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DIRECT AND INVERSE PROBLEMS FOR RESTRICTED SIGNED SUMSETS IN INTEGERS

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Abstract. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a nonempty finite subset of an additive abelian group G. For a positive integer h ($\leq k$), we let

$h^{\wedge}_{\pm}A = \{\Sigma_{i=0}^{k-1}\lambda_i a_i : \lambda_i \in \{-1,0,1\}$ for $i = 0,1,\ldots,k-1$, $\Sigma_{i=0}^{k-1}|\lambda_i| = h\}$,

be the h-fold *restricted signed sumset* of A. The *direct problem* for the restricted signed sumset is to find the minimum number of elements in $h^{\wedge}_{\pm}A$ in terms of $|A|$, where $|A|$ is the cardinality of A. The *inverse problem* for the restricted signed sumset is to determine the structure of the finite set A for which the minimum value of $|h^{\wedge}_{\pm}A|$ is achieved. In this article, we solve some cases of both direct and inverse problems for $h^{\wedge}_{\pm}A$ in the group of integers. In this connection, we also mention some conjectures in the remaining cases.

1. Introduction

Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a nonempty finite subset of an additive abelian group G. Let h be a positive integer. The h-fold *sumset* hA, the h-fold *restricted sumset* $h^{\wedge}A$, and the h-fold *signed sumset* $h_{\pm}A$ of the set A are defined respectively by (see [4, 5, 19])

$$hA = \left\{\sum_{i=0}^{k-1}\lambda_i a_i : \lambda_i \in \mathbb{N} \text{ for } i = 0,1,\ldots,k-1 \text{ and } \sum_{i=0}^{k-1}\lambda_i = h\right\},$$

$$h^{\wedge}A = \left\{\sum_{i=0}^{k-1}\lambda_i a_i : \lambda_i \in \{0,1\} \text{ for } i = 0,1,\ldots,k-1 \text{ and } \sum_{i=0}^{k-1}\lambda_i = h\right\},$$

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and

$$h_{\pm}A = \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \mathbb{Z} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} |\lambda_i| = h \right\},$$

where $\mathbb{N}$ denotes the set of all nonnegative integers, $\mathbb{Z}$ denotes the set of all integers, and $h \leq k$ in case of $h^\wedge A$.

The study of sumsets has more than a two-hundred-year old history. In 1813, Cauchy [8] found the minimum cardinality of the sumset $A+B$, where A and B are nonempty subsets of the group of residue classes modulo a prime. Later, Davenport [9] rediscovered Cauchy's result in 1935. The result is now known as the Cauchy–Davenport theorem.

Theorem 1.1 (Cauchy–Davenport Theorem)**.** *Let A and B be nonempty subsets of the group $\mathbb{Z}/p\mathbb{Z}$ of prime order p. Then*

$$|A + B| \geq \min\{p, |A| + |B| - 1\}.$$

The h-fold generalization of this theorem is the following theorem.

Theorem 1.2. *Let A be a nonempty subset of the group $\mathbb{Z}/p\mathbb{Z}$ of prime order p. Then*

$$|hA| \geq \min\{p, h|A| - h + 1\}.$$

Several results about the minimum cardinality of the sumset and its inverse that if the minimum cardinality is achieved by the sumset, then the characterization of the set, have been obtained in the past. A comprehensive list of references may be found in Mann [17], Freiman [14], Nathanson [19], and Tao [21]. Plagne [20] (see also [12]) settled the general case obtaining the minimum cardinality of sumset in an abelian group. The theorem of Plagne is mentioned below.

Theorem 1.3 (Plagne)**.** *Let G be an abelian group of order n. Let A be a nonempty subset of G with cardinality k. Then*

$$|hA| \geq \min\{(h\lceil k/d \rceil - h + 1) \cdot d : \ d \in D(n)\},$$

where $D(n)$ is the set of positive divisors of n.

On the other hand, the study on h-fold restricted sumset is not very well settled. In the case of groups of integers, the minimum size of the restricted sumset was given by Nathanson [18] in the following theorem.

Theorem 1.4. *Let A be a finite set of k integers, and let $1 \leq h \leq k$ be a positive integer. Then*

$$|h^\wedge A| \geq hk - h^2 + 1.$$

Nathanson [18] also classified the sets of integers which give the exact lower bound. Among other extremal sets, an important class of the extremal sets is mentioned in the following theorem.

Theorem 1.5. *Let $2 \leq h \leq k-2$. Let A be a finite set of k (≥ 5) integers. Then $|h^\wedge A| = hk - h^2 + 1$ if and only if A is a k-term arithmetic progression.*

The minimum size of the restricted sumset in the group $\mathbb{Z}/p\mathbb{Z}$ was actually asked in a conjecture by Erdős and Heilbronn [13] in 1964. The conjecture was first confirmed by Dias-Da Silva and Hamidoune [10] in 1994 using some ideas from the exterior algebra. Later, it was reproved by Alon, Nathanson and Ruzsa [1, 2] using the polynomial method.

Theorem 1.6 (Da Silva–Hamidoune Theorem)**.** *Let A be a set of k distinct residues modulo a prime p. Then*

$$|h^\wedge A| \geq \min\{p, hk - h^2 + 1\}.$$

Finding the minimum size of restricted sumset for the general finite abelian group seems to be much more difficult problem than the usual h-fold sumset as the minimum size of restricted sumset heavily depends on the structure of the group rather than its size.

Unlike the usual and restricted sumsets, the h-fold signed sumset $h_{\pm}A$ has been appeared recently. The signed sumset first appeared in the work of Bajnok and Ruzsa [6] in the context of the "independence number" of a subset A of a group and in the work of Klopsch and Lev [15, 16] in the context of the "diameter" of a group with respect to the subset A. The first systematic and point centric study appeared in the work of Bajnok and Matzke [4] (see also [3] for many related questions and results) in which, they studied the minimum cardinality of h-fold signed sumset $h_{\pm}A$ of subsets of a finite abelian group. In particular, they proved that the minimum cardinality of $h_{\pm}A$ is the same as the minimum cardinality of hA, when A is a subset of a finite cyclic group. A year later, they [5] classified all possible values of k for which the minimum cardinality of $h_{\pm}A$ coincides with the minimum cardinality of hA, when A is a subset of a particular elementary abelian group.

Along the line of the signed sumset $h_{\pm}A$, we define the h-fold *restricted signed sumset* of A for $1 \leq h \leq k$, denoted by $h^\wedge_{\pm}A$, by

$$h^\wedge_{\pm}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \{-1, 0, 1\} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} |\lambda_i| = h \right\}.$$

Clearly,

$$h^\wedge A \cup h^\wedge(-A) \subseteq h^\wedge_{\pm}A.$$

Also, for an integer α, we have

$$h^\wedge_{\pm}(\alpha * A) = \alpha * (h^\wedge_{\pm}A),$$

where $\alpha * A = \{\alpha \cdot a \mid a \in A\}$ is the α-*dilation* of the set A.

The *direct problem* associated with the sumsets is a problem in which we try to determine the structure and properties of the sumset and an *inverse problem* associated with the sumsets is a problem in which we attempt to deduce the properties of the individual sets from the properties of their sumset. Thus, the direct problem for $h^\wedge_{\pm}A$ is to find the minimum number of elements in $h^{\wedge}_{\pm}A$ in terms of $|A|$ and the inverse problem for $h^{\wedge}_{\pm}A$ is to determine the structure of the finite set A for which $|h^{\wedge}_{\pm}A|$ is minimal.

Very recently, Bhanja and Pandey [7] gave some direct and inverse results for the sumset $h_{\pm}A$ in the group of integers. In this article, we study similar direct and inverse problems for restricted signed sumset $h^{\wedge}_{\pm}A$, when A is a finite set of integers. More specifically, in Section 2, we study the problem when A contains only positive integers and in Section 3, we study the problem when A contains nonnegative integers with $0 \in A$. In both sections 2 and 3, we also mention some open problems as conjectures.

Notation. For integers a, b with $b \geq a$, we let $[a, b] = \{n \in \mathbb{Z} : a \leq n \leq b\}$. Let $\binom{a}{b} = \frac{a!}{b!(a-b)!}$, if $a \geq b$, otherwise $\binom{a}{b} = 0$. We say that a set S is symmetric, if $-s \in S$ for all $s \in S$.

2. A contains only positive integers

Theorem 2.1. *Let A be a set of k positive integers, and let $1 \leq h \leq k$ be an integer. Then*

$$|h^{\wedge}_{\pm}A| \geq 2(hk - h^2) + \binom{h+1}{2} + 1. \tag{2.1}$$

This lower bound is best possible for $h = 1$, 2, and k.

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$. For $i = 0, 1, \ldots, k - h - 1$ and $j = 0, 1, \ldots, h$, let

$$s_{i,j} := \sum_{\substack{l=0 \\ l \neq h-j}}^{h} a_{i+l}. \tag{2.2}$$

Let

$$s_{k-h,0} := \sum_{l=0}^{h-1} a_{k-h+l}. \tag{2.3}$$

Each $s_{i,j}$ is a sum of h distinct elements of A, and hence it is in $h^{\wedge}_{\pm}A$. Moreover, for $i = 0, 1, \ldots, k - h - 1$ and $j = 0, 1, \ldots, h - 1$, we have

$$s_{i,j} < s_{i,j+1} \text{ and } s_{i,h} = s_{i+1,0}.$$

Thus, we get at least $hk - h^2 + 1$ positive integers in $h^{\wedge}_{\pm}A$. Since $h^{\wedge}_{\pm}A$ is symmetric, the inverses of these $hk - h^2 + 1$ integers are also in $h^{\wedge}_{\pm}A$ with $-s_{0,0} < s_{0,0}$. So, we counted $2(hk - h^2 + 1)$ integers in $h^{\wedge}_{\pm}A$.

For $i = 0, 1, \ldots, h - 1$ and $j = 0, 1, \ldots, h - i - 1$, define

$$t_{i,j} := \sum_{\substack{l=0 \\ l \neq j}}^{h-i-1} (-a_l) + a_j + \sum_{m=1}^{i} a_{h-m}. \tag{2.4}$$

Clearly, $t_{i,j} \in h^{\wedge}_{\pm}A$ for all i and j. Moreover, for $j = 0, 1, \ldots, h-i-2$, we have

$$t_{i,j} < t_{i,j+1},$$

and for $i = 0, 1, \ldots, h-2$, we have

$$t_{i,h-i-1} < t_{i+1,0}.$$

We also have

$$-s_{0,0} < t_{0,0} \text{ and } t_{h-1,0} = s_{0,0}.$$

So, we get $\binom{h+1}{2} - 1$ more integers in $h^{\wedge}_{\pm}A$ which are given by (2.4). Further, these elements are different from the elements in (2.2) and (2.3). Hence, we get

$$|h^{\wedge}_{\pm}A| \geq 2(hk - h^2) + \binom{h+1}{2} + 1.$$

This establishes (2.1).

Next, we show that the lower bound in (2.1) is best possible for $h = 1$, 2 and k.

Let $h = 1$. Then for any finite set A of k positive integers, we have $|1^{\wedge}_{\pm}A| = 2k = 2(hk - h^2) + \binom{h+1}{2} + 1$.

Now, let $h = 2$ and $A = \{1, 3, 5, \ldots, 2k-1\}$. Then

$$2^{\wedge}_{\pm}A = \{-(4k-4), -(4k-6), \ldots, -2, 2, 4, \ldots, 4k-4\},$$

and hence $|2^{\wedge}_{\pm}A| = 4k - 4 = 2(hk - h^2) + \binom{h+1}{2} + 1$.

Finally, let $h = k$ and $A = [1, k]$. It is easy to see that $k^{\wedge}_{\pm}A$ contains either odd integers or even integers. Since $k^{\wedge}_{\pm}A \subseteq \left[-\binom{k+1}{2}, \binom{k+1}{2}\right]$, we get

$$|k^{\wedge}_{\pm}A| \leq \binom{k+1}{2} + 1.$$

This together with (2.1) gives $|k^{\wedge}_{\pm}A| = \binom{k+1}{2} + 1 = 2(hk - h^2) + \binom{h+1}{2} + 1$. This completes the proof of the theorem. $\square$

The next two theorems give the inverse results for the cases $h = 2$ and $h = k$, respectively. For $h = 1$, any set with k elements is extremal. Also, for $h = 2$, any set of two positive integers is an extremal set. So, in both the inverse theorems, we assume that $k \geq 3$.

Theorem 2.2. *Let A be a set of k (≥ 3) positive integers such that $|2^{\wedge}_{\pm}A| = 4k - 4$. Then, $A = d * \{1, 3, \ldots, 2k-1\}$ for some positive integer d.*

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$. Let

$$|2^{\wedge}_{\pm}A| = 4k - 4.$$

First, let $k = 3$. Then

$$\begin{aligned}2^{\wedge}_{\pm}A = \{&a_0 + a_1, a_0 - a_1, -a_0 + a_1, -a_0 - a_1, a_0 + a_2, a_0 - a_2, -a_0 + a_2,\\ &- a_0 - a_2, a_1 + a_2, a_1 - a_2, -a_1 + a_2, -a_1 - a_2\},\end{aligned}$$

where

$$\begin{aligned}(2.5)\qquad -a_1 - a_2 < -a_0 - a_2 < -a_0 - a_1 < a_0 - a_1 < -a_0 + a_1 < a_0 + a_1 \\ < a_0 + a_2 < a_1 + a_2.\end{aligned}$$

If $|2^{\wedge}_{\pm}A| = 4k - 4 = 8$, then $2^{\wedge}_{\pm}A$ contains precisely the integers listed in (2.5). Since

$$-a_0 - a_2 < a_0 - a_2 < a_0 - a_1,$$

we get $a_0 - a_2 = -a_0 - a_1$, i.e., $a_2 - a_1 = 2a_0$.

Similarly, since

$$a_0 - a_1 < a_2 - a_1 < a_2 - a_0 = a_0 + a_1,$$

we have $a_2 - a_1 = a_1 - a_0$. Hence, $A = a_0 * \{1, 3, 5\}$.

Now, let $k = 4$. Then

$$\begin{aligned}2^{\wedge}_{\pm}A = \{&a_0 + a_1, a_0 - a_1, -a_0 + a_1, -a_0 - a_1, a_0 + a_2, a_0 - a_2, -a_0 + a_2, \\ &- a_0 - a_2, a_0 + a_3, a_0 - a_3, -a_0 + a_3, -a_0 - a_3, a_1 + a_2, a_1 - a_2, \\ &- a_1 + a_2, -a_1 - a_2, a_1 + a_3, a_1 - a_3, -a_1 + a_3, -a_1 - a_3, a_2 + a_3, \\ &a_2 - a_3, -a_2 + a_3, -a_2 - a_3\},\end{aligned}$$

where

$$\begin{aligned}(2.6)\qquad -a_2 - a_3 < -a_1 - a_3 < -a_1 - a_2 < -a_0 - a_2 < -a_0 - a_1 < a_0 - a_1 \\ < -a_0 + a_1 < a_0 + a_1 < a_0 + a_2 < a_1 + a_2 < a_1 + a_3 < a_2 + a_3.\end{aligned}$$

If $|2^{\wedge}_{\pm}A| = 4k - 4 = 12$, then $2^{\wedge}_{\pm}A$ contains precisely the integers listed in (2.6). Since

$$a_0 + a_2 < a_0 + a_3 < a_1 + a_3,$$

from (2.6) it follows that $a_0 + a_3 = a_1 + a_2$, which is equivalent to $a_3 - a_2 = a_1 - a_0$.

Similarly, since

$$-a_0 + a_1 < -a_0 + a_2 < a_0 + a_2,$$

we have $-a_0 + a_2 = a_0 + a_1$, equivalently, $a_2 - a_1 = 2a_0$.

We also have

$$-a_1 - a_2 = -a_0 - a_3 < a_0 - a_3 < a_0 - a_2 = -a_0 - a_1.$$

So, $a_0 - a_3 = -a_0 - a_2$, which is equivalent to $a_3 - a_2 = 2a_0$. Hence, $A = a_0 * \{1, 3, 5, 7\}$.

Finally, let $k \geq 5$, and $|2^{\wedge}_{\pm}A| = 4k - 4$. From Theorem 2.1, it follows that the sumset $h^{\wedge}_{\pm}A$ contains precisely the integers listed in (2.2), (2.3), and (2.4), for $h = 2$. Since $2^{\wedge}A \subseteq [a_0 + a_1, a_{k-2} + a_{k-1}]$ and there are exactly $2k - 3$ integers in (2.2) and (2.3) between $a_0 + a_1$ and $a_{k-2} + a_{k-1}$, Theorem 1.5 implies that the set A is in arithmetic progression.

Again, since

$$-a_0 - a_2 < -a_0 - a_1 < a_0 - a_1,$$

and

$$-a_0 - a_2 < a_0 - a_2 < a_0 - a_1,$$

we have $a_2 - a_1 = 2a_0$. Hence $A = a_0 * \{1, 3, \ldots, 2k-1\}$.
This completes the proof of the theorem. $\square$

Theorem 2.3. *Let A be a set of k (≥ 3) positive integers such that $|k^{\wedge}_{\pm}A| = \binom{k+1}{2} + 1$. Then*

$$A = \begin{cases} \{a_0, a_1, a_0 + a_1\} \text{ with } 0 < a_0 < a_1, & \text{if } k = 3; \\ d * [1, k] \text{ for some positive integer } d, & \text{if } k \geq 4. \end{cases}$$

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$. Let

$$|k^{\wedge}_{\pm}A| = \binom{k+1}{2} + 1.$$

First, let $k = 3$. Then

$$\begin{aligned} 3^{\wedge}_{\pm}A = \{&a_0 + a_1 + a_2, a_0 + a_1 - a_2, a_0 - a_1 + a_2, a_0 - a_1 - a_2, \\ &- a_0 + a_1 + a_2, -a_0 + a_1 - a_2, -a_0 - a_1 + a_2, -a_0 - a_1 - a_2\}, \end{aligned}$$

where

$$\begin{aligned} -a_0 - a_1 - a_2 &< a_0 - a_1 - a_2 < -a_0 + a_1 - a_2 < -a_0 - a_1 + a_2 \\ &< a_0 - a_1 + a_2 < -a_0 + a_1 + a_2 < a_0 + a_1 + a_2. \end{aligned} \tag{2.7}$$

So, if $|3^{\wedge}_{\pm}A| = \binom{4}{2} + 1 = 7$, then $3^{\wedge}_{\pm}A$ contains precisely the seven integers of (2.7). Since

$$-a_0 + a_1 - a_2 < a_0 + a_1 - a_2 < a_0 - a_1 + a_2,$$

we have $a_0 + a_1 - a_2 = -a_0 - a_1 + a_2$, i.e., $a_2 - a_1 = a_0$. Hence, $A = \{a_0, a_1, a_0 + a_1\}$.

Next, let $k = 4$. If $|4^{\wedge}_{\pm}A| = \binom{5}{2} + 1 = 11$, then the sumset $4^{\wedge}_{\pm}A$ contains precisely the following eleven integers written in an increasing order.

$$\begin{aligned} &- a_0 - a_1 - a_2 - a_3 < a_0 - a_1 - a_2 - a_3 < -a_0 + a_1 - a_2 - a_3 \\ &< -a_0 - a_1 + a_2 - a_3 < -a_0 - a_1 - a_2 + a_3 < a_0 - a_1 - a_2 + a_3 \\ &< -a_0 + a_1 - a_2 + a_3 < -a_0 - a_1 + a_2 + a_3 < a_0 - a_1 + a_2 + a_3 \\ &< -a_0 + a_1 + a_2 + a_3 < a_0 + a_1 + a_2 + a_3. \end{aligned} \tag{2.8}$$

Since the sumset $4^{\wedge}_{\pm}A$ is symmetric, from (2.8) it follows that

$$-a_0 - a_1 + a_2 - a_3 = -(-a_0 - a_1 + a_2 + a_3),$$

$$-a_0 - a_1 - a_2 + a_3 = -(-a_0 + a_1 - a_2 + a_3),$$

and

$$a_0 - a_1 - a_2 + a_3 = 0.$$

The above three relations give $a_3 - a_2 = a_2 - a_1 = a_1 - a_0 = a_0$. Hence, $A = a_0 * \{1, 2, 3, 4\}$.

Finally, let $k \geq 5$, and $|k^{\wedge}_{\pm}A| = \binom{k+1}{2} + 1$. Then, $k^{\wedge}_{\pm}A$ contains precisely the integers listed in (2.4), with one more integer $-a_0 - a_1 - \cdots - a_{k-1}$. For $j = 1, 2, \ldots, k-1$, set

$$u_j = a_0 + \sum_{\substack{l=1 \\ l \neq j}}^{k-1} (-a_l) + a_j. \tag{2.9}$$

Clearly,

$$t_{0,1} < u_1 < u_2 < \cdots < u_{k-2} < u_{k-1} = t_{1,0}.$$

So, there are exactly $k-2$ distinct integers in (2.9) between $t_{0,1}$ and $t_{1,0}$. Therefore, by (2.4) and (2.9) we get

$$t_{0,j+1} = u_j,$$

for $j = 1, 2, \ldots, k-2$. This is equivalent to $a_{j+1} - a_j = a_0$, for $j = 1, 2, \ldots, k-2$. That is

$$a_{k-1} - a_{k-2} = \cdots = a_3 - a_2 = a_2 - a_1 = a_0.$$

Again, since $k^{\wedge}_{\pm}A$ is symmetric, we have $-t_{0,0} = t_{k-3,0}$, i.e.,

$$-(-a_0 - a_1 - a_2 - a_3 + a_4 - \cdots - a_{k-1}) = a_0 - a_1 - a_2 + a_3 + a_4 + \cdots + a_{k-1}.$$

Equivalently,

$$a_4 = a_1 + a_2.$$

Since $a_3 - a_2 = a_0$, we get $a_4 - a_3 = a_1 - a_0$. Hence, $A = a_0 * [1, k]$.
This completes the proof of the theorem. □

For $h \geq 3$, we believe that the sumset $h^{\wedge}_{\pm}A$ contains at least $2hk - h^2 + 1$ integers. So, we formulate the following conjecture.

Conjecture 2.4. *Let A be a set of k (≥ 4) positive integers and let $3 \leq h \leq k-1$. Then*

$$|h^{\wedge}_{\pm}A| \geq 2hk - h^2 + 1. \tag{2.10}$$

The lower bound in (2.10) is best possible.

The following example confirms the conjecture in a very special case. Also in Theorem 2.5, we prove Conjecture 2.4 for $h = 3$. Furthermore, we also give the inverse result in this case.

Example 1 (Super increasing sequence). Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $k \geq 6$, $a_0 > 0$, and $a_i > \sum_{j=0}^{i-1} a_j$ for $i = 1, 2, \ldots, k-1$.

Let $h \geq 5$. Clearly, the sumset $h^{\wedge}_{\pm}A$ contains at least $2(hk - h^2) + \binom{h+1}{2} + 1$ integers, which are listed in (2.2), (2.3), and (2.4).

For $j = 1, 2, \ldots, h-2$, consider the integers $-2a_0 + s_{0,j}$. Clearly

$$-2a_0 + s_{0,j} = -a_0 + \sum_{\substack{l=1 \\ l \neq h-j}}^{h} a_l \in h^{\wedge}_{\pm}A,$$

and

$$s_{0,j-1} < -2a_0 + s_{0,j} < s_{0,j}.$$

So, we get $h-2$ extra positive integers $h^{\wedge}_{\pm}A$, which are not already present in (2.2), (2.3), and (2.4). Since

$$-s_{0,j} < -(-2a_0 + s_{0,j}) < -s_{0,j-1},$$

we get $h-2$ further extra integers in $h^{\wedge}_{\pm}A$.

Also, for $j = 2, 3, \ldots, h-3$, consider the integers

$$t_{0,h-j-1} < -t_{j,h-j-2} < -t_{j,h-j-3} < \cdots < -t_{j,0} < -t_{j-1,h-j} < t_{0,h-j}. \tag{2.11}$$

Then, for $j = 2, 3, \ldots, h-3$, we get $h-j$ extra integers. Therefore, we get $3+4+\cdots+(h-2) = \binom{h}{2}-h-2$ more integers in $h^{\wedge}_{\pm}A$ which are listed in (2.11) and never counted before. We also get one more integer, i.e., $-t_{h-3,2}$ such that $t_{0,1} < -t_{h-3,2} < t_{0,2}$. So, we get $2(h-2)+\binom{h}{2}-h-2+1 = \binom{h}{2}+(h-5)$ extra integers. Hence, by and large, we have

$$|h^{\wedge}_{\pm}A| \geq 2hk - h^2 + h - 4 \geq 2hk - h^2 + 1.$$

Theorem 2.5. *Let A be a set of k (≥ 4) positive integers. Then*

$$|3^{\wedge}_{\pm}A| \geq 6k - 8. \tag{2.12}$$

Moreover, if $|3^{\wedge}_{\pm}A| = 6k-8$, then $A = d * \{1, 3, 5, \ldots, 2k-1\}$ for some positive integer d.

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$. From Theorem 2.1, we have $|3^{\wedge}_{\pm}A| \geq 6k - 11$.

Next, we show that there exist at least three extra integers in $3^{\wedge}_{\pm}A$ which are not counted in Theorem 2.1. Consider the following thirteen integers of $3^{\wedge}_{\pm}A$:

$$\begin{aligned}
&-a_1 - a_2 - a_3 < -a_0 - a_2 - a_3 < -a_0 - a_1 - a_3 < -a_0 - a_1 - a_2 \\
&\quad < a_0 - a_1 - a_2 < -a_0 + a_1 - a_2 < -a_0 - a_1 + a_2 < a_0 - a_1 + a_2 \\
&\quad < -a_0 + a_1 + a_2 < a_0 + a_1 + a_2 < a_0 + a_1 + a_3 < a_0 + a_2 + a_3 \\
&\quad < a_1 + a_2 + a_3.
\end{aligned} \tag{2.13}$$

We exhibit at least three extra integers between $-a_1 - a_2 - a_3$ and $a_1 + a_2 + a_3$ in all possible cases.

Case 1: $a_3 - a_2 < a_3 - a_1 < 2a_0$.

We get at least two extra positive integers $-a_0+a_1+a_3$ and $-a_0+a_2+a_3$ which are not present in (2.13) such that

$$-a_0 + a_1 + a_2 < -a_0 + a_1 + a_3 < -a_0 + a_2 + a_3 < a_0 + a_1 + a_2.$$

Case 2: $a_3 - a_2 < 2a_0 < a_3 - a_1$.

Again, we get at least two extra positive integers $-a_0 - a_1 + a_3$ and $-a_0 + a_1 + a_3$ which are not present in (2.13) such that

$$-a_0 - a_1 + a_2 < -a_0 - a_1 + a_3 < a_0 - a_1 + a_2 < -a_0 + a_1 + a_2$$
$$< -a_0 + a_1 + a_3 < a_0 + a_1 + a_2.$$

CASE 3: $2a_0 < a_3 - a_2 < a_3 - a_1$.

We get one extra positive integer $-a_0 + a_1 + a_3$ such that

$$a_0 + a_1 + a_2 < -a_0 + a_1 + a_3 < a_0 + a_1 + a_3.$$

To exhibit one more extra positive integer consider the following subcases: *Subcase (i):* $a_2 - a_1 < 2a_0$. We get one more extra positive integer $-a_0 + a_2 + a_3$ such that

$$a_0 + a_1 + a_2 < -a_0 + a_1 + a_3 < -a_0 + a_2 + a_3 < a_0 + a_1 + a_3.$$

Subcase (ii): $a_2 - a_1 > 2a_0$. We get one more extra positive integer $-a_0 + a_2 + a_3$ such that

$$a_0 + a_1 + a_2 < -a_0 + a_1 + a_3 < a_0 + a_1 + a_3 < -a_0 + a_2 + a_3 < a_0 + a_2 + a_3.$$

Subcase (iii): $a_2 - a_1 = 2a_0$. In this subcase, we get two positive integers $a_0 - a_1 + a_3$ and $a_0 - a_2 + a_3$ such that

$$a_0 - a_1 + a_2 = 3a_0 < a_0 - a_2 + a_3 < a_0 - a_1 + a_3 < -a_0 + a_1 + a_3 < a_0 + a_1 + a_3.$$

But, we already have

$$a_0 - a_1 + a_2 < -a_0 + a_1 + a_2 < a_0 + a_1 + a_2 < -a_0 + a_1 + a_3 < a_0 + a_1 + a_3.$$

Thus, except in the case $a_0 - a_2 + a_3 = -a_0 + a_1 + a_2$ and $a_0 - a_1 + a_3 = a_0 + a_1 + a_2$, we get at least one extra positive integer and hence we are done.

So, let

$$a_0 - a_2 + a_3 = -a_0 + a_1 + a_2,$$

and

$$a_0 - a_1 + a_3 = a_0 + a_1 + a_2.$$

These two relations imply that

$$2(a_2 - a_0) = a_3 - a_1 = a_1 + a_2.$$

Consider the integer $-a_0 - a_2 + a_3$. We have

$$-a_0 - a_1 + a_2 = a_0 < -a_0 - a_2 + a_3 < -a_0 - a_1 + a_3 = -a_0 + a_1 + a_2.$$

If $-a_0 - a_2 + a_3 \neq a_0 - a_1 + a_2$, then we are done, as we get one extra positive integer. Otherwise, let

$$-a_0 - a_2 + a_3 = a_0 - a_1 + a_2.$$

This is equivalent to

$$a_3 - a_2 = a_2 - a_1 + 2a_0 = 4a_0.$$

Therefore, we have

$$a_3 - a_1 = a_3 - a_2 + a_2 - a_1 = 6a_0,$$

and
$$a_2 - a_0 = \frac{1}{2}(a_3 - a_1) = 3a_0.$$
Solving these relations we get $a_1 = 2a_0$, $a_2 = 4a_0$ and $a_3 = 8a_0$. Thus, we get one extra positive integer $-a_1 + a_2 + a_3$ such that
$$-a_0 + a_1 + a_3 = 9a_0 < 10a_0 = -a_1 + a_2 + a_3 < 11a_0 = a_0 + a_1 + a_3.$$
Hence, we get at least two extra positive integers in every case.

Case 4: $a_3 - a_2 < a_3 - a_1 = 2a_0$.
We get at least two extra positive integers $-a_0 - a_1 + a_3$ and $-a_0 + a_1 + a_3$ which are not present in (2.13) such that
$$\begin{aligned}-a_0 - a_1 + a_2 < -a_0 - a_1 + a_3 = a_0 < a_0 - a_1 + a_2 < -a_0 + a_1 + a_2 \\ < -a_0 + a_1 + a_3 < a_0 + a_1 + a_2.\end{aligned}$$

Case 5: $2a_0 = a_3 - a_2 < a_3 - a_1$.
We consider the following three subcases:
Subcase (i): $a_2 - a_1 < 2a_0$. We get at least two extra positive integers $-a_0 - a_2 + a_3$ and $-a_0 + a_2 + a_3$ such that
$$\begin{aligned}-a_0 - a_1 + a_2 < a_0 = -a_0 - a_2 + a_3 < a_0 - a_1 + a_2 < -a_0 + a_1 + a_2 \\ < a_0 + a_1 + a_2 < -a_0 + a_2 + a_3 < a_0 + a_1 + a_3.\end{aligned}$$
Subcase (ii): $a_2 - a_1 > 2a_0$. Again, we get two extra positive integers $-a_0 - a_2 + a_3$ and $-a_0 + a_2 + a_3$ such that
$$\begin{aligned}a_0 + a_1 - a_2 < -a_0 < a_0 = -a_0 - a_2 + a_3 < -a_0 - a_1 + a_2 < a_0 - a_1 + a_2 \\ < -a_0 + a_1 + a_2 < a_0 + a_1 + a_2 < a_0 + a_1 + a_3 \\ < -a_0 + a_2 + a_3 < a_0 + a_2 + a_3.\end{aligned}$$
Subcase (iii): $a_2 - a_1 = 2a_0$. We get one extra positive integer $a_1 - a_2 + a_3$ such that
$$a_0 - a_1 + a_2 = 3a_0 < 2a_0 + a_1 = a_1 - a_2 + a_3 < a_0 + 2a_1 = -a_0 + a_1 + a_2.$$
If $a_1 - a_0 > 2a_0$, then we get one more extra positive integer $a_0 - a_1 + a_3$ such that
$$a_0 - a_1 + a_2 < a_0 - a_1 + a_3 < -2a_0 + a_3 = a_1 - a_2 + a_3 < -a_0 + a_1 + a_2.$$
If $a_1 - a_0 < 2a_0$, then also we get one more extra positive integer $-a_1 + a_2 + a_3$ such that
$$\begin{aligned}a_0 - a_1 + a_2 < a_1 - a_2 + a_3 < -a_0 + a_1 + a_2 < a_0 + a_1 + a_2 \\ < -a_1 + a_2 + a_3 < a_0 + a_1 + a_3.\end{aligned}$$
Let $a_1 - a_0 = 2a_0$. Then, the integer $-a_0 - a_1 + a_2 = a_0$ is positive. So, the inverse of this integer gives one more extra integer with
$$-a_0 + a_1 - a_2 < a_0 + a_1 - a_2 < -a_0 - a_1 + a_2 < a_0 - a_1 + a_2.$$

From the above discussion, we conclude that except in the case $a_1 - a_0 = a_2 - a_1 = a_3 - a_2 = 2a_0$, we get at least two extra positive integers in $3^{\wedge}_{\pm}A$, which are not present in (2.13). Since, the inverses of these integers are negative, we get two more extra integers. So, total we get at least four extra integers in $3^{\wedge}_{\pm}A$, which are not included in (2.13). In case $a_1-a_0 = a_2-a_1 = a_3 - a_2 = 2a_0$, we get at least three extra integers. Therefore, in each case we get at least three extra integers in $3^{\wedge}_{\pm}A$, which are not present in (2.13). Hence, $|3^{\wedge}_{\pm}A| \geq 6k - 8$. This establishes (2.12).

Moreover, if $|3^{\wedge}_{\pm}A| = 6k - 8$, then $a_1 - a_0 = a_2 - a_1 = a_3 - a_2 = 2a_0$.

Now, let $|3^{\wedge}_{\pm}A| = 6k-8$. If $k = 4$, then we are done, as $A = \{a_0, 3a_0, 5a_0, 7a_0\} = a_0 * \{1, 3, 5, 7\}$.

Let $k \geq 5$, and let $A' = A \setminus \{a_0\}$. Then A' is a finite set of $k - 1$ positive integers such that $3^{\wedge}A' \subseteq [a_1 + a_2 + a_3, a_{k-3} + a_{k-2} + a_{k-1}]$. Since $|3^{\wedge}_{\pm}A| = 6k-8$, from the above proof, it follows that $|3^{\wedge}A'| = 3k-11$. Thus, Theorem 1.5 implies that the set A' is in arithmetic progression, i.e.,

$$a_{k-1} - a_{k-2} = a_{k-2} - a_{k-3} = \cdots = a_2 - a_1 = d.$$

Hence

$$A = a_0 * \{1, 3, 5, \ldots, 2k - 1\}.$$

This completes the proof of the theorem. □

Now, we formulate the following conjecture for the inverse problem.

Conjecture 2.6. *Let A be a set of k (≥ 4) positive integers and let $3 \leq h \leq k-1$. If $|h^{\wedge}_{\pm}A| = 2hk - h^2 + 1$, then $A = d * \{1, 3, \ldots, 2k - 1\}$ for some positive integer d.*

Remark: Theorem 2.5 confirms Conjecture 2.6 for $h = 3$.

3. A contains nonnegative integers with $0 \in A$

Theorem 3.1. *Let A be a set of k nonnegative integers with $0 \in A$. Let $1 \leq h \leq k$ be an integer. Then*

$$|h^{\wedge}_{\pm}A| \geq 2(hk - h^2) + \binom{h}{2} + 1. \tag{3.1}$$

This lower bound is best possible for $h = 1$, 2 and k.

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 = a_0 < a_1 < \cdots < a_{k-1}$. If $h = 1$, then by (2.2) and (2.3), the sumset $h^{\wedge}_{\pm}A$ contains at least $hk - h^2$ positive integers and 0. Hence, including their inverses, $h^{\wedge}_{\pm}A$ contains at least $2(hk - h^2) + 1$ integers. This establishes (3.1) for $h = 1$.

So, let $h \geq 2$. Then, again from (2.2) and (2.3), it follows that $h^{\wedge}_{\pm}A$ contains at least $hk - h^2 + 1$ positive integers and hence including their inverses, $h^{\wedge}_{\pm}A$ contains at least $2(hk - h^2 + 1)$ integers.

Now, since $a_0 = 0$, from (2.4) it follows that $-s_{0,0} = t_{0,0}$, $t_{h-1,0} = s_{0,0}$ and $t_{i,h-i-1} = t_{i+1,0}$ for $i = 0, 1, \dots, h-2$. Thus, we get $\binom{h}{2} - 1$ extra integers in $h^{\wedge}_{\pm}A$ from the list (2.4). Hence

$$|h^{\wedge}_{\pm}A| \geq 2(hk - h^2) + \binom{h}{2} + 1.$$

Next, we show that the lower bound in (3.1) is best possible for $h = 1$, 2, and k.

If $h = 1$, then for any finite set A of k nonnegative integers with $0 \in A$, we have $|1^{\wedge}_{\pm}A| = 2k - 1 = 2(hk - h^2) + \binom{h}{2} + 1$.

Now, let $h = 2$ and $A = [0, k-1]$. Then

$$2^{\wedge}_{\pm}A = [-(2k-3), (2k-3)] \setminus \{0\}.$$

So, $|2^{\wedge}_{\pm}A| = 4k - 6 = 2(hk - h^2) + \binom{h}{2} + 1$.

Finally, let $h = k$ and $A = [0, k-1]$. Then, it is easy to see that $k^{\wedge}_{\pm}A$ contains either all odd integers or all even integers. Since $k^{\wedge}_{\pm}A \subseteq \left[-\binom{k}{2}, \binom{k}{2}\right]$, we get

$$|k^{\wedge}_{\pm}A| \leq \binom{k}{2} + 1.$$

This together with (3.1) give $|k^{\wedge}_{\pm}A| = \binom{k}{2} + 1 = 2(hk - h^2) + \binom{h}{2} + 1$. This completes the proof of the theorem. □

We now give inverse results for $h = 2$ and $h = k$ in Theorem 3.2 and Theorem 3.3, respectively. For $h = 2$, any set of the form $\{0, a\}$ with $a > 0$ is an extremal set. For $h = 3$, any set of the form $\{0, a, b\}$ with $0 < a < b$ is an extremal set.

Theorem 3.2. *Let A be a set of k (≥ 3) nonnegative integers with $0 \in A$ such that $|2^{\wedge}_{\pm}A| = 4k - 6$. Then, $A = d * [0, k-1]$ for some positive integer d.*

Proof. Let $A = \{0, a_1, a_2, \dots, a_{k-1}\}$, where $0 < a_1 < a_2 < \cdots < a_{k-1}$. Let

$$|2^{\wedge}_{\pm}A| = 4k - 6.$$

First, let $k = 3$. Then

$$2^{\wedge}_{\pm}A = \{a_1, -a_1, a_2, -a_2, a_1 + a_2, a_1 - a_2, -a_1 + a_2, -a_1 - a_2\},$$

where

$$-a_1 - a_2 < -a_2 < -a_1 < a_1 < a_2 < a_1 + a_2. \tag{3.2}$$

If $|2^{\wedge}_{\pm}A| = 4k - 6 = 6$, then $2^{\wedge}_{\pm}A$ contains precisely the integers listed in (3.2). Since

$$-a_2 < a_1 - a_2 < a_1,$$

we have from (3.2) that $a_1 - a_2 = -a_1$, i.e., $a_2 - a_1 = a_1$. Hence, $A = \{0, a_1, 2a_1\} = a_1 * [0, 2]$.

Now, let $k = 4$. Then

$$\begin{aligned} 2^{\wedge}_{\pm}A = \{&a_1, -a_1, a_2, -a_2, a_3, -a_3, a_1 + a_2, a_1 - a_2, -a_1 + a_2, -a_1 - a_2, \\ &a_1 + a_3, a_1 - a_3, -a_1 + a_3, -a_1 - a_3, a_2 + a_3, a_2 - a_3, -a_2 + a_3, \\ &- a_2 - a_3\}, \end{aligned}$$

where

$$\begin{aligned} -a_2 - a_3 < -a_1 - a_3 < -a_1 - a_2 < -a_2 < -a_1 < a_1 < a_2 \\ < a_1 + a_2 < a_1 + a_3 < a_2 + a_3. \end{aligned} \tag{3.3}$$

If $|2^{\wedge}_{\pm}A| = 4k - 6 = 10$, then $2^{\wedge}_{\pm}A$ contains precisely the integers listed in (3.3). Since

$$a_2 < a_3 < a_1 + a_3,$$

we have from (3.3) that $a_3 = a_1 + a_2$, equivalently, $a_3 - a_2 = a_1$.

Similarly,

$$-a_2 < a_1 - a_2 < a_1$$

implies that $a_1 - a_2 = -a_1$, equivalently, $a_2 - a_1 = a_1$. Hence, $A = \{0, a_1, 2a_1, 3a_1\} = a_1 * [0, 3]$.

Finally, let $k \geq 5$ and $|2^{\wedge}_{\pm}A| = 4k - 6$. From Theorem 1.4, we know that $|2^{\wedge}A| \geq 2k - 3$. Since $2^{\wedge}A \cap 2^{\wedge}(-A) = \emptyset$, we have $|2^{\wedge}A| = 2k - 3$. So, by Theorem 1.5, the set A is in arithmetic progression with the common difference $a_{k-1} - a_{k-2} = a_{k-2} - a_{k-3} = \cdots = a_1 - a_0 = a_1$. Hence, $A = a_1 * [0, k - 1]$.

This completes the proof of the theorem. □

Theorem 3.3. *Let A be a set of k (≥ 4) nonnegative integers with $0 \in A$ such that $|k^{\wedge}_{\pm}A| = \binom{k}{2} + 1$. Then*

$$A = \begin{cases} \{0, a_1, a_2, a_1 + a_2\} \text{ with } 0 < a_1 < a_2, & \text{if } k = 4; \\ d * [0, k-1] \text{ for some positive integer } d, & \text{if } k \geq 5. \end{cases}$$

Proof. Let $A = \{0, a_1, a_2, \ldots, a_{k-1}\}$, where $0 < a_1 < a_2 < \cdots < a_{k-1}$. Let

$$|k^{\wedge}_{\pm}A| = \binom{k}{2} + 1.$$

First, let $k = 4$. Then

$$\begin{aligned} 4^{\wedge}_{\pm}A = \{&a_1 + a_2 + a_3, a_1 + a_2 - a_3, a_1 - a_2 + a_3, a_1 - a_2 - a_3, \\ &- a_1 + a_2 + a_3, -a_1 + a_2 - a_3, -a_1 - a_2 + a_3, -a_1 - a_2 - a_3\}, \end{aligned}$$

where

$$\begin{aligned} -a_1 - a_2 - a_3 < a_1 - a_2 - a_3 < -a_1 + a_2 - a_3 < -a_1 - a_2 + a_3 \\ < a_1 - a_2 + a_3 < -a_1 + a_2 + a_3 < a_1 + a_2 + a_3. \end{aligned} \tag{3.4}$$

If $|4^{\wedge}_{\pm}A| = \binom{4}{2} + 1 = 7$, then $4^{\wedge}_{\pm}A$ contains precisely the above seven integers in (3.4). Since

$$-a_1 + a_2 - a_3 < a_1 + a_2 - a_3 < a_1 - a_2 + a_3,$$

we have $a_1 + a_2 - a_3 = -a_1 - a_2 + a_3$, i.e., $a_3 - a_2 = a_1$. Hence, $A = \{0, a_1, a_2, a_1 + a_2\}$.

Now, let $k \geq 5$ and let $A' = A \setminus \{0\}$. So, A' is a set of $k-1$ positive integers with $k^{\wedge}_{\pm}A = (k-1)^{\wedge}_{\pm}A'$. Since $|(k-1)^{\wedge}_{\pm}A'| = |k^{\wedge}_{\pm}A| = \binom{k}{2} + 1$, Theorem 2.3 implies that the set A' is in arithmetic progression with the common difference a_1, the smallest element in A'. Hence $A = a_1 * \{0, 1, 2, \ldots, k-1\} = a_1 * [0, k-1]$.
This completes the proof of the theorem. □

For $h \geq 3$, we believe that the sumset $h^{\wedge}_{\pm}A$ contains at least $2hk - h(h+1) + 1$ integers. So, we formulate the following conjecture.

Conjecture 3.4. *Let A be a set of k (≥ 5) nonnegative integers with $0 \in A$. Let $3 \leq h \leq k-1$ be a positive integer. Then*

$$|h^{\wedge}_{\pm}A| \geq 2hk - h(h+1) + 1. \tag{3.5}$$

The lower bound in (3.5) is best possible.

We confirm Conjecture 3.4 for $h = 3$ in the next theorem. Furthermore, we also give the inverse result in this case.

Theorem 3.5. *Let A be a set of k (≥ 5) nonnegative integers with $0 \in A$. Then*

$$|3^{\wedge}_{\pm}A| \geq 6k - 11. \tag{3.6}$$

Furthermore, if $|3^{\wedge}_{\pm}A| = 6k - 11$, then $A = d * [0, k-1]$.

Proof. Let $A = \{0, a_1, a_2, \ldots, a_{k-1}\}$, where $0 < a_1 < a_2 < \cdots < a_{k-1}$. From Theorem 3.1, it follows that $|3^{\wedge}_{\pm}A| \geq 6k - 14$.

Next, we show that there exists at least three extra integers in $3^{\wedge}_{\pm}A$ which are not counted in Theorem 3.1. Consider the following twelve integers of $3^{\wedge}_{\pm}A$:

$$\begin{aligned} -a_1 - a_2 - a_4 &< -a_1 - a_2 - a_3 < -a_2 - a_3 < -a_1 - a_3 < -a_1 - a_2 \\ &< a_1 - a_2 < -a_1 + a_2 < a_1 + a_2 < a_1 + a_3 < a_2 + a_3 \\ &< a_1 + a_2 + a_3 < a_1 + a_2 + a_4. \end{aligned} \tag{3.7}$$

We exhibit at least three extra integers between $-a_1 - a_2 - a_4$ and $a_1 + a_2 + a_4$ in all cases.

Case 1: $a_3 - a_2 < a_1$.

We have

$$a_1 - a_2 < -a_2 + a_3 < -a_1 + a_3 < a_1 + a_2,$$

and

$$a_1 - a_2 < -a_1 + a_2 < -a_1 + a_3.$$

If $-a_2 + a_3 \neq -a_1 + a_2$, then we get two extra positive integers $-a_2 + a_3$ and $-a_1 + a_3$.

So, let $-a_2 + a_3 = -a_1 + a_2$. If $a_3 - a_1 < a_1$, then we get two extra positive integers $-a_1 + a_3$ and $-a_1 + a_2 + a_3$ such that

$$-a_1 + a_2 < -a_1 + a_3 < -a_1 + a_2 + a_3 < a_1 + a_2.$$

If $a_3 - a_1 > a_1$, then we get two extra positive integers $-a_1 + a_3$ and $-a_1 + a_2 + a_3$ such that

$$-a_1 + a_2 < -a_1 + a_3 < a_1 + a_2 < -a_1 + a_2 + a_3 < a_1 + a_3.$$

If $a_3 - a_1 = a_1$, then also we get two extra positive integers $-a_1 + a_3$ and $a_1 - a_2 + a_3$ such that

$$-a_1 + a_2 < -a_1 + a_3 < a_1 - a_2 + a_3 < a_1 + a_2.$$

Case 2: $a_3 - a_2 = a_1$.

By the similar arguments to Case 1, we get two extra positive integers $-a_2 + a_3$ and $-a_1 + a_3$ except in the situation $-a_2 + a_3 = -a_1 + a_2$.
So, let $-a_2 + a_3 = -a_1 + a_2$. Then, we get an extra positive integer $-a_1 + a_3$ such that

$$-a_1 + a_2 < -a_1 + a_3 < a_1 + a_2.$$

Further, we get one more extra integer $-a_1 - a_2 + a_3 = 0$ such that

$$a_1 - a_2 < -a_1 - a_2 + a_3 < -a_1 + a_2.$$

Case 3: $a_3 - a_2 > a_1$. So, $a_3 - a_1 > a_1$.

Subcase (i): $-a_1 + a_3 < a_1 + a_2$. We get two extra positive integers $-a_2 + a_3$ and $-a_1 + a_3$ which are not included in (3.7) except in the situation $-a_2 + a_3 = -a_1 + a_2$.
So let $-a_2 + a_3 = -a_1 + a_2$. Then also we get two extra positive integers $-a_1 + a_3$ and $-a_1 + a_2 + a_3$ such that

$$-a_1 + a_2 < -a_1 + a_3 < a_1 + a_2 < a_1 + a_3 < -a_1 + a_2 + a_3 < a_2 + a_3.$$

Subcase (ii): $-a_1 + a_3 > a_1 + a_2$. We get an extra positive integer $-a_1 + a_3$ such that

$$a_1 + a_2 < -a_1 + a_3 < a_1 + a_3.$$

If $-a_2 + a_3 = -a_1 + a_2$, then we get an extra positive integer $-a_1 - a_2 + a_3$ such that

$$a_1 - a_2 < -a_1 - a_2 + a_3 < -a_1 + a_2.$$

If $-a_2 + a_3 = a_1 + a_2$, then also we get an extra positive integer $-a_1 - a_2 + a_3$ such that

$$-a_1 + a_2 < -a_1 - a_2 + a_3 < a_1 + a_2.$$

If neither $-a_2 + a_3 = -a_1 + a_2$ nor $-a_2 + a_3 = a_1 + a_2$, then also we get one more extra positive integer $-a_2 + a_3$.
Subcase (iii): $-a_1 + a_3 = a_1 + a_2$. If $-a_2 + a_3 < -a_1 + a_2$, then we get two extra positive integers $-a_2 + a_3$ and $-a_1 - a_2 + a_3$ such that

$$a_1 - a_2 < -a_1 - a_2 + a_3 < -a_2 + a_3 < -a_1 + a_2.$$

If $-a_2 + a_3 = -a_1 + a_2$, then $a_2 = 3a_1$ and $a_3 = 5a_1$. We get two extra positive integers $-a_1 - a_2 + a_3$ and $a_1 - a_2 + a_3$ such that

$$a_1 - a_2 < -a_1 - a_2 + a_3 < -a_1 + a_2 < a_1 - a_2 + a_3 < a_1 + a_2.$$

Now, let $-a_2 + a_3 > -a_1 + a_2$. Then, we get an extra positive integer $-a_2 + a_3$ such that

$$-a_1 + a_2 < -a_2 + a_3 < -a_1 + a_3 = a_1 + a_2.$$

If $a_2 - a_1 \neq a_1$, then $-a_1 + a_2 + a_3 \neq a_1 + a_3$. So, we get one more extra positive integer $-a_1 + a_2 + a_3$ such that

$$a_1 + a_2 = -a_1 + a_3 < -a_1 + a_2 + a_3 < a_2 + a_3.$$

Let $a_2 - a_1 = a_1$. So, $a_2 = 2a_1$ and $a_3 = 4a_1$. If $a_4 - a_3 > a_1$, then we get an extra positive integer $a_2 + a_4$ such that

$$a_1 + a_2 + a_3 < a_2 + a_4 < a_1 + a_2 + a_4.$$

If $a_4 - a_3 < a_1$, then we get an extra positive integer $a_2 + a_4$ such that

$$a_2 + a_3 < a_2 + a_4 < a_1 + a_2 + a_3.$$

If $a_4 - a_3 = a_1$, then also we get an extra positive integer $a_1 - a_2 + a_4$ such that

$$a_1 + a_2 < a_1 - a_2 + a_4 < a_1 + a_3.$$

Thus, in Cases 1 and 3, we get at least two extra positive integers. As the inverses of these extra integers are also in $3^{\wedge}_{\pm}A$, so we get at least four extra integers in these two cases, which are not present in (3.7). In Case 2, we get at least three extra integers. Therefore, in each case we get at least three extra integers in $3^{\wedge}_{\pm}A$ which are not present in (3.7). Hence

$$|3^{\wedge}_{\pm}A| \geq 6k - 11.$$

This establishes (3.6).

Now, let $|3^{\wedge}_{\pm}A| = 6k - 11$. From the above discussion it is clear that we are in Case 2 with $a_3 - a_2 = a_2 - a_1 = a_1$.

Let $A' = A \setminus \{0\}$. Then, A' is a finite set of $k - 1$ positive integers such that $3^{\wedge}A' \subseteq [a_1 + a_2 + a_3, a_{k-3} + a_{k-2} + a_{k-1}]$. Since $|3^{\wedge}_{\pm}A| = 6k - 11$, it follows from the above discussion that $|3^{\wedge}A'| = 3k - 11$. Thus, Theorem 1.5 implies that the set A' is in arithmetic progression, i.e.,

$$a_{k-1} - a_{k-2} = a_{k-2} - a_{k-3} = \cdots = a_2 - a_1 = d.$$

Hence, $A = a_1 * \{0, 1, 2, \ldots, k - 1\} = a_1 * [0, k - 1]$.
This completes the proof of the theorem. $\square$

We observe in the following theorem that the minimum requirement of five elements in the set A in Theorem 3.5 is the best possible.

Theorem 3.6. *Let A be a set of four nonnegative integers with $0 \in A$. Then*

$$|3^{\wedge}_{\pm}A| \geq 12. \tag{3.8}$$

Furthermore, if $|3^{\wedge}_{\pm}A| = 12$, then $A = d * \{0, 1, 2, 4\}$.

Proof. Let $A = \{0, a_1, a_2, a_3\}$, where $0 < a_1 < a_2 < a_3$. From Theorem 3.1, it follows that $3^{\wedge}_{\pm}A$ contains at least the following ten integers.

$$\begin{aligned} -a_1 - a_2 - a_3 < -a_2 - a_3 < -a_1 - a_3 < -a_1 - a_2 < a_1 - a_2 \\ < -a_1 + a_2 < a_1 + a_2 < a_1 + a_3 < a_2 + a_3 < a_1 + a_2 + a_3. \end{aligned} \tag{3.9}$$

Again, from the proof of Theorem 3.5, it follows that the sumset $3^{\wedge}_{\pm}A$ contains at least three extra integers, except when $a_2 = 2a_1$ and $a_3 = 4a_1$. In the case $a_2 = 2a_1$ and $a_3 = 4a_1$, we get two extra integers. Therefore, we always get two extra integers in $3^{\wedge}_{\pm}A$ which are not present in (3.9). Hence, $|3^{\wedge}_{\pm}A| \geq 12$. This establishes (3.8). Moreover, if $|3^{\wedge}_{\pm}A| = 12$, then we have $a_2 = 2a_1$ and $a_3 = 4a_1$. Hence $A = a_1 * \{0, 1, 2, 4\}$. □

Finally, we formulate the following conjecture for the inverse problem.

Conjecture 3.7. *Let A be a set of k (≥ 5) nonnegative integers with $0 \in A$. Let $3 \leq h \leq k - 1$ be an integer. If $|h^{\wedge}_{\pm}A| = 2hk - h(h + 1) + 1$, then $A = d * [0, k - 1]$ for some positive integer d.*

Remark: Theorem 3.5 confirms Conjecture 3.7 for $h = 3$.

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References

1. N. Alon, M. B. Nathanson, and I. Z. Ruzsa, *Adding distinct congruence classes modulo a prime*, Amer. Math. Monthly **102** (1995), 250–255.
2. ______, *The polynomial method and restricted sums of congruence classes*, J. Number Theory **56** (1996), 404–417.
3. B. Bajnok, *Additive combinatorics: A menu of research problems*, Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 2018.
4. B. Bajnok and R. Matzke, *The minimum size of signed sumsets*, Electron. J. Comb. **22** (2015), #P2.50.
5. ______, *On the minimum size of signed sumsets in elementary abelian groups*, J. Number Theory **159** (2015), 384–401.
6. B. Bajnok and I. Ruzsa, *The independence number of a subset of an abelian group*, Integers **3** (2003), #A2.
7. J. Bhanja and R. K. Pandey, *Direct and inverse theorems on signed sumsets of integers*, J. Number Theory **196** (2019), 340–352.
8. A. L. Cauchy, *Recherches sur les nombres*, J. École Polytech. **9** (1813), 99–116.
9. H. Davenport, *On the addition of residue classes*, J. Lond. Math. Soc. **10** (1935), 30–32.
10. J. A. Dias da Silva and Y. O. Hamidoune, *Cyclic spaces for Grassmann derivatives and additive theory*, Bull. London Math. Soc. **26** (1994), 140–146.
11. S. Eliahou and M. Kervaire, *Restricted sumsets in finite vector spaces: the case $p = 3$*, Integers **1** (2001), #A2.
12. S. Eliahou, M. Kervaire, and A. Plagne, *Optimally small sumsets in finite abelian groups*, J. Number Theory **101** (2003), 338–348.
13. P. Erdős and H. Heilbronn, *On the addition of residue classes (mod p)*, Acta Arith. **9** (1964), 149–159.

14. G. A. Freiman, *Foundations of structural theory of set sddition*, vol. **37**, Translations of Mathematical Monographs, American Mathematical Society, Provedience, R.I., 1973.
15. B. Klopsch and V.F. Lev, *How long does it take to generate a group*?, J. Algebra **261** (2003), 145–171.
16. ———, *Generating abelian groups by addition only*, Forum Math. **21** (2009), 23–41.
17. H. B. Mann, *Addition Theorems: the addition theorems of group theory and number theory*, Wiley-Interscience, New York, 1965.
18. M. B. Nathanson, *Inverse theorems for subset sums*, Trans. Amer. Math. Soc. **347** (1995), 1409–1418.
19. ———, *Additive Number Theory: inverse problems and the geometry of sumsets*, Springer, 1996.
20. A. Plagne, *Optimally small sumsets in groups, I. The supersmall sumset property, the $\mu_G^{(k)}$ and the $\nu_G^{(k)}$ functions*, Unif. Distrib. Theory **1** (2006), 27–44.
21. T. Tao and V. H. Vu, *Additive combinatorics*, Cambridge Studies in Advanced Mathematics, vol. **105**, Cambridge, 2010.

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