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BOUNDS OF CHARACTERISTIC POLYNOMIALS OF REGULAR MATROIDS

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Abstract. A regular chain group N is the set of integral vectors orthogonal to rows of a matrix representing a regular matroid, i.e., a totally unimodular matrix. Introducing canonical forms of an equivalence relation generated by N and a special basis of N, we improve several results about polynomials counting elements of N and find new bounds and formulas for these polynomials.

1. Introduction

Regular matroids are representable by totally unimodular matrices. Other equivalent characterizations are in [22, Chapter 13] or [8, 23, 28, 27]. A regular chain group N on a finite set E consists of integral vectors (called chains) indexed by E and orthogonal to rows of a totally unimodular matrix (i.e., a representative matrix of a regular matroid). Suppose that $Q(N;k)$ denotes the number of chains from N with values from $\{\pm 1, \ldots, \pm(k-1)\}$. Moreover, if the coordinates with negative values are indexed by elements of $X \subseteq E$ (resp. are considered mod k), denote this number by $Q(N, X; k)$ (resp. $P(N;k)$). It is known that $Q(N;k)$ and $P(N;k)$ are polynomials in variable k and are sums of $Q(N, X; k)$ where X runs through different subsets of E (determined by an equivalence relation $\sim$ on the powerset of E). Furthermore $Q(N, X; k)$ is an Ehrhard polynomial of an integral polytope and $P(N;k)$ is the characteristic polynomial of the dual of the matroid accompanied with N.

Basic properties of regular chain groups are surveyed in the second section. In the last section we introduce a canonical representation of equivalence classes of the relation $\sim$, characterize k for which all nontrivial $Q(N, X; k)$ are nonzero and find a basis satisfying a triangular condition and consisting of chains such that all coordinates with negative values are covered by a

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fixed set. Using these results we establish inequalities between polynomials $Q(N;k)$ and $P(N;k)$ and introduce formulas expressing growth of $Q(N;k)$, $P(N;k)$, and the Tutte polynomial of regular matroids.

2. Preliminaries

In this section we recall some basic properties of regular matroids and regular chain groups presented in [2, 22, 24, 25, 26, 28].

Throughout this paper, E denotes a finite nonempty set. The collection of mappings from E to a set S is denoted by S^E. If R is a ring, the elements of R^E are considered as vectors indexed by E and we will use the notation $f+g$, $-f$, and sf for $f,g \in R^E$ and $s \in R$. A *chain* on E (over R, or simply an *R-chain*) is $f \in R^E$ and the *support* of f is $\sigma(f) = \{e \in E; f(e) \neq 0\}$. We say that f is *proper* if $\sigma(f) = E$. The *zero chain* (denoted by 0) has null support. Given $X \subseteq E$ and $f \in R^E$, let $\rho_X(f) \in R^E$ be defined so that for each $e \in E$,

$$[\rho_X(f)](e) = \begin{cases} -f(e) & \text{if } e \in X, \\ f(e) & \text{if } e \notin X, \end{cases}$$

and let $\rho_X(Y) = \{\rho_X(f), f \in Y\}$ for any $Y \subseteq R^E$. Furthermore, define by $f^{\setminus X} \in R^{E\setminus X}$ such that $f^{\setminus X}(e) = f(e)$ for each $e \in E \setminus X$.

A matroid M on E of rank $r(M)$ is *regular* if there exists an $r \times n$ ($r = r(M)$, $n = |E|$) totally unimodular matrix D (called a *representative* matrix of M) such that independent sets of M correspond to independent sets of columns of D. For any basis B of M, D can be transformed to a form $(I_r|U)$ such that I_r corresponds to B and U is totally unimodular. The dual of M is a regular matroid M^* with a representative matrix $(-U^T|I_{n-r})$ (where I_{n-r} corresponds to $E \setminus B$).

By a *regular chain group* N on E (*associated with* D) we mean a set of chains on E over $\mathbb{Z}$ that are orthogonal to each row of D (i.e., are integral combinations of rows of a representative matrix of M^*). The set of chains orthogonal to every chain of N is a chain group called *orthogonal* to N and denoted by $N^\perp$ (clearly, $N^\perp$ is the set of integral combinations of rows of D). By *rank* of N we mean $r(N) = n - r(M) = r^*(M)$. Then $r(N^\perp) = n - r(N) = r(M)$.

Throughout this paper, we always assume that a regular chain group N is associated with a matrix $D = D(N)$ representing a matroid $M = M(N)$.

For any $X \subseteq E$, define by

$$\begin{aligned} N{-}X &= \left\{ f^{\setminus X}; f \in N,\ \sigma(f) \cap X = \emptyset \right\}, \\ N/X &= \left\{ f^{\setminus X}; f \in N \right\}. \end{aligned} \tag{2.1}$$

Clearly, $M(N{-}X) = M - X$ and $D(N{-}X)$ arises from $D(N)$ after deleting the columns corresponding to X. Furthermore $(N{-}X)^\perp = N^\perp/X$, $(N/X)^\perp = N^\perp{-}X$, and $M(N/X) = M/X$.

A chain f of N is *elementary* if there is no nonzero g of N such that $\sigma(g) \subset \sigma(f)$. An elementary chain f is called a *primitive* chain of N if the coefficients of f are restricted to the values 0, 1, and -1. (Notice that the set of supports of primitive chains of N is the set of circuits of $M(N)$.) We say that a chain g *conforms* to a chain f, if $g(e)$ and $f(e)$ are nonzero and have the same sign for each $e \in E$ such that $g(e) \neq 0$. By [25, Section 6.1]),

(2.2) every chain f of N can be expressed as a sum of primitive chains in N that conform to f.

Let A be an Abelian group with additive notation. We shall consider A as a (right) $\mathbb{Z}$-module such that the scalar multiplication $a \cdot z$ of $a \in A$ by $z \in \mathbb{Z}$ is equal to 0 if $z = 0$, $\sum_1^z a$ if $z > 0$, and $\sum_1^{-z}(-a)$ if $z < 0$. Similarly if $a \in A$ and $f \in \mathbb{Z}^E$ then define $a \cdot f \in A^E$ so that $(a \cdot f)(e) = a \cdot f(e)$ for each $e \in E$. If N is a regular chain group on E, define by

$$A(N) = \left\{ \sum_{i=1}^{m} a_i \cdot f_i; a_i \in A, f_i \in N, m \geq 1 \right\},$$
$$A[N] = \{ f \in A(N);\, \sigma(f) = E \}.$$

Notice that $A(N) = N$ if $A = \mathbb{Z}$. By [2, Proposition 1],

(2.3) $g \in A^E$ is from $A(N)$ if and only if for each $f \in N^{\perp}$, $\sum_{e \in E} g(e) \cdot f(e) = 0$.

Let $P(N, A) = |A[N]|$ and denote by $P(N; k) = P(N, \mathbb{Z}_k) = |\mathbb{Z}_k[N]|$.

We will denote by $\mathbb{Z}_+$ the set of positive integers. Define by

$$N_k = \{f \in N; 1 \leq |f(e)| \leq k-1 \text{ for each } e \in E\},$$
$$N_k(X) = \{f \in N_k;\, \rho_X(f) \in \mathbb{Z}_+^E\}, \quad X \subseteq E.$$

Let $Q(N; k) = |N_k|$ and $Q(N, X; k) = |N_k(X)|$, $X \subseteq E$. Clearly, N_k is equal to the (disjoint) union of $N_k(X)$ where X runs through all subsets of E.

We denote by $\mathcal{P}(E)$ the set of subsets of E. For any $X \subseteq E$ denote by $\chi_X \in \mathbb{Z}^E$ such that $\chi_X(e) = 1$ (resp. $\chi_X(e) = 0$) for each $e \in E$ (resp. $e \in E \setminus X$).

Define the equivalence relation $\sim$ on $\mathcal{P}(E)$ by: $X, X' \in \mathcal{P}(E)$ satisfies $X \sim X'$ if and only if $\chi_X - \chi_{X'} \in N$. The set of the equivalence classes will be denoted by $\mathcal{P}(E)/\sim$. By [2, Proposition 3(a)],

(2.4) for each $\mathcal{X} \in \mathcal{P}(E)/\sim$ and $X, X' \in \mathcal{X}$, $Q(N, X; k) = Q(N, X'; k)$.

Thus we can define $Q(N, \mathcal{X}; k)$ to be equal $Q(N, X; k)$ for some $X \in \mathcal{X}$.

We say that $X \subseteq E$ is *positive* if $\rho_X(N) \cap \mathbb{Z}_+^E \neq \emptyset$. We say $\mathcal{X} \in \mathcal{P}(E)/\sim$ is *positive* whenever some element of $\mathcal{X}$ is positive (because by (2.4) every element of $\mathcal{X}$ will be positive). We shall denote the set of positive elements of $\mathcal{P}(E)$ (resp. of $\mathcal{P}(E)/\sim$) by $O(N)^+$ (resp. $\mathcal{O}(N)^+$). By Propositions 3, 7, 10, and 15 from [2] we have

(2.5)

$$\begin{aligned}
&\text{if } X \in O(N)^+, \text{ then } Q(N,X;k) \text{ is a polynomial in } k \text{ of degree } r(N),\\
&|O(N)^+| = (-1)^{r(N)}P(N;-1),\\
&|\mathcal{O}(N)^+| = (-1)^{r(N)}P(N;0),\\
&P(N;k) = \sum_{\mathcal{X}\in\mathcal{P}(E)/\sim} Q(N,\mathcal{X};k) = \sum_{\mathcal{X}\in\mathcal{O}(N)^+} Q(N,\mathcal{X};k).
\end{aligned}$$

For a chain f of N we shall call it a *k-chain* if $0 \le f(e) \le k-1$ for each $e \in E$. For any $X \subseteq E$, denote by $\overline{N}_k(X)$ the set of all k-chains of $\rho_X(N)$ and define $\overline{Q}(N,X;k) = |\overline{N}_k(X)|$. Furthermore, $\overline{Q}(N,X;k) = \overline{Q}(N,X';k)$ if $X, X' \in \mathcal{X} \in \mathcal{P}(E)/\sim$, and we can define $\overline{Q}(N,\mathcal{X};k) = \overline{Q}(N,X;k)$ for some $X \in \mathcal{X}$. By Propositions 9 and 12–14 from [2] we have

$$\begin{aligned}
Q(N,X;-k) &= (-1)^{r(N)}\overline{Q}(N,X;k+1), \text{ for each } X \in O(N)^+,\\
|\mathcal{X}| &= \overline{Q}(N,\mathcal{X};2), \text{ for each } \mathcal{X} \in \mathcal{O}(N)^+,\\
P(N;k) &= \sum_{X\in O(N)^+} \frac{Q(N,X;k)}{\overline{Q}(N,X;2)},\\
P(N;-k) &= (-1)^{r(N)} \sum_{X\in O(N)^+} \frac{\overline{Q}(N,X;k+1)}{\overline{Q}(N,X;2)}\\
&= (-1)^{r(N)} \sum_{\mathcal{X}\in\mathcal{O}(N)^+} \overline{Q}(N,\mathcal{X};k+1).
\end{aligned} \tag{2.6}$$

The reciprocity law expressed in the first row of (2.6) follows from the fact that $Q(N,X;k)$ is an Ehrhart polynomial of an integral polytope (for more details see [2, Section III.2] and [12, Theorem 5.1, Corollary B.1, p.23]). From definition of $Q(N;k)$, (2.4), and (2.6) we have

$$\begin{aligned}
Q(N;k) &= \sum_{X\in O(N)^+} Q(N,X;k) = \sum_{\mathcal{X}\in\mathcal{O}(N)^+} Q(N,\mathcal{X};k)\overline{Q}(N,\mathcal{X};2),\\
Q(N;-k) &= (-1)^{r(N)} \sum_{X\in O(N)^+} \overline{Q}(N,X;k+1)\\
&= (-1)^{r(N)} \sum_{\mathcal{X}\in\mathcal{O}(N)^+} \overline{Q}(N,\mathcal{X};k+1)\overline{Q}(N,\mathcal{X};2).
\end{aligned} \tag{2.7}$$

3. Properties of chain polynomials

Let N be a regular chain group on E. Then e is called a *loop* (resp. *isthmus*) of N if $\chi_e \in N$ (resp. $\chi_e \in N^{\perp}$), i.e., if e is a loop (resp. isthmus) of $M(N)$.

Lemma 3.1. *$P(N;k) = P(N,A)$ for any Abelian group of order k. If N has an isthmus, then $P(N;k) = 0$ and $P(N;k)$ has degree $r(N)$ otherwise. Furthermore,*

$$\begin{aligned} P(N;k) &= (k-1)P(N-e;k) && \text{if } e \in E \text{ is a loop in } N, \\ P(N;k) &= P(N/e;k) - P(N-e;k) && \text{if } e \in E \text{ is not a loop in } N. \end{aligned}$$

Proof. We use induction on $|E|$. Formally we allow $E = \emptyset$ and define $P(N;k) = P(N,A) = 1$ in this case. If e is a loop (resp. isthmus) of N then from (2.3), $P(N,A) = (k-1)P(N-e,A)$ (resp. $P(N,A) = P(N/e,A) - P(N-e,A)$), $N-e = N/e$, $r(N-e) = r(N/e) = r(N)-1$, and the statement holds true by the induction hypothesis.

If e is neither an isthmus nor a loop of N, then there exists $f \in N^\perp$ such that $f(e) \neq 0$ and $f \neq \chi_e$. Given $g \in A^{E\setminus e}$ and $x \in A$ let $g_x \in A^E$ be defined so that $g_x^{\setminus e} = g$ and $g_x(e) = x$. If $g \in A[N/e]$, then by (2.1) there exists $a \in A$ such that $g_a \in A(N)$. By (2.3), g_a must be orthogonal to f, whence a is unique. Furthermore, if $a = 0$ (resp. $a \neq 0$) then by (2.1), $g \in A[N-e]$ (resp. $g_a \in A[N]$), i.e., $g \mapsto g_a$ is a bijection from $A[N/e]$ to the disjoint union of $A[N]$ and $A[N-e]$. Thus $P(N/e,A) = P(N,A) + P(N-e,A)$, $r(N/e) = r(N) = r(N-e)+1$, and the statement holds true by the induction hypothesis. □

By Lemma 3.1, $P(N;k)$ is the *characteristic polynomial* of $M(N)^*$ (see [1, 29]). Thus the characteristic polynomial of regular matroid $M(N)^*$ counts the number of proper A-chains for any Abelian group A of order k.

Let $H \subseteq E$ and ℓ be a labeling of elements of E by pairwise different integers. For any $f \in N$ denote by $e_f \in E$ such that

$$\ell(e_f) = \min\{\ell(e); e \in \sigma(f)\}.$$

We say that f is *(H,ℓ)-compatible* if $f(e_f)$ and $(\chi_H - \chi_{E\setminus H})(e_f)$ have the same sign. Denote by $O_{H,\ell}(N)$ the subset of $O(N)^+$ consisting of sets X such that each $c \in \overline{N}_2(X)$ is (H,ℓ)-compatible.

Lemma 3.2. *$|O_{H,\ell}(N) \cap \mathcal{X}| = 1$ for each $\mathcal{X} \in \mathcal{O}(N)^+$.*

Proof. For each $X \in O(N^+)$ define

$$\ell_X = \min\{\ell(e_c); c \in \overline{N}_2(X),\ c(e_c) \neq (\chi_H - \chi_{E\setminus H})(e_c)\}$$

and write $\ell_X = \infty$ if each $c \in \overline{N}_2(X)$ is (H,ℓ)-compatible. If $\mathcal{X} \in \mathcal{O}(N)^+$, choose $X \in \mathcal{X}$ with the maximal ℓ_X. If $\ell_X < \infty$, there exists $c \in \overline{N}_2(X)$ such that $\ell(e_c) = \ell_X$ and (applying the definition of $\sim$) choose $X' \in \mathcal{X}$ such that $\chi_{X'} - \chi_X = c$. Thus $-c \in \overline{N}_2(X')$ is (H,ℓ)-compatible, whence $\ell_{X'} > \ell_X$, a contradiction with the choice of X. Therefore $\ell_X = \infty$, i.e., $X \in O_{H,\ell}(G)$. On the other hand if $X'' \in \mathcal{X}$ and $X'' \neq X$, then $c'' = \chi_{X''} - \chi_X \in \overline{N}_2(X)$ is (H,ℓ)-compatible and $-c \in \overline{N}_2(X'')$ is not (H,ℓ)-compatible. Therefore $O_{H,\ell}(G) \cap \mathcal{X} = \{X\}$. □

Notice that $\mathbb{Z}_k[N,X] \neq \emptyset$ for each $X \in O_{H,\ell}(N)$ if and only if $k \geq |\sigma(\tilde{g})|$ where $\tilde{g}$ is a primitive chain in $N^\perp$ with the maximal $|\sigma(\tilde{g})|$. This follows from the following statement.

Proposition 3.3. *$Q(N,X;k) \neq 0$ for each $X \in O(N)^+$ if and only if $k \geq |\sigma(g)|$ for each primitive chain g of $N^\perp$.*

Proof. The proof is trivial if $O(N)^+ = \emptyset$. Assume that $O(N)^+ \neq \emptyset$.

For $f \in \mathbb{Z}^E$, denote by $\sigma^+(f)$ ($\sigma^-(f)$) the number of positive (negative) coefficients of f, i.e., $\sigma^+(f) + \sigma^-(f) = |\sigma(f)|$. Assume that $D(N)$ has the form $(I_r|U)$ and let D_X be the matrix arising from $(I_r|U)$ after changing the signs of all entries from the columns corresponding to X. Then $\rho_X(N_k(X))$ is the set of chains on E that are orthogonal to all rows of D_X and have coordinates from 1 to $k-1$. In other words, $|N_k(X)| = |\rho_X(N_k(X))| \neq 0$ if and only if the integral polyhedron

$$\mathbf{P}_k = \{\mathbf{y}; \mathbf{1} \leq \mathbf{y} \leq \mathbf{k{-}1}, D_X\mathbf{y} = \mathbf{0}\}$$

is nonempty. With respect to the construction of D_X, (arising from $(I_r|U)$ after changing the signs of all entries from the columns corresponding to X), $\mathbf{u}$, $\mathbf{v}$ are $\{0, \pm 1\}$-vectors satisfying $\mathbf{u}D_X = \mathbf{v}$ if and only if $\mathbf{v}$ is from the set

$$C_1 = \{\rho_X(f);\ f \in N^\perp,\ |f(e)| \leq 1 \text{ for each } e \in E\}.$$

Thus by [24, Corollary 21.3a] (see also [14, Section 3]), $\mathbf{P}_k \neq \emptyset$ if and only if $\sigma^-(c) \leq (k{-}1)\sigma^+(c)$ for every $c \in C_1$. Hence by (2.2), $\mathbf{P}_k \neq \emptyset$ if and only if $\sigma^-(\rho_X(g)) \leq (k{-}1)\sigma^+(\rho_X(g))$ for each primitive chain g in $N^\perp$ (notice that by (2.3), $\sigma^+(\rho_X(g))$, $\sigma^-(\rho_X(g))$ must be nonzero for $X \in O(N)^+$). Thus if $\tilde{g}$ is a primitive chain in $N^\perp$ with the maximal $|\sigma(\tilde{g})|$, then $N_k(X) \neq \emptyset$ for each $k \geq |\sigma(\tilde{g})|$ and $X \in O(N)^+$.

Denote by $\tilde{E} = \sigma(\tilde{g})$, $E' = E \setminus \tilde{E}$, $Y = \{e \in E; \tilde{g}(e) < 0\}$, and $\tilde{N} = N/E'$. Then $\tilde{N}^\perp = N^\perp{-}E'$, whence by (2.1), $\tilde{g}^{\setminus E'}$ is the unique primitive chain in $\tilde{N}^\perp$ and thus by (2.3), $\{(\rho_Y(\chi_e{-}\chi_{e'}))^{\setminus E'}; e, e' \in \tilde{E}, e \neq e')\} \subseteq \tilde{N}$. Choose $\tilde{e} \in \tilde{E}$ and define by $\tilde{X} = Y\setminus\tilde{e} \cup \tilde{e}\setminus Y$. Then $(\rho_{\tilde{X}}(\tilde{g}))^{\setminus E'} = (\chi_{\tilde{E}\setminus\tilde{e}}{-}\chi_{\tilde{e}})^{\setminus E'}$,

$$\{(\rho_{\tilde{X}}(\chi_{\{\tilde{e},e\}}))^{\setminus E'}; e \in \tilde{E}\setminus\tilde{e}\} \subseteq \tilde{N},$$

and $\tilde{f} = \rho_{\tilde{X}}\left(\sum_{e \in \tilde{E}\setminus\tilde{e}} \chi_{\{\tilde{e},e\}}\right)$ satisfies $\sigma(\tilde{f}) = \tilde{E}$ and $\tilde{f}^{\setminus E'} \in \tilde{N}$. Hence $\tilde{X} \in O(\tilde{N})^+$. Consider a proper chain $\bar{f} \in N$ (that exists because $O(N)^+ \neq \emptyset$). Then $f' = \bar{f} + \left(1{+}\sum_{e \in E} |\bar{f}(e)|\right)\tilde{f}$ is proper and $(\rho_{\tilde{X}}(f'))^{\setminus E'} \in \mathbb{Z}_+^{\tilde{E}}$, whence $X' = \{e \in E; f'(e) < 0\} \in O(N)^+$ and $X' \cap \tilde{E} = \tilde{X}$. If $f \in N_k(X')$, then $\rho_{X'}(f) \in \mathbb{Z}_+^E$ and by (2.3), $\rho_{X'}(f)$ is orthogonal to $\rho_{X'}(\tilde{g}) = \chi_{\tilde{E}\setminus\tilde{e}}{-}\chi_{\tilde{e}}$, i.e., $|f(\tilde{e})| \geq |\tilde{E}\setminus\tilde{e}| = |\sigma(\tilde{g})|{-}1$. Thus $N_k(X') = \emptyset$ for each $k \leq |\sigma(\tilde{g})| - 1$, concluding the proof. □

We say that a sequence of primitive chains $c_1, \ldots, c_r \in \overline{N}_2(X)$ ($r = r(N)$, $X \in O(N)^+$) is a *triangular X-basis* of N if there exist $e_1, \ldots, e_r \in E$ such that $e_i \in \sigma(c_i)$, $e_i \notin \sigma(c_j)$ for each $i, j \in \{1, \ldots, r\}$, $i < j$. Clearly, any triangular X-basis is a basis of the linear hull of N. Therefore for each $f \in N$ there are numbers $z_1, \dots, z_r$ such that $f = \sum_{i=1}^r z_i c_i$ and thus $z_1 = f(e_1)$, $z_2 = f(e_2) - z_1, \dots, z_r = f(e_r) - z_1 - \dots - z_{r-1}$ are integral.

Lemma 3.4. *For each regular chain group N on E and $X \in O(N)^+$ there exists a triangular X-basis of N.*

Proof. Choose a primitive chain $c_1 \in \overline{N}_2(X)$ and $e_1 \in E$ such that $c_1(e_1) = 1$. Let $E' \subseteq E$ be defined so that $e_1 \in E'$ and $E' \setminus e_1$ is the set of isthmuses in $N - e_1$. Notice that N has no isthmus because $O(N)^+ \neq \emptyset$. Thus by (2.1), $N^\perp$ must contain a chain of form $\chi_{e_1} \pm \chi_e$ for each $e \in E' \setminus e_1$. Since each chain from $N^\perp$ is orthogonal to c_1, we have $\rho_X(\chi_{e_1} - \chi_e) \in N^\perp$. Then by (2.3), $[\rho_X(f)](e_1) = [\rho_X(f)](e)$ for every $e \in E' \setminus e_1$ and $f \in N$, whence $r(N - E') = r(N) - 1$. Thus applying the induction hypothesis on $N - E'$ and $X \setminus E' \in O(N - E')^+$ we can extend c_1 and e_1 into a triangular X-basis of N (considering chains from $N - E'$ as chains from N after setting the undefined coordinates to be 0). □

We claim that for each regular chain group N and $X \in O(N)^+$,

$$r(N) + 1 \le \overline{Q}(N, X; 2) \le 2^{r(N)}. \tag{3.1}$$

Clearly, $\overline{N}_2(X)$ contains the zero chain and at least $r(N)$ nonzero chains by Lemma 3.4. This implies the left hand side. The right hand side follows from the fact that each $c \in \overline{N}_2(X)$ is a linear combination of rows of a representative matrix of $M(N)^*$ having form $(-U^T | I_{r(N)})$ such that $I_{r(N)}$ corresponds to a base B^* of $M(N)^*$, $|B^*| = r(N)$, and that $c(e) \in \{0, [\rho_X(\chi_E)](e)\}$ for every $e \in B^*$.

For example let g be a chain on E such that $g(\tilde{e}) = -1$ for a fixed $\tilde{e} \in E$ and $g(e) = 1$ for $e \in E$, $e \neq \tilde{e}$. Consider N so that g is the unique primitive chain of $N^\perp$. By (2.3), $\{\pm\rho_{\tilde{e}}(\chi_e - \chi_{e'}), e, e' \in E, e \neq e'\}$ is the set of primitive chains of N, whence $\overline{N}_2(\{\tilde{e}\}) = \{\chi_\emptyset\} \cup \{\chi_{e,\tilde{e}}; e \in E, e \neq \tilde{e}\}$. Thus $\overline{Q}(N, \{\tilde{e}\}; 2) = |E| = r(N) + 1$, i.e., the left hand side of (3.1) is tense.

If N has $|E|$ loops, then $\overline{N}_2(X) = \{\rho_X(\chi_Y); Y \subseteq E\}$, whence $\overline{Q}(N, X; 2) = 2^{|E|} = 2^{r(N)}$ for each $X \subseteq E$. Thus the right hand side of (3.1) is tense.

Lemma 3.5. *For each regular chain group N and each integer $k > 0$,*

$$(r(N) + 1)P(N; k) \le Q(N; k) \le 2^{r(N)} P(N; k).$$

Proof. The proof follows from the third row of (2.6), the first row of (2.7), and (3.1). □

Proposition 3.6. *For each regular chain group N on E and $k \ge 2$,*

$$\begin{aligned} Q(N, X; k + 1) &\ge Q(N, X; k)\, k(k-1)^{-1}, \\ P(N; k + 1) &\ge P(N; k)\, k(k-1)^{-1}, \\ Q(N; k + 1) &\ge Q(N; k)\, k(k-1)^{-1}. \end{aligned}$$

Proof. Let $X \in O(N)^+$. For each $f \in N_k(X)$ and each $c \in \overline{N}_2(X)$, c not equal to the zero chain, there exists a unique integer $r > 0$ such that $f + rc \in$

$N_{k+1}(X)\backslash N_k(X)$. We shall call this chain an (f,c)-*lift* (shortly a *lift*). In this way we can construct $s_X = Q(N,X;k)\overline{Q}(N,X;2)$ (not necessary different) lifts. On the other hand each $f' \in N_{k+1}(X) \setminus N_k(X)$ could be an $(f'-ic,c)$-lift for $i = 1,\dots,s$, $0 \le s \le k-1$ ($s = 0$ if $f'-ic \notin N_k(X)$ for each $i \ge 1$). Thus f' can be constructed as a lift at most $s'_X = (k-1)\overline{Q}(N,X;2)$ times. Hence

$$\begin{aligned} Q(N,X;k{+}1) - Q(N,X;k) &= |N_{k+1}(X) \setminus N_k(X)| \\ &\ge s_X/s'_X = Q(N,X;k)(k{-}1)^{-1} \end{aligned}$$

This implies the first row of the formula for $X \in O(N)^+$. If $X \notin O(N)^+$, the first row of the formula is trivial because then $Q(N,X;k{+}1) = Q(N,X;k) = 0$. Hence the second and the third rows follow from the third row of (2.6) and the first row of (2.7), respectively. □

Proposition 3.7. *For each regular chain group N on E and $k \ge 2$,*

$$\begin{array}{lll} Q(N,X;k+1) > Q(N,X;k) & \textit{if} & Q(N,X;k+1) > 0, \\ P(N;k+1) > P(N;k) & \textit{if} & P(N;k+1) > 0, \\ Q(N;k+1) > Q(N;k) + r(N) & \textit{if} & Q(N;k+1) > 0. \end{array}$$

Proof. The first two rows follows from Proposition 3.6 and the fact that $k(k-1)^{-1} > 1$. The last row follows from the first one, the third row of (2.6), and (3.1). □

Propositions 3.6 and 3.7 generalize [2, Proposition 6]. Polynomial $P(N;k)$ (resp. $Q(N;k)$) corresponds to a flow (resp. integral flow) polynomial if $N(M)$ is a graphic matroid and corresponds to a tension (resp. integral tension) polynomial if $N(M)$ is a congraphic matroid. Flow and tension polynomials (and their integral variants) were studied in [15, 16] where we proved Lemmas 3.1, 3.5, and Propositions 3.6, 3.7 for flows and tensions on graphs. Similar versions of Lemmas 3.2, 3.4, and Proposition 3.3 were proved in [16, 17, 18, 20]. Several other generalizations of flow and tension polynomials are presented in [3, 4, 5, 6, 7, 9, 10, 11, 13].

We can generalize Propositions 3.6 and 3.7 for $\overline{Q}(N,X;k)$ and the Tutte polynomial of regular matroids. Assume that N is a regular chain group on E and $X \subseteq E$. Using (2.1) and the definitions of $\overline{N}_k(X)$ and $N_k(X)$, it is easy to check that $\overline{N}_k(X)$ equals the disjoint union of $[N{-}Y]_k(X\backslash Y)$ where Y runs through the powerset of E. Therefore by the definitions of $\overline{Q}(N,X;k)$ and $Q(N,X;k)$,

$$\overline{Q}(N,X;k) = \sum_{Y \subseteq E} Q(N{-}Y, X\backslash Y;k). \tag{3.2}$$

By Proposition 3.6, for each $k \ge 2$ we have

$$\sum_{Y \subseteq E} Q(N{-}Y, X\backslash Y;k+1) \ge \sum_{Y \subseteq E} Q(N{-}Y, X\backslash Y;k)\, k(k{-}1)^{-1},$$

whence by (3.2)

$$\overline{Q}(N,X;k+1) \geq \overline{Q}(N,X;k)\,k(k-1)^{-1}, \tag{3.3}$$

and thus

$$\overline{Q}(N,X;k+1) > \overline{Q}(N,X;k) \quad \text{if} \quad \overline{Q}(N,X;k+1) > 0. \tag{3.4}$$

The Tutte polynomial $T(M;x,y)$ of a matroid M on E is (see cf. [9, 21])

$$T(M;x,y) = \sum_{X\subseteq E} (x-1)^{r(E)-r(X)}(y-1)^{|X|-r(X)}.$$

If $M = M(N)$ is regular, ℓ is a labeling of elements of E by the numbers $1,\dots,|E|$, $H \subseteq E$, and $x,y \geq 2$ are integers, then by [21, Equation 16],

$$T(M(N);x,y) = \sum_{X\subseteq E}\left(\sum_{Y\in O_{H\setminus X,\ell}(N^{\perp}-X)} \overline{Q}(N^{\perp}-X,Y;x)\right)\left(\sum_{Y'\in O_{H\cap X,\ell}(N|X)} \overline{Q}(N|X,Y';y)\right).$$

Applying (3.3) on the right hand side of this equation we get that

$$\begin{aligned}
&T(M(N);x+1,y) \geq T(M(N);x,y)\,x(x-1)^{-1},\\
&T(M(N);x,y+1) \geq T(M(N);x,y)\,y(y-1)^{-1},\\
&T(M(N);x+1,y) > T(M(N);x,y) \text{ if } T(M(N);x+1,y) > 0,\\
&T(M(N);x,y+1) > T(M(N);x,y) \text{ if } T(M(N);x,y+1) > 0,
\end{aligned} \tag{3.5}$$

for any regular chain group N and any pair of integers $x,y \geq 2$.

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