Contributions to Discrete Mathematics

Volume 19, Number 3, Pages 178–183
ISSN 1715-0868

CONSTRUCTION OF THE PROJECTIVE PLANE PG$(2, q^2)$ FROM THE UNITARY GROUP PSU$(3, q)$

DEAN CRNKOVIĆ, VEDRANA MIKULIĆ CRNKOVIĆ, FRANCESCO PAVESE, AND ANDREA ŠVOB

Abstract. In 2013, the first and the second author of this paper described a construction of the projective plane PG$(2, q^2)$ from the unitary group PSU$(3, q)$, for $q = 3, 4, 5, 7$. The construction is obtained by using a computer. In the same paper, it is conjectured that in a similar way one can construct the projective plane PG$(2, q^2)$ from the unitary group PSU$(3, q)$, for every prime power q. In this paper, we give a construction of a Desarguesian projective plane from a unitary group that confirms this conjecture.

1. Introduction

We assume that the reader is familiar with the basic facts of group theory, design theory and finite geometry. We refer the reader to [1, 8] for relevant background reading in design theory, to [2, 7] for relevant background reading in group theory and to [5] for relevant background reading in finite geometry.

An *incidence structure* is an ordered triple $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are non-empty disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$. The elements of the set $\mathcal{P}$ are called *points*, the elements of the set $\mathcal{B}$ are called *blocks* and $\mathcal{I}$ is called an *incidence relation*. If $|\mathcal{P}| = |\mathcal{B}|$, then the incidence structure is called *symmetric*. The *incidence matrix* of an incidence structure is a $b \times v$ matrix $[m_{ij}]$ where v and b are the numbers of points and blocks respectively, such that $m_{ij} = 1$ if the point P_j and the block x_i are incident, and $m_{ij} = 0$ otherwise. An *isomorphism* from one incidence structure to another is a bijective mapping of points to points and blocks to blocks which preserves incidence. An isomorphism from an incidence structure $\mathcal{D}$ onto itself is called

Received by the editors August 22, 2019, and in revised form August 9, 2022.

1991 *Mathematics Subject Classification.* 05B05, 51E15, 05E18.

Key words and phrases. projective plane, unitary group, block design, Hermitian unital.

This work has been partially supported by Croatian Science Foundation under the projects 6732 and 5713. D. Crnković and A. Švob were supported by the University of Rijeka under the project uniri-prirod-18-51, and V. Mikulić Crnković was supported by the University of Rijeka under the project uniri-prirod-18-111-1249.

an *automorphism* of $\mathcal{D}$. The set of all automorphisms forms a group called the *full automorphism group* of $\mathcal{D}$ and is denoted by $Aut(\mathcal{D})$.

A t-(v,k,λ) *design* or a t-*design* is a finite incidence structure $\mathcal{D} = (\mathcal{P},\mathcal{B},\mathcal{I})$ satisfying the following requirements: $|\mathcal{P}| = v$, every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$, and every t elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$. Blocks can be regarded as subsets of the set of points. 2-designs are called *block designs*. A symmetric design with parameters 2-$(v,k,1)$ is called a *projective plane*. A classical projective plane over the finite field GF(q) (see [5]) is denoted by PG$(2,q)$ and called a *Desarguesian projective plane*. A *unital* of order n is a design with parameters 2-$(n^3+1,n+1,1)$. The *Hermitian unital* of order q, q a prime power, consists of the absolute points and non-absolute lines of a unitary polarity in the Desarguesian plane PG$(2,q^2)$. A *semi-symmetric* $(v,k,(\lambda))$ *design* is a finite incidence structure with v points and b blocks such that every point (block) is incident with exactly k blocks (points) and every pair of points (blocks) are incident with 0 or λ blocks (points).

In [3], with the aid of a computer, the authors constructed PG$(2,q^2)$ using the groups PSU$(3,q)$, $q = 3,4,5,7$, respectively, and conjectured that the construction can be generalized for every prime power q. In this paper, we confirm this conjecture by giving the construction described in Theorem 3.1.

The paper is organized as follows. In Section 2, we briefly describe the method of construction used in [3] to construct with a computer the Desarguesian plane PG$(2,q^2)$ from the group PSU$(3,q)$, $q = 3,4,5,7$, and give the conjecture proposed in [3]. In Section 3, we give a computer-free construction of PG$(2,q^2)$ from the group PSU$(3,q)$ for any prime power q, that confirms the proposed conjecture.

2. Construction of the projective plane PG$(2,q^2)$ from the unitary group PSU$(3,q)$, for $q = 3,4,5,7$

The method for constructing 1-designs from a primitive action of a group was introduced in [3], and further generalized in [4] for any transitive action. That generalized method of construction is given in Theorem 2.1.

Theorem 2.1 ([4]). *Let G be a finite permutation group acting transitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^{s} \delta_i G_\alpha$, where $G_\alpha = \{g \in G \mid \alpha g = \alpha\}$ is the stabilizer of α and $\delta_1,...,\delta_s \in \Omega_2$ are representatives of distinct G_α-orbits on Ω_2. If $\Delta_2 \neq \Omega_2$ and $\mathcal{B} = \{\Delta_2 g : g \in G\}$, then $\mathcal{D}(G,\alpha,\delta_1,...,\delta_s) = (\Omega_2,\mathcal{B})$ is a 1-$(n, |\Delta_2|, \frac{|G_\alpha|}{|G_{\Delta_2}|}\sum_{i=1}^{s}|\alpha G_{\delta_i}|)$ design with $\frac{m\cdot|G_\alpha|}{|G_{\Delta_2}|}$ blocks. The group $H \cong G/\bigcap_{x\in\Omega_2} G_x$ acts as an automorphism group on $(\Omega_2,\mathcal{B})$, transitively on points and blocks of the design.*

If $\Delta_2 = \Omega_2$ then the set $\mathcal{B}$ consists of one block, and $\mathcal{D}(G,\alpha,\delta_1,...,\delta_s)$ is a design with parameters 1-$(n,n,1)$.

Taking into consideration each of the groups $\mathrm{PSU}(3,q)$, $q = 3,4,5,7$, by applying Theorem 2.1 one can construct a Hermitian unital 2-$(q^3+1, q+1, 1)$ and a semi-symmetric design $(q^4-q^3+q^2, q^2-q, (1))$, having $\mathrm{PSU}(3,q)$ as an automorphism group, for $q = 3,4,5,7$. With $\mathcal{D}_1$ we denote the trivial design with parameters 1-$(q^3+1, 1, 1)$, with $\mathcal{D}_2$ the Hermitian unital with parameters 2-$(q^3+1, q+1, 1)$ and with $\mathcal{D}_3$ the semi-symmetric design with parameters 2-$(q^4-q^3+q^2, q^2-q, (1))$. Further with I, M_2 and M_3 we denote their incidence matrices, respectively. As it is shown in [3], the matrix

$$M = \left[\begin{array}{c|c} I & M_2^t \\ \hline M_2 & M_3 \end{array}\right]$$

is the incidence matrix of the Desarguesian projective plane $\mathrm{PG}(2,q^2)$, $q = 3,4,5,7$.

In [3], the following conjecture was proposed. From any group $\mathrm{PSU}(3,q)$, using the construction described above, one can construct the incidence matrix M_2 of a Hermitian unital and the incidence matrix M_3 of a semi-symmetric design which build the incidence matrix M of the Desarguesian projective plane $\mathrm{PG}(2,q^2)$. In Section 3, we give the construction that proves the proposed conjecture.

3. Construction of a projective plane from conjugacy classes of maximal subgroups

Let $\mathrm{PG}(2,q^2)$ be the finite projective plane over the finite field $\mathrm{GF}(q^2)$, equipped with homogeneous projective coordinates X_1, X_2, X_3. First we summarize some properties of a Hermitian curve, see [5, Chapter II]. Let $\mathcal{H}$ be the Hermitian curve of $\mathrm{PG}(2,q^2)$ having equation $X_1^{q+1}+X_2^{q+1}+X_3^{q+1} = 0$ and let $\perp$ be the unitary polarity of $\mathrm{PG}(2,q^2)$ defining $\mathcal{H}$. In particular, for a point $P = (x_1, x_2, x_3)$ in $\mathrm{PG}(2,q^2)$, we have that $P^\perp$ is the line of $\mathrm{PG}(2,q^2)$ having equation $x_1^q X_1 + x_2^q X_2 + x_3^q X_3 = 0$. Moreover, $\perp$ is an involutory bijection between points and lines of $\mathrm{PG}(2,q^2)$ reversing incidences, i.e., $P \in \ell$ if and only if $\ell^\perp \in P^\perp$. The Hermitian curve $\mathcal{H}$ has q^3+1 points, hence, if $\mathcal{S} := \mathrm{PG}(2,q^2) \setminus \mathcal{H}$, we have that $|\mathcal{S}| = q^4 - q^3 + q^2$. A line of $\mathrm{PG}(2,q^2)$ either meets $\mathcal{H}$ in one point and it is called a *tangent line*, or meets $\mathcal{H}$ in $q+1$ points and it is called a *secant line*. Through a point of $\mathcal{H}$ there pass one tangent line and q^2 secant lines, whereas through a point of $\mathcal{S}$ there pass $q+1$ tangent lines and q^2-q secant lines. Let $P \in \mathcal{H}$ and $R \in \mathcal{S}$. Then $P^\perp$ is the line of $\mathrm{PG}(2,q^2)$ tangent to $\mathcal{H}$ at P, whereas $R^\perp$ is a secant line. It follows that the q^2 secant lines through P are $\{T^\perp \mid T \in P^\perp, T \neq P\}$ and that the $q+1$ tangent lines containing R are those obtained by joining R with $R^\perp \cap \mathcal{H}$.

Let U be the subgroup of $\mathrm{PGL}(3,q^2)$ isomorphic to $\mathrm{PSU}(3,q)$ leaving $\mathcal{H}$ invariant. The group U, of size

$$\frac{q^3(q^3+1)(q^2-1)}{M(q+1,3)},$$

has a maximal subgroup of index q^3+1 (denote that subgroup with S_1) and a maximal subgroup of index $q^4-q^3+q^2$ (denote that subgroup with S_2). The following holds:

- for $P_1, P_2 \in ccl_U(S_1)$ the group $P_1 \cap P_2$ is isomorphic to $Z_{\frac{q^2-1}{M(q+1,3)}}$,
- for $P_1 \in ccl_U(S_1)$ and $P_2 \in ccl_U S_2$ the group $P_1 \cap P_2$ is isomorphic to $Z_{\frac{q+1}{M(q+1,3)}}$ or $Z_q : Z_{\frac{q^2-1}{M(q+1,3)}}$,
- for $P_1, P_2 \in ccl_U(S_2)$, $P_1 \neq P_2$, the group $P_1 \cap P_2$ is isomorphic to E_q, $Z_{\frac{q+1}{M(q+1,3)}}$, or $Zq+1 \times Z_{\frac{q+1}{M(q+1,3)}}$.

Consider the following incidence structures.

- The incidence structure $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1)$, where $\mathcal{P}_1 = ccl_U(S_1)$ and $\mathcal{B}_1 = ccl_U(S_1)$, and a point S_1^g is incident with a block S_1^h if and only if $S_1^g = S_1^h$.
- The incidence structure $\mathcal{D}_2 = (\mathcal{P}_2, \mathcal{B}_2)$, where $\mathcal{P}_2 = ccl_U(S_1)$ and $\mathcal{B}_2 = ccl_U(S_2)$, and a point S_1^g is incident with a block S_2^h if and only if $S_1^g \cap S_2^h \cong Z_q : Z_{\frac{q^2-1}{M(q+1,3)}}$.
- The incidence structure $\mathcal{D}_3 = (\mathcal{P}_3, \mathcal{B}_3)$, where $\mathcal{P}_3 = ccl_U(S_2)$ and $\mathcal{B}_3 = ccl_U(S_2)$, and a point S_2^g is incident with a block S_2^h if and only if $S_2^g \cap S_2^h \cong Z_{q+1} \times Z_{\frac{q+1}{M(q+1,3)}}$.

Theorem 3.1. *$\mathcal{D}_1$ is the trivial design with parameters* 1-$(q^3+1, 1, 1)$*, $\mathcal{D}_2$ is a Hermitian unital with parameters* 2-$(q^3+1, q+1, 1)$ *and $\mathcal{D}_3$ is a semi-symmetric design with parameters* 2-$(q^4-q^3+q^2, q^2-q, (1))$*. Moreover, if M_2 is the incidence matrix of the incidence structure $\mathcal{D}_2$ and M_3 is the incidence matrix of the incident structure $\mathcal{D}_3$, then the matrix*

$$M = \left[\begin{array}{c|c} I & M_2^t \\ \hline M_2 & M_3 \end{array}\right]$$

is the incidence matrix of the projective plane PG$(2, q^2)$.

Proof. Let us denote with $P_1, \ldots, P_{q^3+1}$ the points of the Hermitian curve $\mathcal{H}$ and with $R_1, \ldots, R_{q^4-q^3+q^2}$ the points of $\mathcal{S}$. From [6, Theorem 2.6, Theorem 2.7], the group S_1 is the stabilizer of a point P_i of $\mathcal{H}$ together with its polar line, whereas the group S_2 is the stabilizer of a point R_j of $\mathcal{S}$ together with its polar line. In particular, there is a one-to-one correspondence between the elements of $ccl_U S_1$ and the points of $\mathcal{H}$ and the elements of $ccl_U S_2$ and the points of $\mathcal{S}$. Moreover, it is not difficult to see that $Stab_U(P_i) \cap Stab_U(R_j)$ is isomorphic either to $Z_{\frac{q+1}{M(q+1,3)}}$ or to $Z_q : Z_{\frac{q^2-1}{M(q+1,3)}}$, depending on whether $P_i \notin R_j^\perp$ or $P_i \in R_j^\perp$, respectively. Similarly, if $j \neq k$, then $Stab_U(R_j) \cap Stab_U(R_k)$ is isomorphic either to E_q or to $Z_{\frac{q+1}{M(q+1,3)}}$ or to $Z_{q+1} \times Z_{\frac{q+1}{M(q+1,3)}}$, depending on whether $R_j \notin R_k^\perp$ and the line R_jR_k is tangent or $R_j \notin R_k^\perp$ and the line R_jR_k is secant or $R_j \in R_k^\perp$, respectively.

Therefore, the incidence structures previously introduced can be described as follows. Here the incidence relation is the containment.

- $\mathcal{D}_1 = (\mathcal{H}, \mathcal{B}_1)$, where $\mathcal{B}_1 = \{P_i^{\perp} \cap \mathcal{H} \mid 1 \leq i \leq q^3 + 1\}$. Then the incidence matrix of $\mathcal{D}_1$ is the following:

$$\begin{array}{c|ccc} & P_1 & \dots & P_{q^3+1} \\ \hline P_1^{\perp} \cap \mathcal{H} & & & \\ \vdots & & I_{q^3+1} & \\ P_{q^3+1}^{\perp} \cap \mathcal{H} & & & \end{array}$$

  where I_{q^3+1} is the identity matrix of order $q^3 + 1$.
- $\mathcal{D}_2 = (\mathcal{H}, \mathcal{B}_2)$, where $\mathcal{B}_2 = \{R_i^{\perp} \cap \mathcal{H} \mid 1 \leq i \leq q^4 - q^3 + q^2\}$. Then the incidence matrix of $\mathcal{D}_2$ is the following:

$$\begin{array}{c|ccc} & P_1 & \dots & P_{q^3+1} \\ \hline R_1^{\perp} \cap \mathcal{H} & & & \\ \vdots & & M_2 & \\ R_{q^4-q^3+q^2}^{\perp} \cap \mathcal{H} & & & \end{array}$$

  where each row of M_2 has exactly $q+1$ ones and each column of M_2 has exactly q^2 ones.

  Note that $P_i \in R_j^{\perp}$ if and only if $R_j \in P_i^{\perp}$. Hence, if we consider $\mathcal{D}_2' = (\mathcal{S}, \mathcal{B}_3)$, where $\mathcal{B}_3 = \{P_i^{\perp} \cap \mathcal{S} \mid 1 \leq i \leq q^3 + 1\}$, we have that the incidence matrix for $\mathcal{D}_2'$ is the following:

$$\begin{array}{c|ccc} & R_1 & \dots & R_{q^4-q^3+q^2} \\ \hline P_1^{\perp} \cap \mathcal{H} & & & \\ \vdots & & M_2^t & \\ P_{q^3+1}^{\perp} \cap \mathcal{H} & & & \end{array}$$

  where M_2^t is the transpose matrix of M_2.
- $\mathcal{D}_4 = (\mathcal{S}, \mathcal{B}_4)$, where $\mathcal{B}_4 = \{R_i^{\perp} \cap \mathcal{S} \mid 1 \leq i \leq q^4 - q^3 + q^2\}$. Then the incidence matrix of $\mathcal{D}_4$ is the following:

$$\begin{array}{c|ccc} & R_1 & \dots & R_{q^4-q^3+q^2} \\ \hline R_1^{\perp} \cap \mathcal{S} & & & \\ \vdots & & M_3 & \\ R_{q^4-q^3+q^2}^{\perp} \cap \mathcal{S} & & & \end{array}$$

  where each row and column of M_3 contains exactly $q^2 - q$ ones. In particular, M_3 is symmetric, indeed $R_i \in R_j^{\perp}$ if and only if $R_j \in R_i^{\perp}$.

Note that $\mathcal{H} \cup \mathcal{S}$ is the whole set of points of $\mathrm{PG}(2, q^2)$ and $|\mathcal{H} \cap \mathcal{S}| = 0$. Hence, for a line ℓ, we have that $\ell = (\ell \cap \mathcal{H}) \cup (\ell \cap \mathcal{S})$. Therefore, if we consider the following matrix

$$\begin{array}{c|ccc|ccc} & P_1 & \dots & P_{q^3+1} & R_1 & \dots & R_{q^4-q^3+q^2} \\ \hline P_1^{\perp} & & & & & & \\ \vdots & & I_{q^3+1} & & & M_2^t & \\ P_{q^3+1}^{\perp} & & & & & & \\ \hline R_1^{\perp} & & & & & & \\ \vdots & & M_2 & & & M_3 & \\ R_{q^4-q^3+q^2}^{\perp} & & & & & & \end{array}$$

we get a square matrix of order $q^4 + q^2 + 1$, whose columns are indexed by the points of PG$(2, q^2)$ and rows by the lines of PG$(2, q^2)$. It follows that this matrix is the incidence matrix of PG$(2, q^2)$. $\square$

References

1. T. Beth, D. Jungnickel and H. Lenz, *Design Theory*, 2nd Edition, Cambridge University Press, Cambridge, 1999.
2. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson and J. G. Thackray, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.
3. D. Crnković, V. Mikulić, *Unitals, projective planes and other combinatorial structures constructed from the unitary groups $U(3, q)$, $q = 3, 4, 5, 7$*, Ars Combin. 110 (2013), 3–13.
4. D. Crnković, V. Mikulić, A. Švob, *On some transitive combinatorial structures constructed from the unitary group $U(3, 3)$*, J. Statist. Plann. Inference 144 (2014), 19–40.
5. D. R. Hughes, F. C. Piper, *Projective planes*, Graduate Texts in Mathematics, Vol. 6. Springer–Verlag, New York-Berlin, 1973.
6. O. H. King, *The subgroup structure of finite classical groups in terms of geometric configurations*, Surveys in combinatorics 2005, 29–56, London Math. Soc. Lecture Note Ser., 327, Cambridge Univ. Press, Cambridge, 2005.
7. D. Robinson, *A Course in the Theory of groups*, Springer-Verlag, New York, Berlin, Heidelberg, 1996.
8. V. D. Tonchev, *Combinatorial Configurations: Designs, Codes, Graphs*, John Willey & Sons, New York, 1988.

Faculty of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia
E-mail address: `deanc@math.uniri.hr`

Faculty of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia
E-mail address: `vmikulic@math.uniri.hr`

Dipartimento di Meccanica Matematica e Management, Politecnico di Bari, Via Orabona 4, I-70125 Bari, Italy
E-mail address: `francesco.pavese@poliba.it`

Faculty of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia
E-mail address: `asvob@math.uniri.hr`