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UNIFORMLY RESOLVABLE $\{P_4, C_k\}$-DECOMPOSITION OF K_n - A COMPLETE SOLUTION

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Abstract. Let K_n, C_n, and P_n respectively denote the complete graph, cycle and path on n vertices. Uniformly resolvable decomposition of K_n is a decomposition of K_n into subgraphs which can be partitioned into factors containing pairwise isomorphic subgraphs. In this paper, we determine necessary and sufficient conditions for the existence of uniformly resolvable decomposition of K_n into P_4 and C_k, $k \geq 3$.

1. Introduction

All graphs considered here are finite. Let P_n, C_n, K_n, and I_n denote the path, cycle, complete graph, and independent set on n vertices, respectively. Let λG denote the λ edge-disjoint copies of G. A complete m-partite graph with partite sets $V_0, V_1, \ldots, V_{m-1}$ consisting of $n_0, n_1, \ldots, n_{m-1}$ vertices respectively is denoted as $K_{n_0,n_1,\ldots,n_{m-1}}$. $K_n - I$ denotes the complete graph with a 1-factor removed when n is even.

For two graphs G and H their *wreath product* $G \otimes H$ has the vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. One can easily observe that $K_m \otimes I_n \cong K_{n,n,\ldots,n}$, the complete m-partite graph in which each partite set has exactly n vertices. We write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_l$, if H_1, $H_2, \ldots, H_l$ are edge-disjoint subgraphs of G and $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_l)$. Note that, by the properties of the wreath product, if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, and $H \cong I_n$ then $G \otimes H = (H_1 \otimes I_n) \oplus (H_2 \otimes I_n) \oplus \cdots \oplus (H_k \otimes I_n)$. For more details on product graphs, see [18].

For a given collection $\mathcal{H}$ containing simple graphs, an $\mathcal{H}$*-decomposition* of a graph G is a set of subgraphs of G whose edge set partition $E(G)$,

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and each subgraph is isomorphic to a graph from $\mathcal{H}$. A *factor* of a graph G is a spanning subgraph of G. A factor is called *uniform H-factor* if each component of the factor is isomorphic to the same graph H. An *r-factor* of G is an r-regular spanning subgraph of G. An $\mathcal{H}$-decomposition of a graph G is called *uniformly resolvable $\mathcal{H}$-decomposition* if the subgraphs in the $\mathcal{H}$-decomposition can be partitioned into uniform H-factors, for some $H \in \mathcal{H}$. Suppose $\mathcal{H} = \{H\}$, uniformly resolvable $\mathcal{H}$-decomposition is called *H-factorization.*

Recently, lots of results have been obtained on uniformly resolvable $\mathcal{H}$-decomposition of a graph K_n. The existence of uniformly resolvable $\mathcal{H}$-decompositions of K_n has been studied in the cases, when $\mathcal{H} = \{K_k\}$ with $k = 3, 4, 5$ (for $k = 5$ there are only four undecided values of n), see [1]; $\mathcal{H} = \{P_k\}$ for any $k \geq 2$ [4, 12, 14]; $\mathcal{H}$ is a set of two complete graphs of order at most five [7, 25, 26, 27, 28, 29]; $\mathcal{H}$ is a set of two paths on two, three, or four vertices [10, 11]; $\mathcal{H} = \{P_3, K_3 + e\}$ [9]; $\mathcal{H} = \{K_3, K_{1,3}\}$ [16]; $\mathcal{H} = \{K_2, K_{1,3}\}$ [15, 6]; $\mathcal{H} = \{C_4, P_3\}$[23]; $\mathcal{H} = \{K_3, P_3\}$ [24]; $\mathcal{H} = \{P_2, P_3, P_4\}$ [22]; $\mathcal{H} = \{C_4, K_{1,3}\}$[8]; $\mathcal{H} = \{K_2, P_{2k}\}, k \geq 2$ [17].

In this paper, we determine necessary and sufficient conditions for the existence of uniformly resolvable decomposition of K_n into P_4 and $C_k,\ k \geq 3$.

2. Preliminary Results

In this section, we give some useful notations, basic results, and necessary conditions for the existence of uniformly resolvable decomposition of K_n into P_4 and $C_k,\ k \geq 3$.

Let (P_4, C_k)–URD$(n; r, s)$ denote the uniformly resolvable decomposition of K_n into r P_4-factors and s C_k-factors. A (P_4, C_k)–URD(r, s) of a graph G is a uniformly resolvable decomposition of graph G into r P_4-factors and s C_k-factors. We denote $P_k,\ k \geq 2$ with vertex set $\{a_1, a_2, \ldots, a_k\}$ and edge set $\{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{k-1}, a_k\}\}$ by $[a_1, a_2, \ldots, a_k]$; $C_k,\ k \geq 3$ with vertex set $\{a_1, a_2, \ldots, a_k\}$ and edge set $\{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{k-1}, a_k\}, \{a_k, a_1\}\}$ by $(a_1, a_2, \ldots, a_k)$. The floor function, $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.

Theorem 2.1 ([2, 3, 13])**.** *Let $n,\ t \geq 3$ be integers. There is a C_t-factorization of K_n (when n is odd) or $K_n - I$ (when n is even and I denotes a 1-factor of K_n) if and only if t divides n, except when $t = 3$ and $n \in \{6, 12\}$.*

Theorem 2.2 ([20, 21])**.** *For $t \geq 3$ and $m \geq 2$, $K_m \otimes I_n$ has a C_t-factorization if and only if mn is divisible by t, $(m-1)n$ is even, t is even if $m = 2$, and $(m, n, t) \neq (3, 2, 3), (3, 6, 3), (6, 2, 3), (2, 6, 6)$.*

Theorem 2.3 ([19])**.** *For $n \geq 1$ and $r \geq 3$, $C_r \otimes I_n$ has a C_{rn}-factorization.*

Theorem 2.4 ([5])**.** *The graph $C_k \otimes I_t$ has a C_k-factorization for all $t \geq 1$ and $k \geq 3$ with the definite exceptions $(t, k) = (6, 3), (2, 2r + 1)$.*

Theorem 2.5. *For $r \geq 3$, $C_r \otimes I_4$ has a C_{2r}-factorization.*

Proof. By Theorem 2.3, let $\{\mathcal{C}_{2r}^1, \mathcal{C}_{2r}^2\}$ be a C_{2r}-factorization of $C_r \otimes I_2$, where each $\mathcal{C}_{2r}^i$ is a C_{2r}-factor of $C_r \otimes I_2$. Then

$$\begin{aligned} C_r \otimes I_4 &\cong (C_r \otimes I_2) \otimes I_2 \\ &\cong (\mathcal{C}_{2r}^1 \oplus \mathcal{C}_{2r}^2) \otimes I_2 \cong (\mathcal{C}_{2r}^1 \otimes I_2) \oplus (\mathcal{C}_{2r}^2 \otimes I_2). \end{aligned}$$

By Theorem 2.4, each $\mathcal{C}_{2r}^i \otimes I_2$ has a C_{2r}-factorization (since $\mathcal{C}_{2r}^i \otimes I_2 \cong C_{2r} \otimes I_2$). Hence $C_r \otimes I_4$ has a C_{2r}-factorization. □

Lemma 2.6. *Let $k \geq 3$. If there exists a (P_4, C_k)–URD$(n; r, s)$ of K_n, then $n \equiv 0 \pmod{l}$, $l = \operatorname{lcm}(4, k)$ and $(r, s) \in J(n) = \{(4x+2, \frac{n-4}{2} - 3x) \mid x = 0, 1, \ldots, \lfloor \frac{n-4}{6} \rfloor\}$.*

Proof. Assume that there exists a (P_4, C_k)–URD$(n; r, s)$ of K_n. Then by resolvability, $n \equiv 0 \pmod{l}$, $l = \operatorname{lcm}(4, k)$ is trivial. (i.e.) if $k \equiv 1 \pmod 2$, then $n \equiv 0 \pmod{4k}$; if $k \equiv 2 \pmod 4$, then $n \equiv 0 \pmod{2k}$ and if $k \equiv 0 \pmod 4$, then $n \equiv 0 \pmod{k}$. Since there are r P_4-factors and s C_k-factors, by edge divisibility,

$$r\frac{n}{4}3 + s\frac{n}{k}k = \frac{n(n-1)}{2} \implies 3r + 4s = 2(n-1).$$

Clearly, $r \equiv 2 \pmod 4$. Let $r = 4x+2,\ x \geq 0$. Then $s = \frac{n-4}{2} - 3x$. Hence $(r, s) \in J(n) = \{(4x+2, \frac{n-4}{2} - 3x) \mid x = 0, 1, \ldots, \lfloor \frac{n-4}{6} \rfloor\}$. This completes the proof. □

3. Constructions

In this section, we give two constructions which we use to prove our main results.

If X and Y are two sets of pairs of nonnegative integers, then $X+Y$ denotes the set $\{(x_1+y_1, x_2+y_2) \mid (x_1, x_2) \in X,\ (y_1, y_2) \in Y\}$. If X is a set of pairs of nonnegative integers and h is a positive integer, then $h*X$ denotes the set of pairs of nonnegative integers which can be obtained by adding any h elements of X together (repetitions of elements of X are allowed).

Theorem 3.1. *Let $m \geq 3$ be an odd integer and t divides m. If there exists*

1. *a (P_4, C_k)–URD(r, s) of $\mathcal{C} \otimes I_4$ with $(r, s) \in \{(4, 1), (0, 4)\}$, where $k \in \{4, t, 2t, 4t\}$ and $\mathcal{C}$ is a C_t-factor of K_m; and*
2. *a (P_4, C_k)–URD$(16, 0)$ of $(\mathcal{C}^a \oplus \mathcal{C}^b \oplus \mathcal{C}^c) \otimes I_4$, where $\mathcal{C}^a$, $\mathcal{C}^b$, and $\mathcal{C}^c$ are any 3 edge-disjoint C_t-factors of K_m,*

then there exists a (P_4, C_k)–URD$(4m; r, s)$ of K_{4m} with $(r, s) \in J(4m) = \{(4x+2, \frac{4m-4}{2} - 3x) \mid x = 0, 1, \ldots, \lfloor \frac{4m-4}{6} \rfloor\}$, where $k \in \{4, t, 2t, 4t\}$.

Proof. Assume that (1) and (2) holds. Let $A = \{(4x+2, \frac{4m-4}{2} - 3x) \mid 0 \leq x \leq \frac{m-1}{2}\}$ and $B = \{(4x+2, \frac{4m-4}{2} - 3x) \mid \frac{m-1}{2} + 1 \leq x \leq \lfloor \frac{4m-4}{6} \rfloor\}$

be the partition of $J(4m)$. By Theorem 2.1, let $\{\mathcal{C}^i \mid 1 \le i \le \frac{m-1}{2}\}$ be a C_t-factorization of K_m.

$$\begin{aligned} K_{4m} &\cong (K_m \otimes I_4) \oplus (I_m \otimes K_4) \\ &\cong ((\mathcal{C}^1 \oplus \mathcal{C}^2 \oplus \cdots \oplus \mathcal{C}^{\frac{m-1}{2}}) \otimes I_4) \oplus (I_m \otimes K_4) \\ &\cong ((\mathcal{C}^1 \otimes I_4) \oplus (\mathcal{C}^2 \otimes I_4) \oplus \cdots \oplus (\mathcal{C}^{\frac{m-1}{2}} \otimes I_4)) \oplus (I_m \otimes K_4). \end{aligned}$$

Now we prove the existence of (P_4, C_k)–URD$(4m; r, s)$ of K_{4m} with $(r, s) \in J(4m) = A \cup B$, where $k \in \{4, t, 2t, 4t\}$ in two cases as follows:

Case 1: $(r, s) \in A$.

By hypothesis (1), for each i, there exists a (P_4, C_k)–URD(r, s) of $\mathcal{C}^i \otimes I_4$, with $(r, s) \in \{(4, 1), (0, 4)\}$, where $k \in \{4, t, 2t, 4t\}$. Since K_4 has 2 P_4-factors, $I_m \otimes K_4 (\cong mK_4)$ has a (P_4, C_k)–URD$(2, 0)$. This gives the existence of (P_4, C_k)–URD$(4m; r, s)$ of K_{4m} with $(r, s) \in \{\frac{m-1}{2} * \{(4, 1), (0, 4)\} + \{(2, 0)\}\}$, where $k \in \{4, t, 2t, 4t\}$. Now consider

$$\begin{aligned} &\{\frac{m-1}{2} * \{(0, 4), (4, 1)\} + \{(2, 0)\}\} \\ &\quad = \{\{(\frac{m-1}{2} - x)(0, 4) + x(4, 1) \mid 0 \le x \le \frac{m-1}{2}\} + \{(2, 0)\}\} \\ &\quad = \{(4x + 2, (\frac{m-1}{2})4 - 4x + x) \mid 0 \le x \le \frac{m-1}{2}\} \\ &\quad = \{(4x + 2, \frac{4m-4}{2} - 3x) \mid 0 \le x \le \frac{m-1}{2}\}. \end{aligned}$$

Hence, there exists a (P_4, C_k)–URD$(4m; r, s)$ of K_{4m} with $(r, s) \in \{(4x + 2, \frac{4m-4}{2} - 3x) \mid 0 \le x \le \frac{m-1}{2}\}$, where $k \in \{4, t, 2t, 4t\}$.

Case 2: $(r, s) \in B$.

By (1), for each i, there exists a (P_4, C_k)–URD$(4, 1)$ of $\mathcal{C}^i \otimes I_4$, where $k \in \{4, t, 2t, 4t\}$. Since K_4 has 2 P_4-factors, $I_m \otimes K_4 (\cong mK_4)$ has a (P_4, C_k)–URD$(2, 0)$. By (2), there exists a (P_4, C_k)–URD$(16, 0)$ of $(\mathcal{C}^a \oplus \mathcal{C}^b \oplus \mathcal{C}^c) \otimes I_4$. This gives the existence of (P_4, C_k)–URD$(4m; r, s)$ of K_{4m} with $(r, s) \in \{\{(\frac{m-1}{2} - 3y) * \{(4, 1)\} + y * \{(16, 0)\} \mid 1 \le y \le \lfloor \frac{m-1}{6} \rfloor\} + \{(2, 0)\}\}$. Now consider

$$\begin{aligned} &\{(\frac{m-1}{2} - 3y) * \{(4, 1)\} + y * \{(16, 0)\} \mid 1 \le y \le \left\lfloor \frac{m-1}{6} \right\rfloor\} + \{(2, 0)\}\} \\ &\quad = \{((\frac{m-1}{2} - 3y)4 + 16y + 2, \frac{m-1}{2} - 3y) \mid 1 \le y \le \left\lfloor \frac{m-1}{6} \right\rfloor\} \\ &\quad = \{(4(\frac{m-1}{2} + y) + 2, \frac{m-1}{2} - 3y) \mid 1 \le y \le \left\lfloor \frac{m-1}{6} \right\rfloor\} \\ &\quad = \{(4x + 2, \frac{4m-4}{2} - 3x) \mid \frac{m-1}{2} + 1 \le x \le \left\lfloor \frac{4m-4}{6} \right\rfloor\}. \end{aligned}$$

Hence, there exists a (P_4, C_k)–URD$(4m; r, s)$ of K_{4m} with $(r, s) \in \{(4x + 2, \frac{4m-4}{2} - 3x) \mid \frac{m-1}{2} + 1 \le x \le \lfloor \frac{4m-4}{6} \rfloor\}$, where $k \in \{4, t, 2t, 4t\}$. This completes the proof.

□

Theorem 3.2. *Let $m \geq 4$ be an even integer and t divides m. If there exists*

1. *a (P_4, C_k)–URD(r, s) of $\mathcal{C} \otimes I_4$ with $(r, s) \in \{(4, 1), (0, 4)\}$, where $k \in \{4, 8, t, 2t, 4t\}$ and $\mathcal{C}$ is a C_t-factor of K_m;*
2. *a (P_4, C_k)–URD(r, s) of $(\mathcal{C} \oplus I) \otimes I_4$ with $(r, s) \in \{(8, 0), (4, 3)\}$, where $\mathcal{C}$ and I are a edge-disjoint C_t-factor and 1-factor of K_m and $k \in \{4, 8, t, 2t, 4t\}$; and*
3. *a (P_4, C_k)–URD$(16, 0)$ of $(\mathcal{C}^a \oplus \mathcal{C}^b \oplus \mathcal{C}^c) \otimes I_4$, where $\mathcal{C}^a$, $\mathcal{C}^b$, and $\mathcal{C}^c$ are any 3 edge-disjoint C_t-factors of K_m,*

then there exists a (P_4, C_k)–URD(r, s) of K_{4m} with $(r, s) \in J(4m) \setminus \{(2, \frac{4m-4}{2})\} = \{(4x + 2, \frac{4m-4}{2} - 3x) \mid x = 1, \ldots, \lfloor \frac{4m-4}{6} \rfloor\}$, where $k \in \{4, 8, t, 2t, 4t\}$, except when $t = 3$ and $m \in \{6, 12\}$.

Proof. Assume that (1) to (3) holds. Let $A = \{(4x+2, \frac{4m-4}{2} - 3x) \mid 1 \leq x \leq \frac{m}{2}\}$ and $B = \{(4x + 2, \frac{4m-4}{2} - 3x) \mid \frac{m}{2} + 1 \leq x \leq \lfloor \frac{4m-4}{6} \rfloor\}$ be the partition of $J(4m) \setminus \{(2, \frac{4m-4}{2})\}$.

By Theorem 2.1, let $\{\mathcal{C}^i \mid 1 \leq i \leq \frac{m-2}{2}\}$ be a C_t-factorization of $K_m - I$, where I is a 1-factor of K_m, except for $t = 3$ and $m \in \{6, 12\}$.

$$\begin{aligned} K_{4m} &\cong (K_m \otimes I_4) \oplus (I_m \otimes K_4) \\ &\cong ((\mathcal{C}^1 \oplus \mathcal{C}^2 \oplus \cdots \oplus \mathcal{C}^{\frac{m-2}{2}} \oplus I) \otimes I_4) \oplus (I_m \otimes K_4) \\ &\cong ((\mathcal{C}^1 \otimes I_4) \oplus \cdots \oplus (\mathcal{C}^{\frac{m-4}{2}} \otimes I_4) \oplus ((\mathcal{C}^{\frac{m-2}{2}} \oplus I) \otimes I_4)) \oplus (I_m \otimes K_4) \end{aligned}$$

Now we prove the existence of (P_4, C_k)–URD(r, s) of K_{4m} with $(r, s) \in A \cup B$, where $k \in \{4, 8, t, 2t, 4t\}$, except when $t = 3$ and $m \in \{6, 12\}$ in two cases as follows:

Case 1: $(r, s) \in A$.

By (1), for each i, $1 \leq i \leq \frac{m-4}{2}$, there exists a (P_4, C_k)–URD(r, s) of $\mathcal{C}^i \otimes I_4$, with $(r, s) \in \{(4, 1), (0, 4)\}$, where $k \in \{4, 8, t, 2t, 4t\}$. By (2), there exists a (P_4, C_k)–URD(r, s) of $(\mathcal{C}^{\frac{m-2}{2}} \oplus I) \otimes I_4$ with $(r, s) \in \{(8, 0), (4, 3)\}$, where $k \in \{4, 8, t, 2t, 4t\}$. Since K_4 has 2 P_4-factors, $I_m \otimes K_4 (\cong mK_4)$ has a (P_4, C_k)–URD$(2, 0)$. This gives the existence of (P_4, C_k)–URD$(4m; r, s)$ with $(r, s) \in \{\frac{m-4}{2} * \{(4, 1), (0, 4)\} + \{(8, 0), (4, 3)\} + \{(2, 0)\}\}$, where

$k \in \{4, 8, t, 2t, 4t\}$, except when $t = 3$ and $m \in \{6, 12\}$. Now consider

$$\begin{aligned}
&\{\frac{m-4}{2} * \{(0,4),(4,1)\} + \{(8,0),(4,3))\} + \{(2,0)\}\} \\
&\quad = \{\{(\frac{m-4}{2} - x)(0,4) + x(4,1) \mid 0 \le x \le \frac{m-4}{2}\} \\
&\qquad + \{(4y+2, 6-3y) \mid 1 \le y \le 2\}\} \\
&\quad = \{\{(4x, (\frac{m-4}{2})4 - 3x) \mid 0 \le x \le \frac{m-4}{2}\} \\
&\qquad + \{(4y+2, 6-3y) \mid 1 \le y \le 2\}\} \\
&\quad = \{(4(x+y)+2, (\frac{m-4}{2})4 + 6 - 3(x+y)) \\
&\qquad \mid 0 \le x \le \frac{m-4}{2} \text{ and } 1 \le y \le 2]\} \\
&\quad = \{(4z+2, \frac{4m-4}{2} - 3z) \mid 1 \le z \le \frac{m}{2}\}.
\end{aligned}$$

Hence, there exists a (P_4, C_k)–URD$(4m; r, s)$ of K_{4m} with $(r, s) \in \{(4z+2, \frac{4m-4}{2} - 3z) \mid 1 \le z \le \frac{m}{2}\}$, where $k \in \{4, 8, t, 2t, 4t\}$, except when $t = 3$ and $m \in \{6, 12\}$.

Case 2: $(r, s) \in B$.

By (1), for each i, $1 \le i \le \frac{m-4}{2}$, there exists a (P_4, C_k)–URD$(4, 1)$ of $\mathcal{C}^i \otimes I_4$, where $k \in \{4, 8, t, 2t, 4t\}$. By (2), there exists a (P_4, C_k)–URD(r, s) of $(\mathcal{C}^{\frac{m-2}{2}} \oplus I) \otimes I_4$ with $(r, s) \in \{(8, 0)\}$. Since K_4 has 2 P_4-factors, $I_m \otimes K_4 (\cong mK_4)$ has a (P_4, C_k)–URD$(2, 0)$. By (3), there exists a (P_4, C_k)–URD$(16, 0)$ of $(\mathcal{C}^a \oplus \mathcal{C}^b \oplus \mathcal{C}^c) \otimes I_4$. This gives the existence of (P_4, C_k)–URD$(4m; r, s)$ with $(r, s) \in \{\{(\frac{m-4}{2} - 3y) * \{(4, 1)\} + y * \{(16, 0)\} \mid 1 \le y \le \lfloor \frac{m-4}{6} \rfloor\} + \{(8, 0)\} + \{(2, 0)\}\}$, except when $t = 3$ and $m \in \{6, 12\}$. Now consider

$$\begin{aligned}
&\{\{(\frac{m-4}{2} - 3y) * \{(4,1)\} + y * \{(16,0)\} \\
&\qquad \mid 1 \le y \le \left\lfloor \frac{m-4}{6} \right\rfloor\} + \{(8,0)\} + \{(2,0)\}\} \\
&\quad = \{\{((\frac{m-4}{2} - 3y)4 + 16y + 10, \frac{m-4}{2} - 3y) \\
&\qquad \mid 1 \le y \le \left\lfloor \frac{m-4}{6} \right\rfloor\} \\
&\quad = \{(4(\frac{m-4}{2} + y) + 10, \frac{m-4}{2} - 3y) \mid 1 \le y \le \left\lfloor \frac{m-4}{6} \right\rfloor\} \\
&\quad = \{\{(4(\frac{m}{2} + y) + 2, \frac{m-4}{2} - 3y) \mid 1 \le y \le \left\lfloor \frac{m-4}{6} \right\rfloor\} \\
&\quad = \{(4z+2, \frac{4m-4}{2} - 3z) \mid \frac{m}{2} + 1 \le z \le \left\lfloor \frac{4m-4}{6} \right\rfloor\}.
\end{aligned}$$

Hence, there exists a (P_4, C_k)–URD$(4m; r, s)$ of K_{4m} with $(r, s) \in \{(4z+2, \frac{4m-4}{2} - 3z) \mid \frac{m}{2} + 1 \le z \le \lfloor \frac{4m-4}{6} \rfloor\}$, where $k \in \{4, 8, t, 2t, 4t\}$, except when $t = 3$ and $m \in \{6, 12\}$. This completes the proof. □

Theorem 3.3. *Let $m \ge 4$ be an even integer and t divides m. Then there exists a (P_4, C_k)–URD$(4m; 2, \frac{4m-4}{2})$, where $k \in \{4, 8, t, 2t, 4t\}$.*

Proof. We construct 2 P_4-factors and $\frac{4m-4}{2}$ C_k-factors, where $k \in \{4,\ 8,\ t,\ 2t,\ 4t\}$ of K_{4m} as follows:

Consider $K_{4m} \cong (K_m \otimes I_4) \oplus (I_m \otimes K_4)$. By Theorem 2.2, $K_m \otimes I_4$ has a C_k-factorization, where $k \in \{4, 8, t, 2t, 4t\}$. Since K_4 has 2 P_4-factors, $I_m \otimes K_4 (\cong mK_4)$ has a P_4-factorization. Therefore, K_{4m} has 2 P_4-factors and $\frac{4m-4}{2}$ C_k-factors, where $k \in \{4, 8, t, 2t, 4t\}$. That is, there exists a (P_4, C_k)–URD$(4m; 2, \frac{4m-4}{2})$, where $k \in \{4, 8, t, 2t, 4t\}$. □

4. (P_2, P_4) and (P_2, C_k)–URD of $C_t \otimes I_4$.

In this section, we prove the existence of uniformly resolvable decomposition of $C_t \otimes I_4$ into P_2 and P_4 or P_2 and C_k, $k \in \{t, 2t, 4t\}$.

Let $K_{n,n}$ be a complete bipartite graph with bipartition (X, Y), where $X = \{x_1,\ x_2,\ \ldots,\ x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$. Now we define a 1-factor of $K_{n,n}$ as $F_i(X,Y) = \{\{x_j, y_{(i+j)}\} \mid 1 \le j \le n$, where addition in the subscript is taken modulo n with residues $1, 2, \ldots, n\}$, $0 \le i \le n-1$, then $E(K_{n,n}) = \bigcup_{i=0}^{n-1} F_i(X, Y)$. Clearly $\{F_i \mid 0 \le i \le n-1\}$ gives a 1-factorization of $K_{n,n}$.

Lemma 4.1. *For any $t \ge 3$, there exists a (P_2, P_4)–URD$(2, 4)$ of $C_t \otimes I_4$.*

Proof. Let $V(C_t \otimes I_4) = \cup_{i \in \mathbb{Z}_t} X_i$, where $X_i = \{i_0, i_1, i_2, i_3\}, i \in \mathbb{Z}_t$. Now we construct a (P_2, P_4)–URD$(2, 4)$ of $C_t \otimes I_4$ in two cases as follows:

Case 1: t odd.

Let

$$\mathcal{P}_2^1 = \{[i_0, (i+1)_1], [i_2, (i+1)_3] \mid 0 \le i \le t-1\};$$

$$\mathcal{P}_2^2 = \{[i_1, (i+1)_2], [i_3, (i+1)_0] \mid 0 \le i \le t-1\};$$

$$\mathcal{P}^1 = \{[(t-1)_0, (t-2)_2, (t-1)_1, (t-2)_3], [(2i)_0, (2i+1)_0, (2i)_1, (2i+1)_1], [(2i-1)_2, (2i)_2, (2i-1)_3, (2i)_3] \mid 0 \le i \le \frac{t-3}{2}\};$$

$$\mathcal{P}^2 = \{[(t-1)_3, 0_1, (t-1)_2, 0_0], [(2i+1)_3, (2i)_3, (2i+1)_2, (2i)_2], [(2i+2)_1, (2i+1)_1, (2i+2)_0, (2i+1)_0] \mid 0 \le i \le \frac{t-3}{2}\};$$

$$\mathcal{P}^3 = \{[(t-1)_3, (t-2)_3, (t-1)_2, (t-2)_2], [(t-1)_0, 0_0, (t-1)_1, 0_1], [(i+1)_0, i_2, (i+1)_1, i_3] \mid 0 \le i \le t-3\};$$

$$\mathcal{P}^4 = \{[(i+1)_2, i_0, (i+1)_3, i_1] \mid 0 \le i \le t-1\},$$

where the additions are taken modulo t.

Case 2: t even.

Let

$$\mathcal{P}_2^1 = \big\{[i_0,(i+1)_1],[i_2,(i+1)_3] \mid 0 \le i \le t-1\big\};$$
$$\mathcal{P}_2^2 = \big\{[i_1,(i+1)_2],[i_3,(i+1)_0] \mid 0 \le i \le t-1\big\};$$
$$\mathcal{P}^1 = \big\{[(2i)_0,(2i+1)_0,(2i)_1,(2i+1)_1], [(2i)_2,(2i+1)_2,(2i)_3,(2i+1)_3] \mid 0 \le i \le \frac{t-2}{2}\big\};$$
$$\mathcal{P}^2 = \big\{[(2i)_1,(2i+1)_3,(2i)_0,(2i+1)_2], [(2i)_3,(2i+1)_1,(2i)_2,(2i+1)_0] \mid 0 \le i \le \frac{t-2}{2}\big\};$$
$$\mathcal{P}^3 = \big\{[(2i+1)_0,(2i+2)_0,(2i+1)_1,(2i+2)_1], [(2i+1)_2,(2i+2)_2,(2i+1)_3,(2i+2)_3] \mid 0 \le i \le \frac{t-2}{2}\big\};$$
$$\mathcal{P}^4 = \big\{[(2i+1)_1,(2i+2)_3,(2i+1)_0,(2i+2)_2], [(2i+1)_3,(2i+2)_1,(2i+1)_2,(2i+2)_0] \mid 0 \le i \le \frac{t-2}{2}\big\},$$

where the additions are taken modulo t.

Clearly, $\mathcal{P}_2^1$ and $\mathcal{P}_2^2$ are P_2-factors of $C_t \otimes I_4$ and each $\mathcal{P}^i$, $i = 1,2,3,4$ is a P_4-factor of $C_t \otimes I_4$. Hence $\{\mathcal{P}_2^1, \mathcal{P}_2^2, \mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4\}$ gives the existence of (P_2, P_4)–URD$(2,4)$ of $C_t \otimes I_4$. □

Lemma 4.2. *For any $t \ge 3$, there exists a (P_2, C_t)–URD$(2,3)$ of $C_t \otimes I_4$.*

Proof. Let $V(C_t \otimes I_4) = \cup_{i\in\mathbb{Z}_t} X_i$, where $X_i = \{i_0, i_1, i_2, i_3\}, i \in \mathbb{Z}_t$. Then $E(C_t \otimes I_4) = \cup_{i\in\mathbb{Z}_t, l\in\mathbb{Z}_4} F_l(X_i, X_{i+1})$. Now we prove the existence of (P_2, C_t)–URD$(2,3)$ of $C_t \otimes I_4$ in two cases as follows:

Case 1: t odd.

Let

$$\mathcal{C}^1 = F_0(X_0, X_1) \cup \bigcup_{i=1}^{\frac{t-1}{2}} F_3(X_{2i}, X_{2i+1}) \cup \bigcup_{i=1}^{\frac{t-1}{2}} F_1(X_{2i-1}, X_{2i});$$
$$\mathcal{C}^2 = F_0(X_{t-1}, X_0) \cup \bigcup_{i=0}^{\frac{t-3}{2}} F_1(X_{2i}, X_{2i+1}) \cup \bigcup_{i=1}^{\frac{t-1}{2}} F_3(X_{2i-1}, X_{2i});$$
$$\mathcal{C}^3 = F_3(X_0, X_1) \cup \bigcup_{i=1}^{t-2} F_0(X_i, X_{i+1}) \cup F_1(X_{t-1}, X_0);$$
$$\mathcal{P}_2^1 = \{[i_0,(i+1)_2],[i_1,(i+1)_3] \mid i \in \mathbb{Z}_t\};$$
$$\mathcal{P}_2^2 = \{[i_2,(i+1)_0],[i_3,(i+1)_1] \mid i \in \mathbb{Z}_t\},$$

where the additions are taken modulo t.

Case 2: t even.

Let

$$\mathcal{C}^1 = \bigcup_{i=0}^{\frac{t-2}{2}} F_3(X_{2i}, X_{2i+1}) \cup \bigcup_{i=0}^{\frac{t-2}{2}} F_1(X_{2i+1}, X_{2i+2});$$

$$\mathcal{C}^2 = \bigcup_{i=0}^{\frac{t-2}{2}} F_1(X_{2i}, X_{2i+1}) \cup \bigcup_{i=0}^{\frac{t-2}{2}} F_3(X_{2i+1}, X_{2i+2});$$

$$\mathcal{C}^3 = \bigcup_{i=0}^{t-1} F_0(X_i, X_{i+1});$$

$$\mathcal{P}_2^1 = \{[i_0, (i+1)_2], [i_1, (i+1)_3] \mid i \in \mathbb{Z}_t\};$$

$$\mathcal{P}_2^2 = \{[i_2, (i+1)_0], [i_3, (i+1)_1] \mid i \in \mathbb{Z}_t\},$$

where the additions are taken modulo t.

Clearly, each $\mathcal{P}^i, i = 1, 2$ is a P_2-factor of $C_t \otimes I_4$ and each $\mathcal{C}^i, i = 1, 2, 3$ is a C_t-factor of $C_t \otimes I_4$. Hence $\{\mathcal{P}_2^1, \mathcal{P}_2^2, \mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^3\}$ gives the existence of (P_2, C_t)–URD$(2, 3)$ of $C_t \otimes I_4$. □

Lemma 4.3. *For any $t \geq 3$, there exists a (P_2, C_{2t})–URD$(2, 3)$ of $C_t \otimes I_4$.*

Proof. Let $V(C_t \otimes I_4) = \cup_{i \in \mathbb{Z}_t} X_i$, where $X_i = \{i_0, i_1, i_2, i_3\}, i \in \mathbb{Z}_t$. Let $U_i = \{i_0, i_1\}$ and $V_i = \{i_2, i_3\}, i \in \mathbb{Z}_t$, then $X_i = U_i \cup V_i, i \in \mathbb{Z}_t$. We write

$$\begin{aligned} E(C_t \otimes I_4) = \{ &\bigcup_{i \in \mathbb{Z}_t, l \in \mathbb{Z}_2} F_l(U_i, U_{i+1})\} \\ \cup \{ &\bigcup_{i \in \mathbb{Z}_t, l \in \mathbb{Z}_2} F_l(U_i, V_{i+1})\} \\ \cup \{ &\bigcup_{i \in \mathbb{Z}_t, l \in \mathbb{Z}_2} F_l(V_i, V_{i+1})\} \\ \cup \{ &\bigcup_{i \in \mathbb{Z}_t, l \in \mathbb{Z}_2} F_l(V_i, U_{i+1})\}. \end{aligned}$$

Now we prove the existence of (P_2, C_{2t})–URD$(2, 3)$ of $C_t \otimes I_4$ in two cases as follows:

Case 1: t odd.

Let

$$\begin{aligned}
\mathcal{C}^1 &= F_1(U_0,U_1)\cup F_1(V_0,V_1)\cup F_1(U_{t-1},V_0)\cup F_1(V_{t-1},U_0)\\
&\quad \cup\Big\{\bigcup_{i=1}^{t-2}F_0(U_i,U_{i+1})\Big\}\cup\Big\{\bigcup_{i=1}^{t-2}F_0(V_i,V_{i+1})\Big\};\\
\mathcal{C}^2 &= F_0(U_0,U_1)\cup F_0(V_0,V_1)\cup F_0(U_{t-1},U_0)\cup F_0(V_{t-1},V_0)\\
&\quad \cup\Big\{\bigcup_{i=1}^{t-2}F_1(U_i,V_{i+1})\Big\}\cup\Big\{\bigcup_{i=1}^{t-2}F_1(V_i,U_{i+1})\Big\};\\
\mathcal{C}^3 &= F_1(U_0,V_1)\cup F_1(V_0,U_1)\cup F_1(U_{t-1},U_0)\cup F_1(V_{t-1},V_0)\\
&\quad \cup\Big\{\bigcup_{i=1}^{t-2}F_1(U_i,U_{i+1})\Big\}\cup\Big\{\bigcup_{i=1}^{t-2}F_1(V_i,V_{i+1})\Big\};\\
\mathcal{P}_2^1 &= \bigcup_{i=0}^{t-1}F_0(U_i,V_{i+1});\ \mathcal{P}_2^2=\bigcup_{i=0}^{t-1}F_0(V_i,U_{i+1}),
\end{aligned}$$

where additions in the subscript are taken modulo t.

Case 2: t even.

Let

$$\begin{aligned}
\mathcal{C}^1 &= F_1(U_{t-2},V_{t-1})\cup F_1(V_{t-2},U_{t-1})\cup F_0(U_{t-1},U_0)\cup F_0(V_{t-1},V_0)\\
&\quad \cup\Big\{\bigcup_{i=0}^{t-3}F_1(U_i,U_{i+1})\Big\}\cup\Big\{\bigcup_{i=0}^{t-3}F_1(V_i,V_{i+1})\Big\};\\
\mathcal{C}^2 &= F_1(U_{t-2},U_{t-1})\cup F_1(V_{t-2},V_{t-1})\cup F_1(U_{t-1},V_0)\cup F_1(V_{t-1},U_0)\\
&\quad \cup\Big\{\bigcup_{i=0}^{t-3}F_1(U_i,V_{i+1})\Big\}\cup\Big\{\bigcup_{i=0}^{t-3}F_1(V_i,U_{i+1})\Big\};\\
\mathcal{C}^3 &= F_1(U_{t-1},U_0)\cup F_1(V_{t-1},V_0)\cup\Big\{\bigcup_{i=0}^{t-2}F_0(U_i,U_{i+1})\Big\}\cup\Big\{\bigcup_{i=0}^{t-2}F_0(V_i,V_{i+1})\Big\};\\
\mathcal{P}_2^1 &= \bigcup_{i=0}^{t-1}F_0(U_i,V_{i+1});\ \mathcal{P}_2^2=\bigcup_{i=0}^{t-1}F_0(V_i,U_{i+1}),
\end{aligned}$$

where additions in the subscript are taken modulo t.

Clearly, each $\mathcal{P}_2^i, i=1,2$ is a P_2-factor of $C_t\otimes I_4$ and each $\mathcal{C}^i, i=1,2,3$ is a C_{2t}-factor of $C_t\otimes I_4$. Hence $\{\mathcal{P}_2^1,\mathcal{P}_2^2,\mathcal{C}^1,\mathcal{C}^2,\mathcal{C}^3\}$ gives the existence of (P_2,C_{2t})–URD$(2,3)$ of $C_t\otimes I_4$. □

Lemma 4.4. *For any $t\geq 3$, there exists a (P_2,C_{4t})–URD$(2,3)$ of $C_t\otimes I_4$.*

Proof. Let $V(C_t \otimes I_4) = \cup_{i\in\mathbb{Z}_t} X_i$, where $X_i = \{i_0, i_1, i_2, i_3\}, i \in \mathbb{Z}_t$. Let $U_i = \{i_0, i_1\}$ and $V_i = \{i_2, i_3\}, i \in \mathbb{Z}_t$, then $X_i = U_i \cup V_i, i \in \mathbb{Z}_t$. We write

$$E(C_t \otimes I_4) = \Big\{ \bigcup_{i\in\mathbb{Z}_t, l\in\mathbb{Z}_2} F_l(U_i, U_{i+1}) \Big\} \cup \Big\{ \bigcup_{i\in\mathbb{Z}_t, l\in\mathbb{Z}_2} F_l(U_i, V_{i+1}) \Big\}$$
$$\cup \Big\{ \bigcup_{i\in\mathbb{Z}_t, l\in\mathbb{Z}_2} F_l(V_i, V_{i+1}) \Big\} \cup \Big\{ \bigcup_{i\in\mathbb{Z}_t, l\in\mathbb{Z}_2} F_l(V_i, U_{i+1}) \Big\}.$$

Now we construct a (P_2, C_{4t})–URD$(2,3)$ of $C_t \otimes I_4$ in two cases as follows:
CASE 1: t odd.

Let

$$\mathcal{C}^1 = F_0(U_0, V_1) \cup F_1(V_0, U_1) \cup \bigcup_{i=1}^{t-1} F_0(U_i, U_{i+1}) \cup \bigcup_{i=1}^{t-1} F_0(V_i, V_{i+1});$$
$$\mathcal{C}^2 = F_0(U_{t-1}, V_0) \cup F_1(V_{t-1}, U_0) \cup \bigcup_{i=0}^{t-2} F_1(U_i, U_{i+1}) \cup \bigcup_{i=0}^{t-2} F_1(V_i, V_{i+1});$$
$$\mathcal{C}^3 = F_0(U_0, U_1) \cup F_0(V_0, V_1) \cup F_1(U_{t-1}, U_0) \cup F_1(V_{t-1}, V_0)$$
$$\cup \bigcup_{i=1}^{t-2} F_0(U_i, V_{i+1}) \cup \bigcup_{i=1}^{t-2} F_1(V_i, U_{i+1});$$
$$\mathcal{P}_2^1 = \bigcup_{i=0}^{t-1} F_1(U_i, V_{i+1});\ \mathcal{P}_2^2 = \bigcup_{i=0}^{t-1} F_0(V_i, U_{i+1}),$$

where additions in the subscript are taken modulo t.
CASE 2: t even.

Let

$$\mathcal{C}^1 = F_0(U_0, U_1) \cup F_0(U_0, V_1) \cup F_1(U_{t-1}, V_0) \cup F_0(V_{t-1}, V_0)$$
$$\cup F_1(U_{t-2}, U_{t-1}) \cup F_1(V_{t-2}, V_{t-1})$$
$$\cup \bigcup_{i=1}^{t-3} F_1(U_i, V_{i+1}) \cup \bigcup_{i=1}^{t-3} F_1(V_i, U_{i+1});$$
$$\mathcal{C}^2 = F_1(V_0, V_1) \cup F_1(V_0, U_1) \cup F_0(U_{t-1}, U_0) \cup F_1(V_{t-1}, U_0)$$
$$\cup \bigcup_{i=1}^{t-2} F_0(U_i, U_{i+1}) \cup \bigcup_{i=1}^{t-2} F_0(V_i, V_{i+1});$$
$$\mathcal{C}^3 = F_0(V_0, V_1) \cup F_1(U_{t-2}, V_{t-1}) \cup F_1(V_{t-2}, U_{t-1}) \cup F_1(U_{t-1}, U_0)$$
$$\cup F_1(V_{t-1}, V_0) \cup \bigcup_{i=0}^{t-3} F_1(U_i, U_{i+1}) \cup \bigcup_{i=1}^{t-3} F_1(V_i, V_{i+1});$$
$$\mathcal{P}_2^1 = F_1(U_0, V_1) \cup \bigcup_{i=1}^{t-1} F_0(U_i, V_{i+1});\ \mathcal{P}_2^2 = \bigcup_{i=0}^{t-1} F_0(V_i, U_{i+1}),$$

where additions in the subscript are taken modulo t.

Clearly, each $\mathcal{P}_2^i, i = 1, 2$ is a P_2-factor of $C_t \otimes I_4$ and each $\mathcal{C}^i, i = 1, 2, 3$ is a C_{4t}-factor of $C_t \otimes I_4$. Hence $\{\mathcal{P}_2^1, \mathcal{P}_2^2, \mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^3\}$ gives the existence of (P_2, C_{4t})–URD$(2, 3)$ of $C_t \otimes I_4$. □

5. (P_4, C_k)–URD(r, s) of $C_t \otimes I_4$.

In this section, we prove the existence of uniformly resolvable decomposition of $C_t \otimes I_4$ into P_4 and C_k, $k \in \{4, 8, t, 2t, 4t\}$.

Lemma 5.1. *For any $t \geq 3$, there exists a (P_4, C_4)–URD(r, s) of $C_t \otimes I_4$ with $(r, s) \in \{(4, 1), (0, 4)\}$.*

Proof. Let $V(C_t \otimes I_4) = \cup_{i \in \mathbb{Z}_t} V_i$, where $V_i = \{i_0, i_1, i_2, i_3\}, i \in \mathbb{Z}_t$. Now we construct the required number of P_4-factor and C_4-factor of $C_t \otimes I_4$ in two cases as follows:

Case 1: $(r, s) = (4, 1)$.

Subcase i: t odd.

Let

$$\begin{aligned}
\mathcal{P}^1 = \big\{&[(2i+1)_0, (2i)_1, (2i+1)_1, (2i)_0],\\
&[(2i)_2, (2i-1)_3, (2i)_3, (2i-1)_2],\\
&[(t-2)_2, (t-1)_1, (t-2)_3, (t-1)_0] \mid 0 \leq i \leq \frac{t-3}{2}\big\};\\
\mathcal{P}^2 = \big\{&[(2i+2)_1, (2i+1)_0, (2i+2)_0, (2i+1)_1],\\
&[(2i+1)_3, (2i)_2, (2i+1)_2, (2i)_3],\\
&[(t-1)_3, 0_0, (t-1)_2, 0_1] \mid 0 \leq i \leq \frac{t-3}{2}\big\};\\
\mathcal{P}^3 = \big\{&[(2i)_3, (2i+1)_1, (2i)_2, (2i+1)_0],\\
&[(2i+1)_2, (2i+2)_0, (2i+1)_3, (2i+2)_1],\\
&[(t-1)_1, 0_0, (t-1)_0, 0_1], [(t-1)_2, (t-2)_3, (t-1)_3, (t-2)_2],\\
&[(t-3)_3, (t-2)_1, (t-3)_2, (t-2)_0] \mid 0 \leq i \leq \frac{t-5}{2}\big\};\\
\mathcal{P}^4 = \big\{&[(2i)_0, (2i+1)_0, (2i)_3, (2i+1)_3],\\
&[(2i+1)_1, (2i+2)_1, (2i+1)_2, (2i+2)_2],\\
&[(t-2)_1, (t-1)_1, 0_1, (t-1)_3], [(t-1)_0, (t-2)_2, (t-1)_2, 0_2],\\
&[(t-3)_0, (t-2)_0, (t-3)_3, (t-2)_3] \mid 0 \leq i \leq \frac{t-5}{2}\big\};\\
\mathcal{C}^1 = \big\{&(i_0, (i+1)_2, i_1, (i+1)_3) \mid 0 \leq i \leq t-1\big\},
\end{aligned}$$

where the additions are taken modulo t.

Subcase ii: t even.
Let

$$\mathcal{P}^1 = \{[(2i+1)_0, (2i)_1, (2i+1)_1, (2i)_0], [(2i+1)_2, (2i)_3, (2i+1)_3, (2i)_2] \mid 0 \le i \le \frac{t-2}{2}\};$$

$$\mathcal{P}^2 = \{[(2i)_1, (2i-1)_0, (2i)_0, (2i-1)_1], [(2i)_3, (2i-1)_2, (2i)_2, (2i-1)_3] \mid 1 \le i \le \frac{t}{2}\};$$

$$\mathcal{P}^3 = \{[(2i+1)_2, (2i)_1, (2i+1)_3, (2i)_0], [(2i+1)_1, (2i)_2, (2i+1)_0, (2i)_3] \mid 0 \le i \le \frac{t-2}{2}\};$$

$$\mathcal{P}^4 = \{[(2i+1)_0, (2i)_0, (2i+1)_2, (2i)_2], [(2i)_1, (2i-1)_1, (2i-2)_3, (2i-3)_3] \mid 0 \le i \le \frac{t-2}{2}\};$$

$$\mathcal{C}^1 = \{((2i-1)_0, (2i)_2, (2i-1)_1, (2i)_3), ((2i-1)_2, (2i)_0, (2i-1)_3, (2i)_1) \mid 1 \le i \le \frac{t}{2}\},$$

where the additions are taken modulo t.
Clearly, each $\mathcal{P}^i, i = 1, 2, 3, 4$ is a P_4-factor of $C_t \otimes I_4$ and $\mathcal{C}^1$ is a C_4-factor of $C_t \otimes I_4$. Hence $\{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4, \mathcal{C}^1\}$ gives the existence of (P_4, C_4)–URD$(4, 1)$ of $C_t \otimes I_4$.

Case 2: $(r, s) = (0, 4)$.
By Theorem 2.3, let $\{\mathcal{C}^1_{2t}, \mathcal{C}^2_{2t}\}$ be a C_{2t}-factorization of $C_t \otimes I_2$, where each $\mathcal{C}^i_{2t}$ is a C_{2t}-factor of $C_t \otimes I_2$. Then

$$\begin{aligned} C_t \otimes I_4 &\cong (C_t \otimes I_2) \otimes I_2 \cong (\mathcal{C}^1_{2t} \oplus \mathcal{C}^2_{2t}) \otimes I_2 \\ &\cong (\mathcal{C}^1_{2t} \otimes I_2) \oplus (\mathcal{C}^2_{2t} \otimes I_2) \cong ((\mathcal{I}^1_1 \oplus \mathcal{I}^1_2) \otimes I_2) \oplus ((\mathcal{I}^2_1 \oplus \mathcal{I}^2_2) \otimes I_2), \\ &\cong (\mathcal{I}^1_1 \otimes I_2) \oplus (\mathcal{I}^1_2 \otimes I_2) \oplus (\mathcal{I}^2_1 \otimes I_2) \oplus (\mathcal{I}^2_2 \otimes I_2), \end{aligned}$$

where each $j,\ j = 1, 2,\ \mathcal{I}^i_j$ is a 1-factor of $\mathcal{C}^i_{2t},\ i = 1, 2$. Since $\mathcal{I}^i_j \otimes I_2 \cong tK_{2,2} \cong tC_4$, $C_t \otimes I_4$ has a C_4-factorization. Hence there exists a (P_4, C_4)–URD$(0, 4)$ of $C_t \otimes I_4$.

□

Lemma 5.2. *For any even $t \ge 4$, there exists a (P_4, C_8)–URD(r, s) of $C_t \otimes I_4$ with $(r, s) \in \{(4, 1), (0, 4)\}$.*

Proof. Let $V(C_t \otimes I_4) = \cup_{i \in \mathbb{Z}_t} V_i$, where $V_i = \{i_0, i_1, i_2, i_3\}, i \in \mathbb{Z}_t$. Now we construct the required number of P_4-factors and C_8-factors of $C_t \otimes I_4$ in two cases as follows:

CASE 1: $(r, s) = (4, 1)$.

Let

$$\mathcal{P}^1 = \big\{[(2i-1)_0, (2i)_1, (2i-1)_2, (2i)_3], \quad [(2i)_0, (2i-1)_1, (2i)_2, (2i-1)_3] \mid 1 \le i \le \frac{t}{2}\big\};$$

$$\mathcal{P}^2 = \big\{[(2i-1)_1, (2i)_3, (2i-1)_0, (2i)_2], \quad [(2i-1)_2, (2i)_0, (2i-1)_3, (2i)_1] \mid 1 \le i \le \frac{t}{2}\big\};$$

$$\mathcal{P}^3 = \big\{[(2i)_1, (2i+1)_3, (2i)_2, (2i+1)_0], \quad [(2i)_3, (2i+1)_1, (2i)_0, (2i+1)_2] \mid 0 \le i \le \frac{t-2}{2}\big\};$$

$$\mathcal{C}^1 = \big\{((2i)_0, (2i+1)_0, (2i)_1, (2i+1)_1, (2i)_2, (2i+1)_2, (2i)_3, (2i+1)_3) \mid 0 \le i \le \frac{t-2}{2}\big\},$$

where the additions are taken modulo t.

Clearly, each $\mathcal{P}^i$ is a P_4-factor of $C_t \otimes I_4$ and $\mathcal{C}^1$ is a C_8-factor of $C_t \otimes I_4$. Hence $\{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4, \mathcal{C}^1\}$ gives the existence of (P_4, C_4)–URD$(4, 1)$ of $C_t \otimes I_4$.

CASE 2: $(r, s) = (0, 4)$.

Let $\{\mathcal{I}_1, \mathcal{I}_2\}$ be a 1-factorization of C_t, since t is even. Then

$$C_t \otimes I_4 \cong (\mathcal{I}_1 \oplus \mathcal{I}_2) \otimes I_4 \cong (\mathcal{I}_1 \otimes I_4) \oplus (\mathcal{I}_2 \otimes I_4).$$

Since $K_{4,4}$ has 2 C_8-factors and each $i,\ i = 1, 2,\ \mathcal{I}_i \otimes I_4 \cong \frac{t}{2} K_{4,4}$, $C_t \otimes I_4$ has a C_8-factorization. Hence there exist a (P_4, C_8)–URD$(0, 4)$ of $C_t \otimes I_4$. □

Lemma 5.3. *For any even $t \ge 4$, there exists a (P_4, C_k)–URD$(4, 1)$ of $C_t \otimes I_4$, where $k \in \{t, 2t, 4t\}$.*

Proof. Let $V(C_t \otimes I_4) = \cup_{i \in \mathbb{Z}_t} V_i$, where $V_i = \{i_0, i_1, i_2, i_3\}, i \in \mathbb{Z}_t$. Now we prove the existence of (P_4, C_k)–URD$(4, 1)$ of $C_t \otimes I_4$ in three cases as follows:

CASE 1: $k = t$.

Let

$$\mathcal{P}^1 = \big\{[(2i)_0, (2i+1)_1, (2i)_2, (2i+1)_3], [(2i+1)_0, (2i)_1, (2i+1)_2, (2i)_3] \mid 0 \le i \le \frac{t-2}{2}\big\};$$

$$\mathcal{P}^2 = \big\{[(2i)_2, (2i+1)_0, (2i)_3, (2i+1)_1], [(2i+1)_2, (2i)_0, (2i+1)_3, (2i)_1] \mid 0 \le i \le \frac{t-2}{2}\big\};$$

$$\mathcal{P}^3 = \big\{[(2i+1)_0, (2i+2)_1, (2i+1)_2, (2i+2)_3], [(2i+2)_0, (2i+1)_1, (2i+2)_2, (2i+1)_3] \mid 0 \le i \le \frac{t-2}{2}\big\};$$

$$\mathcal{P}^4 = \big\{[(2i+1)_2, (2i+2)_0, (2i+1)_3, (2i+2)_1], [(2i+2)_2, (2i+1)_0, (2i+2)_3, (2i+1)_1] \mid 0 \le i \le \frac{t-2}{2}\big\};$$

$$\mathcal{C}^1 = \big\{(0_i, 1_i, 2_i, \ldots, (t-2)_i, (t-1)_i) \mid 0 \le i \le 3\big\},$$

where the additions are taken modulo t.

Clearly, each $\mathcal{P}^i, i = 1, 2, 3, 4$ is a P_4-factor of $C_t \otimes I_4$ and $\mathcal{C}^1$ is a C_t-factor of $C_t \otimes I_4$. Hence $\{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4, \mathcal{C}^1\}$ gives the existence of (P_4, C_t)–URD$(4, 1)$ of $C_t \otimes I_4$.

CASE 2: $k = 2t$.

Let

$$\mathcal{P}^1 = \big\{[0_0, 1_3, 0_2, 1_1], [0_3, 1_0, 0_1, 1_2], [(2i)_0, (2i+1)_1, (2i)_2, (2i+1)_3], [(2i+1)_0, (2i)_1, (2i+1)_2, (2i)_3] \mid 1 \le i \le \frac{t-2}{2}\big\};$$

$$\mathcal{P}^2 = \big\{[0_1, 1_1, 0_0, 1_0], [0_2, 1_2, 0_3, 1_3], [(2i)_2, (2i+1)_0, (2i)_3, (2i+1)_1], [(2i+1)_2, (2i)_0, (2i+1)_3, (2i)_1] \mid 1 \le i \le \frac{t-2}{2}\big\};$$

$$\mathcal{C}^1 = \big\{(0_0, 1_2, 2_2, \ldots, (t-1)_2, 0_2, 1_0, 2_0, \ldots, (t-1)_0) \; (0_1, 1_3, 2_3, \ldots, (t-1)_3, 0_3, 1_1, 2_1, \ldots, (t-1)_1)\big\},$$

where the additions are taken modulo t.

Take $\mathcal{P}^3$ and $\mathcal{P}^4$ are as in case 1. Clearly, each $\mathcal{P}^i, i = 1, 2, 3, 4$ is a P_4-factor of $C_t \otimes I_4$ and $\mathcal{C}^1$ is a C_{2t}-factor of $C_t \otimes I_4$. Hence $\{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4, \mathcal{C}^1\}$ gives the existence of (P_4, C_{2t})–URD$(4, 1)$ of $C_t \otimes I_4$.

CASE 3: $k = 4t$.

Let

$$\mathcal{P}^1 = \big\{[0_0, 1_0, 0_1, 1_1], [0_2, 1_2, 0_3, 1_3], [(2i)_0, (2i+1)_1, (2i)_2, (2i+1)_3], [(2i+1)_0, (2i)_1, (2i+1)_2, (2i)_3] \mid 1 \le i \le \frac{t-2}{2}\big\};$$

$$\mathcal{P}^2 = \big\{[0_1, 1_3, 0_0, 1_2], [0_3, 1_1, 0_2, 1_0], [(2i)_2, (2i+1)_0, (2i)_3, (2i+1)_1], [(2i+1)_2, (2i)_0, (2i+1)_3, (2i)_1] \mid 1 \le i \le \frac{t-2}{2}\big\},$$

$$\mathcal{C}^1 = \big\{(0_0, 1_1, 2_1, \ldots, (t-1)_1, 0_1, 1_2, 2_2, \ldots, (t-1)_2, 0_2, 1_3, 2_3, \ldots, (t-1)_3, 0_3, 1_0, 2_0, \ldots, (t-1)_0)\big\}$$

where the additions are taken modulo t.

$\mathcal{P}^3$ and $\mathcal{P}^4$ are same as in case 1. Clearly, each $\mathcal{P}^i, i = 1, 2, 3, 4$ is a P_4-factor of $C_t \otimes I_4$ and $\mathcal{C}^1$ is a C_{4t}-factor of $C_t \otimes I_4$. Hence $\{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4, \mathcal{C}^1\}$ gives the existence of (P_4, C_{4t})–URD$(4, 1)$ of $C_t \otimes I_4$. Hence the Lemma is proved.

□

Lemma 5.4. *For any odd* $t \ge 3$, *there exists a* (P_4, C_k)–URD$(4, 1)$ *of* $C_t \otimes I_4$, *where* $k \in \{t, 2t, 4t\}$.

Proof. Let $V(C_t \otimes I_4) = \cup_{i \in \mathbb{Z}_t} V_i$, where $V_i = \{i_0, i_1, i_2, i_3\}, i \in \mathbb{Z}_t$. Now we prove the existence of (P_4, C_k)–URD$(4, 1)$ of $C_t \otimes I_4$ in three cases as follows:

CASE 1: $k = t$.

Let

$$\mathcal{P}^1 = \big\{[i_0, (i-1)_1, (i-2)_2, (i-3)_3] \mid 0 \le i \le t-1\big\};$$

$$\mathcal{P}^2 = \big\{[i_0, (i+1)_1, (i+2)_2, (i+3)_3] \mid 0 \le i \le t-1\big\};$$

$$\mathcal{P}^3 = \big\{[i_1, (i+1)_3, i_0, (i+1)_2] \mid 0 \le i \le t-1\big\};$$

$$\mathcal{P}^4 = \big\{[i_2, (i+1)_0, i_3, (i+1)_1] \mid 0 \le i \le t-1\big\};$$

$$\mathcal{C}^1 = \big\{(0_i, 1_i, 2_i, \ldots, (t-2)_i, (t-1)_i) \mid 0 \le i \le 3\big\},$$

where the additions are taken modulo t.

Clearly, each $\mathcal{P}^i, i = 1, 2, 3, 4$ is a P_4-factor of $C_t \otimes I_4$ and $\mathcal{C}^1$ is a C_t-factor of $C_t \otimes I_4$. Hence $\{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4, \mathcal{C}^1\}$ gives the existence of (P_4, C_t)–URD$(4, 1)$ of $C_t \otimes I_4$.

CASE 2: $k = 2t$.

Let

$$\mathcal{P}^1 = \big\{[(t-2)_2, (t-1)_1, (t-2)_3, (t-1)_0], [(2i)_2, (2i-1)_3, (2i)_3, (2i-1)_2],$$
$$[(2i+1)_0, (2i)_1, (2i+1)_1, (2i)_0] \mid 0 \leq i \leq \frac{t-3}{2}\big\};$$
$$\mathcal{P}^2 = \big\{[(t-1)_2, 0_1, (t-1)_3, 0_0], [(2i+1)_2, (2i)_3, (2i+1)_3, (2i)_2],$$
$$[(2i+2)_0, (2i+1)_1, (2i+2)_1, (2i+1)_0] \mid 0 \leq i \leq \frac{t-3}{2}\big\};$$
$$\mathcal{P}^3 = \big\{[(t-1)_3, (t-2)_2, (t-1)_2, (t-2)_3], [(t-1)_1, 0_0, (t-1)_0, 0_1],$$
$$[(i+1)_1, i_2, (i+1)_0, i_3] \mid 0 \leq i \leq t-3\big\};$$
$$\mathcal{P}^4 = \big\{[(i+1)_3, i_0, (i+1)_2, i_1] \mid 0 \leq i \leq t-1\big\};$$
$$\mathcal{C}^1 = \big\{(0_0, 1_0, 2_0, \dots, (t-2)_0, (t-1)_0, (t-2)_2, (t-3)_2, \dots, 3_2, 2_2, 1_2, 0_2,$$
$$(t-1)_2), (0_1, 1_3, 2_1, 3_3, 4_1 \dots, (t-3)_1, (t-2)_3,$$
$$(t-1)_3, (t-2)_1, (t-3)_3, (t-4)_1, \dots, 4_3,$$
$$3_1, 2_3, 1_1, 0_3, (t-1)_1)\big\},$$

where the additions are taken modulo t.

Clearly, each $\mathcal{P}^i, i = 1, 2, 3, 4$ is a P_4-factor of $C_t \otimes I_4$ and $\mathcal{C}^1$ is a C_{2t}-factor of $C_t \otimes I_4$. Hence $\{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4, \mathcal{C}^1\}$ gives the existence of (P_4, C_{2t})–URD$(4, 1)$ of $C_t \otimes I_4$.

CASE 3: $k = 4t$.

The proof of this case follows from the proof of case 1 of Lemma 4.1 by taking $\mathcal{C}^1 = \mathcal{P}_2^1 \cup \mathcal{P}_2^2$. Clearly $\mathcal{C}^1$ is a C_{4t}-factor of $C_t \otimes I_4$. Hence $\{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4, \mathcal{C}^1\}$ gives the existence of (P_4, C_{4t})–URD$(4, 1)$ of $C_t \otimes I_4$. Hence the Lemma is proved.

□

6. (P_4, C_k)–URD(r, s) OF SOME PRODUCT GRAPHS

In this section, we prove the existence of uniformly resolvable decomposition of some product graphs into P_4 and C_k, $k \geq 3$.

We arrange the vertex set of K_{4m} in a $m \times 4$ array. The vertices of each row form a copy of K_4 and the vertices of 4 columns together form a $K_m \otimes I_4$. Let $K_{4,4}$ be a complete bipartite graph with bipartition (X, Y), where $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$. Then let $\mathcal{P}^1 = \{[x_1, y_4, x_4, y_1], [x_2, y_3, x_3, y_2]\}$, $\mathcal{P}^2 = \{[x_3, y_1, x_1, y_3], [x_4, y_2, x_2, y_4]\}$ and $\mathcal{P}_2^1 = \{[x_1, y_2], [x_2, y_1], [x_3, y_4], [x_4, y_3]\}$. Clearly $\{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}_2^1\}$ gives the existence of (P_2, P_4)–URD$(1, 2)$ of $K_{4,4}$.

Theorem 6.1. *Let $t \geq 3$, $m \equiv 0 \pmod t$ and let $\mathcal{C}$ be any C_t-factor of K_m. Then there exists a (P_4, C_k)–URD(r, s) of $\mathcal{C} \otimes I_4$ with $(r, s) \in \{(4, 1), (0, 4)\}$, where $k \in \{t, 2t, 4t\}$.*

Proof. Let $m = tx$, $x \geq 1$. Since $\mathcal{C}$ is a C_t-factor of K_m, $\mathcal{C} \otimes I_4 \cong x(C_t \otimes I_4)$. By Lemmas 5.3 and 5.4, $C_t \otimes I_4$ has 4 P_4-factors and a C_k-factor, where $k \in \{t, 2t, 4t\}$. That is, $C_t \otimes I_4$ has a (P_4, C_k)–URD$(4, 1)$. Also by Theorems 2.3, 2.4, and 2.5, $C_t \otimes I_4$ has a C_k-factorization where $k \in \{t, 2t, 4t\}$. That is, (P_4, C_k)–URD$(0, 4)$ exists for $C_t \otimes I_4$. Hence $\mathcal{C} \otimes I_4$ has a (P_4, C_k)–URD(r, s) with $(r, s) \in \{(4, 1), (0, 4)\}$, where $k \in \{t, 2t, 4t\}$. □

Theorem 6.2. *Let $t \geq 3$, $m \equiv 0 \pmod t$ is even. Let $\mathcal{C}$ and I be the edge-disjoint C_t-factor and 1-factor of K_m. Then there exists a (P_4, C_k)–URD(r, s) of $(\mathcal{C} \oplus I) \otimes I_4$ with $(r, s) \in \{(4, 3), (8, 0)\}$, where $k \in \{t, 2t, 4t\}$.*

Proof. Let $m = tx$, $x \geq 1$. Consider the graph $G = (\mathcal{C} \oplus I) \otimes I_4 \cong (\mathcal{C} \otimes I_4) \oplus (I \otimes I_4)$. Now we prove the existence of (P_4, C_k)–URD(r, s) of G with $(r, s) \in \{(4, 3), (8, 0)\}$, where $k \in \{t, 2t, 4t\}$ in two cases as follows:

Case 1: $(r, s) = (4, 3)$.

Since $\mathcal{C}$ is a C_t-factor of K_m, $\mathcal{C} \otimes I_4 \cong x(C_t \otimes I_4)$. By Lemmas 4.2, 4.3, and 4.4, $C_t \otimes I_4$ has 2 P_2-factors and 3 C_k-factors, where $k \in \{t, 2t, 4t\}$ and hence $\mathcal{C} \otimes I_4$ has 2 P_2-factors, say H_1, H_2 and 3 C_k-factors, where $k \in \{t, 2t, 4t\}$. Since $I \otimes I_4 \cong \frac{m}{2}(K_{4,4})$, and $K_{4,4}$ has a P_2-factor and 2 P_4-factors, the graph $I \otimes I_4$ has a P_2-factor and 2 P_4-factors.

Therefore, each C_k-factor of $\mathcal{C} \otimes I_4$ is also a C_k-factor of G, where $k \in \{t, 2t, 4t\}$ and each P_4-factor of $I \otimes I_4$ is also a P_4-factor of G. There are 3 C_k-factors and 2 P_4 factors of G. The remaining 2 P_4 factors of G can be constructed from 2 P_2-factors of $\mathcal{C} \otimes I_4$ and a P_2-factor of $I \otimes I_4$.

Each $H_i, i = 1, 2$, is a P_2-factor between the set of vertices in the 1st and 2nd columns and the set of vertices in the 3rd and 4th columns (as per Lemmas 4.2–4.4). A P_2-factor of $I \otimes I_4$ is a union of a P_2-factor between the vertices in the 1st and 2nd columns and a P_2-factor between the vertices in the 3rd and 4th columns.

The graph obtained by joining the P_2-factor between the vertices in the 1st and 2nd columns of $I \otimes I_4$ with H_1 gives a P_4-factor of G and the graph obtained by joining the P_2-factor between the vertices in the 3rd and 4th columns of $I \otimes I_4$ and H_2 gives a P_4-factor of G.

In total, there are 3 C_k-factors and 4 P_4 factors of G. That is, there exists a (P_4, C_k)–URD$(4, 3)$ of $G = (\mathcal{C} \oplus I) \otimes I_4$.

Case 2: $(r, s) = (8, 0)$.

By Lemma 4.1, $C_t \otimes I_4$ has 2 P_2-factors and 4 P_4-factors. Hence $\mathcal{C} \otimes I_4$ has 2 P_2-factors and 4 P_4-factors. Since $I \otimes I_4 \cong \frac{m}{2}(K_{4,4})$ and $K_{4,4}$ has a P_2-factor and 2 P_4-factors, the graph $I \otimes I_4$ has a P_2-factor and 2 P_4-factors.

Each P_4-factor of $\mathcal{C} \otimes I_4$ is also a P_4-factor of G and each P_4-factor of $I \otimes I_4$ is also a P_4-factor of G. So, there are 6 P_4-factors of G. The remaining 2 P_4-factors of G can be constructed from 2 P_2-factors of $\mathcal{C} \otimes I_4$ and a P_2-factor of $I \otimes I_4$.

Note that one P_2-factor of $\mathcal{C} \otimes I_4$ is a union of a P_2-factor between the vertices in the 1st and 2nd columns and a P_2-factor between the vertices

in 3rd and 4th columns. Another P_2-factor of $\mathcal{C} \otimes I_4$ is a union of a P_2-factor between the vertices in the 2nd and 3rd columns and a P_2-factor between the vertices in the 4th and 1st columns. Also P_2-factor of $I \otimes I_4$ is a union of a P_2-factor between the vertices in the 1st and 2nd columns and a P_2-factor between the vertices in the 3rd and 4th columns.
The union of 2 P_2-factors of $\mathcal{C} \otimes I_4$ gives a 2-factor of $\mathcal{C} \otimes I_4$, say H, such that it has a P_2-factor between any two consecutive columns. Now remove the P_2-factor between the vertices in the 2nd and 3rd columns of H, then the existing graph is a P_4-factor of G. Joining the removed edges from H with the P_2-factor of $I \otimes I_4$, gives a P_4-factor of G.
In total, there are 8 P_4-factors of G. That is, there exists a (P_4, C_k)–URD$(8, 0)$ of $G = (\mathcal{C} \oplus I) \otimes I_4$.

□

Theorem 6.3. *Let $t \geq 3$, $m \equiv 0 \pmod t \geq 7$ and let $\mathcal{C}^1$, $\mathcal{C}^2$, and $\mathcal{C}^3$ be any three edge-disjoint C_t-factors of K_m. Then there exists a (P_4, C_k)–*URD$(16, 0)$ *of $(\oplus_{i=1}^{3}\mathcal{C}^i) \otimes I_4$.*

Proof. Let $m = tx$, $x \geq 1$. Consider the graph $G = (\oplus_{i=1}^{3}\mathcal{C}) \otimes I_4 \cong (\mathcal{C}^1 \otimes I_4) \oplus (\mathcal{C}^2 \otimes I_4) \oplus (\mathcal{C}^3 \otimes I_4)$. Now we prove the existence of (P_4, C_k)–URD$(16, 0)$ of G as follows:

Since each $\mathcal{C}^i, i = 1, 2, 3$ is a C_t-factor of K_m, $\mathcal{C}^i \otimes I_4 \cong x(C_t \otimes I_4)$. By Lemma 4.1, $C_t \otimes I_4$ has 2 P_2-factors and 4 P_4-factors. Hence each $\mathcal{C}^i \otimes I_4, i = 1, 2, 3$ has 2 P_2-factors and 4 P_4-factors. These 4 P_4-factors of each $\mathcal{C}^i \otimes I_4$, $i = 1, 2, 3$ together gives 12 P_4-factors of G. The remaining 4 P_4-factors of G can be constructed from 2 P_2-factors of each $\mathcal{C}^i \otimes I_4, i = 1, 2, 3$.

For each i, $i = 1, 2, 3$, adding 2 P_2-factors of $\mathcal{C}^i \otimes I_4$ gives a 2-factor, say H_i, of $\mathcal{C}^i \otimes I_4$ such that it has a P_2-factor between any two consecutive columns.

Now remove the P_2-factor between the vertices in the i and $(i+1)$th column of each H_i, $i = 1, 2, 3$, then the remaining graph gives a P_4-factor of G. Form a new graph by adding the removed edges from each H_i, $i = 1, 2, 3$, then the resulting graph itself is a P_4-factor of G. Hence we get 4 P_4-factors of G.

In total, there are 16 P_4-factors of G. (i.e.) there exists a (P_4, C_k)–URD$(16, 0)$ of $G = (\oplus_{i=1}^{3}\mathcal{C}^i) \otimes I_4$. □

Theorem 6.4. *Let $m \geq 4$ is even. Let $\mathcal{C}$ and I be the edge-disjoint C_m-factor and 1-factor of K_m. Then there exists a (P_4, C_k)–*URD(r, s) *of $(\mathcal{C} \oplus I) \otimes I_4$ with $(r, s) \in \{(8, 0), (4, 3), (0, 6)\}$, where $k \in \{4, 8\}$.*

Proof. Let $G = (\mathcal{C} \oplus I) \otimes I_4 \cong (\mathcal{C} \otimes I_4) \oplus (I \otimes I_4)$. Now we prove the existence of (P_4, C_k)–URD(r, s) of G with $(r, s) \in \{(8, 0), (4, 3), (0, 6)\}$, where $k \in \{4, 8\}$ in two cases as follows:

Case 1: $(r, s) \in \{(4,3),(0,6)\}$.

Since $I \otimes I_4 \cong \frac{m}{2}(K_{4,4})$ and $K_{4,4}$ has 2 C_k-factors, $k \in \{4,8\}$. Hence (P_4, C_k)–URD$(0,2)$ of $I \otimes I_4$ exists, where $k \in \{4,8\}$. By Lemmas 5.1 and 5.2, $\mathcal{C} \otimes I_4$ has a (P_4, C_k)–URD(r,s) with $(r,s) \in \{(4,1),(0,4)\}$, where $k \in \{4,8\}$.

Therefore there exists a (P_4, C_k)–URD(r,s) of G with $(r,s) \in \{(4,1),(0,4)\} + \{(0,2)\} = \{(4,3),(0,6)\}$, where $k \in \{4,8\}$.

Case 2: $(r,s) = (8,0)$.

This case follows from case 2 of Theorem 6.2.

□

7. Main Results

In this section, we prove our main results.

Theorem 7.1. *There exists a* (P_4, C_4)–URD$(n; r, s)$ *if and only if* $n \equiv 0 \pmod 4$ *and* $(r,s) \in J(n)$.

Proof. Necessity follows from Lemma 2.6. Conversely, let $n = 4m$, $m \geq 1$. Since K_4 has 2 P_4-factors, (P_4, C_4)–URD$(4; 2, 0)$ exists. We know that $K_8 \cong K_{4,4} \oplus 2K_4$, $K_{4,4}$ has 2 C_4-factor and K_4 has 2 P_4-factor. Hence (P_4, C_4)–URD$(8; 2, 2)$ exists.

Let $m \geq 3$ and let $\mathcal{C}$ be any C_m-factor of K_m. Since $\mathcal{C}$ is a C_m-factor of K_m, $\mathcal{C} \otimes I_4 \cong C_m \otimes I_4$. By Lemma 5.1, $C_m \otimes I_4$ has a (P_4, C_4)–URD(r,s) with $(r,s) \in \{(4,1),(0,4)\}$. Hence $\mathcal{C} \otimes I_4$ has a (P_4, C_4)–URD(r,s) with $(r,s) \in \{(4,1),(0,4)\}$.

By Theorem 6.3, there exists a (P_4, C_4)–URD$(16,0)$ of $(\mathcal{C}^a \oplus \mathcal{C}^b \oplus \mathcal{C}^c) \otimes I_4$, where $\mathcal{C}^a$, $\mathcal{C}^b$, and $\mathcal{C}^c$ are any 3 edge-disjoint C_m-factors of K_m. When m is even, $(\mathcal{C} \oplus I) \otimes I_4$ has a (P_4, C_4)–URD(r,s) with $(r,s) \in \{(8,0),(4,3)\}$ by Theorem 6.4, where I is a 1-factor of K_m edge-disjoint from $\mathcal{C}$.

Applying Theorem 3.1 (when m is odd) and Theorems 3.2 and 3.3, (when m is even) with $t = m$ and $k = 4$, we obtain a (P_4, C_4)–URD$(4m; r, s)$ with $(r,s) \in J(4m)$. This completes the proof. □

Theorem 7.2. *There exists a* (P_4, C_8)–URD$(n; r, s)$ *if and only if* $n \equiv 0 \pmod 8$ *and* $(r,s) \in J(n)$.

Proof. Necessity follows from Lemma 2.6. Conversely, let $n = 8x = 4m$, $m \geq 2$ is even. We know that $K_8 \cong K_{4,4} \oplus 2K_4$, $K_{4,4}$ has 2 C_8-factors and K_4 has 2 P_4-factors. Hence (P_4, C_8)–URD$(8; 2, 2)$ exists.

Let $m \geq 4$ is even and let $\mathcal{C}$ be any C_m-factor of K_m. Since $\mathcal{C}$ is a C_m-factor of K_m, $\mathcal{C} \otimes I_4 \cong C_m \otimes I_4$. By Lemma 5.2, $C_m \otimes I_4$ has a (P_4, C_8)–URD(r,s) with $(r,s) \in \{(4,1),(0,4)\}$. Hence $\mathcal{C} \otimes I_4$ has a (P_4, C_8)–URD(r,s) with $(r,s) \in \{(4,1),(0,4)\}$.

By Theorem 6.3, there exists a (P_4, C_8)–URD$(16,0)$ of $(\mathcal{C}^a \oplus \mathcal{C}^b \oplus \mathcal{C}^c) \otimes I_4$, where $\mathcal{C}^a$, $\mathcal{C}^b$, and $\mathcal{C}^c$ are any 3 edge-disjoint C_m-factors of K_m. $(\mathcal{C} \oplus I) \otimes I_4$ has a (P_4, C_8)–URD(r,s) with $(r,s) \in \{(8,0),(4,3)\}$ by Theorem 6.4, where I is a 1-factor of K_m edge-disjoint from $\mathcal{C}$.

Applying Theorems 3.2 and 3.3 with $t = m$ and $k = 8$, we obtain a (P_4, C_8)–URD$(4m; r, s)$ with $(r, s) \in J(4m)$. This completes the proof. □

Theorem 7.3. *Let $k \geq 3$ be an odd integer. Then there exists a (P_4, C_k)–URD$(n; r, s)$ if and only if $n \equiv 0 \pmod{4k}$ and $(r, s) \in J(n)$, except for $k = 3$ and $n \in \{24, 48\}$.*

Proof. Necessity follows from Lemma 2.6. Conversely, let $n = 4kx$, $x \geq 1$.

There exists a (P_4, C_k)–URD(r, s) of $\mathcal{C} \otimes I_4$, with $(r, s) \in \{(4, 1), (0, 4)\}$, by Theorem 6.1, where $\mathcal{C}$ is a C_k-factor of K_{kx}. By Theorem 6.3, there exists a (P_4, C_k)–URD$(16, 0)$ of $(\mathcal{C}^a \oplus \mathcal{C}^b \oplus \mathcal{C}^c) \otimes I_4$, where $\mathcal{C}^a$, $\mathcal{C}^b$, and $\mathcal{C}^c$ are any 3 edge-disjoint C_k-factors of K_{kx}. When kx is even, $(\mathcal{C} \oplus I) \otimes I_4$ has a (P_4, C_k)–URD(r, s) with $(r, s) \in \{(8, 0), (4, 3)\}$ by Theorem 6.2, where I is a 1-factor of K_{kx} edge-disjoint from $\mathcal{C}$.

Applying Theorem 3.1 (when kx is odd) and Theorems 3.2 and 3.3, (when kx is even) with $m = kx$ and $t = k$, we obtain a (P_4, C_k)–URD$(4kx; r, s)$ with $(r, s) \in J(4kx)$ except when $k = 3$ and $4kx \in \{24, 48\}$.

That is, there exists a (P_4, C_k)–URD$(n; r, s)$ with $(r, s) \in J(n)$ except when $k = 3$ and $n \in \{24, 48\}$, where $k \equiv 1 \pmod 2 \geq 3$ and $n \equiv 0 \pmod{4k}$. □

Theorem 7.4. *Let $k \equiv 2 \pmod 4 \geq 6$. Then there exists a (P_4, C_k)–URD$(n; r, s)$ if and only if $n \equiv 0 \pmod{2k}$ and $(r, s) \in J(n)$, except for $k = 6$ and $n \in \{24, 48\}$.*

Proof. Necessity follows from Lemma 2.6. Conversely, let $n = 2kx = 4(\frac{k}{2})x$, $x \geq 1$. Let $\frac{k}{2} = k'$, then $k' \geq 3$ is an odd integer.

Let $\mathcal{C}$ be any $C_{k'}$-factor of $K_{k'x}$. Then there exists a $(P_4, C_{2k'})$–URD(r, s) of $\mathcal{C} \otimes I_4$, with $(r, s) \in \{(4, 1), (0, 4)\}$, by Theorem 6.1. By Theorem 6.3, there exists a $(P_4, C_{2k'})$–URD$(16, 0))$ of $(\mathcal{C}^a \oplus \mathcal{C}^b \oplus \mathcal{C}^c) \otimes I_4$, where $\mathcal{C}^a$, $\mathcal{C}^b$, and $\mathcal{C}^c$ are any 3 edge-disjoint $C_{k'}$-factors of $K_{k'x}$. $(\mathcal{C} \oplus I) \otimes I_4$ has a $(P_4, C_{2k'})$–URD(r, s) with $(r, s) \in \{(8, 0), (4, 3)\}$ by Theorem 6.2, when $k'x$ is even, where I is a 1-factor of $K_{k'x}$ edge-disjoint from $\mathcal{C}$.

Applying Theorem 3.1 (when $k'x$ is odd) and Theorems 3.2 and 3.3, (when $k'x$ is even) with $m = k'x$, $t = k'$ and $k = 2t$, we obtain a $(P_4, C_{2k'})$–URD$(4k'x; r, s)$ with $(r, s) \in J(4k'x)$ except when $k' = 3$ and $4k'x \in \{24, 48\}$.

That is, there exists a (P_4, C_k)–URD$(n; r, s)$ with $(r, s) \in J(n)$ except when $k = 6$ and $n \in \{24, 48\}$, where $k \equiv 2 \pmod 4 \geq 6$ and $n \equiv 0 \pmod{2k}$. □

Theorem 7.5. *Let $k \equiv 0 \pmod 4 \geq 12$. Then there exists a (P_4, C_k)–URD$(n; r, s)$ if and only if $n \equiv 0 \pmod k$ and $(r, s) \in J(n)$.*

Proof. Necessity follows from Lemma 2.6. Conversely, let $n = kx = 4(\frac{k}{4})x$, $x \geq 1$. Let $\frac{k}{4} = k'$ Then $n = 4k'x$, $x \geq 1$.

Let $\mathcal{C}$ be any $C_{k'}$-factor of $K_{k'x}$. There exists a $(P_4, C_{4k'})$–URD(r, s) of $\mathcal{C} \otimes I_4$, with $(r, s) \in \{(4, 1), (0, 4)\}$, by Theorem 6.1. By Theorem 6.3, there exists a $(P_4, C_{4k'})$–URD$(16, 0)$ of $(\mathcal{C}^a \oplus \mathcal{C}^b \oplus \mathcal{C}^c) \otimes I_4$, where $\mathcal{C}^a$, $\mathcal{C}^b$, and $\mathcal{C}^c$ are any 3 edge-disjoint $C_{k'}$-factors of $K_{k'x}$. The graph $(\mathcal{C} \oplus I) \otimes I_4$ has a $(P_4, C_{4k'})$–URD(r, s) with $(r, s) \in \{(8, 0), (4, 3)\}$ by Theorem 6.2, when $k'x$ is even, where I is a 1-factor of $K_{k'x}$ edge-disjoint from $\mathcal{C}$.

Applying Theorem 3.1 (when $k'x$ is odd) and Theorems 3.2 and 3.3, (when $k'x$ is even) with $m = k'x$, $t = k'$ and $k = 4t$, we obtain a $(P_4, C_{4k'})$–URD$(4kn; r, s)$ with $(r, s) \in J(4k'x)$ except when $k' = 3$ and $4k'x \in \{24, 48\}$.

That is, there exists a (P_4, C_k)–URD$(n; r, s)$ with $(r, s) \in J(n)$ except when $k = 12$ and $n \in \{24, 48\}$, where $k \equiv 0 \pmod 4 \geq 12$ and $n \equiv 0 \pmod k$. □

Theorem 7.6. *There exists a* (P_4, C_3)–URD$(24; r, s)$ *with* $(r, s) \in \{(4x + 2, 10 - 3x) \mid x = 0, 1, 2, 3\}$.

Proof. We prove the existence of (P_4, C_3)–URD$(24; r, s)$ with $(r, s) \in \{(4x + 2, 10 - 3x) \mid x = 0, 1, 2, 3\}$ in three cases as follows:

Case 1: $(r, s) = (2, 10)$.

By Theorem 3.3, there exists a (P_4, C_3)–URD$(24; 2, 10)$.

Case 2: $(r, s) \in \{(14, 1), (10, 4)\}$.

Let $V(K_6) = \{0, 1, 2, 3, 4, 5\}$. Then $K_6 = \mathcal{C}^1 \oplus \mathcal{C}^2 \oplus I$, where $\mathcal{C}^1 = \{(0, 1, 2), (3, 4, 5)\}$, $\mathcal{C}^2 = \{(0, 3, 1, 5, 2, 4)\}$ and $I = \{[2, 3], [1, 4], [0, 5]\}$. Clearly $\mathcal{C}^1$ is a C_3-factor, $\mathcal{C}^2$ is a C_6-factor and I is a 1-factor of K_6. Consider

$$\begin{aligned} K_{24} &\cong (K_6 \otimes I_4) \oplus (I_6 \otimes K_4) \\ &\cong ((\mathcal{C}^1 \otimes I_4) \oplus ((\mathcal{C}^2 \oplus I) \otimes I_4) \oplus (I_6 \otimes K_4). \end{aligned}$$

Since $\mathcal{C}^1 \otimes I_4 \cong 2(C_3 \otimes I_4)$, there exists a (P_4, C_3)–URD(r, s) of $\mathcal{C}^1 \otimes I_4$ with $(r, s) \in \{(4, 1), (0, 4)\}$, by Theorem 6.1. The graph $(\mathcal{C}^2 \oplus I) \otimes I_4$ has a (P_4, C_3)–URD(r, s) with $(r, s) = (8, 0)$, by Theorem 6.2. Since K_4 has 2 P_4-factors, $I_6 \otimes K_4$ has a (P_4, C_3)–URD(r, s) with $(r, s) = (2, 0)$. This gives the existence of (P_4, C_3)–URD$(v; r, s)$ of K_{24} with $(r, s) \in \{\{(4, 1), (0, 4)\} + \{(8, 0)\} + \{(2, 0)\}\} = \{(14, 1), (10, 4)\}$.

Case 3: $(r, s) = (6, 7)$.

Consider $K_{24} \cong (K_3 \otimes I_8) \oplus (I_3 \otimes K_8)$. Let $V(K_3 \otimes K_8) = \cup_{i \in \mathbb{Z}_3} X_i$, where $X_i = \{i_j \mid j \in \mathbb{Z}_8\}, i \in \mathbb{Z}_3$. $E(K_3 \otimes I_8) = \cup_{i \in \mathbb{Z}_3, l \in \mathbb{Z}_8} F_l(X_i, X_{i+1})$ and $E(I_3 \otimes K_8) = \cup_{i \in \mathbb{Z}_3, 0 \leq a < b \leq 7} \{i_a, i_b\}$. Now we construct 6 P_4-factors of K_{24} as follows: Let

$$\mathcal{P}^1 = \{[i_1, i_4, i_0, i_5], [i_3, i_6, i_2, i_7] \mid i \in \mathbb{Z}_3\};$$

$$\mathcal{P}^2 = \{[i_0, i_7, i_3, i_4], [i_2, i_5, i_1, i_6] \mid i \in \mathbb{Z}_3\};$$

$$\mathcal{P}^3 = \{[i_0, i_6, i_5, i_3], [i_4, i_2, i_1, i_7] \mid i \in \mathbb{Z}_3\};$$

$$\mathcal{P}^4 = \{[i_1, i_3, i_0, i_2], [i_6, i_4, i_7, i_5] \mid i \in \mathbb{Z}_3\};$$

$$\mathcal{P}^5 = \{[(i+1)_4, i_0, i_1, (i+1)_5], [(i+1)_6, i_2, i_3, (i+1)_7] \mid i \in \mathbb{Z}_3\};$$

$$\mathcal{P}^6 = \{[(i+1)_0, i_4, i_5, (i+1)_1], [(i+1)_2, i_6, i_7, (i+1)_3] \mid i \in \mathbb{Z}_3\},$$

where the additions are taken modulo 3. Now we construct 7 C_3-factors of K_{24} as follows: Let

$$\begin{aligned}
\mathcal{C}^1 &= F_0(X_0,X_1) \cup F_0(X_1,X_2) \cup F_0(X_2,X_0);\\
\mathcal{C}^2 &= F_1(X_0,X_1) \cup F_5(X_1,X_2) \cup F_2(X_2,X_0);\\
\mathcal{C}^3 &= F_2(X_0,X_1) \cup F_1(X_1,X_2) \cup F_5(X_2,X_0);\\
\mathcal{C}^4 &= F_3(X_0,X_1) \cup F_7(X_1,X_2) \cup F_6(X_2,X_0);\\
\mathcal{C}^5 &= F_5(X_0,X_1) \cup F_2(X_1,X_2) \cup F_1(X_2,X_0);\\
\mathcal{C}^6 &= F_6(X_0,X_1) \cup F_3(X_1,X_2) \cup F_7(X_2,X_0);\\
\mathcal{C}^7 &= F_7(X_0,X_1) \cup F_6(X_1,X_2) \cup F_3(X_2,X_0),
\end{aligned}$$

Hence $\{\mathcal{P}^i, \mathcal{C}^j \mid 1 \le i \le 6, 1 \le j \le 7\}$ gives a (P_4, C_3)–URD$(6,7)$ of K_{24}. Hence there exists a (P_4, C_3)–URD$(24; r, s)$ with $(r,s) \in \{(2,10), (6,7), (10,4), (14,1)\}$.

□

Theorem 7.7. *There exists a* (P_4, C_3)–URD$(48; r, s)$ *with* $(r,s) \in \{(4x + 2, 22 - 3x) \mid x = 0,1,2,3,4,5,6,7\}$.

Proof. We prove the existence of (P_4, C_3)–URD$(48; r, s)$ with $(r,s) \in \{(4x+2, 22-3x) \mid x = 0,1,2,3,4,5,6,7\}$ in three cases as follows:
Case 1: $(r,s) = (2,22)$.
By Theorem 3.3, there exists a (P_4, C_3)–URD$(48; (2,22))$.
Case 2: $(r,s) \in \{(10,16), (14,13), (18,10), (22,7), (26,4), (30,1)\}$.
Let $V(K_{12}) = \{0,1,2,3,4,5,7,8,9,10,11\}$. Then $K_{12} = \mathcal{C}^1 \oplus \mathcal{C}^2 \oplus \mathcal{C}^3 \oplus \mathcal{C}^4 \oplus \mathcal{C}^5 \oplus I$, where $\mathcal{C}^1 = \{(0,2,4), (1,3,5), (6,8,10), (7,9,11)\}$; $\mathcal{C}^2 = \{(0,3,6), (1,2,7), (4,8,11), (5,9,10)\}$; $\mathcal{C}^3 = \{(0,5,11), (1,4,10), (2,6,9), (3,7,8)\}$; $\mathcal{C}^4 = \{(0,7,10), (1,6,11), (2,8,5), (3,9,4)\}$; $\mathcal{C}^5 = \{(0,8,1,9), (2,10,3,11), (4,6,5,7)\}$ and $I = \{[0,1], [2,3], [5,4], [6,7].[8,9], [10,11]\}$. Clearly each $\mathcal{C}^i$, $i = 1,2,3,4$ is a C_3-factor, $\mathcal{C}^5$ is a C_4-factor and I is a 1-factor of K_{12}. Now

$$\begin{aligned}
K_{48} &\cong (K_{12} \otimes I_4) \oplus (I_{12} \otimes K_4)\\
&\cong ((\mathcal{C}^1 \otimes I_4) \oplus (\mathcal{C}^2 \otimes I_4) \oplus (\mathcal{C}^3 \otimes I_4)\\
&\qquad \oplus (\mathcal{C}^4 \otimes I_4) \oplus ((\mathcal{C}^5 \oplus I) \otimes I_4) \oplus (I_{12} \otimes K_4).
\end{aligned}$$

There exists a (P_4, C_3)–URD(r,s) of each $\mathcal{C}^i \otimes I_4$, $i = 1,2,3,4$ with $(r,s) \in \{(4,1), (0,4)\}$ by Theorem 6.1. $(\mathcal{C}^5 \oplus I) \otimes I_4$ has a (P_4, C_3)–URD$(8,0)$ by Theorem 6.2. There exists a (P_4, C_3)–URD$(16,0)$ of $(\oplus_{i=1}^{3} \mathcal{C}^i) \otimes I_4$ by Theorem 6.3. Since K_4 has 2 P_4-factor, $I_{12} \otimes K_4$ has a (P_4, C_3)–URD$(2,0)$. This gives the existence of (P_4, C_3)–URD$(48; r, s)$ with $(r,s) \in \{(4-3x) * \{(4,1), (0,4)\} + x * \{(16,0)\} + \{(8,0)\} + \{(2,0)\} \mid 0 \le x \le 1\} = \{(10,16), (14,13), (18,10), (22,7), (26,4), (30,1)\}$.

Case 3: $(r,s)=(6,19)$.

Consider $K_{48}\cong(K_4\otimes I_{12})\oplus(I_4\otimes K_{12})$. By Theorem 2.2, $K_4\otimes I_{12}$ has a (P_4,C_3)–URD$(0,18)$. There exists a (P_4,C_3)–URD$(12;6,1)$ by Theorem 7.3 and $I_4\otimes K_{12}\cong 4K_{12}$, $I_4\otimes K_{12}$ has a (P_4,C_3)–URD$(6,1)$. This gives the existence of (P_4,C_3)–URD$(48;6,19)$.

Therefore, there exists a (P_4,C_3)–URD$(48;r,s)$ with $(r,s)\in\{(2,22),(6,19),(10,16),(14,13),(18,10),(22,7),(26,4),(30,1)\}$.

□

Theorem 7.8. *There exists a* (P_4,C_k)–URD$(n;r,s)$ *with* $(r,s)\in J(n)=\{(4x+2,\frac{n-4}{2}-3x)\mid x=0,1,\dots,\lfloor\frac{n-4}{6}\rfloor\}$, *where* $k\in\{6,12\}$ *and* $n\in\{24,48\}$.

Proof. We prove the existence of (P_4,C_k)–URD$(n;r,s)$ with $(r,s)\in J(n)$, where $k\in\{6,12\}$ and $n\in\{24,48\}$ in two cases as follows:

Case 1: $n=24$.

Let $\mathcal{C}$ be a C_6-factor of K_6-I, where I is a 1-factor of K_6. There exists a (P_4,C_k)–URD(r,s) of $\mathcal{C}\otimes I_4$, with $(r,s)\in\{(4,1),(0,4)\}$, by Theorem 6.1 and $(\mathcal{C}\oplus I)\otimes I_4$ has a (P_4,C_k)–URD(r,s) with $(r,s)\in\{(8,0),(4,3)\}$ by Theorem 6.2, where $k\in\{6,12\}$.

Applying Theorems 3.2 and 3.3, with $t=m=6$ and $k\in\{6,12\}$, we obtain a (P_4,C_k)–URD$(24;r,s)$ with $(r,s)\in J(24)$, where $k\in\{6,12\}$.

Case 2: $n=48$.

Let $\mathcal{C}$ and I be edge-disjoint C_6-factor and 1-factor of K_{12}. There exists a (P_4,C_k)–URD(r,s) of $\mathcal{C}\otimes I_4$, with $(r,s)\in\{(4,1),(0,4)\}$, by Theorem 6.1 and $(\mathcal{C}\oplus I)\otimes I_4$ has a (P_4,C_k)–URD(r,s) with $(r,s)\in\{(8,0),(4,3)\}$ by Theorem 6.2, where $k\in\{6,12\}$. By Theorem 6.3, there exists a (P_4,C_k)–URD$(16,0)$ of $(\mathcal{C}^a\oplus\mathcal{C}^b\oplus\mathcal{C}^c)\otimes I_4$, where $\mathcal{C}^a$, $\mathcal{C}^b$, and $\mathcal{C}^c$ are any 3 edge-disjoint C_6-factors of K_{12}.

Applying Theorems 3.2 and 3.3, with $t=6,\ m=12$ and $k\in\{6,12\}$, we obtain a (P_4,C_k)–URD$(48;r,s)$ with $(r,s)\in J(48)$, where $k\in\{6,12\}$.

From cases 1 and 2, there exists a (P_4,C_k)–URD$(n;r,s)$ with $(r,s)\in J(n)$, where $k\in\{6,12\}$ and $n\in\{24,48\}$. □

8. Conclusion

Combining Theorems 7.1 to 7.8, we have completely settled the existence of (P_4,C_k)–URD$(n;r,s)$ for any admissible parameters $n,\ r,$ and s, where $k\geq 3$.

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