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FURTHER GENERALIZATIONS OF THE PARALLELOGRAM LAW

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Abstract. In a recent work of Alessandro Fonda, a generalization of the parallelogram law in any dimension $N \geq 2$ was given by considering the ratio of the quadratic mean of the measures of the $(N-1)$-dimensional diagonals to the quadratic mean of the measures of the faces of a parallelotope. In this paper, we provide a further generalization considering not only $(N-1)$-dimensional diagonals and faces, but the k-dimensional ones for every $1 \leq k \leq N-1$.

1. Introduction

If we consider the usual Euclidean space $(\mathbb{R}^n, \|\cdot\|)$, the well-known identity

$$\|a+b\|^2 + \|a-b\|^2 = 2(\|a\|^2 + \|b\|^2) \tag{1.1}$$

is called the *parallelogram law*.

This identity can be extended to higher dimensions in several ways. For example, it is straightforward to see that

$$\|a+b+c\|^2+\|a+b-c\|^2+\|a-b+c\|^2+\|a-b-c\|^2 = 4(\|a\|^2+\|b\|^2+\|c\|^2) \tag{1.2}$$

with the subsequent analoguous identities arising inductively. There are many works devoted to provide generalizations of (1.1) in many different contexts [1, 3, 4].

Note that if we rewrite (1.1) as

$$\frac{\|a+b\|^2 + \|a-b\|^2}{2} = 2\frac{(\|a\|^2 + \|b\|^2 + \|a\|^2 + \|b\|^2)}{4} \tag{1.3}$$

this means that in any parallelogram, the ratio of the quadratic mean of the lengths of its diagonals to the quadratic mean of the lengths of its sides equals $\sqrt{2}$. With this interpretation in mind, Alessandro Fonda [2] recently proved the following interesting generalization.

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Theorem 1.1. *Given linearly independent vectors $a_1, \dots, a_N \in \mathbb{R}^n$, it holds that*

$$\sum_{i<j} \left(\left\| (a_i + a_j) \wedge \bigwedge_{k \neq i,j} a_k \right\|^2 + \left\| (a_i - a_j) \wedge \bigwedge_{k \neq i,j} a_k \right\|^2 \right) =$$
$$= (N-1) \sum_{k=1}^{N} 2 \left\| a_1 \wedge \dots \wedge \widehat{a_k} \wedge \dots \wedge a_N \right\|^2 .$$

In other words, for any N-dimensional parallelotope, the ratio of the quadratic mean of the $(N-1)$-dimensional measures of its diagonals to the quadratic mean of the $(N-1)$-dimensional measures of its faces is equal to $\sqrt{2}$.

In this work we extend this result to faces of dimension k for every $1 \leq k \leq N-1$ and to a suitable definition of the k-dimensional diagonal of a parallelotope. Then Theorem 1.1 will be a particular case of our result for $k = N-1$. Indeed, our result can be stated as follows.

Theorem 1.2. *Let us consider an N-dimensional parallelotope and let $1 \leq k \leq N-1$. The ratio of the quadratic mean of the k-dimensional measures of its k-dimensional diagonals to the quadratic mean of the k-dimensional measures of its k-dimensional faces is equal to $\sqrt{N-k+1}$.*

In fact, our generalization follows in line with the work [3] but instead considers the definition of a diagonal face given in [2].

2. Notation and preliminaries

In this section, we introduce some notation and present some basic facts that will be useful in the sequel. Let us consider a parallelotope $\mathcal{P}$ generated by a family of linearly independent vectors $\mathcal{B} = \{a_1, a_2, \dots, a_N\} \subseteq \mathbb{R}^n$. This means that

$$\mathcal{P} = \left\{ \sum_{i=1}^{N} \alpha_i a_i : \alpha_i \in [0,1] \right\}.$$

Let us fix $1 \leq k \leq N-1$. Given k different vectors $\mathcal{S} = \{a_{i_1}, \dots, a_{i_k}\} \subseteq \mathcal{B}$, we can consider the face generated by them:

$$\mathcal{F}(\mathcal{S}) = \left\{ \sum_{v \in \mathcal{S}} \alpha_v v : \alpha_v \in [0,1] \right\}.$$

This face can now be translated by one or more of the remaining vectors thus obtaining a face

$$\mathcal{F}^I(\mathcal{S}) = \left\{ \sum_{v \in \mathcal{S}} \alpha_v a_v + \sum_{w \in \mathcal{B} \setminus \mathcal{S}} \alpha_w w \in \mathcal{P} : \alpha_w \in \{0,1\} \right\},$$

where $I = (\alpha_v)_{v \notin \mathcal{S}} \in \{0,1\}^{N-k}$. Since each choice of a set $\mathcal{S} \subseteq \mathcal{B}$ and a vector $I \in \{0,1\}^{N-k}$ leads to a different face and every face can be obtained in this way, it follows that $\mathcal{P}$ has exactly $2^{N-k}\binom{N}{k}$ k-dimensional faces. Moreover, it is clear that all the 2^{N-k} different faces $\mathcal{F}^I(\mathcal{S})$ are congruent to the set generated by $\mathcal{S}$, $\mathcal{F}(\mathcal{S})$.

Now, we focus on the k-dimensional diagonals which will be defined following the ideas in [2]. Let us consider $N-k+1$ different vectors $\mathcal{T} = \{a_{i_1}, \ldots, a_{i_{N-k+1}}\} \subseteq \mathcal{B}$ and let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ be a decomposition of $\mathcal{T}$ into two disjoint sets (either $\mathcal{T}_1$ or $\mathcal{T}_2$ could be empty). Then, the following set

$$\mathcal{D}_{\mathcal{T}}(\mathcal{T}_1, \mathcal{T}_2) = \left\{ \alpha \sum_{v \in \mathcal{T}_1} v + (1-\alpha) \sum_{v \in \mathcal{T}_2} v + \sum_{w \in \mathcal{B} \setminus \mathcal{T}} \alpha_w w : \alpha, \alpha_w \in [0,1] \right\}$$

is called the k-dimensional diagonal associated to $(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2)$. Clearly each choice of a set $\mathcal{T} \subseteq \mathcal{B}$ and a decomposition $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ allows us to define a diagonal. Since it is clear that $\mathcal{D}_{\mathcal{T}}(\mathcal{T}_1, \mathcal{T}_2) = \mathcal{D}_{\mathcal{T}}(\mathcal{T}_2, \mathcal{T}_1)$, it readily follows that $\mathcal{P}$ has exactly $2^{N-k}\binom{N}{N-k+1}$ different k-dimensional diagonals. Moreover, if we define the vector

$$V_{\mathcal{T}}(\mathcal{T}_1, \mathcal{T}_2) = \sum_{v \in \mathcal{T}_1} v - \sum_{v \in \mathcal{T}_2} v,$$

we have that

$$\mathcal{D}_{\mathcal{T}}(\mathcal{T}_1, \mathcal{T}_2) = \left\{ \alpha V_{\mathcal{T}}(\mathcal{T}_1, \mathcal{T}_2) + \sum_{v \in \mathcal{T}_2} v + \sum_{w \in \mathcal{B} \setminus \mathcal{T}} \alpha_w w : \alpha, \alpha_w \in [0,1] \right\}$$

and consequently, it is clear that the diagonal $\mathcal{D}_{\mathcal{T}}(\mathcal{T}_1, \mathcal{T}_2)$ is a translation of the set generated by $\{V_{\mathcal{T}}(\mathcal{T}_1, \mathcal{T}_2), w : w \in \mathcal{B} \setminus \mathcal{T}\}$ and hence it is congruent to it.

Example. Let us see how the definition of $\mathcal{D}(\mathcal{T}_1, \mathcal{T}_2)$ applies in the case of lower dimensions; i.e, if $N = 2, 3$.

In the case $N = 2$, we only consider $k = 1$. If we consider the parallelogram $\mathcal{P}$ generated by $\mathcal{B} = \{a_1, a_2\} \subseteq \mathbb{R}^N$, then clearly $\mathcal{T} = \mathcal{B}$ (because $k = 1$) and $\mathcal{P}$ has two different diagonals which are defined by the two possible decompositions $\mathcal{T} = \{a_1\} \cup \{a_2\}$ and $\mathcal{T} = \mathcal{T} \cup \emptyset$. In fact,

$$\begin{aligned}
\mathcal{D}_{\mathcal{B}}(\{a_1\}, \{a_2\}) &= \{\alpha a_1 + (1-\alpha) a_2 : \alpha \in [0,1]\} \\
&= a_2 + \{\alpha(a_1 - a_2) : \alpha \in [0,1]\}, \\
\mathcal{D}_{\mathcal{B}}(\mathcal{B}, \emptyset) &= \{\alpha(a_1 + a_2) : \alpha \in [0,1]\}.
\end{aligned}$$

Figure 1 shows how we obtain the two diagonals of the parallelogram. Note that, in this case, $V_{\mathcal{B}}(\mathcal{B}, \emptyset) = a_1 + a_2$ and $V_{\mathcal{B}}(\{a_1\}, \{a_2\}) = a_1 - a_2$.

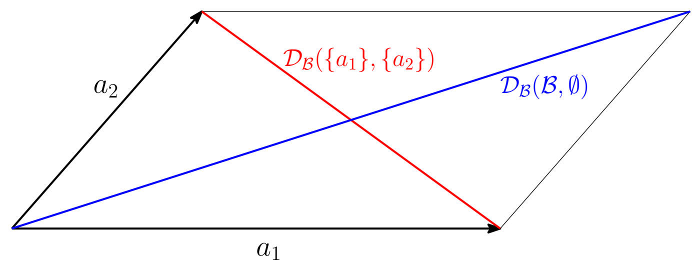

FIGURE 1. The case $N = 2$, $k = 1$.

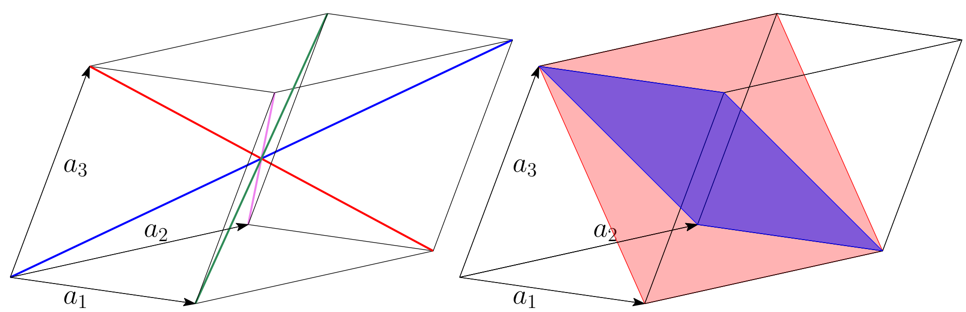

FIGURE 2. The case $N = 3$, $k = 1$ (left) and $k = 2$ (right).

Now, if $N = 3$ and $k = 1$, let us consider the parallelepiped $\mathcal{P}$ generated by $\mathcal{B} = \{a_1, a_2, a_3\} \subseteq \mathbb{R}^N$. Again, $\mathcal{T} = \mathcal{B}$ but in this case there are four different 1-dimensional diagonals which are defined by the decompositions $\mathcal{T} = \{a_1, a_2\} \cup \{a_3\}$, $\mathcal{T} = \{a_1, a_3\} \cup \{a_2\}$, $\mathcal{T} = \{a_2, a_3\} \cup \{a_1\}$, and $\mathcal{T} = \mathcal{T} \cup \emptyset$. In fact,

$$\begin{aligned}
\mathcal{D}_{\mathcal{B}}(\{a_1, a_2\}, \{a_3\}) &= \{\alpha(a_1 + a_2) + (1-\alpha)a_3 : \alpha \in [0,1]\} \\
&= a_3 + \{\alpha(a_1 + a_2 - a_3) : \alpha \in [0,1]\}, \\
\mathcal{D}_{\mathcal{B}}(\{a_1, a_3\}, \{a_2\}) &= \{\alpha(a_1 + a_3) + (1-\alpha)a_2 : \alpha \in [0,1]\} \\
&= a_2 + \{\alpha(a_1 - a_2 + a_3) : \alpha \in [0,1]\}, \\
\mathcal{D}_{\mathcal{B}}(\{a_2, a_3\}, \{a_1\}) &= \{\alpha(a_2 + a_3) + (1-\alpha)a_1 : \alpha \in [0,1]\} \\
&= a_1 + \{\alpha(-a_1 + a_2 + a_3) : \alpha \in [0,1]\}, \\
\mathcal{D}_{\mathcal{B}}(\mathcal{B}, \emptyset) &= \{\alpha(a_1 + a_2 + a_3) : \alpha \in [0,1]\}.
\end{aligned}$$

On the left hand side of Figure 2, we can see the above four 1-dimensional diagonals of $\mathcal{P}$ (in red, purple, green, and blue, respectively). Note that, in this case, $V_{\mathcal{B}}(\mathcal{B}, \emptyset) = a_1 + a_2 + a_3$, $V_{\mathcal{B}}(\{a_1, a_2\}, \{a_3\}) = a_1 + a_2 - a_3$, $V_{\mathcal{B}}(\{a_1, a_3\}, \{a_2\}) = a_1 - a_2 + a_3$, and $V_{\mathcal{B}}(\{a_2, a_3\}, \{a_1\}) = -a_1 + a_2 + a_3$.

In the same way, if $N = 3$ and $k = 2$, we could define the six 2-dimensional diagonals of $\mathcal{P}$. On the right hand side of Figure 2 we see, for instance, $\mathcal{D}_{\{a_1,a_3\}}(\{a_1\},\{a_3\})$ in red and $\mathcal{D}_{\{a_2,a_3\}}(\{a_2\},\{a_2\})$ in blue.

3. Proof of Theorem 1.2

After introducing the notation and the basic objects involved in this work, we are now ready to prove the main result of the paper.

Let $\mathcal{P}$ be a parallelotope generated by $\mathcal{B} = \{a_1, a_2, \dots, a_N\} \subseteq \mathbb{R}^n$. We first compute the quadratic mean of the k-dimensional measures of its k-dimensional faces. We first note that for every $\mathcal{S} = \{a_{i_1}, \dots, a_{i_k}\} \subseteq \mathcal{B}$, the k-dimensional measure of the face $\mathcal{F}(\mathcal{S})$ is $\|a_{i_1} \wedge \cdots \wedge a_{i_k}\|$. In the previous section we have seen that $\mathcal{P}$ has exactly $2^{N-k}\binom{N}{k}$ k-dimensional faces and moreover, there are exactly 2^{N-k} copies of each face $\mathcal{F}(\mathcal{S})$. Consequently, the quadratic mean of the k-dimensional measures of the k-dimensional faces of $\mathcal{P}$ is:

$$\sqrt{\frac{2^{N-k}\sum \|a_{i_1} \wedge \cdots \wedge a_{i_k}\|^2}{2^{N-k}\dbinom{N}{k}}}. \tag{3.1}$$

Now we have to compute the quadratic mean of the k-dimensional measures of the k-dimensional diagonals of $\mathcal{P}$. First of all, recall that $\mathcal{P}$ has exactly $2^{N-k}\binom{N}{N-k+1}$ different k-dimensional diagonals. Each of them is a translation of the set generated by $\{V_{\mathcal{T}}(\mathcal{T}_1,\mathcal{T}_2), w : w \in \mathcal{B} \setminus \mathcal{T}\}$ for exactly one choice of $(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2)$. The k-dimensional measure of this latter set is $\left\|V_{\mathcal{T}}(\mathcal{T}_1,\mathcal{T}_2) \wedge \bigwedge_{w\in\mathcal{B}\setminus\mathcal{T}} w\right\|$. Consequently, the quadratic mean of the k-dimensional measures of the k-dimensional diagonals of $\mathcal{P}$ is:

$$\sqrt{\frac{\displaystyle\sum_{\mathcal{T},\mathcal{T}_1,\mathcal{T}_2}\left\|V_{\mathcal{T}}(\mathcal{T}_1,\mathcal{T}_2) \wedge \bigwedge_{w\in\mathcal{B}\setminus\mathcal{T}} w\right\|^2}{2^{N-k}\dbinom{N}{N-k+1}}}. \tag{3.2}$$

Using the bilinearity of the scalar product and taking into account the definition of $V_{\mathcal{T}}(\mathcal{T}_1,\mathcal{T}_2)$, it can be easily seen that when we vary $(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2)$, we get the term $\|a_{i_1} \wedge \cdots \wedge a_{i_k}\|^2$ exactly $2^{N-k}k$ times for every possible choice of $\{a_{i_1}, \dots, a_{i_k}\} \subseteq \mathcal{B}$. This implies that the quadratic mean of the k-dimensional measures of the k-dimensional diagonals of $\mathcal{P}$ (3.2) can be written as:

$$\sqrt{\frac{2^{N-k}k\sum \|a_{i_1} \wedge \cdots \wedge a_{i_k}\|^2}{2^{N-k}\dbinom{N}{N-k+1}}}. \tag{3.3}$$

Finally to obtain Theorem 1.2, it is enough to divide (3.3) by (3.1) to get

$$\sqrt{\frac{k\binom{N}{k}}{\binom{N}{N-k+1}}} = \sqrt{N-k+1}.$$

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