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THE ASYMMETRIC INDEX OF A GRAPH

ALEJANDRA BREWER CASTANO, EMMA FARNSWORTH, NATALIE GOMEZ, ADAM GREGORY, QUINDEL JONES, HERLANDT LINO, BRENDAN ROONEY, AND DARREN A. NARAYAN

Abstract. A graph G is *asymmetric* if its automorphism group is trivial. Asymmetric graphs were introduced by Erdős and Rényi (1963). They suggested the problem of starting with an asymmetric graph and removing some number r of edges and/or adding some number s of edges so that the resulting graph is nonasymmetric. Erdős and Rényi defined the *degree of asymmetry* of a graph to be the minimum value of $r+s$. In this paper, we define another property that measures how close a given nonasymmetric graph is to being asymmetric. We define the *asymmetric index* of a graph G, denoted $ai(G)$, to be the minimum of $r+s$ so that the resulting graph G is asymmetric. We investigate the asymmetric index of both connected and disconnected graphs. We prove that for any nonnegative integer k, there exists a graph G where $ai(G)=k$. We show that the asymmetric index of a cycle with at least six vertices is two, and provide a complete characterization of all possible pairs of edges that can be added to a cycle to create an asymmetric graph. In addition we determine the asymmetric index of paths, certain circulant graphs, Cartesian products involving paths and cycles, and bounds for complete graphs, and complete bipartite graphs.

1. Introduction

We consider undirected graphs without multiple edges or loops. A graph G is *asymmetric* if its automorphism group is trivial. To avoid confusion with *symmetric* (or arc-transitive) graphs where the automorphism group acts transitively on ordered pairs of adjacent vertices, a graph with a nontrivial automorphism group of vertices will be referred to as nonasymmetric. Asymmetric graphs were introduced by Erdős and Rényi [1] in 1963. Any asymmetric graph can be made nonasymmetric by removing some r number of edges and adding some s number of edges. Erdős and Rényi defined the *degree of asymmetry* $A(G)$ of a graph G to be the minimum of $r+s$. In this paper, we define a property that measures how close a nonasymmetric graph is to being asymmetric. We define the *asymmetric index* of a graph G, denoted $ai(G)$, to be the minimum of $r+s$ so that the resulting graph G is asymmetric. At first glance it might appear that calculating the degree of asymmetry of a graph is the inverse problem of calculating the asymmetric index: one might think that adding (removing) edges to (from) an asymmetric graph to

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obtain a nonasymmetric graph would be the same as removing (adding) edges from (to) a nonasymmetric graph to obtain an asymmetric graph. However, Erdős and Rényi [1] sought the minimum value of $r+s$ to create symmetry, while in this paper we are seeking the minimum value of $r+s$ to eliminate symmetry. In problems involving the degree of asymmetry, graphs that are far from being asymmetric, such as complete graphs, are not encountered. In fact, if we let n denote the number of vertices in a graph, we will show that when $n \geq 6$, $ai\,(K_n) \geq \frac{6n}{7}$ but Erdős and Rényi showed that the degree of asymmetry of a graph with n vertices is less than or equal to $\lceil \frac{n-1}{2} \rceil$ for all graphs G.

For a graph G we will use $V(G)$ to denote the set of vertices, and $E(G)$ to denote the set of edges. The edge between vertices u and v will be denoted uv. Two graphs G and H are *isomorphic* if there is a bijection $f : G \to H$ where $uv \in E(G) \Leftrightarrow f(u)f(v) \in E(H)$. Recall that f is an automorphism if it is an isomorphism from a graph to itself, and the set of all automorphisms of a graph forms a permutation group under function composition. We will use $\mathrm{Aut}(G)$ to denote the automorphism group of a graph G. The *complement* of a graph G will be denoted $\overline{G}$.

A vertex will be referred to as *fixed* if it is fixed under every automorphism of G. The *degree* of a vertex v is the number of edges incident to v. The *distance* between two vertices u and v is the number of edges in a shortest path between u and v and will be denoted $d(u, v)$. For graphs G and H with disjoint vertex sets, $G + H$ will simply be the disjoint union of the two graphs. For graphs G and H with disjoint vertex sets, the *join* of G and H is denoted $G \vee H$ and is a graph with the vertices and edges of G and H, along with edges between each vertex of G and each vertex of H. Given two graphs H and K, with vertex sets $V(H)$ and $V(K)$ the Cartesian product $G = H \Box K$ is a graph where $V(G) = \{(u_i, v_j)$ where $u_i \in V(H)$ and $v_j \in V(K)\}$, and $E(G) = \{(u_i, v_j), (u_k, v_l)\}$ if and only if $i = k$ and v_j and v_l are adjacent in K or $j = l$ and u_i and u_k are adjacent in H. For any undefined notation, please see the text [7] by West.

Many papers on asymmetric graphs have followed the seminal paper by Erdős and Rényi [1]. These include papers by Schweitzer and Schweitzer [6], and L. Quintas [5]. A comprehensive treatment of asymmetric graphs is given in the text by Godsil and Royle [2].

In this paper we investigate the asymmetric index of a graph for several families of graphs.

We prove that in some cases vertex-transitive graphs and asymmetric graphs are separated by as few as two edges. We show that the asymmetric index of a cycle with at least six vertices is two, and provide a complete characterization of all possible pairs of edges that can be added to a cycle to create an asymmetric graph. In addition, we obtain the asymmetric index for certain circulant graphs, Cartesian products involving paths and cycles, and bounds for complete graphs.

2. The Asymmetric Index

We begin by restating an elementary property regarding asymmetric graphs.

Proposition 2.1. *Given any graph G, $Aut(G) = Aut(\overline{G})$.*

As a consequence, if G is an asymmetric graph, then the complementary graph $\overline{G}$ is also asymmetric. This leads to the following proposition.

Proposition 2.2. *Given any graph G, $ai(G) = ai(\overline{G})$.*

Proof. Suppose G can be made into an asymmetric graph by removing some set R with r edges and adding some set S with s edges. Then by definition of $\overline{G}$, if we now add those same r edges in R to $\overline{G}$ and remove the same s edges in S from $\overline{G}$, we produce an asymmetric graph. $\square$

We continue by presenting two elementary results involving the join and disjoint union of two nonisomorphic asymmetric graphs.

Proposition 2.3. *If G and H are nonisomorphic asymmetric graphs then $G \vee H$ is asymmetric.*

Proof. Since G is an asymmetric graph, and in $G \vee H$, each vertex u in G has the same adjacencies to vertices in H, each vertex of G will be fixed in $G \vee H$. Similarly each vertex in H will be fixed in $G \vee H$. $\square$

Proposition 2.4. *If G and H are nonisomorphic asymmetric graphs then $G + H$ is asymmetric.*

Proof. By Proposition 2.1, if G and H are asymmetric then $\overline{G}$ and $\overline{H}$ are asymmetric. Then by Proposition 2.3, $\overline{G} \vee \overline{H}$ is asymmetric. Since $\overline{G} \vee \overline{H} = \overline{G + H}$, by Proposition 2.1, $G + H$ is asymmetric. $\square$

Other than the trivial case of a single vertex, it was shown by Erdős and Rényi [1] that the next smallest asymmetric graph has six vertices. Hence any graph with five or fewer vertices cannot be made asymmetric by removing or adding edges.

There is only one asymmetric tree on six vertices [1] only one asymmetric tree on seven vertices [5] and only one asymmetric tree on eight vertices. In a graph G, two vertices u and v can be *transposed* if there exists an automorphism $\sigma : G \to G$ in which $\sigma(u) = v$ and $\sigma(v) = u$. In a graph G, a vertex is *fixed* if and only if it is fixed under every automorphism. Note that a graph G is asymmetric, if all of its vertices are fixed.

Lemma 2.5. *Every asymmetric graph on $n \geq 6$ vertices can be extended to an asymmetric graph on $n + 1$ vertices by adding a single vertex and a single edge.*

Proof. Let G be an asymmetric graph without a pendant vertex. Let G' be a graph obtained by adding a new vertex u and an edge uv, where v is a vertex of maximum degree in G. We claim that G' is asymmetric. If G' is not asymmetric then there exists an automorphism f where two vertices in G' can be transposed. We note that any automorphism of $V(G')$ must send v to itself since it is the only vertex of degree $\Delta(G) + 1$ and f must send u to itself since it is the only vertex of degree 1. Let v_i and v_j be two vertices that can be transposed by the automorphism f. Since removing the vertex u will impact v_i and v_j in exactly the same way, then there exists an automorphism of $V(G)$ in which v_i and v_j could be switched. This would contradict the fact that G is asymmetric.

If G has a vertex of degree one, then we choose a vertex u with degree one that has a greatest distance d from a vertex of degree greater than or equal to 3. Then we can create a new graph G^* where a vertex z and edge uz are added to G. We next show that G^* is asymmetric. Since G is asymmetric, each vertex in $G - u$ has a property that each of the other vertices does not. The vertex u is the only vertex in G^* that has degree 2 and is adjacent to z which has the greatest distance $d + 1$

to a vertex of degree of 3 or more. Hence u and z will both be fixed in G^*, making G^* asymmetric. □

The above observations can be stated in the following theorem.

Theorem 2.6. *The asymmetric index is well-defined for any graph G consisting of a single vertex or having six or more vertices.*

In the next theorem we show that there exist graphs where the asymmetric index is arbitrarily large.

Theorem 2.7. *For any positive integer k, there exists a graph G where $ai(G) = k$.*

Proof. We first begin with small cases. Let T_7 be the asymmetric tree with 7 vertices. Starting with $8K_1$ and adding the six edges of T_7 shows that $ai(8K_1) = 6$. Successively removing pendant edges will create graphs with asymmetric index i for $2 \leq i \leq 5$. It is clear that to obtain asymmetric graphs in each of these cases edges must only be added and adding any smaller positive number of edges will result in a graph with at most five edges which cannot be asymmetric.

Starting with tK_1 where $9 \leq t \leq 15$ and first adding the edges of T_7 and then extending the longest path incident to the vertex of degree 3 will create graphs with asymmetric index i for $7 \leq i \leq 13$. We note that in each of these cases the number of edges in the resulting tree equals the asymmetric index. Starting with tK_1 and adding fewer than $t - 2$ edges will either leave at least two isolated vertices or a component with between 1 and 5 edges which cannot be asymmetric. It is tempting to continue in this manner, starting with $16K_1$ and adding the edges of T_7 and then extend the longest path incident to a vertex of degree 3 by eight edges. This will create a graph which shows $ai(16K_1) \leq 14$. However, it is possible to improve this bound by using the disjoint union of two different nontrivial asymmetric trees. If we start with $16K_1$ and add the edges of the asymmetric tree with 7 vertices, and the edges of the asymmetric tree with 8 vertices, and leave one isolated vertex, we have an asymmetric graph with 13 edges (see the figure below). Since the trees have a total of 13 edges, we have that $ai(15K_1) \leq 13$. To show equality, we note that 12 or fewer edges will either be a graph with at least two isolated vertices or a component that has between 1 and 5 edges, which cannot be asymmetric.

To create graphs G with $ai(G) \geq 14$, we use the disjoint union of nonisomorphic asymmetric trees. We first note that $16K_1$ is the smallest graph for which we can add the edges of two nontrivial nonisomorphic asymmetric trees and result in an asymmetric graph. Since there is a unique asymmetric tree on 7 vertices and a unique asymmetric tree on 8 vertices, the smallest graph for which we can add the edges of three nontrivial nonisomorphic asymmetric trees will contain asymmetric trees on 7, 8, and 9 vertices along with an isolated vertex. This graph will have 25 vertices and contain 21 edges. Hence $ai(25K_1) = 21$. Above we showed that $ai(16K_1) = 13$, where we added the edges of T_7 and T_8. To create graphs with asymmetric indices between 14 and 20, we start with tK_1 where $17 \leq t \leq 23$ respectively. Then we successively add edges to extend the longest path in T_8 incident to the vertex of degree 3 to create trees with j vertices where $9 \leq j \leq 15$. In each of these graphs the largest tree will be asymmetric by Proposition 1.

We will use T_r to denote a tree with r vertices. Consider the set of nonisomorphic asymmetric trees $T_{a_1}, T_{a_2}, \ldots, T_{a_k}$ where $a_1 \leq a_2 \leq \cdots \leq a_n$. In all cases $a_1 = 1$ and $a_2 \geq 6$. There must exist a smallest integer n_1 such that the edges of the trees

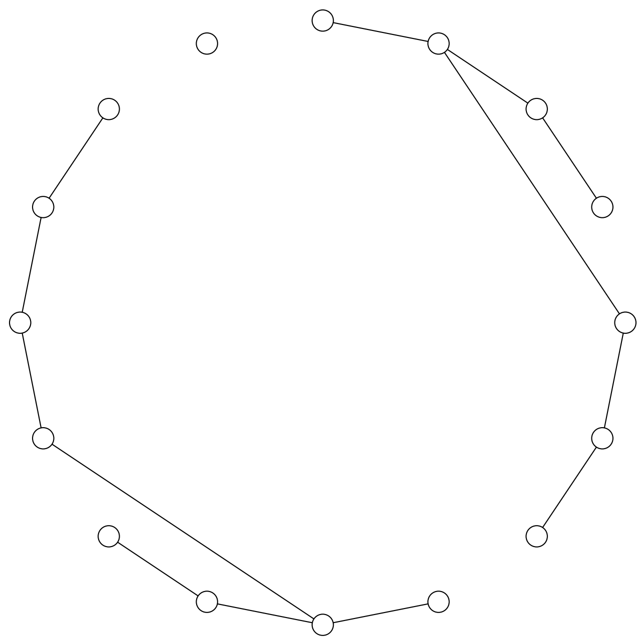

FIGURE 1. An asymmetric graph with 16 vertices

$T_{a_1}, T_{a_2}, \ldots, T_{a_k}$ can be added to n_1K_1 so that the resulting graph G_1 is asymmetric. This graph will have $\sum_{i=1}^{k} a_i$ vertices and $(\sum_{i=1}^{k} a_i) - k$ edges and hence $ai(G_1) \le n_1 - k$. There must also exist a smallest integer n_2 such that the edges of the trees $T_{a_1}, T_{a_2}, \ldots, T_{a_k}, T_{a_{k+1}}$ can be added to n_2K_1 so that the resulting graph G_2 is asymmetric. This graph will have $\sum_{i=1}^{k+1} a_i$ vertices and $(\sum_{i=1}^{k+1} a_i) - (k+1)$ edges and hence $ai(G_2) \le n_2 - (k+1)$. We note that in both cases any smaller set of edges will result in a graph with more than one isolated vertex or a tree with fewer than or equal to five edges. Hence $ai(G_1) = n_1 - k$ and $ai(G_2) = n_2 - (k+1)$. Next we can construct graphs with asymmetric index j for all $n_1 - k < j < n_2 - (k+1)$ by appending a path with q edges for $1 \le q \le (a_{k+1}) - 2$, to the longest path incident to a vertex of degree 3 in G_1. This completes the proof. □

We note that we can also create connected graphs with $ai(G) = k$ for any nonnegative integer k. For $1 \le k \le 6$ we use a similar construction to the one described above starting with K_7 and removing edges of G_i $1 \le i \le 6$. For cases when $k \ge 7$ we start with K_n where $n \ge 7$ and remove the edges shown in Figure 2.

3. Calculating the asymmetric index of a graph

In this section we investigate the asymmetric index of several families of graphs. We begin with the family of paths.

3.1. Paths. We consider paths, P_n where $n \ge 6$. Removing any number of edges will leave a nonasymmetric graph, in the form of shorter paths. We will show that the addition of a single edge can make the resulting graph asymmetric.

Proposition 3.1. *For $n \geq 6$, $ai(P_n) = 1$.*

Proof. Consider a path on $n \geq 6$ vertices with consecutive labels $v_1, v_2, \ldots, v_n$. Adding the edge v_2v_4 will produce a cycle with two pendant paths of different lengths, which is asymmetric. This implies that $ai(P_n) = 1$. □

An example of this fact can be seen in Figure 2.

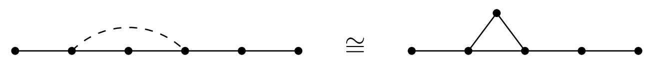

FIGURE 2. Adding a single edge to a path on six vertices

We next investigate the different possibilities for a single edge to be added to a path to make the resulting graph asymmetric.

Theorem 3.2. *The number of asymmetric graphs obtained by adding an edge to a path P_n is $\left\lfloor \frac{(n-4)^2}{4} \right\rfloor$.*

Proof. Let the vertices of P_n be $v_0, v_1, ..., v_{(n-1)}$. We first note that adding an edge incident to either v_0 or v_{n-1} will result in a graph that is not asymmetric. We first consider when n is even, $n = 2k$. Then v_1 has $n-5$ possibilities excluding edges to v_0, v_1, v_2, v_{n-2}, and v_{n-1}. Similarly $v_{\frac{n}{2}-1}$ has $n-5$ possibilities. Then v_i for all $1 \leq i \leq \frac{n}{2} - 2$ have $n-6$ possibilities excluding edges to $v_0, v_{i-1}, v_i, v_{i+1}$, and v_{n-1}. Hence the total number of cases is $2 + \sum_{i=1}^{\frac{n}{2}-1}(n-6) = 2 + \sum_{i=1}^{k-1}(2k-6) = 2k^2 - 8k + 8$. After dividing by 2 to account for reflections, the total number of graphs is $k^2 - 4k + 4$.

Next, we consider when n is odd, $n = 2k+1$. Then v_1 has $n-5$ possibilities excluding edges to v_0, v_1, v_2, v_{n-2}, and v_{n-1}. Similarly $v_{\frac{n-1}{2}}$ has $n-5$ possibilities. Then v_i for all $1 \leq i \leq \frac{n-1}{2} - 1$ have $n-6$ possibilities excluding edges to $v_0, v_{i-1}, v_i, v_{i+1}$, and v_{n-1}. Hence the total number of cases is $2 + \sum_{i=1}^{\frac{n-1}{2}}(n-6) = 2 + \sum_{i=1}^{k}(2k+1-6) = 2k^2 - 5k + 2$. Then we add in the cases for the remaining vertices, which gives $1 + \sum_{i=k+1}^{2k-1}(2k-5) = 2k^2 - 7k + 5$. This gives a total of $4k^2 - 12k + 7$. Then to adjust for double counting all but one of the cases we have $2k^2 - 6k + 4$. Then the total number of graphs is $k^2 - 3k + 2$. Noting that $\frac{(2k-4)^2}{4} = k^2 - 4k + 4$ and $\frac{(2k+1-4)^2}{4} = k^2 - 3k + \frac{9}{4}$ we combine both cases to obtain $\left\lfloor \frac{(n-4)^2}{4} \right\rfloor$. □

3.2. **Cycles.** A cycle, C_n, is both vertex and edge transitive and $\mathrm{Aut}(C_n)$ is the dihedral group D_n. We first note that $ai(C_n) > 1$, since deletion of a single edge will result in a path, which is nonasymmetric and adding a single edge will result in a graph with a reflective line of symmetry that bisects the added edge.

We will show next that for $n \geq 6$, $ai(C_n) = 2$. We consider adding two edges to C_n where $n \geq 6$. The following three theorems give necessary and sufficient conditions for two edges to be added to a cycle to make the resulting graph asymmetric. There are three ways to add a pair of edges to a cycle to create an asymmetric graph: (i) Adding two noncrossing edges that are incident, (ii) two noncrossing edges that are not incident, and (iii) two crossing edges. These three cases are considered in the following three theorems.

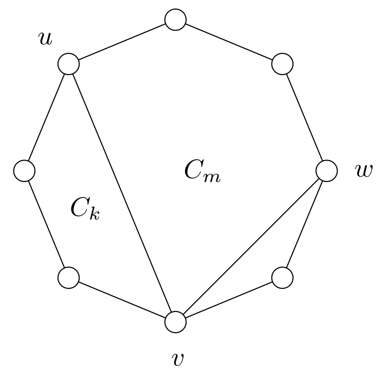

FIGURE 3. A cycle with two chords that are incident to the same vertex

Recall that a vertex v is called fixed if v is fixed under every automorphism of G.

Theorem 3.3. *Let G be the graph where two noncrossing edges are added to the cycle C_n so that the resulting graph has three chordless cycles C_k, C_m, and C_l, where C_m is the subgraph sharing edges with both C_k and C_l and $k+m+l=n+4$. Then the resulting graph is asymmetric if and only if $k \neq l$.*

Proof. Assume that $k \neq l$. Let v be the vertex that is incident to both chords, let u be the vertex that is part of both C_k and C_m, and let w be the vertex that is part of both C_l and C_m. The vertices u,v, and w are fixed since they are the only vertices contained in two of the three cycles C_k, C_m, and C_l. Each vertex x in C_k is fixed since the pairs $(d(x,u), d(x,v))$ are different for each vertex x. Each vertex y in C_l is fixed since the pairs $(d(y,u), d(y,w))$ are different for each vertex y. Each vertex z in C_l is fixed since the pairs $(d(z,v), d(z,w))$ are different for each vertex y. For the converse note that if $k = l$ then there is an axis of symmetry that passes through the vertex of degree 4 and the middle of the arc opposite this vertex. Hence G is nonasymmetric. □

In the following theorems for vertices x and y on the cycle C_n we use (x, y) to denote the vertices and edges on the minor arc between x and y and $l(x, y)$ to denote the number of edges on the minor arc of the cycle C_n between x and y.

Theorem 3.4. *Let G be the graph where two noncrossing, nonincident edges are added to the cycle C_n and the resulting graph has three chordless C_k, C_m, and C_l , where C_m is the subgraph between C_k and C_l and $k+m+l=n+4$. Let t and w be the vertices in both C_k and C_m and let u and v be the vertices in both C_m and C_l. Then the resulting graph is asymmetric if and only if $k \neq l$ and $l\,(v, w) \neq l\,(t, u)$.*

Proof. First assume that $k \neq l$ and $l\,(v, w) \neq l\,(t, u)$. We will proceed to show that every vertex in G is fixed. We first note that vertices t and w are the only vertices of degree 3 that are contained in both C_k and C_m. Since $l\,(v, w) \neq l\,(t, u)$, t and w have different distances to a vertex in C_l. Hence t and w are fixed. Next note that vertices u and v are the only vertices of degree 3 that are contained in both C_m and C_l. Since $l\,(v, w) \neq l\,(t, u)$, u and v have different distances to a vertex in

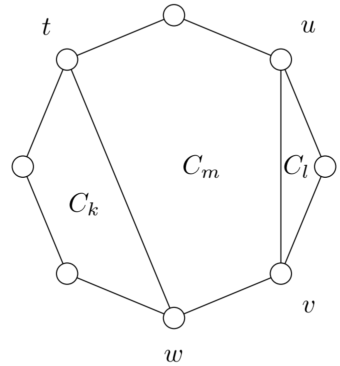

FIGURE 4. A cycle with two chords that do not cross

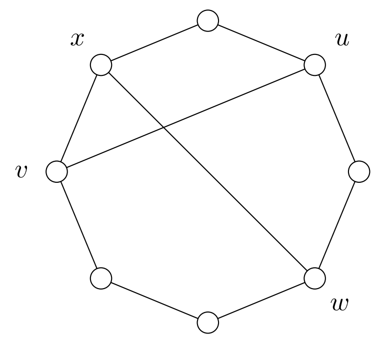

FIGURE 5. A cycle with two chords that cross

C_k. Hence u and v are fixed. Then all of the other vertices are fixed since no two vertices on any of the arcs t, u, u, v, v, w, and w, t have the same pair of distances to the end vertices of the arcs they are on. For the reverse direction first note that if $k = l$ then there is an axis of symmetry that passes through the middle of the arc u, w and the middle of the arc $w - t$. If $k \neq l$ and $l\,(v, w) = l\,(t, u)$ then G has an axis of symmetry that passes through the middle of the arc v, w and the middle of the arc t, u. Hence G is asymmetric. □

Theorem 3.5. *Let G be the graph with two crossing chords so that the resulting graph has three chordless C_k, C_m, and C_l , where C_m is the subgraph that shares edges with both C_k and C_l and $k + m + l = n + 4$. Furthermore the two edges have vertices u and v, and w and x, and the resulting graph has four arcs along the cycle, u, w, w, v, v, x, and x, u. Then the resulting graph is asymmetric if and only if:*
(i) $l(v, x) \neq l(x, u)$ or $l(w, v) \neq l(u, w)$, and
(ii) $l\,(u, w) \neq l\,(v, x)$ and $l\,(w, v) \neq l\,(x, u)$.

Proof. We first show that if (i) and (ii) both hold the graph is asymmetric. We will show that each vertex is fixed. If (i) holds the graph cannot have any reflectional symmetry about the chord from x to w. Then since $l(u, w) \neq l(v, x)$ and $l(w, v) \neq l(x, u)$ we have that u, v, w, and x are all fixed. Then all of the other vertices are fixed since no two vertices on the same arc have the same pair of distances to the two end vertices on the arcs. Hence G is asymmetric.

We next consider if either (i) or (ii) does not hold. If (i) does not hold then $l(v,x) = l(x,u)$ and $l(w,v) = l(u,w)$ then the graph has a line of reflection and is therefore nonasymmetric. Next we consider if (ii) does not hold. If $l\,(u,w) = l\,(v,x)$, then the graph has an axis of symmetry through the middle of the arcs x, u and v, w. The other case involving arcs w, v and x, u is similar. □

Next we will investigate the asymmetric index of wheel graphs. A wheel graph has n vertices and is formed with a universal vertex and an outer cycle.

Theorem 3.6. *For* $n \geq 6$, $ai(W_n) = 2$.

Proof. We will first show that $ai(W_n) > 1$. If we remove an edge incident to the vertex of degree $n-1$, this results in a vertex of degree two and the neighbors of this vertex are not fixed. Similarly, if we remove an edge whose endpoints are both degree three, we are left with two vertices of degree two that are each not fixed. Now if an edge is added to the graph, it must be added between two vertices of degree three. These vertices now have degree four and are each not fixed. It has now been established that $ai(W_n)$ is at least two.

Next we show the bound is tight. Removing an edge that is incident to one of these vertices of degree two, as well as incident to the vertex of degree $n-1$, then the resulting graph is asymmetric (see Figure 3). □

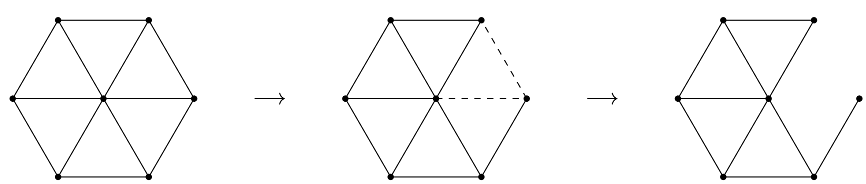

FIGURE 6. Changing a wheel into an asymmetric graph

3.3. **A Class of Circulant Graphs.** We extend our observations about cycles to a specific class of circulant graphs. The graph $C_n(L)$ has vertex set $\mathbb{Z}_n$, and connection set $L \subseteq \mathbb{Z}_n \setminus \{0\}$ satisfying $-L = L$. Two vertices $i, j \in \mathbb{Z}_n$ are adjacent in $C_n(L)$ if and only if $i - j \in L$ (since $-L = L$, $i - j \in L$ if and only if $j - i \in L$, so this relation is symmetric). The cycle C_n can alternatively be described as the circulant $C_n(\{1, -1\})$.

We note the following observation from [3].

Lemma 3.7. *If a graph G has order n, and $Aut(G)$ contains an element of order n, then G is a circulant graph.*

In particular, any graph G on n vertices with $\mathrm{Aut}(G) \cong D_n$ is a circulant. The converse does not hold in general, as the automorphism group of $C_n(\mathbb{Z}_n \setminus \{0\}) \cong K_n$ is isomorphic to S_n. Our goal is to show that every graph G with $\mathrm{Aut}(G) \cong D_n$ has asymmetric index 2. We will establish this by showing that for most such graphs, there is a "wedge" of nonedges that can be added to break all of the symmetries. We begin by defining these "wedges".

Suppose G has order n, and $\mathrm{Aut}(G)$ contains an element σ of order n. Choose an arbitrary vertex $v \in V(G)$, and consider the bijection $\varphi : V(G) \to \mathbb{Z}_n$ defined by $\varphi(u) = i$ where $u = \sigma^i(v)$. Note that the statement of Lemma 3.7 now follows as φ is an isomorphism from G to $C_n(L)$ where $L = \{i : \sigma^i(v) \leftrightarrow v\}$. Moreover, since $\sigma \in \mathrm{Aut}(G)$ has order n, L must contain some i that is relatively prime to n. So the edges defined by i give a subgraph of G isomorphic to C_n. This now implies that the map $k \mapsto k \cdot i$ is an automorphism of $C_n(L)$ that sends i to 1. So without loss of generality, we can assume that $1 \in L$.

A *wedge of nonedges* for G is a pair of nonedges $\{a, b\}$ and $\{b, c\}$ so that $|\varphi(a) - \varphi(b)| \neq |\varphi(b) - \varphi(c)|$. That is, the distance between the images of a and b along the cycle defined by σ is different than the distance between the images of b and c. The following lemma guarantees the existence of a wedge of nonedges in a class of circulants.

Lemma 3.8. *If G has order $n \geq 6$, $\overline{G} \ncong C_n$, and Aut$(G) \cong D_n$, then G has a wedge of nonedges.*

Proof. Let $v \in V(G)$, $\sigma \in \mathrm{Aut}(G)$, and φ be as described above, and let $C_n(L) \cong G$ be the resulting circulant. We show that there is a wedge of nonedges for $C_n(L)$ of the form $\{0, i\}$, $\{0, j\}$ for $1 \leq i < j \leq \lfloor n/2 \rfloor$.

First, note that since $\mathrm{Aut}(G) \ncong S_n$, $C_n(L)$ is not complete. So there is some $i \in \mathbb{Z}_n$ so that 0 and i are not adjacent. Since $C_n(L)$ is a circulant, this implies that there is some $1 \leq i \leq \lfloor n/2 \rfloor$ so that 0 and i are not adjacent. If there is any $j \neq i$ not adjacent to 0 with $1 \leq j \leq \lfloor n/2 \rfloor$, then we have our desired wedge.

Suppose no such j exists. Then $L = \mathbb{Z}_n \setminus \{0, j, -j\}$. Thus the complement of $C_n(L)$ is $C_n(\{j, -j\})$. Since $\overline{G} \ncong C_n$, we have $C_n(\{j, -j\}) \ncong C_n$. So $\gcd(j, n) > 1$, and $C_n(\{j, -j\})$ is isomorphic to a disjoint collection of j copies of $C_{n/j}$ (unless $j = n/2$, in which case $C_n(\{j, -j\}) = C_n(\{n/2\})$ is a disjoint collection of $n/2$ copies of K_2). So the automorphism group of $C_n(\{j, -j\})$ is the direct product of j copies of $D_{n/j}$ (or if $j = n/2$, the direct product of $n/2$ copies of S_2). In any case we see that $\mathrm{Aut}(C_n(\{j, -j\})) \ncong D_n$. However, since $\mathrm{Aut}(G) \cong \mathrm{Aut}(\overline{G})$ we have a contradiction. $\square$

We are now prepared to prove the main theorem in this section.

Theorem 3.9. *If G has order $n \geq 6$, and Aut$(G) \cong D_n$, then the asymmetric index of G is 2. If $\overline{G} \ncong C_n$, then some addition of two edges to G results in an asymmetric graph. If $\overline{G} \cong C_n$, then some deletion of two edges from G results in an asymmetric graph.*

Proof. Suppose $\overline{G} \ncong C_n$. From Lemma 3.8, G has a wedge of nonedges $\{a, b\}$ and $\{b, c\}$. Again, we consider the circulant $C_n(L)$ isomorphic to G constructed by applying $\sigma \in \mathrm{Aut}(G)$ of order n repeatedly to the vertex b. Recall that without loss of generality, $1 \in L$. So there is a subgraph H of G with $H \cong C_n$, and since $\{a, b\}$ and $\{b, c\}$ form a wedge, the distance between a and b on H is distinct from the distance between b and c. Now from Theorem 3.3, $H + \{\{a, b\}, \{b, c\}\}$ is asymmetric. Thus $G + \{\{a, b\}, \{b, c\}\}$ is asymmetric.

Now suppose $\overline{G} \cong C_n$. Since the graph obtained from C_n by deleting any two edges is never asymmetric, adding any pair of edges to G never results in an asymmetric graph. However, if e, f is any pair of edges so that $\overline{G} + \{e, f\}$ is asymmetric, then $G \setminus \{e, f\}$ is asymmetric. Theorem 3.3 (for instance) guarantees the existence of such a pair of edges. $\square$

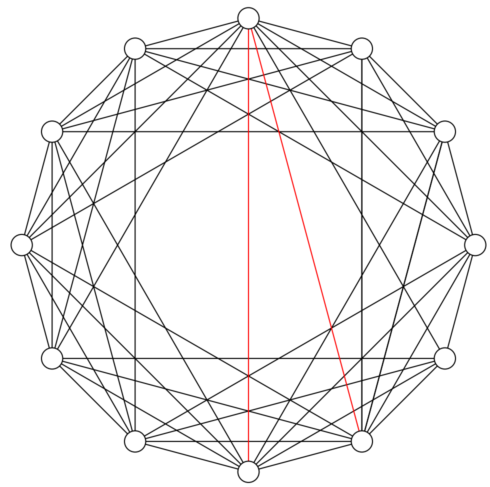

FIGURE 7. A wedge of nonedges (in red) is added to C_{12} to create an asymmetric graph

For example, Figure 7 gives a demonstration of Theorem 3.9. We also note that the Möbius ladder graphs satisfy the hypotheses of Theorem 3.9.

Corollary 3.10. *Let $n \geq 4$ and let ML_{2n} be the Möbius ladder graph with $2n$ vertices. Then $ai\,(ML_{2n}) = 2$.* $\square$

3.4. **Complete graphs.** We next investigate complete graphs K_n (and their complements nK_1 known as null graphs) and show they have higher asymmetric indices than other graphs we have encountered. This is expected as the automorphism group of K_n is S_n and the automorphism group of an asymmetric graph is trivial. We note that $ai\,(K_1) = 0$, and $ai\,(K_n)$ is not defined when $2 \leq n \leq 5$. To determine $ai\,(K_6)$ we start with six isolated vertices and add the edges of the asymmetric graph on six vertices (shown in Figure 2). To determine $ai\,(K_7)$ we start with

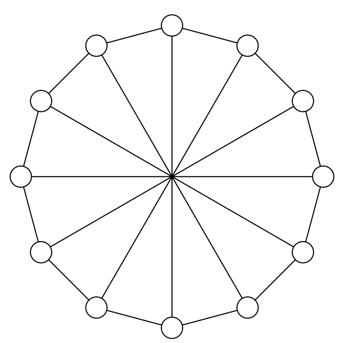

FIGURE 8. A möbius ladder graph

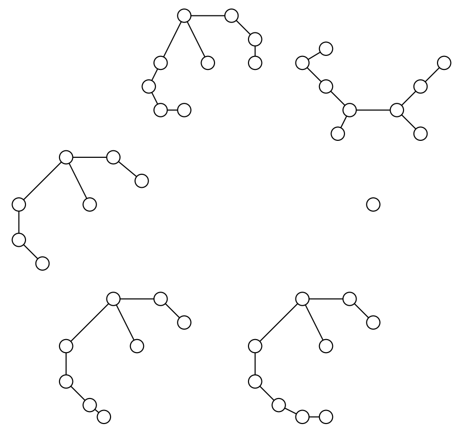

FIGURE 9. An asymmetric graph with 43 vertices and 37 edges

seven isolated vertices and add the edges of the asymmetric graph on six vertices. In both cases using any fewer edges results in a graph that is not asymmetric. Hence $ai\,(6K_1) = 6$ and $ai\,(7K_1) = 6$. By Proposition 2.1, $ai\,(K_6) = 6$ and $ai\,(K_7) = 6$. We have shown in the proof of Theorem 2.7 when $8 \leq n \leq 15$, $ai\,(K_n) = n - 2$. It is not too difficult to establish an upper bound for $ai(K_n)$. There exists an asymmetric tree H with seven vertices and six edges. By Proposition 2.4 and Lemma 2.5 H can be extended to an asymmetric graph H_n consisting of a tree with $n - 1$ vertices along with an isolated vertex. Then by Proposition 2.1, $K_n - H_n$ will be asymmetric. We have shown that for $n \geq 8$, $ai(K_n) \leq n - 2$. For larger cases this bound can be improved, as there exists a larger number of nonisomorphic trees with a total of n vertices. As mentioned in the proof of Theorem 2.7, $ai\,(K_{16}) = 13$ since there exist three distinct asymmetric trees with a total of 16 vertices.

We continue with a larger example. Consider $ai(K_{43})$. It is known that there is a single asymmetric tree with 7 vertices and a single asymmetric tree with 8 vertices and there are three nonisomorphic asymmetric trees on 9 vertices. By starting with 43 isolated vertices we can add the edges necessary to construct each of these trees. This graph has five nontrivial trees and an isolated vertex, and a total of 37 edges (see Figure 9).

Since removing any edge will result in two vertices that are each not fixed, it follows that $ai(43K_1) = 37$ which by Proposition 2.1 implies that $ai(K_{43}) = 37$. What led to this improvement over $ai(K_n) = n - 2$ is the presence of multiple nonisomorphic asymmetric graphs of the same order. The more of these graphs the better the bound will be. In general when $n \geq 15$, $ai\,(K_n) = n - t_n$ where t_n is the number of distinct asymmetric trees that exist on a total of n vertices. We note that in general, t_n is not known. In fact, t_n relies on the number of asymmetric trees on a fixed number of vertices, which is unknown. Noe and Heinz computed the number of asymmetric trees on n vertices for $n = 1, \ldots, 1000$ (OEIS A000220 [4]). We can create a rough general lower bound using multiple copies of the asymmetric tree with seven vertices and six edges. This bound can be improved by taking graphs that are not isomorphic (if they can be identified). Let G be a graph with n vertices. We first isolate a single vertex. Then we construct $\left\lfloor \frac{n-1}{7} \right\rfloor$

sets with seven vertices and one remaining set with $n-1-7\left\lfloor\frac{n-1}{7}\right\rfloor$ vertices. From this we create $\left\lfloor\frac{n-1}{7}\right\rfloor-1$ asymmetric trees with seven vertices and one asymmetric tree with $n-1-7\left(\left\lfloor\frac{n-1}{7}\right\rfloor-1\right)$ vertices. The total number of edges in this graph will be $6\left\lfloor\frac{n-1}{7}\right\rfloor-1+n-1-7\left(\left\lfloor\frac{n-1}{7}\right\rfloor-1\right)-1=n-\left\lfloor\frac{1}{7}n-\frac{1}{7}\right\rfloor+4$.

Hence we have proved the following general formula which for specific cases can be improved.

Theorem 3.11. *For $n \geq 16$, $n-\left\lfloor\frac{1}{7}n-\frac{1}{7}\right\rfloor+4 \leq ai(K_n) \leq n-2$.*

By taking the disjoint union of nonisomorphic asymmetric trees we can construct graphs that have a relatively larger asymmetric index. For example, if we were to take the disjoint union of all nonisomorphic asymmetric trees up to 17 vertices (quantities given by Noe and Heinz [4]) we would create a graph with 43,914 vertices and 41,196 edges. Here $\frac{ai(G)}{n} \approx 0.938$. It appears that by taking the disjoint union of all nonisomorphic asymmetric trees with larger orders, $\frac{ai(G)}{n} \to 1$.

3.5. **Complete bipartite graphs.** We begin with a lemma which will be useful in our next two theorems.

Lemma 3.12. *If a graph G has a set of t vertices where any pair can be transposed, then $ai(G) \geq \left\lfloor\frac{t-1}{2}\right\rfloor$.*

Proof. Let T be a set of t vertices, any two of which can be transposed. To eliminate all symmetries in G we must either add or remove an edge incident to each vertex in T. Since the addition of an edge can be incident to two vertices, the minimum number of edges that needs to be added or removed is equal to $\left\lfloor\frac{t-1}{2}\right\rfloor$. □

We next present bounds for complete bipartite graphs, $K_{a,b}$.

Theorem 3.13. *For $n \geq 6$, $\left\lfloor\frac{n-1}{2}\right\rfloor \leq ai\left(K_{1,n-1}\right) \leq n-1$.*

Proof. The lower bound follows from Lemma 3.12. For the upper bound, It may be helpful to refer to the third graph in Figure 6. We start by adding edges so that $n-1$ of the vertices that are not the universal vertex will be connected by a path with $n-2$ edges. Then removing the edge between an end vertex on the path and the universal vertex will result in an asymmetric graph. □

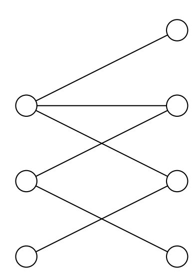

FIGURE 10. An asymmetric tree with seven vertices

Next we present bounds for complete bipartite graphs with at least 6 vertices.

Theorem 3.14. *Let $G = K_{a,b}$. Then*

(1) *If $a = 2$ then $1+\left\lfloor\frac{b}{2}\right\rfloor \leq ai(K_{2,i}) \leq 2i-4$*

(2) *If* $a = 3$ *and* $b = 3$, $2 \leq ai(K_{3,3}) \leq 6$.
(3) *When* $a \geq 3$ *and* $b \geq 4$, $\left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{b}{2} \right\rfloor \leq ai\left(K_{a,b}\right) \leq ab - 6$.

Proof. The lower bounds follow from Lemma 3.12.

We next prove 1. Starting with $K_{2,i}$ remove but five edges and then add in the dashed edge to form an asymmetric graph with six edges.

For 2, we remove three edges so that the resulting graph is P_6. Then we add a single edge to create the asymmetric graph with six vertices and six edges. A graph for the case where $i = 4$ is shown in Figure 11.

For 3, the lower bound follows Lemma 3.12. □

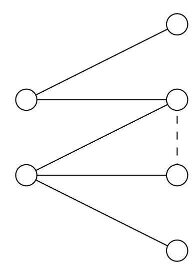

FIGURE 11. A construction for $K_{2,4}$ showing that $ai(K_{2,4}) \leq 4$

3.6. **Cartesian products of paths and cycles.** We next investigate the asymmetric index for grids and cylinders.

Theorem 3.15. *Let* $G = (P_s \Box P_t)$. *Then*
(1) $ai(P_2 \Box P_3) = 2$.
(2) *If* $s, t \geq 3$ *then* $ai(P_s \Box P_t) = 1$.

Proof. For 1, we first note that the removal of any single edge results in a graph that has either horizontal or vertical symmetry. Next we show that the asymmetric index of this graph equals 2. We first remove the edge between $(2, 0)$ and $(2, 1)$. Then we add an edge between $(1, 0)$ and $(2, 1)$ to create an asymmetric graph.

For 2, note that all automorphisms of a grid graph $P_s \Box P_t$ are compositions of horizontal or vertical reflections about the midlines of the grid. As a result we can remove the edge between (0.0) and $(1, 0)$ and the resulting graph will be asymmetric. □

We next consider cylinders, the Cartesian product of paths and cycles.

Theorem 3.16. *For all* $s \geq 2$ *and* $t \geq 3$, $ai(P_s \Box C_t) = 2$.

Proof. The only automorphisms of a cylinder are reflections and rotations. For the lower bound note that if we remove a single edge uv from G then there is a line of symmetry passing through the missing edge. Hence $ai(P_s \Box C_t) \geq 2$.

It may be helpful to refer to Figure 12. For the upper bound we remove the edges uv and uw from $P_s \Box C_t$. This creates a graph without reflections or rotational symmetries. □

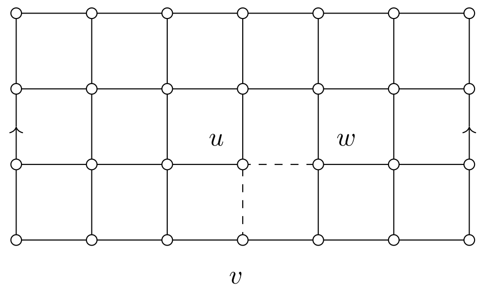

FIGURE 12. A cylinder (where the left side and right side are the same)

4. CONCLUSION

We have shown that there exist infinite families of graphs with a small asymmetric index, including those with an asymmetric index of 1. However the largest possible asymmetric index for a graph of order n is not known. We pose this as the following open problem

Problem. *Determine for graphs with n vertices, which graphs have the highest asymmetric index.*

In Section 3.3 we investigated a specialized class of circulant graphs. It would be interesting to investigate the asymmetric index of other families of circulant graphs. For more information on automorphism groups of circulant graphs the reader is referred to [3].

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Department of Mathematics and Statistics, Auburn University, Auburn, AL
E-mail address: `breweralie@gmail.com`

Department of Mathematics, University of Rochester, Rochester, NY
E-mail address: `farnsworthej@gmail.com`

Department of Mathematics, Texas State University, Round Rock, TX
E-mail address: `12nat26@gmail.com`

Mathematics Department, University of Florida, Gainesville, FL
E-mail address: `adamgregory@ufl.edu`

Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, VA
E-mail address: `quindel.d.jones@gmail.com`

Department of Exercise Science and Athletic Training, Springfield College, Springfield, MA
E-mail address: `hxl1240@rit.edu`

School of Mathematics and Statistics, Rochester Institute of Technology, Rochester, NY
E-mail address: `brsma@rit.edu`

School of Mathematics and Statistics, Rochester Institute of Technology, Rochester, NY
E-mail address: `dansma@rit.edu`