there are infinitely many. Hence, for the age algebra to be indeed stable by multiplication, one shall consider infinite linear combinations of types.

References

1. J. Balogh, B. Bollobás, and R. Morris, *Hereditary properties of partitions, ordered graphs and ordered hypergraphs*, European J. Combin. **27** (2006), no. 8, 1263–1281. MR 2260120 (2007f:05126)
2. ______, *Hereditary properties of combinatorial structures: posets and oriented graphs*, J. Graph Theory **56** (2007), no. 4, 311–332. MR 2360508 (2008j:05290)
3. R. Bogvad and T. Meyer, *On algorithmically checking whether a hilbert series comes from a complete intersection*, Research Reports in Mathematics 5, Department of Mathematics, Stockholm University, may 2004.
4. Y. Boudabbous and M. Pouzet, *The morphology of infinite tournaments; application to the growth of their profile*, European J. Combin. **31** (2010), no. 2, 461–481. MR 2565340 (2010m:05130)
5. W. Bruns and J. Gubeladze, *Polytopes, rings, and k-theory*, Springer, 2009.
6. X. Buchwalder, *Sur l'algèbre et la combinatoire des sous-graphes d'un graphe*, Ph.D. thesis, Université Claude Bernard Lyon 1, Nov. 2009.
7. P. J. Cameron, *Transitivity of permutation groups on unordered sets*, Math. Z. **148** (1976), no. 2, 127–139. MR 0401885
8. ______, *Orbits of permutation groups on unordered sets. II*, J. London Math. Soc. (2) **23** (1981), no. 2, 249–264. MR 609105
9. ______, *Oligomorphic permutation groups*, London Mathematical Society Lecture Note Series, vol. 152, Cambridge University Press, Cambridge, 1990. MR MR1066691 (92f:20002)
10. ______, *The algebra of an age*, Model theory of groups and automorphism groups (Blaubeuren, 1995), London Math. Soc. Lecture Note Ser., vol. 244, Cambridge Univ. Press, Cambridge, 1997, pp. 126–133. MR MR1689855 (2000c:13005)
11. D. Cox, J. Little, and D. O'Shea, *Ideals, varieties, and algorithms*, second ed., Springer-Verlag, New York, 1997, An introduction to computational algebraic geometry and commutative algebra.
12. H. Derksen and G. Kemper, *Computational invariant theory*, Invariant Theory and Algebraic Transformation Groups, I, Springer-Verlag, Berlin, 2002, Encyclopaedia of Mathematical Sciences, 130. MR 2003g:13004
13. V. Drensky and L. Gerritzen, *Nonassociative exponential and logarithm*, J. Algebra **272** (2004), no. 1, 311–320. MR MR2029036 (2004i:17003)

14. G. Duchamp, F. Hivert, and J.-Y. Thibon, *Noncommutative symmetric functions. VI. Free quasi-symmetric functions and related algebras*, Internat. J. Algebra Comput. **12** (2002), no. 5, 671–717. MR 1 935 570
15. D. Eisenbud, *Commutative algebra*, Springer-Verlag, New York, 1995, With a view toward algebraic geometry.
16. J. Falque, *Classification of p-oligomorphic groups; conjectures of cameron and macpherson*, Ph.D. thesis, Université Paris-Sud, November 2019, 136 pages.
17. J. Falque and N. M. Thiéry, *Macpherson's conjecture holds: The orbit algebra of a permutation group with polynomial profile is finitely generated*, 30th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2018, Hanover), February 2018.
18. R. Fraïssé, *Cours de logique mathématique. Tome 1: Relation et formule logique*, Gauthier-Villars Éditeur, Paris, 1971, Deuxième édition revue et modifiée, Collection de "Logique Mathématique", Série A, No. 23. MR MR0345782 (49 #10513)
19. ———, *Theory of relations*, revised ed., Studies in Logic and the Foundations of Mathematics, vol. 145, North-Holland Publishing Co., Amsterdam, 2000, With an appendix by Norbert Sauer. MR MR1808172 (2002d:03084)
20. R. Fraïssé, *Sur l'extension aux relations de quelques propriétés des ordres*, Ann. Sci. Ecole Norm. Sup. (3) **71** (1954), 363–388. MR 0069239
21. A. M. Garsia and D. Stanton, *Group actions of Stanley - Reisner rings and invariants of permutation groups*, Adv. in Math. **51** (1984), no. 2, 107–201.
22. A. M. Garsia and N. Wallach, *Qsym over Sym is free*, J. Combin. Theory Ser. A **104** (2003), no. 2, 217–263. MR 2 019 274
23. L. Gerritzen, *Automorphisms of the planar tree power series algebra and the non-associative logarithm*, Serdica Math. J. **30** (2004), no. 2-3, 135–158. MR MR2098329 (2005g:17009)
24. ———, *Planar rooted trees and non-associative exponential series*, Adv. in Appl. Math. **33** (2004), no. 2, 342–365. MR MR2074402 (2005c:05018)
25. I. M. Gessel, *Multipartite P-partitions and inner products of skew Schur functions*, Combinatorics and algebra (Boulder, Colo., 1983), Contemp. Math., vol. 34, Amer. Math. Soc., Providence, RI, 1984, pp. 289–317. MR MR777705 (86k:05007)
26. D. Grinberg and V. Reiner, *Hopf algebras in combinatorics*, Mathematisches Forschungsinstitut Oberwolfach gGmbH, 2020.
27. B. Herwig and D. Lascar, *Extending partial automorphisms and the profinite topology on free groups*, Trans. Amer. Math. Soc. **352** (2000), no. 5, 1985–2021. MR 1621745
28. G. Higman, *Ordering by divisibility in abstract algebras*, Proc. London Math. Soc. (3) **2** (1952), 326–336. MR 0049867 (14,238e)
29. F. Hivert, *An Introduction to Combinatorial Hopf Algebras - Examples and Realizations*, Nato Advanced Study Institute School on Physics and Computer Science (Cargèse, France) (Gazeau J.-P., ed.), 2005, p. 10pp. (Anglais).
30. ———, *Local actions of the symmetric group and generalisations of quasi-symmetric functions*, 2005.
31. F. Hivert and N. M. Thiéry, *Deformation of symmetric functions and the rational Steenrod algebra*, Invariant theory in all characteristics, CRM Proc. Lecture Notes, vol. 35, Amer. Math. Soc., Providence, RI, 2004, arXiv:0812.3056v1 [math.CO], pp. 91–125. MR 2 066 460
32. I. M. Hodkinson and H. D. Macpherson, *Relational structures determined by their finite induced substructures*, J. Symbolic Logic **53** (1988), no. 1, 222–230. MR MR929387 (89e:03047)
33. D. Kapur and K. Madlener, *A completion procedure for computing a canonical basis for a k-subalgebra*, Computers and mathematics (Cambridge, MA, 1989), Springer, New York, 1989, pp. 1–11. MR 90g:13001

34. M. Klazar, *Some general results in combinatorial enumeration*, Permutation patterns, London Math. Soc. Lecture Note Ser., vol. 376, Cambridge Univ. Press, Cambridge, 2010, pp. 3–40. MR 2732822
35. M. V. Lawson, *Inverse semigroups*, World Scientific Publishing Co., Inc., River Edge, NJ, 1998, The theory of partial symmetries. MR 1694900
36. H. D. Macpherson, *Growth rates in infinite graphs and permutation groups*, Proceedings of the London Mathematical Society **3** (1985), no. 2, 285–294.
37. T. Monteil and M. Pouzet, *From the complexity of infinite permutations to the profile of bichains*, ROGICS'08: International Conference on Relations, Orders and Graphs: Interaction with Computer Science, 2008, pp. 1–6.
38. D. Oudrar, *Sur l'énumération de structures discrètes, une approche par la théorie des relations*, Ph.D. thesis, Université d'Alger USTHB à Bab Ezzouar, Sept. 2015, arXiv:1604.05839 [math.CO].
39. D. Oudrar and M. Pouzet, *Profile and hereditary classes of ordered relational structures*, J. Mult.-Valued Logic Soft Comput. **27** (2016), no. 5-6, 475–500. MR 3548054
40. M. Pouzet, *Modèle universel d'une théorie n-complète: Modèle uniformément préhomogène*, C. R. Acad. Sci. Paris Sér. A-B **274** (1972), A695–A698. MR 292661
41. ———, *Application d'une propriété combinatoire des parties d'un ensemble aux groupes et aux relations*, Math. Z. **150** (1976), no. 2, 117–134. MR 57 #425
42. ———, *Sur la théorie des relations*, 1978, Thèse d'état, Université Claude-Bernard, Lyon 1.
43. ———, *The profile of relations*, Glob. J. Pure Appl. Math. **2** (2006), no. 3, 237–272. MR 2311852 (2008g:05012)
44. ———, *When is the orbit algebra of a group an integral domain? Proof of a conjecture of P. J. Cameron*, Theor. Inform. Appl. **42** (2008), no. 1, 83–103. MR MR2382545
45. M. Pouzet and I. G. Rosenberg, *Sperner properties for groups and relations*, European J. Combin. **7** (1986), no. 4, 349–370.
46. M. Pouzet and H. Si Kaddour, *Isomorphy up to complementation*, J. Comb. **7** (2016), no. 2-3, 285–305. MR 3465955
47. M. Pouzet and N. M. Thiéry, *Some relational structures with polynomial growth and their associated algebras*, Proceedings of FPSAC'05 Taormina, 2005, arXiv:0601256 [math.CO].
48. ———, *Some relational structures with polynomial growth and their associated algebras II*, 2008, arXiv:0801.4404v1 [math.CO].
49. ———, *Some relational structures with polynomial growth and their associated algebras I: Quasi-polynomiality of the profile*, Electron. J. Combin. **20** (2013), no. 2, Paper 1, 35. MR 3066340
50. J.-B. Priez, *A lattice of combinatorial Hopf algebras: Binary trees with multiplicities*, DMTCS Proceedings (2013), 1167–1180, AS - 25th International Conference on Formal Power Series and Algebraic Combinatorics, arXiv preprint arXiv:1303.5538.
51. L. Robbiano and M. Sweedler, *Subalgebra bases*, Commutative algebra (Salvador, 1988), Springer, Berlin, 1990, pp. 61–87.
52. R. P. Stanley, *Combinatorics and invariant theory*, Relations between combinatorics and other parts of mathematics (Proc. Sympos. Pure Math., Ohio State Univ., Columbus, Ohio, 1978), Amer. Math. Soc., Providence, R.I., 1979, pp. 345–355.
53. ———, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997, With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original. MR MR1442260 (98a:05001)
54. B. Steinberg, *Möbius functions and semigroup representation theory*, Journal of Combinatorial Theory, Series A **113** (2006), no. 5, 866–881.
55. B. Sturmfels, *Algorithms in invariant theory*, Springer-Verlag, Vienna, 1993.
56. N. M. Thiéry, *Computing minimal generating sets of invariant rings of permutation groups with SAGBI-Gröbner basis*, Discrete models: combinatorics, computation, and

geometry (Paris, 2001), Discrete Math. Theor. Comput. Sci. Proc., AA, Maison Inform. Math. Discrèt., Paris, 2001, pp. 315–328 (electronic). MR 1 888 782

57. N. M. Thiéry and S. Thomassé, *Convex cones and SAGBI bases of permutation invariants*, Invariant theory in all characteristics, CRM Proc. Lecture Notes, vol. 35, Amer. Math. Soc., Providence, RI, 2004, arXiv:0607380 [math.AC], pp. 259–263. MR MR2066473 (2005e:13006)
58. N. M. Thiéry, *Algebraic invariants of graphs: a study based on computer exploration*, SIGSAM Bulletin (ACM Special Interest Group on Symbolic and Algebraic Manipulation) **34** (2000), no. 3, 9–20, arXiv:0812.3082v1 [math.CO].
59. V. Weispfenning, *Admissible orders and linear forms*, ACM Sigsam Bulletin **21** (1987), no. 2, 16–18.
60. Wikipedia, 2020, `http://en.wikipedia.org/wiki/InverseSemigroup`.

Univ. Lyon, Université Claude-Bernard Lyon1, CNRS UMR 5208, Institut Camille Jordan, 43 bd. 11 Novembre 1918, 69622 Villeurbanne Cedex, France and Mathematics & Statistics Department, University of Calgary, Calgary, Alberta, Canada T2N-1N4

E-mail address: `pouzet@univ-lyon1.fr, maurice.pouzet@ucalgary.ca`

Université Paris-Saclay, CNRS, Laboratoire de recherche en informatique, 91405, Orsay, France.

E-mail address: `Nicolas.Thiery@universite-paris-saclay.fr`