BOUNDS OF CHARACTERISTIC POLYNOMIALS OF
REGULAR MATROIDS

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ABSTRACT. A regular chain group $N$ is the set of integral vectors orthogonal to rows of a matrix representing a regular matroid, i.e., a totally unimodular matrix. Introducing canonical forms of an equivalence relation generated by $N$ and a special basis of $N$, we improve several results about polynomials counting elements of $N$ and find new bounds and formulas for these polynomials.

1. INTRODUCTION

Regular matroids are representable by totally unimodular matrices. Other equivalent characterizations are in [22, Chapter 13] or [8, 23, 28, 27]. A regular chain group $N$ on a finite set $E$ consists of integral vectors (called chains) indexed by $E$ and orthogonal to rows of a totally unimodular matrix (i.e., a representative matrix of a regular matroid). Suppose that $Q(N; k)$ denotes the number of chains from $N$ with values from $\{\pm 1, \ldots, \pm (k-1)\}$. Moreover, if the coordinates with negative values are indexed by elements of $X \subseteq E$ (resp. are considered mod $k$), denote this number by $Q(N, X; k)$ (resp. $P(N; k)$). It is known that $Q(N; k)$ and $P(N; k)$ are polynomials in variable $k$ and are sums of $Q(N, X; k)$ where $X$ runs through different subsets of $E$ (determined by an equivalence relation $\sim$ on the powerset of $E$). Furthermore $Q(N, X; k)$ is an Ehrhart polynomial of an integral polytope and $P(N; k)$ is the characteristic polynomial of the dual of the matroid accompanied with $N$.

Basic properties of regular chain groups are surveyed in the second section. In the last section we introduce a canonical representation of equivalence classes of the relation $\sim$, characterize $k$ for which all nontrivial $Q(N, X; k)$ are nonzero and find a basis satisfying a triangular condition and consisting of chains such that all coordinates with negative values are covered by a
fixed set. Using these results we establish inequalities between polynomials $Q(N; k)$ and $P(N; k)$ and introduce formulas expressing growth of $Q(N; k)$, $P(N; k)$, and the Tutte polynomial of regular matroids.

2. Preliminaries

In this section we recall some basic properties of regular matroids and regular chain groups presented in [2, 22, 24, 25, 26, 28].

Throughout this paper, $E$ denotes a finite nonempty set. The collection of mappings from $E$ to a set $S$ is denoted by $S^E$. If $R$ is a ring, the elements of $R^E$ are considered as vectors indexed by $E$ and we will use the notation $f + g$, $-f$, and $sf$ for $f, g \in R^E$ and $s \in R$. A \textit{chain} on $E$ (over $R$, or simply an \textit{R-chain}) is $f \in R^E$ and the \textit{support} of $f$ is $\sigma(f) = \{e \in E; f(e) \neq 0\}$. We say that $f$ is \textit{proper} if $\sigma(f) = E$. The \textit{zero chain} (denoted by $0$) has null support.

Given $X \subseteq E$ and $f \in R^E$, let $\rho_X(f) \in R^E$ be defined so that for each $e \in E$,

$$\rho_X(f)(e) = \begin{cases} -f(e) & \text{if } e \in X, \\ f(e) & \text{if } e \notin X, \end{cases}$$

and let $\rho_X(Y) = \{\rho_X(f), f \in Y\}$ for any $Y \subseteq R^E$. Furthermore, define by $f^\perp \in R^{E \setminus X}$ such that $f^\perp(e) = f(e)$ for each $e \in E \setminus X$.

A matroid $M$ on $E$ of rank $r(M)$ is \textit{regular} if there exists an $r \times n$ ($r = r(M), n = |E|$) totally unimodular matrix $D$ (called a \textit{representative} matrix of $M$) such that independent sets of $M$ correspond to independent sets of columns of $D$. For any basis $B$ of $M$, $D$ can be transformed to a form $(I_r | U)$ such that $I_r$ corresponds to $B$ and $U$ is totally unimodular. The dual of $M$ is a regular matroid $M^*$ with a representative matrix $(-U^T | I_{n-r})$ (where $I_{n-r}$ corresponds to $E \setminus B$).

By a \textit{regular chain group} $N$ on $E$ (\textit{associated with $D$}) we mean a set of chains on $E$ over $\mathbb{Z}$ that are orthogonal to each row of $D$ (i.e., are integral combinations of rows of a representative matrix of $M^*$). The set of chains orthogonal to every chain of $N$ is a chain group called \textit{orthogonal} to $N$ and denoted by $N^\perp$ (clearly, $N^\perp$ is the set of integral combinations of rows of $D$). By \textit{rank} of $N$ we mean $r(N) = n - r(M) = r^*(M)$. Then $r(N^\perp) = n - r(N) = r(M)$.

Throughout this paper, we always assume that a regular chain group $N$ is associated with a matrix $D = D(N)$ representing a matroid $M = M(N)$.

For any $X \subseteq E$, define by

$$N-X = \left\{ f^\perp; f \in N, \sigma(f) \cap X = \emptyset \right\},$$

$$N/X = \left\{ f^\perp; f \in N \right\}. \tag{2.1}$$

Clearly, $M(N-X) = M - X$ and $D(N-X)$ arises from $D(N)$ after deleting the columns corresponding to $X$. Furthermore $(N-X)^\perp = N^\perp/X$, $(N/X)^\perp = N^\perp - X$, and $M(N/X) = M/X$. 
A chain \( f \) of \( N \) is called elementary if there is no nonzero \( g \) of \( N \) such that \( \sigma(g) \subset \sigma(f) \). An elementary chain \( f \) is called a primitive chain of \( N \) if the coefficients of \( f \) are restricted to the values 0, 1, and \(-1\). (Notice that the set of supports of primitive chains of \( N \) is the set of circuits of \( M(N) \).) We say that a chain \( g \) conforms to a chain \( f \), if \( g(e) \) and \( f(e) \) are nonzero and have the same sign for each \( e \in E \) such that \( g(e) \neq 0 \). By [25, Section 6.1],

\[
\text{(2.2)} \quad \text{every chain } f \text{ of } N \text{ can be expressed as a sum of primitive chains in } N \text{ that conform to } f.
\]

Let \( A \) be an Abelian group with additive notation. We shall consider \( A \) as a (right) \( \mathbb{Z} \)-module such that the scalar multiplication \( a \cdot z \) of \( a \in A \) by \( z \in \mathbb{Z} \) is equal to 0 if \( z = 0 \), \( \sum_{i=1}^{n} a_i \) if \( z > 0 \), and \( \sum_{i=1}^{n} (-a_i) \) if \( z < 0 \). Similarly if \( a \in A \) and \( f \in \mathbb{Z}^E \) then define \( a \cdot f \in A^E \) so that \( (a \cdot f)(e) = a \cdot f(e) \) for each \( e \in E \). If \( N \) is a regular chain group on \( E \), define by

\[
A(N) = \left\{ \sum_{i=1}^{m} a_i \cdot f_i; a_i \in A, f_i \in N, m \geq 1 \right\},
\]

\[
A[N] = \{ f \in A(N); \sigma(f) = E \}.
\]

Notice that \( A(N) = N \) if \( A = \mathbb{Z} \). By [2, Proposition 1],

\[
\text{(2.3)} \quad g \in A^E \text{ is from } A(N) \text{ if and only if for each } f \in N^+, \, \sum_{e \in E} g(e) \cdot f(e) = 0.
\]

Let \( P(N, A) = |A[N]| \) and denote by \( P(N; k) = P(N, \mathbb{Z}_k) = |\mathbb{Z}_k[N]| \).

We will denote by \( \mathbb{Z}_+ \) the set of positive integers. Define by

\[
N_k = \{ f \in N; 1 \leq |f(e)| \leq k - 1 \text{ for each } e \in E \},
\]

\[
N_k(X) = \{ f \in N_k; \rho_X(f) \in \mathbb{Z}_+^E \}, \quad X \subseteq E.
\]

Let \( Q(N; k) = |N_k| \) and \( Q(N, X; k) = |N_k(X)| \), \( X \subseteq E \). Clearly, \( N_k \) is equal to the (disjoint) union of \( N_k(X) \) where \( X \) runs through all subsets of \( E \).

We denote by \( \mathcal{P}(E) \) the set of subsets of \( E \). For any \( X \subseteq E \) denote by \( \chi_X \in \mathbb{Z}^E \) such that \( \chi_X(e) = 1 \) (resp. \( \chi_X(e) = 0 \)) for each \( e \in E \) (resp. \( e \in E \setminus X \)).

Define the equivalence relation \( \sim \) on \( \mathcal{P}(E) \) by: \( X, X' \in \mathcal{P}(E) \) satisfies \( X \sim X' \) if and only if \( \chi_X - \chi_{X'} \in N_k \), \( X \subseteq E \). The set of the equivalence classes will be denoted by \( \mathcal{P}(E)/\sim \). By [2, Proposition 3(a)],

\[
\text{(2.4)} \quad \text{for each } \mathcal{X} \in \mathcal{P}(E)/\sim \, \text{ and } X, X' \in \mathcal{X}, \, Q(N, X; k) = Q(N, X'; k).
\]

Thus we can define \( Q(N, \mathcal{X}; k) \) to be equal \( Q(N, X; k) \) for some \( X \in \mathcal{X} \).

We say that \( X \subseteq E \) is positive if \( \rho_X \cap \mathbb{Z}_+^E \neq \emptyset \). We say \( \mathcal{X} \in \mathcal{P}(E)/\sim \) is positive whenever some element of \( \mathcal{X} \) is positive (because by (2.4) every element of \( \mathcal{X} \) will be positive). We shall denote the set of positive
elements of \( \mathcal{P}(E) \) (resp. of \( \mathcal{P}(E)/\sim \)) by \( O(N)^+ \) (resp. \( O(N)^+ \)). By Propositions 3, 7, 10, and 15 from [2] we have
\[
\text{(2.5)}
\]
if \( X \in O(N)^+ \), then \( Q(N, X; k) \) is a polynomial in \( k \) of degree \( r(N) \),
\[
|O(N)^+| = (-1)^{r(N)}P(N; -1),
\]
\[
|O(N)^+| = (-1)^{r(N)}P(N; 0),
\]
\[
P(N; k) = \sum_{\mathcal{X} \in \mathcal{O}(N)^+} Q(N, \mathcal{X}; k) = \sum_{\mathcal{X} \in \mathcal{O}(N)^+} Q(N, \mathcal{X}; k).
\]

For a chain \( f \) of \( N \) we shall call it a \( k \)-chain if \( 0 \leq f(e) \leq k - 1 \) for each \( e \in E \). For any \( X \subseteq E \), denote by \( \overline{N}_k(X) \) the set of all \( k \)-chains of \( \rho_X(N) \) and define \( \overline{Q}(N, X; k) = |N_k(X)| \). Furthermore, \( \overline{Q}(N, X; k) = \overline{Q}(N, X'; k) \) if \( X, X' \in \mathcal{X} \in \mathcal{P}(E)/\sim \), and we can define \( \overline{Q}(N, \mathcal{X}; k) = \overline{Q}(N, X; k) \) for some \( X \in \mathcal{X} \). By Propositions 9 and 12–14 from [2] we have
\[
Q(N, X; -k) = (-1)^{r(N)}\overline{Q}(N, X; k + 1), \text{ for each } X \in O(N)^+,
\]
\[
|\mathcal{X}'| = \overline{Q}(N, \mathcal{X}; 2), \text{ for each } \mathcal{X} \in \mathcal{O}(N)^+,
\]
\[
P(N; k) = \sum_{\mathcal{X} \in \mathcal{O}(N)^+} \frac{Q(N, X; k)}{\overline{Q}(N, X; 2)},
\]
\[
(2.6)
\]
\[
P(N; -k) = (-1)^{r(N)} \sum_{\mathcal{X} \in \mathcal{O}(N)^+} \frac{\overline{Q}(N, X; k + 1)}{\overline{Q}(N, X; 2)}
\]
\[
= (-1)^{r(N)} \sum_{\mathcal{X} \in \mathcal{O}(N)^+} \overline{Q}(N, \mathcal{X}; k + 1).
\]

The reciprocity law expressed in the first row of (2.6) follows from the fact that \( Q(N, X; k) \) is an Ehrhart polynomial of an integral polytope (for more details see [2, Section III.2] and [12, Theorem 5.1, Corollary B.1, p.23]). From definition of \( Q(N; k) \), (2.4), and (2.6) we have
\[
Q(N; k) = \sum_{\mathcal{X} \in \mathcal{O}(N)^+} Q(N, X; k) = \sum_{\mathcal{X} \in \mathcal{O}(N)^+} Q(N, X; k)\overline{Q}(N, \mathcal{X}; 2),
\]
\[
(2.7)
\]
\[
Q(N; -k) = (-1)^{r(N)} \sum_{\mathcal{X} \in \mathcal{O}(N)^+} \overline{Q}(N, X; k + 1)
\]
\[
= (-1)^{r(N)} \sum_{\mathcal{X} \in \mathcal{O}(N)^+} \overline{Q}(N, \mathcal{X}; k + 1)\overline{Q}(N, \mathcal{X}; 2).
\]

3. \textbf{Properties of chain polynomials}

Let \( N \) be a regular chain group on \( E \). Then \( e \) is called a \textit{loop} (resp. \textit{isthmus}) of \( N \) if \( \chi_e \in N \) (resp. \( \chi_e \in N^\perp \)), i.e., if \( e \) is a loop (resp. isthmus) of \( M(N) \).
Lemma 3.1. $P(N; k) = P(N, A)$ for any Abelian group of order $k$. If $N$ has an isthmus, then $P(N; k) = 0$ and $P(N; k)$ has degree $r(N)$ otherwise. Furthermore,

$$P(N; k) = (k-1)P(N-e; k) \quad \text{if } e \in E \text{ is a loop in } N,$$

$$P(N; k) = P(N/e; k) - P(N-e; k) \quad \text{if } e \in E \text{ is not a loop in } N.$$  

Proof. We use induction on $|E|$. Formally we allow $E = \emptyset$ and define $P(N; k) = P(N, A) = 1$ in this case. If $e$ is a loop (resp. isthmus) of $N$ then from (2.3), $P(N, A) = (k-1)P(N-e, A)$ (resp. $P(N, A) = P(N/e, A) - P(N-e, A)$), $N-e = N/e, r(N-e) = r(N/e) = r(N)-1$, and the statement holds true by the induction hypothesis.

If $e$ is neither an isthmus nor a loop of $N$, then there exists $f \in N^\perp$ such that $f(e) \neq 0$ and $f \neq \chi_e$. Given $g \in A^E \setminus e$ and $x \in A$ let $g_x \in A^E$ be defined so that $g_x(e) = g$ and $g_x(e) = x$. If $g \in A[N/e]$, then by (2.1) there exists $a \in A$ such that $g_a \in A(N)$. By (2.3), $g_a$ must be orthogonal to $f$, whence $a$ is unique. Furthermore, if $a = 0$ (resp. $a \neq 0$) then by (2.1), $g \in A[N-e]$ (resp. $g_a \in A[N]$), i.e., $g \mapsto g_a$ is a bijection from $A[N/e]$ to the disjoint union of $A[N]$ and $A[N-e]$. Thus $P(N/e, A) = P(N, A) + P(N-e, A)$, $r(N/e) = r(N) = r(N-e)+1$, and the statement holds true by the induction hypothesis. \hfill \Box

By Lemma 3.1, $P(N; k)$ is the characteristic polynomial of $M(N)^*$ (see [1, 29]). Thus the characteristic polynomial of regular matroid $M(N)^*$ counts the number of proper $A$-chains for any Abelian group $A$ of order $k$.

Let $H \subseteq E$ and $\ell$ be a labeling of elements of $E$ by pairwise different integers. For any $f \in N$ denote by $e_f \in E$ such that

$$\ell(e_f) = \min\{\ell(e); c \in \sigma(f)\}.$$  

We say that $f$ is $(H, \ell)$-compatible if $f(e_f)$ and $(\chi_H - \chi_{E\setminus H})(e_f)$ have the same sign. Denote by $O_{H,\ell}(N)$ the subset of $O(N)^+$ consisting of sets $X$ such that each $c \in \overline{N}_2(X)$ is $(H, \ell)$-compatible.

Lemma 3.2. $|O_{H,\ell}(N) \cap \mathcal{X}| = 1$ for each $\mathcal{X} \in \mathcal{O}(N)^+$.  

Proof. For each $X \in \mathcal{O}(N)^+$ define

$$\ell_X = \min\{\ell(c_e); c \in \overline{N}_2(X), c(e_c) \neq (\chi_H - \chi_{E\setminus H})(e_c)\}$$  

and write $\ell_X = \infty$ if each $c \in \overline{N}_2(X)$ is $(H, \ell)$-compatible. If $\mathcal{X} \in \mathcal{O}(N)^+$, choose $X \in \mathcal{X}$ with the maximal $\ell_X$. If $\ell_X < \infty$, there exists $c \in \overline{N}_2(X)$ such that $\ell(c_e) = \ell_X$ and (applying the definition of $\sim$) choose $X' \in \mathcal{X}$ such that $\chi_{X'} - \chi_X = c$. Thus $-c \in \overline{N}_2(X')$ is $(H, \ell)$-compatible, whence $\ell_X < \ell_X$, a contradiction with the choice of $X$. Therefore $\ell_X = \infty$, i.e., $X \in O_{H,\ell}(G)$. On the other hand if $X'' \in \mathcal{X}$ and $X'' \neq X$, then $c'' = \chi_{X''} - \chi_X \in \overline{N}_2(X)$ is $(H, \ell)$-compatible and $-c \in \overline{N}_2(X'')$ is not $(H, \ell)$-compatible. Therefore $O_{H,\ell}(G) \cap \mathcal{X} = \{X\}$. \hfill \Box
Notice that \( \mathbb{Z}_k[N, X] \neq \emptyset \) for each \( X \in O_{H,k}(N) \) if and only if \( k \geq |\sigma(\tilde{g})| \)
where \( \tilde{g} \) is a primitive chain in \( N^\perp \) with the maximal \( |\sigma(\tilde{g})| \). This follows from the following statement.

**Proposition 3.3.** \( Q(N, X; k) \neq 0 \) for each \( X \in O(N)^+ \) if and only if \( k \geq |\sigma(g)| \) for each primitive chain \( g \) of \( N^\perp \).

**Proof.** The proof is trivial if \( O(N)^+ = \emptyset \). Assume that \( O(N)^+ \neq \emptyset \).

For \( f \in \mathbb{Z}_k \), denote by \( \sigma^+(f) (\sigma^-(f)) \) the number of positive (negative) coefficients of \( f \), i.e., \( \sigma^+(f) = \sigma(f) \). Assume that \( D(N) \) has the form \((I_r|U)\) and let \( D_X \) be the matrix arising from \((I_r|U)\) after changing the signs of all entries from the columns corresponding to \( X \). Then \( \rho_X(N_k(X)) \) is the set of chains on \( E \) that are orthogonal to all rows of \( D_X \) and have coordinates from \( 1 \) to \( k - 1 \). In other words, \( |N_k(X)| = |\rho_X(N_k(X))| \neq 0 \) if and only if the integral polyhedron

\[
P_k = \{ y; 1 \leq y \leq k-1, D_X y = 0 \}
\]
is nonempty. With respect to the construction of \( D_X \), (arising from \((I_r|U)\) after changing the signs of all entries from the columns corresponding to \( X \)), \( u, v \) are \( \{0, \pm 1\} \)-vectors satisfying \( uD_X = v \) if and only if \( v \) is from the set

\[
C_1 = \{ \rho_X(f); f \in N^\perp, |f(e)| \leq 1 \text{ for each } e \in E \}.
\]

Thus by [24, Corollary 21.3a] (see also [14, Section 3]), \( P_k \neq \emptyset \) if and only if \( \sigma^-(f) \leq (k-1)\sigma^+(f) \) for every \( f \in C_1 \). Hence by (2.2), \( P_k \neq \emptyset \) if and only if \( \sigma^-(\rho_X(g)) \leq (k-1)\sigma^+(\rho_X(g)) \) for each primitive chain \( g \) in \( N^\perp \) (notice that by (2.3), \( \sigma^+(\rho_X(g)) \), \( \sigma^-(\rho_X(g)) \) must be nonzero for \( X \in O(N)^+ \)). Thus if \( \tilde{g} \) is a primitive chain in \( N^\perp \) with the maximal \( |\sigma(\tilde{g})| \), then \( N_k(X) \neq \emptyset \) for each \( k \geq |\sigma(\tilde{g})| \) and \( X \in O(N)^+ \).

Denote by \( \tilde{E} = \sigma(\tilde{g}), \tilde{E}' = \tilde{E} \setminus \tilde{E}, Y = \{ e \in E; \tilde{g}(e) < 0 \} \), and \( \tilde{N} = N/E' \). Then \( \tilde{N}^\perp = N^\perp \setminus E' \), whence by (2.1), \( \tilde{g}^{\tilde{E}'} \) is the unique primitive chain in \( \tilde{N}^\perp \) and thus by (2.3), \( \{ (\rho_Y(\chi_e-x_{e'})^e)\}|^{\tilde{E}'}; e, e' \in \tilde{E}, e \neq e' \} \subseteq \tilde{N} \). Choose \( \tilde{e} \in \tilde{E} \) and define by \( \tilde{X} = Y \setminus \tilde{e} \cup \tilde{e} \setminus Y \). Then \( \rho_X(\tilde{g})^\tilde{E}' = (\chi_{\tilde{E} \setminus \tilde{e}}-\chi_{\tilde{e}})^{\tilde{E}'} \),

\[
\{(\rho_X(\chi_{\tilde{e}}^e))^{\tilde{E}'}; e \in \tilde{E} \}\subseteq \tilde{N},
\]

and \( \tilde{f} = \rho_X \left( \sum_{e \in \tilde{E} \setminus \tilde{e}} \chi_{\tilde{e}, e} \right) \) satisfies \( \sigma(\tilde{f}) = \tilde{E} \) and \( \tilde{f}^{\tilde{E}'} \in \tilde{N} \). Hence \( \tilde{X} \in O(\tilde{N})^+ \). Consider a proper chain \( \tilde{f} \in N \) (that exists because \( O(N)^+ \neq \emptyset \)). Then \( f' = f + (1+\sum_{e \in E} |\tilde{f}(e)|) \tilde{f} \) is proper and \( (\rho_X(f'))^{\tilde{E}'} \) \( \in \mathbb{Z}_{\tilde{E}'}^+ \), whence \( X' = \{ e \in E; f'(e) < 0 \} \in O(N)^+ \) and \( X' \cap \tilde{E} = \tilde{X} \). If \( f \in N_k(X') \), then \( \rho_X(f) \in \mathbb{Z}_{\tilde{E}}^+ \) and by (2.3), \( \rho_X(f) \) is orthogonal to \( \rho_{X'}(\tilde{g}) = \chi_{\tilde{E} \setminus \tilde{e}}-\chi_{\tilde{e}} \), i.e., \( |f(\tilde{e})| \geq |\tilde{E}| \tilde{f} = |\sigma(\tilde{g})|-1 \). Thus \( N_k(X') = \emptyset \) for each \( k \leq |\sigma(\tilde{g})| - 1 \), concluding the proof.

We say that a sequence of primitive chains \( c_1, \ldots, c_r \in N_2(X) \) (\( r = r(N) \), \( X \in O(N)^+ \)) is a **triangular \( X \)-basis** of \( N \) if there exist \( e_1, \ldots, e_r \in E \) such that \( e_i \in \sigma(c_i) \), \( e_i \notin \sigma(c_j) \) for each \( i, j \in \{1, \ldots, r\}, i < j \). Clearly, any
triangular \( X \)-basis is a basis of the linear hull of \( N \). Therefore for each \( f \in N \) there are numbers \( z_1, \ldots, z_r \) such that \( f = \sum_{i=1}^r z_i c_i \) and thus \( z_1 = f(e_1), z_2 = f(e_2) - z_1, \ldots, z_r = f(e_r) - z_1 - \cdots - z_{r-1} \) are integral.

**Lemma 3.4.** For each regular chain group \( N \) on \( E \) and \( X \in O(N)^+ \) there exists a triangular \( X \)-basis of \( N \).

**Proof.** Choose a primitive chain \( c_1 \in \overline{N}_2(X) \) and \( e_1 \in E \) such that \( c_1(e_1) = 1 \). Let \( E' \subseteq E \) be defined so that \( e_1 \in E' \) and \( E' \setminus e_1 \) is the set of isthmuses in \( N - e_1 \). Notice that \( N \) has no isthmus because \( O(N)^+ \neq \emptyset \). Thus by (2.1), \( N^\perp \) must contain a chain of form \( \chi_{e_1} \pm \chi_e \) for each \( e \in E' \setminus e_1 \). Since each chain from \( N^\perp \) is orthogonal to \( c_1 \), we have \( \rho_X(\chi_{e_1} - \chi_e) \in N^\perp \). Then by (2.3), \( \rho_X(f)(e_1) = \rho_X(f)(e) \) for every \( e \in E' \setminus e_1 \) and \( f \in N \), whence \( r(N - E') = r(N) - 1 \). Thus applying the induction hypothesis on \( N - E' \) and \( X \setminus E' \in O(N - E')^+ \) we can extend \( c_1 \) and \( e_1 \) into a triangular \( X \)-basis of \( N \) (considering chains from \( N - E' \) as chains from \( N \) after setting the undefined coordinates to be 0).

We claim that for each regular chain group \( N \) and \( X \in O(N)^+ \),

\[
(3.1) \quad r(N) + 1 \leq \overline{Q}(N, X; 2) \leq 2^{r(N)}.
\]

Clearly, \( \overline{N}_2(X) \) contains the zero chain and at least \( r(N) \) nonzero chains by Lemma 3.4. This implies the left hand side. The right hand side follows from the fact that each \( c \in \overline{N}_2(X) \) is a linear combination of rows of a representative matrix of \( M(N)^* \) having form \((-U^T|I_{r(N)})\) such that \( I_{r(N)} \) corresponds to a base \( B^* \) of \( M(N)^* \), \(|B^*| = r(N)\), and that \( c(e) \in \{0, [\rho_X(\chi_e)](e)\} \) for every \( e \in B^* \).

For example let \( g \) be a chain on \( E \) such that \( g(\tilde{e}) = -1 \) for a fixed \( \tilde{e} \in E \) and \( g(e) = 1 \) for \( e \in E, e \neq \tilde{e} \). Consider \( N \) so that \( g \) is the unique primitive chain of \( N^\perp \). By (2.3), \( \{\pm \rho_c(\chi_e - \chi_{e'}) \in E, e \neq e' \} \) is the set of primitive chains of \( N \), whence \( \overline{N}_2(\{\tilde{e}\}) = \{\chi_{\tilde{e}}\} \cup \{\chi_{e, e'} \in E, e, e' \neq \tilde{e}\} \). Thus \( \overline{Q}(N, \{\tilde{e}\}; 2) = |E| = r(N) + 1 \), i.e., the left hand side of (3.1) is tense. If \( N \) has \(|E| \) loops, then \( \overline{N}_2(X) = \{\rho_X(\chi_Y); Y \subseteq E\} \), whence \( \overline{Q}(N, X; 2) = 2^{|E|} = 2^{r(N)} \) for each \( X \subseteq E \). Thus the right hand side of (3.1) is tense.

**Lemma 3.5.** For each regular chain group \( N \) and each integer \( k > 0 \),

\[
(r(N) + 1)P(N; k) \leq Q(N; k) \leq 2^{r(N)}P(N; k).
\]

**Proof.** The proof follows from the third row of (2.6), the first row of (2.7), and (3.1).

**Proposition 3.6.** For each regular chain group \( N \) on \( E \) and \( k \geq 2 \),

\[
Q(N, X; k + 1) \geq Q(N, X; k) k(k-1)^{-1},
\]

\[
P(N; k + 1) \geq P(N; k) k(k-1)^{-1},
\]

\[
Q(N; k + 1) \geq Q(N; k) k(k-1)^{-1}.
\]

**Proof.** Let \( X \in O(N)^+ \). For each \( f \in N_k(X) \) and each \( c \in \overline{N}_2(X) \), \( c \) not equal to the zero chain, there exists a unique integer \( r > 0 \) such that \( f + rc \in \).
We shall call this chain an \((f, c)\)-lift (shortly a lift). In this way we can construct \(s_X = Q(N, X; k)Q(N, X; 2)\) (not necessary different) lifts. On the other hand each \(f' \in N_{k+1}(X) \setminus N_k(X)\) could be an \((f' - ic, c)\)-lift for \(i = 1, \ldots, s\), \(0 \leq s \leq k-1\) \((s = 0\) if \(f' - ic \notin N_k(X)\) for each \(i \geq 1\)). Thus \(f'\) can be constructed as a lift at most \(s_X' = (k-1)Q(N, X; 2)\) times. Hence

\[
Q(N, X; k+1) - Q(N, X; k) = |N_{k+1}(X) \setminus N_k(X)| \\
\geq s_X / s_X' = Q(N, X; k)(k-1)^{-1}
\]

This implies the first row of the formula for \(X \in O(N)^+\). If \(X \notin O(N)^+\), the first row of the formula is trivial because then \(Q(N, X; k+1) = Q(N, X; k) = 0\). Hence the second and the third rows follow from the third row of (2.6) and the first row of (2.7), respectively.

**Proposition 3.7.** For each regular chain group \(N\) on \(E\) and \(k \geq 2\),

\[
Q(N, X; k+1) > Q(N, X; k) \quad \text{if} \quad Q(N, X; k+1) > 0,
\]

\[
P(N; k+1) > P(N; k) \quad \text{if} \quad P(N; k+1) > 0,
\]

\[
Q(N; k+1) > Q(N; k) + r(N) \quad \text{if} \quad Q(N; k+1) > 0.
\]

**Proof.** The first two rows follow from Proposition 3.6 and the fact that \(k(k-1)^{-1} > 1\). The last row follows from the first one, the third row of (2.6), and (3.1). □

Propositions 3.6 and 3.7 generalize [2, Proposition 6]. Polynomial \(P(N; k)\) (resp. \(Q(N; k)\)) corresponds to a flow (resp. integral flow) polynomial if \(N(M)\) is a graphic matroid and corresponds to a tension (resp. integral tension) polynomial if \(N(M)\) is a congraphic matroid. Flow and tension polynomials (and their integral variants) were studied in [15, 16] where we proved Lemmas 3.1, 3.5, and Propositions 3.6, 3.7 for flows and tensions on graphs. Similar versions of Lemmas 3.2, 3.4, and Proposition 3.3 were proved in [16, 17, 18, 20]. Several other generalizations of flow and tension polynomials are presented in [3, 4, 5, 6, 7, 9, 10, 11, 13].

We can generalize Propositions 3.6 and 3.7 for \(\overline{Q}(N, X; k)\) and the Tutte polynomial of regular matroids. Assume that \(N\) is a regular chain group on \(E\) and \(X \subseteq E\). Using (2.1) and the definitions of \(N_k(X)\) and \(N_k(X)\), it is easy to check that \(\overline{N}_k(X)\) equals the disjoint union of \([N-Y]_k(X\setminus Y)\) where \(Y\) runs through the powerset of \(E\). Therefore by the definitions of \(\overline{Q}(N, X; k)\) and \(Q(N, X; k)\),

\[
\overline{Q}(N, X; k) = \sum_{Y \subseteq E} Q(N-Y, X\setminus Y; k).
\]

By Proposition 3.6, for each \(k \geq 2\) we have

\[
\sum_{Y \subseteq E} Q(N-Y, X\setminus Y; k+1) \geq \sum_{Y \subseteq E} Q(N-Y, X\setminus Y; k)(k-1)^{-1},
\]
whence by (3.2)

\[ \overline{Q}(N, X; k + 1) \geq \overline{Q}(N, X; k) k(k-1)^{-1}, \]

and thus

\[ \overline{Q}(N, X; k + 1) > \overline{Q}(N, X; k) \quad \text{if} \quad \overline{Q}(N, X; k + 1) > 0. \]

The Tutte polynomial \( T(M; x, y) \) of a matroid \( M \) on \( E \) is (see cf. [9, 21])

\[ T(M; x, y) = \sum_{X \subseteq E} (x - 1)^{r(E) - r(X)} (y - 1)^{|X| - r(X)}. \]

If \( M = M(N) \) is regular, \( \ell \) is a labeling of elements of \( E \) by the numbers \( 1, \ldots, |E|, H \subseteq E \), and \( x, y \geq 2 \) are integers, then by [21, Equation 16],

\[ T(M(N); x, y) = \sum_{X \subseteq E} \left( \sum_{Y \in O_{H \setminus L}(N^L \setminus X)} \overline{Q}(N^L \setminus X, Y; x) \right) \left( \sum_{Y' \in O_{H \cap L}(N \setminus X)} \overline{Q}(N, Y'; y) \right). \]

Applying (3.3) on the right hand side of this equation we get that

\[ T(M(N); x + 1, y) \geq T(M(N); x, y) x(x-1)^{-1}, \]

\[ T(M(N); x, y + 1) \geq T(M(N); x, y) y(y-1)^{-1}, \]

\[ T(M(N); x + 1, y) > T(M(N); x, y) \quad \text{if} \quad T(M(N); x + 1, y) > 0, \]

\[ T(M(N); x, y + 1) > T(M(N); x, y) \quad \text{if} \quad T(M(N); x, y + 1) > 0, \]

for any regular chain group \( N \) and any pair of integers \( x, y \geq 2 \).

**References**


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