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ON THE UNIFORMITY OF THE APPROXIMATION FOR r-ASSOCIATED STIRLING NUMBERS OF THE SECOND KIND

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ABSTRACT. The r-associated Stirling numbers of the second kind are a natural extension of Stirling numbers of the second kind. A combinatorial interpretation of r-associated Stirling numbers of the second kind is the number of ways to partition n elements into m subsets such that each subset contains at least r elements. Calculating the associated Stirling numbers is typically done with a recurrence relation or a generating function that are computationally expensive or alternatively with a closed-form that is practical for only a limited parameter range. In 1994 Hennecart proposed an approximation for the r-associated Stirling numbers that is fast to compute, is amenable to analysis over a wide range of parameters, and is conjectured to be asymptotically tight. There are a few other approximations for the associated Stirling numbers, but none of them are as general as Hennecart's. However, until this work, Hennecart's approximation had been utilized without a proper justification due to the absence of a rigorous proof. This work provides a proof of the uniformity of the Hennecart approximation.

1. INTRODUCTION

In a 1994 paper [11], Hennecart proposed an approximation for the r-associated Stirling numbers of the second kind, and conjectured that the approximation was asymptotically tight for a wide range of parameters. This approximation has been successfully used in a number of applications (see, for example, [5, 7]). However, a proper justification has never appeared in literature.

Let $\left\{ {n \atop m} \right\}$ denote the Stirling number of the second kind. Stirling numbers of the second kind have been studied extensively with applications to combinatorics, probability, and analysis. (See [1] for a short history of their discovery.) Stirling numbers of the second kind can be defined in a number of ways. Combinatorially, we can define them as the number of ways to

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partition n labeled elements into m nonempty unlabeled subsets. A more common definition defines them as the coefficients of the generating function

$$\frac{(\exp(x)-1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} \frac{x^n}{n!}.$$

Other definitions include the closed-form formula

$$\left\{ {n \atop m} \right\} = \frac{1}{m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

and the recurrence relation

$$\left\{ {n \atop m} \right\} = m \left\{ {n-1 \atop m} \right\} + \left\{ {n-1 \atop m-1} \right\}$$

with the initial conditions $\left\{ {n \atop m} \right\} = 0$ if $m = 0$ or $n < m$ and $\left\{ {n \atop m} \right\} = 1$ if $n = m$.

A large number of generalizations and restrictions to Stirling numbers have been created. One natural generalization is $\left\{ {n \atop m} \right\}_{\geq r}$, the *$r$-associated* Stirling number of the second kind. Combinatorially, we define $\left\{ {n \atop m} \right\}_{\geq r}$ as the number of ways to partition n labeled elements into m unlabeled subsets such that each subset contains at least r elements. This definition gives the recurrence relation

$$\left\{ {n \atop m} \right\}_{\geq r} = m \left\{ {n-1 \atop m} \right\}_{\geq r} + \binom{n-1}{r-1} \left\{ {n-r \atop m-1} \right\}_{\geq r} \tag{1.1}$$

with the initial conditions $\left\{ {n \atop m} \right\}_{\geq r} = 0$ if $m = 0$ or $n < rm$, and $\left\{ {n \atop m} \right\}_{\geq r} = 1$ if $n = rm$.

Just as with the usual Stirling numbers, the r-associated Stirling numbers of the second kind appear as the coefficients of a generating function

$$\frac{\left(\exp(x) - \sum_{k=0}^{r-1} \frac{x^k}{k!}\right)^m}{m!} = \sum_{n=mr}^{\infty} \left\{ {n \atop m} \right\}_{\geq r} \frac{x^n}{n!}. \tag{1.2}$$

There are several closed-form formulas for $\left\{ {n \atop m} \right\}_{\geq r}$ including

$$\left\{ {n \atop m} \right\}_{\geq r} = \sum_{\substack{u_1+\ldots+u_m=n \\ u_i \geq r}} \frac{n!}{m! u_1! u_2! \ldots u_m!}, \tag{1.3}$$

and

$$\left\{ {n \atop m} \right\}_{\geq r} = \frac{1}{m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \sum_{l_1, l_2, \ldots, l_k = 0}^{r-1} \frac{n!(m-k)^{n-l_1-l_1-\cdots-l_k}}{l_1! l_2! \ldots l_k! (n-l_1-l_2-\cdots-l_k)!}. \tag{1.4}$$

The last equation can be found in [11]. The case $r = 1$ provides the closed-form equation for a usual Stirling number of the second kind.

The r-associated Stirling numbers of the second kind first appear in [20] where both equations (1.2) and (1.1) are given, and in [23] where they were used to compute unbiased estimators for truncated Poisson distributions. Prior to these works, 2-associated Stirling numbers of the second kind were used in [26] and [16] for analyzing Stirling numbers. For a survey of the associated Stirling numbers, see [13]. The r-associated Stirling numbers of the second kind are related to several combinatorial series including generalized Bell numbers [8, 25], Bernoulli numbers [12, 28], and incomplete poly-Bernoulli numbers [17].

In practical applications, the r-associated Stirling numbers of the second kind show up in series expansions of the Lambert W function [15, 6], in the analysis of error correcting codes for flash memory [10], and in the study of threshold behavior in both spin-glass models of statistical physics [4], as well as constraint satisfaction problems of computer science [7, 5].

As with other Stirling number definitions, computing r-associated Stirling numbers exactly is computationally expensive. For example, the recent works [14, 9] compare the computational cost of equations (1.2) and (1.1) for various values of the parameters. Equation (1.3), though, is generally more efficient than the other two approaches, but it is still impractical for either large r or large m. An alternative to computing the exact value of $\left\{ {n \atop m} \right\}_{\geq r}$ is to utilize an approximation that is sufficiently precise and is faster to compute than either the closed-form or the recurrence relation. Ideally, such an approximation would be more practical than the exact definitions for both analysis and numerical evaluation.

In a 1994 paper [11], Hennecart developed such an approximation for the r-associated Stirling numbers of the second kind, and conjectured that the approximation was asymptotically tight for a wide range of parameters. The proposed approximation is:

$$\left\{ {n \atop m} \right\}_{\geq r} \sim \frac{n!}{m!(n-mr)!} \left(\frac{n-mr}{\mathrm{e}} \right)^{n-mr} \frac{B^m(z_0, r)}{z_0^{n+1}} \sqrt{\frac{mt_0}{\phi''(z_0)}} \tag{1.5}$$

where

$$B(x,r) = \exp(x) - \sum_{k=0}^{r-1} \frac{x^k}{k!},$$
$$\phi(z) = -n \ln z + m \ln B(z,r),$$
$$t_0 = (n-mr)/m,$$

and z_0 is the positive real solution of the equation

$$\frac{z_0 B'(z_0, r)}{B(z_0, r)} = \frac{n}{m}.$$

This approximation has been used in several applications [5, 7] that analyze the asymptotic behavior of constraint satisfaction problems. Unfortunately, [11] does not give a rigorous proof that this approximation (1.5) is uniform.

Instead, [11] calculates the approximation for a number of different positive integer values for n, m, and r to demonstrate a uniform behavior. Without a rigorous proof, the approximation (1.5) is of limited utility, and the works [7, 5] that rely on (1.5) are conditionally valid.

Since [11], there has been some limited additional work on approximations for the associated Stirling numbers. In particular, [3] shows that 2-associated Stirling numbers of the second kind are asymptotically close to a normal distribution, and [28] gives asymptotically tight approximations to summations involving 2-associated Stirling numbers of the second kind. However, (1.5) is the only approximation we have found for all positive r. The purpose of this paper is to make (1.5) rigorous.

Theorem 1.1. *Let r be a fixed positive integer. Approximation (1.5) holds uniformly as $n \to \infty$ for all $\delta_1 n < m < (1-\delta_2)n/r$ where n and m are integers and δ_1 and δ_2 are any positive constants.*

We note that [5, 7] both assume r is a fixed integer and m and n are both integers with $m = \Theta(n)$. Therefore, Theorem 1.1 confirms both of these applications.

In Section 2, we give an approximation for the r-associated Stirling numbers of the second kind that is more general than (1.5), and we prove that this more general approximation is uniform. In Section 3 we prove some technical lemmas needed in the proof of Section 2, and a couple of these concern the behavior of the tail of the Maclaurin series for $\exp(z)$, $z \in \mathbb{C}$. Finally, in Section 4, we give the proof of Theorem 1.1 by successfully truncating the more general approximation of Section 2.

2. The main approximation lemma

The Hennecart [11] approximation (1.5) for $\left\{ {n \atop m} \right\}_{\geq r}$ is based on an approximation for $\left\{ {n \atop m} \right\}$ proposed by Temme [24] and proven uniform in Chelluri, Richmond, and Temme [2]. The proof in [2] takes the error bounds for integer n and m of a slightly different approximation for $\left\{ {n \atop m} \right\}$ by Moser and Wyman [19] and proves that these bounds also apply to the approximation of [24]. In addition, [2] extends the proof of [19] to prove that the [24] approximation is also uniform for real and complex parameters.

This paper follows a strategy analogous to [2]. We develop an approximation for the r-associated Stirling numbers of the second kind that is more general though less computationally efficient than (1.5) but also closer to the approximation of [19] for regular Stirling numbers of the second kind. We adapt the [19] proof to work with our approximation, and we perform similar modifications as [2]. However, we stop once we prove the approximation is uniform for integer parameters, and we do not make a similar extension in this paper as [2] does to noninteger parameters. We believe that such an extension to real and complex parameters should hold, but the location of the zeros of the function $B(z,r)$ will make the extension very tedious to calculate. We give a more detailed explanation at the end of this section.

We extend the [19] approximation for $\left\{ {n \atop m} \right\}$ to $\left\{ {n \atop m} \right\}_{\geq r}$ by following the framework of the [19] proof and with the help of findings from [18], [21], and [27] plus several new lemmas. The result is as follows.

Lemma 2.1.

$$\left\{ {n \atop m} \right\}_{\geq r} \sim \frac{n!B^m(z_0, r)}{2\pi m! z_0^n \sqrt{mz_0 H(z_0)}} \left(\sum_{k=0}^{\infty} \left(\int_{-\infty}^{\infty} \exp(-\eta^2) b_{2k} d\eta \right) / (mz_0)^k \right) \tag{2.1}$$

where $B(x, r) = \exp(x) - \sum_{k=0}^{r-1} x^k/k!$,

$$H(x) = \frac{B(x, r-1)B(x, r) + xB(x, r-2)B(x, r) - xB^2(x, r-1)}{2B^2(x, r)}$$

and b *is a polynomial in* η *with* $b_0 = 1$. *This approximation holds uniformly over integer* n *and* m *with* $m = \Theta(n)$.

Proof. A standard technique in the analysis of Stirling numbers is to convert the generating function

$$\frac{(\exp(x) - 1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} \frac{x^n}{n!},$$

to the contour integral

$$\left\{ {n \atop m} \right\} = \frac{n!}{m!} \frac{1}{2\pi i} \int_C \frac{(\exp(x) - 1)^m}{x^{n+1}} dx.$$

We follow the same technique here. From the generating function (1.2), we have the contour integral

$$\left\{ {n \atop m} \right\}_{\geq r} = \frac{n!}{m!} \frac{1}{2\pi i} \int_C \frac{B^m(x, r)}{x^{n+1}} dx. \tag{2.2}$$

Because n and m are positive integer values, we can let C be a circle about the origin.

Using the technique of [19] we let $x = R\exp(i\theta)$ and this replacement gives

$$\left\{ {n \atop m} \right\}_{\geq r} = A \int_{-\pi}^{\pi} \exp[mg(\theta, R)] d\theta$$

where

$$A = \frac{n!(B(R, r))^m}{2\pi m! R^n}$$

and

$$g(\theta, R) = \ln(B(R\exp(i\theta), r)) - \ln(B(R, r)) - \frac{n}{m} i\theta.$$

As done in [19], we let $F(i\theta, R)$ be the first derivative of $g(\theta, R)$ with respect to R, letting $z = i\theta$ to shorten the presentation:

$$F(z, R) = \frac{\exp(z)B(R\exp(z), r-1)}{B(R\exp(z), r)} - \frac{B(R, r-1)}{B(R, r)}.$$

Let

$$G(z,R)=\int_0^R F(z,x)dx.$$

As $\lim_{R\to 0} B(R\exp(z),r)/B(R,r)=\exp(rz)$,

$$G(z,R)=\ln(B(R\exp(z),r))-\ln(B(R,r))-rz,$$

and so

$$g(\theta,R)=G(i\theta,R)+i\theta\left(r-\frac{n}{m}\right).$$

Notice that for any $r\in\mathbb{N}$ there exists $\alpha_r>0$ such that $F(z,R)$ is a regular function of z in the domains $|z|\le\alpha_r$ and $R\ge 0$. This is a corollary of the fact that function $B(x,r)$ does not have zeros in the domains $\{x\in\mathbb{C}:|x\exp(1-x)|>1\}$ [27] and $\{x\in\mathbb{C}:\mathrm{Im}(x/r)^2<4(\mathrm{Re}(x/r)+1)\}$ [21].

Next, we note that

$$\begin{aligned}\lim_{R\to 0}F(z,R)&=\frac{\left(\binom{2r}{r}-\binom{2r}{r-1}\right)\exp((r+1)z)-\left(\binom{2r}{r}-\binom{2r}{r-1}\right)\exp(rz)}{\binom{2r}{r}\exp(rz)}\\&=\frac{\exp(z)-1}{r+1}\end{aligned}$$

and, if $|z|\le\alpha_r$, $\mathrm{Re}(\exp(z))>0$ and $\lim_{R\to\infty}F(z,R)=\exp(z)-1$. Thus, we know that $F(z,R)$ is a bounded function in the domains $|z|\le\alpha_r$ and $R\ge 0$. As a result, we can apply the proof of Lemma 3.2 of [19] to prove that, given the MacLaurin expansion of G about $z=0$,

$$G(z,R)=\sum_{k=1}^{\infty}C_k(R)z^k,$$

and there exists a constant M independent of k and R such that

$$|C_k(R)/R|\le\frac{M}{\alpha_r^k}\tag{2.3}$$

for all values of k and R.

The first derivative of $g(\theta,R)$ with respect to θ is

$$iR\exp(i\theta)\frac{B(R\exp(i\theta),r-1)}{B(R\exp(i\theta),r)}-i\frac{n}{m},$$

and the second derivative with respect to θ is

$$\begin{aligned}&-R\exp(i\theta)\frac{B(R\exp(i\theta),r-1)}{B(R\exp(i\theta),r)}-R^2\exp(2i\theta)\frac{B(R\exp(i\theta),r-2)}{B(R\exp(i\theta),r)}\\&\qquad+R^2\exp(2i\theta)\frac{B(R\exp(i\theta),r-1)^2}{B(R\exp(i\theta),r)^2}.\end{aligned}$$

The MacLaurin expansion of $g(\theta,R)$ about $\theta=0$ is

$$g(\theta,R)=iB^*\theta-RH(R)\theta^2+\sum_{k=3}^{\infty}C_k(R)(i\theta)^k$$

where

$$B^* = \frac{RB(R, r-1)}{B(R, r)} - \frac{n}{m}$$

and

$$H(x) = \frac{B(x, r-1)B(x, r) + xB(x, r-2)B(x, r) - xB(x, r-1)^2}{2B(x, r)^2}.$$

We let $R = z_0$ be the solution to $B^* = 0$. We have

$$g(\theta, z_0) = -z_0\theta^2 H(z_0) + \sum_{k=3}^{\infty} C_k(z_0)(i\theta)^k.$$

Next, we proceed as in [19] and show that the value of $\int_{-\pi}^{\pi} \exp\left[g(\theta, z_0)\right] d\theta$ is concentrated about $\theta = 0$. Consider the integral J defined as

$$J = \int_{\epsilon}^{\pi} \exp(mg(\theta, z_0))d\theta$$

where we define ϵ as $\epsilon = (mz_0)^{-3/8}$. From Lemma 3.2, $RB(R, r-1)/B(R, r)$ is a strictly increasing function of R. Therefore the same reasoning as the proof of Lemma 3.1 of [19] shows that $\lim_{n\to\infty} mz_0 = \infty$ (and thus $\epsilon \to 0$) for m between $\delta_1 n$ and $(1-\delta_2)n/r$ for any positive constants δ_1 and δ_2.

We prove that J is exponentially small.

$$\begin{aligned}
J &= \int_{\epsilon}^{\pi} \exp(mg(\theta, z_0))d\theta \\
&= \int_{\epsilon}^{\pi} \exp\left(m\left(G(i\theta, z_0) + i\theta\left(r - \frac{n}{m}\right)\right)\right) d\theta \\
&= \int_{\epsilon}^{\pi} \exp\left(m\left(\ln B(z_0\exp(i\theta), r) - \ln B(z_0, r) - i\theta\frac{n}{m}\right)\right) d\theta \\
&= \int_{\epsilon}^{\pi} \frac{\exp(m\ln B(z_0\exp(i\theta), r))}{\exp(m\ln B(z_0, r)) \cdot \exp(i\theta n)} d\theta \\
&= \int_{\epsilon}^{\pi} \left(\frac{B(z_0\exp(i\theta), r)}{B(z_0, r)}\right)^m \cdot \frac{1}{\exp(i\theta n)} d\theta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|J| &\le \int_{\epsilon}^{\pi} \left|\frac{B(z_0\exp(i\theta), r)}{B(z_0, r)}\right|^m d\theta \\
&= \int_{\epsilon}^{\pi} \left|\frac{\exp(-az_0\exp(i\theta))B(z_0\exp(i\theta), r)}{B(z_0, r)} \cdot \exp(az_0\exp(i\theta))\right|^m d\theta
\end{aligned}$$

for some real $a \in (0, 1)$.

From Lemma 3.1 below, for any complex number x, there exists a real number $a \in (0, 1)$ such that

$$|\exp(-ax)B(x, r)| \le \exp(-a|x|)B(|x|, r).$$

Thus,

$$|J| \leq \int_{\epsilon}^{\pi} \left(\frac{\exp(-az_0)B(z_0,r)}{B(z_0,r)} \cdot |\exp(az_0 \exp(i\theta))| \right)^m d\theta$$
$$= \int_{\epsilon}^{\pi} \left(\exp\left(-az_0\right) \cdot \left|\exp\left(az_0(\cos\theta + i\sin\theta)\right)\right|\right)^m d\theta$$
$$= \int_{\epsilon}^{\pi} \exp\left(maz_0(\cos\theta - 1)\right) d\theta.$$

Since $\cos\theta$ in the domain $[\epsilon, \pi]$ has a maximum at $\theta = \epsilon$, we have

$$|J| \leq \pi \exp\left(maz_0(\cos\epsilon - 1)\right) = \pi \exp\left(-maz_0 2\sin^2\left(\epsilon/2\right)\right).$$

As $|\sin x| \leq |x|$ and $\epsilon = (mz_0)^{-3/8}$,

$$|J| \leq \pi \exp\left(-maz_0(mz_0)^{-3/4}\right) = -K\pi \exp((mz_0)^{1/4})$$

for some constant $K > 0$.

As a result, we have

$$\left\{ n \atop m \right\}_{\geq r} \sim A \int_{-\epsilon}^{\epsilon} \exp(mg(\theta, z_0)) d\theta \tag{2.4}$$

where

$$g(\theta, z_0) = -z_0\theta^2 H(z_0) + \sum_{k=3}^{\infty} C_k(z_0)(i\theta)^k.$$

The next step of the conversion is completely analogous to the one in [19]:

$$A\int_{-\epsilon}^{\epsilon} \exp(mg(\theta, z_0)) d\theta$$
$$= A\int_{-\epsilon}^{\epsilon} \exp\left(-\theta^2 mz_0 H(z_0) + \sum_{k=1}^{\infty} C_{k+2}(z_0) m (i\theta)^{k+2}\right) d\theta$$
$$= A\int_{-\epsilon}^{\epsilon} \exp\Bigg(- \theta^2 mz_0 H(z_0) + \sum_{k=1}^{\infty} C_{k+2}(z_0) m (i\theta)^{k+2} \cdot \frac{m^{k/2}(z_0 H(z_0))^{(k+2)/2}}{m^{k/2}(z_0 H(z_0))^{(k+2)/2}}\Bigg) d\theta$$
$$= A\int_{-\epsilon}^{\epsilon} \exp\Bigg(- \theta^2 mz_0 H(z_0) + \sum_{k=1}^{\infty} \frac{C_{k+2}(z_0) i^{k+2} (\theta^2 mz_0 H(z_0))^{(k+2)/2}}{z_0 H(z_0)^{(k+2)/2}} (mz_0)^{-k/2}\Bigg) d\theta$$
$$= \frac{A}{\sqrt{mz_0 H(z_0)}} \int_{-h}^{h} \exp\left(-\eta^2 + \sum_{k=1}^{\infty} a_k \zeta^k\right) d\eta$$

where $\eta = \theta\sqrt{mz_0H(z_0)}$, $h = \epsilon\sqrt{mz_0H(z_0)}$, $a_k = \frac{C_{k+2}(z_0)(i\eta)^{k+2}}{z_0H(z_0)^{(k+2)/2}}$, and $\zeta = (mz_0)^{-1/2}$.

As done in [19], we temporarily consider $\exp\left(\sum_{k=1}^{\infty} a_k\zeta^k\right)$ as a function of three independent variables: ζ, z_0, and η. In this context, we can write the function as a MacLaurin series about $\zeta = 0$:

$$\exp\left(\sum_{k=1}^{\infty} a_k\zeta^k\right) = \sum_{k=0}^{\infty} b_k\zeta^k \tag{2.5}$$

where b_k is a function of z_0 and η. Note that $b_0 = 1$, $b_1 = a_1$, and in general $b_k = \sum_{j=1}^{k} c_j a_1^{r_{1,j}} \dots a_k^{r_{k,j}}$ for some nonnegative constants c_j, and $r_{i,j}$ with $1 \le i, j \le k$. Applying the chain rule to the summation for b_k gives $b_{k+1} = \sum_{j=1}^{k}\left(a_1 + \sum_{l=1}^{k} r_{l,j}a_{l+1} \,/\, a_l\right) c_j a_1^{r_{1,j}} \dots a_k^{r_{k,j}}$, and we can form b_{k+1} by multiplying each term in the summation for b_k by a polynomial containing only odd powers of η. As a result, b_{2k} is a polynomial containing only even powers of η and b_{2k+1} is a polynomial containing only odd powers of η.

For large enough n, $\zeta = (mz_0)^{-1/2}$ lies within the domain of convergence for $\sum_{k=0}^{\infty} b_k\zeta^k$. The proof of this fact is similar to the proof of Lemma 3.3 below. Alternatively, one may use the root test with the fact that $\eta < h$. Therefore, (2.5) is valid, and we have

$$\left\{ {n \atop m} \right\}_{\ge r} \sim \frac{A}{\sqrt{mz_0H(z_0)}} \int_{-h}^{h} \left(\exp(-\eta^2)\sum_{k=0}^{\infty} b_k\zeta^k\right) d\eta.$$

To achieve the approximation (2.1) we need to move the summation outside of the integral. From Lemma 3.3 below, $\int_{-h}^{h} \exp(-\eta^2)\sum_{k=s}^{\infty} b_k\zeta^k d\eta = \mathrm{O}(\zeta^s)$. Thus,

$$\left\{ {n \atop m} \right\}_{\ge r} \sim \frac{A}{\sqrt{mz_0H(z_0)}} \left(\int_{-h}^{h} \left(\exp(-\eta^2)\sum_{k=0}^{s-1} b_k\zeta^k\right) d\eta + \mathrm{O}(\zeta^s)\right).$$

Lemma 3.3 is valid for any h, and as h is $\Theta(m^{1/8})$ extending the integration to the entire real line leads to

$$\left\{ {n \atop m} \right\}_{\ge r} \sim \frac{A}{\sqrt{mz_0H(z_0)}} \left(\int_{-\infty}^{\infty} \left(\exp(-\eta^2)\sum_{k=0}^{s-1} b_k\zeta^k\right) d\eta + \mathrm{O}(\zeta^s)\right).$$

Lemma 3.3 also implies that the integral on the right hand side is absolutely convergent. Thus we can switch the summation and integral:

$$\left\{ {n \atop m} \right\}_{\ge r} \sim \frac{A}{\sqrt{mz_0H(z_0)}} \left(\sum_{k=0}^{s-1}\int_{-\infty}^{\infty} \left(\exp(-\eta^2)b_k\zeta^k\right) d\eta + \mathrm{O}(\zeta^s)\right).$$

Recall that the original integration (2.4) is performed with respect to θ. While ζ and η are not mutually independent, ζ is independent of θ, and so we can treat ζ as a scalar.

$$\left\{ n \atop m \right\}_{\geq r} \sim \frac{A}{\sqrt{mz_0 H(z_0)}} \left(\sum_{k=0}^{s-1} \int_{-\infty}^{\infty} \left(\exp(-\eta^2) b_k d\eta \right) \zeta^k + \mathrm{O}(\zeta^s) \right).$$

Now, as $s \to \infty$,

$$\left\{ n \atop m \right\}_{\geq r} \sim \frac{A}{\sqrt{mz_0 H(z_0)}} \left(\sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \left(\exp(-\eta^2) b_k d\eta \right) \zeta^k \right).$$

Recall that b_{2k+1} is a polynomial containing only odd powers of η. Thus, its integral is 0.

$$\left\{ n \atop m \right\}_{\geq r} \sim \frac{A}{\sqrt{mz_0 H(z_0)}} \left(\sum_{k=0}^{\infty} \left(\int_{-\infty}^{\infty} \exp(-\eta^2) b_{2k} d\eta \right) / (mz_0)^k \right).$$

□

This completes the proof for integer n and m. While it may be possible to further generalize this approximation to noninteger n and m, similar to the result of [2] for Stirling numbers of the second kind, the calculations involved appear to be very tedious. To generalize the [24] approximation, [2] deforms the contour of integration in order to avoid zeros of $B(x, 1)$. It is straightforward to prove that there are no zeros of $B(x, 1)$ with $|x| > 0$ and $|\arg(x)| < \pi/2$. As a result, the contour can include a nearly complete semicircle in the right half plane. Using a semicircle lets [2] complete the rest of the contour with $\mathrm{Re}(x) < \epsilon_1$ and $|x| > \pi - \epsilon_2$ for arbitrarily small positive constants ϵ_1 and ϵ_2, and thus the integral over the rest of the contour is exponentially small. Such a contour is not possible with (2.2). As r grows, there may exist zeros of $B(x, r)$ with $|x| > 0$ and $|\arg(x)| < \pi/2$. Worse, [22] proves that the roots of $B(x, r)$ converge toward a curve that intersects with the region $|x| > 0$ and $|\arg x| < 1$, and we have numerically identified such a zero with $r = 4$.

3. Technical Lemmas

In this section, we present a number of lemmas used in the proofs of Lemma 2.1 of Section 2 and Theorem 1.1 in Section 4.

Lemma 3.1. *For any complex number z, there exists real $a \in (0, 1)$ such that*

$$|\exp(-az)B(z, r)| \leq \exp(-a|z|)B(|z|, r). \tag{3.1}$$

Proof. Notice that whenever $B(z, r) = 0$ any $a \in (0, 1)$ provides (3.1). Let us assume that $B(z, r) \neq 0$. Proving (3.1) is equivalent to proving

$$e^{a(|z| - \mathrm{Re} z)} \leq \frac{B(|z|, r)}{|B(z, r)|},$$

or

$$a(|z| - \mathrm{Re} z) \leq \ln \frac{B(|z|, r)}{|B(z, r)|},$$

where $B(|z|, r)/|B(z, r)| \geq 1$ by the triangle inequality.

We show that $B(|z|, r)/|B(z, r)| = 1$ if and only if z is positive and real. Implication in one direction is obvious, so assume $B(|z|, r)/|B(z, r)| = 1$. Then $\left|\sum_{k=r}^{\infty} z^k/k!\right| = \sum_{k=r}^{\infty} |z|^k/k!$, and repeated applications of the triangle inequality give

$$\left|\sum_{k=r}^{\infty} \frac{z^k}{k!}\right| = \frac{|z|^r}{r!} + \left|\sum_{k=r+1}^{\infty} \frac{z^k}{k!}\right| = \cdots = \sum_{k=r}^{K} \frac{|z|^k}{k!} + \left|\sum_{k=K+1}^{\infty} \frac{z^k}{k!}\right| = \sum_{k=r}^{\infty} \frac{|z|^k}{k!}.$$

Furthermore, for this sequence of equalities to hold, with any two terms $z^{k_1}/k_1!$ and $z^{k_2}/k_2!$ the triangle inequality turns into the equality $|z^{k_1}/k_1! + z^{k_2}/k_2!| = \left|z^{k_1}/k_1!\right| + \left|z^{k_2}/k_2!\right|$, and $z^{k_1}/k_1!$ and $z^{k_2}/k_2!$ must lie on the same line through the origin. For all the terms to lie on the same line through the origin, we require $k\mathrm{Arg} z = \Theta + \pi n_k$ for some angle Θ and integer n_k, i.e. $(k\mathrm{Arg} z - \Theta)/\pi \in \mathbb{Z}$. As this condition is true for any two consecutive terms we see that $\mathrm{Arg} z$ and Θ must be integer multiples of π, requiring z to be real. Negative reals lead to sign alternating in power series, and thus the triangle equality cannot be achieved. Hence, z is a nonnegative real number. Therefore, if $B(|z|, r)/|B(z, r)| = 1$, then $\mathrm{Re} z > 0$ and $\mathrm{Im} z = 0$ and inequality (3.1) is true for any $a \in (0, 1)$.

If $B(|z|, r)/|B(z, r)| > 1$, we have $|z| - \mathrm{Re} z > 0$, and it suffices to pick any

$$0 < a < \min \left\{1, \frac{1}{|z| - \mathrm{Re} z} \ln \frac{B(|z|, r)}{|B(z, r)|}\right\}$$

to satisfy (3.1).

□

Lemma 3.2. *Let* $B(x, r) = \exp(x) - \sum_{k=0}^{r-1} x^k/k!$, *and let* $Q(x, r) = xB(x, r-1)/B(x, r)$. *For positive integer* r *and* $x > 0$, $Q(x, r)$ *is a strictly increasing function of* x, *and* $1/(r+1) \leq Q'(x, r) \leq 1$.

Proof. That Q is monotone increasing follows directly from the positive lower bound on Q'. The proof for $r = 1$ is straightforward: $Q'(x, 1) = \exp(x)(\exp(x) - 1 - x)/(\exp(x) - 1)^2$. Thus, $Q'(x, 1) > 0$, and $\lim_{x\to 0} Q'(x, 1) = 1/2$ and $\lim_{x\to\infty} Q'(x, 1) = 1$.

Let $r > 1$. Note that

$$Q'(x, r) = \frac{B(x, r-1)B(x, r) + xB(x, r-2)B(x, r) - xB(x, r-1)^2}{B(x, r)^2}.$$

Further note that $B(x, r-1) = B(x, r) + x^{r-1}/(r-1)!$ and $B(x, r-2) = B(x, r) + x^{r-1}/(r-1)! + x^{r-2}/(r-2)!$. Therefore,

$$\begin{aligned}
&Q'(x,r) \\
&\quad = \frac{B(x,r) + x^{r-1}/(r-1)! + x(B(x,r) + x^{r-1}/(r-1)! + x^{r-2}/(r-2)!)}{B(x,r)} \\
&\qquad\qquad - \frac{x(B(x,r) + x^{r-1}/(r-1)!)^2}{B(x,r)^2} \\
&\quad = 1 + \frac{x^{r-1}/(r-1)! + x(-x^{r-1}/(r-1)! + x^{r-2}/(r-2)!)}{B(x,r)} \\
&\qquad\qquad - \frac{x(x^{r-1}/(r-1)!)^2}{B(x,r)^2} \\
&\quad = 1 + \left(\frac{r}{x} - 1\right) \frac{x^r/(r-1)!}{B(x,r)} - \frac{1}{x} \frac{(x^r/(r-1)!)^2}{B(x,r)^2}.
\end{aligned}$$

Denote $v(x,r) = \frac{x^r/(r-1)!}{B(x,r)}$, then

$$Q'(x,r) = 1 + \left(\frac{r}{x} - 1\right) v(x,r) - \frac{v(x,r)^2}{x},$$

and note that

$$v(x,r) = \frac{r}{r! \sum_{k=r}^{\infty} \frac{x^{k-r}}{k!}} = \frac{r}{\left(1 + \frac{x}{(r+1)} + \frac{x^2}{(r+1)(r+2)} + \dots\right)}. \tag{3.2}$$

Notice that $\lim_{x \to 0} v(x,r) = r$, while $v(x,r) \le r$ for any $x > 0$, and v is decreasing with respect to x and decreases to 0 as $x \to \infty$. Since v is continuously differentiable with respect to x when $x \in [0, +\infty)$, $v'(x,r) \le 0$ for any $x \in [0, +\infty)$.

We observe that

$$\frac{dv}{dx} = \frac{r}{x} v(x,r) - v(x,r) - \frac{v(x,r)^2}{x} = Q'(x,r) - 1.$$

Thus $Q'(x,r) \le 1$, and we have the upper bound of the lemma.

We make an aside remark that the derivative $v'(0)$ is indeed negative even though we only stated above that the function decreases for values of x in $[0, +\infty)$. In fact, $v'(0) = -r/(r+1)$, and this is easy to see from the formulas appearing below in the proof of the positive lower bound.

To prove that $Q'(x,r)$ has a positive lower bound, it suffices to show that there exists a constant $\alpha > 0$ such that $v'(x) > -1 + \alpha$ for all x. Using (3.2),

$v'(x) = -\dfrac{rD'(x)}{D^2(x)}$, where

$$\begin{aligned}
D(x) &= r!\sum_{k=r}^{\infty}\frac{x^{k-r}}{k!} = 1+\sum_{k=1}^{\infty}\frac{x^k}{(r+1)\cdot\dots\cdot(r+k)}\\
&= 1+\frac{x}{(r+1)}+\frac{x^2}{(r+1)(r+2)}+\dots.\\
D'(x) &= \sum_{k=1}^{\infty}\frac{kx^{k-1}}{(r+1)\cdot\dots\cdot(r+k)} = \frac{1}{(r+1)}+\frac{2x}{(r+1)(r+2)}+\dots\\
\frac{1}{r+1}D^2(x) &= \frac{1}{r+1}\left(1+\sum_{k=1}^{\infty}\frac{x^k}{(r+1)\cdot\dots\cdot(r+k)}\right)^2\\
&= \frac{1}{r+1}\left(1+\sum_{k=1}^{\infty}x^k\sum_{q=0}^{k}\frac{1}{\prod_{l=1}^{q}(r+l)}\frac{1}{\prod_{l=1}^{k-q}(r+l)}\right)\\
&= \frac{1}{r+1}+\sum_{k=1}^{\infty}x^k\sum_{q=0}^{k}\frac{1}{(r+1)\prod_{l=1}^{q}(r+l)\prod_{l=1}^{k-q}(r+l)}.
\end{aligned}$$

Here we assume the convention that $\prod_{t=1}^{0} t = 1$.

Comparing the terms of respective coefficients of the power series representing $D'(x)$ and $D^2(x)/(r+1)$, we prove that $D'(x) \le D^2(x)/(r+1)$. For both series, the first term is $1/(x+1)$. For the next two terms of the power series, it is straightforward to verify that $2/((r+1)(r+2)) \le 2/(r+1)^2$ and $3/((r+1)(r+2)(r+3)) \le 1/(r+1)^3 + 2/\left((r+1)^2(r+2)\right)$. Consider the coefficient of the x^k term in $D^2(x)(r+1)$:

$$\sum_{q=0}^{k}\left((r+1)\prod_{l=1}^{q}(r+l)\prod_{l=1}^{k-q}(r+l)\right)^{-1}.$$

The sum has exactly $(k+1)$ terms, and each denominator can be rewritten in the form $(r+1)^{p_1+1}(r+2)^{p_2}\dots(r+k)^{p_k}$, where $p_1+\dots+p_k = k$ and $p_1 \ge p_2 \ge \dots \ge p_k$, and so $p_1 \ge 1, \dots, p_k \le 1$. Regrouping the multiples, we can state that the right hand side contains exactly $k+1$ fractions of the form $((r+r_1)(r+r_2)\dots(r+r_{k+1}))^{-1}$ with $r_j \le j$. Therefore

$$\frac{k+1}{(r+1)\cdot\dots\cdot(r+k+1)} \le \sum_{q=0}^{k}\frac{1}{(r+1)\prod_{l=1}^{q}(r+l)\prod_{l=1}^{k-q}(r+l)},$$

and we have $D'(x) \le D^2(x)/(r+1)$. As a result,

$$\frac{D'(x)}{\frac{1}{r}D^2(x)} \le \frac{r}{r+1},$$

and that implies

$$v'(x) = -\frac{rD'(x)}{D^2(x)} \geq -\frac{r}{r+1} = -1 + \frac{1}{r+1}.$$

In other words, the desired estimate is true with $\alpha = 1/(r+1)$. Hence, $Q'(x,r) \geq 1/(r+1)$ when $x \in [0, +\infty)$. □

Lemma 3.3.

$$\int_{-h}^{h} \exp(-\eta^2) \sum_{k=s}^{\infty} b_k \zeta^k d\eta = \mathrm{O}(\zeta^s),$$

where b_k is a function of η and z_0. (Please see the exposition above equation (2.5) for the definitions of the terms.)

Proof. The proof uses the framework of an analogous proof in [19] but with adjustments required by the differences in the definition of H and its neighborhood of regularity near $z = 0$. Recall that $a_k = \frac{C_{k+2}(z_0)(i\eta)^{k+2}}{z_0 H(z_0)^{(k+2)/2}}$. From (2.3),

$$|a_k| \leq \left| \frac{M(i\eta)^{k+2}}{H(z_0)^{(k+2)/2}} \right|.$$

From Lemma 3.2 with $2H(z_0) = Q'$, we have

$$|a_k| \leq M\sqrt{2(r+1)} \frac{|\eta|^{k+2}}{\alpha_r^k}.$$

From [18],

$$|b_k| \leq M \frac{\left|\sqrt{2(r+1)}\eta\right|^{k+2}}{\alpha_r^k} \left(1 + \frac{2(r+1)M\eta^2}{\alpha_r^2}\right)^{k-1}.$$

Let $T = \left|\frac{\sqrt{2(r+1)}\eta}{\alpha_r}\right| \left(1 + \frac{2(r+1)M\eta^2}{\alpha_r^2}\right)$. Thus

$$\begin{aligned} \left|\sum_{k=s}^{\infty} b_k \zeta^k\right| &\leq \sum_{k=s}^{\infty} |b_k| \zeta^k \\ &\leq \left(\frac{M2(r+1)\eta^2}{\alpha_r^2 + 2(r+1)M\eta^2}\right) \sum_{k=s}^{\infty} T^k \zeta^k. \end{aligned}$$

Note that $T\zeta = \mathrm{O}(|\eta|\zeta + |\eta|^3\zeta)$. From the limits of the integral, $|\eta| \leq h$, and we have $|\eta|\zeta + |\eta|^3\zeta \leq (h + h^3)\zeta = \mathrm{O}\left((mz_0)^{-1/8}\right)$. Thus for large enough n,

$T\zeta < 1/2$, and

$$\begin{aligned}
&\left(\frac{M2(r+1)\eta^2}{\alpha_r^k + 2(r+1)M\eta^2}\right)\sum_{k=s}^{\infty} T^k\zeta^k \\
&\qquad\qquad = \left(\frac{M2(r+1)\eta^2}{\alpha_r^k + 2(r+1)M\eta^2}\right)\frac{T^s\zeta^s}{1-T\zeta} \\
&\qquad\qquad < \left(\frac{M2(r+1)\eta^2}{\alpha_r^k + 2(r+1)M\eta^2}\right) 2(T\zeta)^s \\
&\qquad\qquad = 2M\left|\frac{\sqrt{2(r+1)}\eta}{\alpha_r}\right|^{s+2}\left(1+\frac{2(r+1)M\eta^2}{\alpha_r^2}\right)^{s-1}\zeta^s \\
&\qquad\qquad = \frac{2M}{\alpha_r^{3s}}\left|\sqrt{2(r+1)}\eta\right|^{s+2}\left(\alpha_r^2 + 2(r+1)M\eta^2\right)^{s-1}\zeta^s.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{-h}^{h}\exp(-\eta^2)\sum_{k=s}^{\infty} b_k\zeta^k d\eta \\
&\leq \int_{-h}^{h}\exp(-\eta^2)\left(\frac{2M}{\alpha_r^{3s}}\left|\sqrt{2(r+1)}\eta\right|^{s+2}\left(\alpha_r^2+2(r+1)M\eta^2\right)^{s-1}\right)\zeta^s d\eta \\
&\leq \zeta^s\int_{-\infty}^{\infty}\exp(-\eta^2)\left(\frac{2M}{\alpha_r^{3s}}\left|\sqrt{2(r+1)}\eta\right|^{s+2}\left(\alpha_r^2+2(r+1)M\eta^2\right)^{s-1}\right) d\eta \\
&= \mathrm{O}(\zeta^s).
\end{aligned}$$

□

4. The Main Result

We are now prepared to prove the main result of this paper, Theorem 1.1, to rigorously show the uniformity of the Hennecart approximation for $\left\{{n \atop m}\right\}_{\geq r}$, as given in [11], and listed above as equation (1.5).

Proof of Theorem 1.1. From Lemma 2.1 we have equation (2.1) that we restate here:

$$\left\{{n \atop m}\right\}_{\geq r} \sim \frac{n!B^m(z_0,r)}{2\pi m! z_0^n\sqrt{mz_0H(z_0)}}\left(\sum_{k=0}^{\infty}\left(\int_{-\infty}^{\infty}\exp(-\eta^2)b_{2k}d\eta\right)/(mz_0)^k\right). \tag{2.1}$$

In Lemma 3.3 we prove

$$\int_{-h}^{h}\exp(-\eta^2)\sum_{k=s}^{\infty} b_k\zeta^k d\eta = \mathrm{O}(\zeta^s)$$

for any h with $h = \Theta\left(m^{1/8}\right)$. Using the same arguments as at the end of Lemma 2.1, the term in the integral is absolutely convergent so we can interchange the summation and integral. Also, while ζ and η are not independent, only η contains θ and so we can treat ζ as a scalar and move it outside the integration. This gives

$$\sum_{k=s}^{\infty}\left(\int_{-\infty}^{\infty}\exp(-\eta^2)b_k d\eta\right)/(mz_0)^{k/2} = \mathrm{O}\left((mz_0)^{-s/2}\right).$$

Since b_{2k+1} only contains odd powers of η, the integral is 0 for odd k. With a change of variables, we have

$$\sum_{k=t}^{\infty}\left(\int_{-\infty}^{\infty}\exp(-\eta^2)b_{2k} d\eta\right)/(mz_0)^{k} = \mathrm{O}\left((mz_0)^{-t}\right).$$

For $t = 1$, this summation is $\mathrm{O}\left((mz_0)^{-1}\right)$. Since $\lim_{n,m\to\infty} z_0 \le n/m$ with m being $\mathrm{O}(n)$, we infer that $(mz_0)^{-1}$ is $\mathrm{O}(n^{-1})$.

Given $b_0 = 1$, and as $\int_{-\infty}^{\infty}\exp(-\eta^2)d\eta = \sqrt{\pi}$, the result of computing one term from the summation of (2.1) is

$$\left\{ n \atop m \right\}_{\ge r} \sim \frac{n! B^m(z_0, r)}{2m! z_0^n \sqrt{m\pi z_0 H(z_0)}}\left(1 + \mathrm{O}(n^{-1})\right). \tag{4.1}$$

We recall that [11] used the following notations (given above immediately following (1.5)), $t_0 = (n - mr)/m$, $\phi(z) = -n\ln z + m\ln B(z, r)$ where $B(x, r) = \exp(x) - \sum_{k=0}^{r-1} x^k/k!$, and z_0 is the positive real solution of the equation $z_0 B'(z_0, r)/B(z_0, r) = n/m$.

Applying Stirling's approximation,

$$(n - mr)! \sim ((n - mr)/e)^{n-mr}\sqrt{2\pi(n - mr)}$$

to (4.1), we get:

$$\begin{aligned}
\left\{ n \atop m \right\}_{\ge r} &\sim \frac{n! B^m(z_0, r)}{2m! z_0^n \sqrt{m\pi z_0 H(z_0)}}\left(\frac{n - mr}{e}\right)^{n-mr}\sqrt{2\pi(n - mr)}\frac{1}{(n - mr)!} \\
&= \frac{n!}{m!(n - mr)!}\left(\frac{n - mr}{e}\right)^{n-mr}\frac{B^m(z_0, r)}{z_0^n}\sqrt{\frac{n - mr}{2mz_0 H(z_0)}} \\
&= \frac{n!}{m!(n - mr)!}\left(\frac{n - mr}{e}\right)^{n-mr}\frac{B^m(z_0, r)}{z_0^{n+1}}\sqrt{\frac{z_0 t_0}{2H(z_0)}}. \qquad (4.2)
\end{aligned}$$

Note that

$$\frac{z_0}{m}\phi''(z_0) = \frac{n}{mz_0} + z_0\frac{B(z_0, r)B(z_0, r - 2) - B^2(z_0, r - 1)}{B^2(z_0, r)},$$

and note that $B'(z_0, r) = B(z_0, r - 1)$. This gives $(z_0/m)\phi''(z_0) = 2H(z_0)$, and if we apply this substitution to (4.2) we get the Hennecart approximation (1.5):

$$\left\{ n \atop m \right\}_{\ge r} \sim \frac{n!}{m!(n - mr)!}\left(\frac{n - mr}{e}\right)^{n-mr}\frac{B^m(z_0, r)}{z_0^{n+1}}\sqrt{\frac{mt_0}{\phi''(z_0)}}.$$

The error term from applying Stirling's approximation to $(n - rm)!$ is $\mathrm{O}((n - rm)^{-1})$, and this approximation is asymptotically tight as $n \to \infty$ with $n - rm > \delta_2 n$ for δ_2 a positive constant. The error term from truncating the summation of (2.1) is $\mathrm{O}(n^{-1})$. Therefore, we can conclude that (1.5) is uniform for all $\delta_1 n < m < (1 - \delta_2)n/r$ as $n \to \infty$ with δ_1 and δ_2 any positive constants. $\square$

5. Conclusion

In summary, Hennecart [11] provides a very useful approximation for $\left\{ {n \atop m} \right\}_{\geq r}$, the r-associated Stirling numbers of the second kind, but does not give a rigorous proof of the approximation. This paper develops and proves an approximation for $\left\{ {n \atop m} \right\}_{\geq r}$ that is similar to the approximation of [19] for regular Stirling numbers of the second kind. The proof that our approximation is uniform follows the general form of [19], but several additional lemmas concerning the behavior of the tails of the Maclaurin series for $\exp(z)$, $z \in \mathbb{C}$, are needed to make that framework valid in the new context. Truncating our new approximation leads to the Hennecart formula, and the error is well controlled. The result is the needed proof that the Hennecart approximation is uniform for any fixed integer r, and integer n and m with $m = \Theta(n)$. All the uses of the Hennecart approximation that we have found in the literature are covered by this range of the parameters. Therefore, our proof confirms the validity of those applications.

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