Contributions to Discrete Mathematics

Volume 15, Number 1, Pages 175–182
ISSN 1715-0868

SOME COMBINATORIAL PROPERTIES OF HEXAGONAL LATTICES

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ABSTRACT. In this paper, we consider the combinatorial properties of the hexagonal lattice. Let $e(n)$ be the number of n-element order ideals in a hexagonal lattice. We give the enumeration of $e(n)$ by showing a bijection between the order ideals and Schröder paths. Further, we get formulae for the flag f- and h-vectors of the hexagonal lattice.

1. INTRODUCTION

Hexagonal systems are very important in mathematics and chemistry [1, 10]. Various topological properties of hexagonal systems are extensively studied by mathematicians and chemists. In this paper, we investigate some combinatorial properties of hexagonal lattices.

Readers may find the poset notations and terminologies we use in this paper from [9, Ch. 3]. Still, we wish to pick out some definitions to emphasize that will frequently appear in what follows. Let P be a finite poset of rank n with rank function $\rho : P \to [0, n]$. An *order ideal* of P is a subset I of P such that if $A \in I$ and $B \leq A$, then $B \in I$. If $S \subseteq [0, n]$, then define the subposet $P_S = \{t \in P : \rho(t) \in S\}$, called the *S-rank-selected* subposet of P. Now define $\tilde{f}_P(S)$ (or simply $\tilde{f}(S)$) to be the number of maximal chains of P_S. For instance, $\tilde{f}(i)$ (short for $\tilde{f}(\{i\})$) is the number of elements of P of rank i. The function $\tilde{f} : 2^{[0,n]} \to Z$ is called the *flag f-vector* of P. Also define $\tilde{h}_P(S) = \tilde{h}(S)$ by $\tilde{h}(S) = \sum_{T \subseteq S}(-1)^{\sharp(S-T)}\tilde{f}(T)$. The function $\tilde{h}$ is called the *flag h-vector* of P. These two functions $\tilde{f}$ and $\tilde{h}$ occur naturally in diverse areas of mathematics and have been widely studied [3, 5, 6, 7, 8].

A *lattice* is a poset L for which every pair of elements has a least upper bound and a greatest lower bound. Let Sq be the *square lattice* determined by the set $N \times N$ (here $N = \{1, 2, \ldots\}$) and the relation $(a', b') \leq (a, b)$ if and only if $a' \leq a$ and $b' \leq b$. Fig. 1 is the Hasse diagram of Sq. The

Received by the editors May 21, 2019, and in revised form September 13, 2019.
2000 *Mathematics Subject Classification.* 06A07; 05D05.
Key words and phrases. Hexagonal lattice; flag f-vector; flag h-vector.
This work was supported in part by the National Natural Science Foundation of China (No.11701249). The idea of this paper is inspired by Prof. Richard Stanley. The author also would like to thank him for his helpful suggestions. We thank the anonymous referee for helpful suggestions and correction.

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square lattice has many interesting combinatorial properties. For example, the number of n-element order ideals of Sq is the number of partitions of n, and the flag h-vector of Sq has a nice combinatorial interpretation under the theory of P-partitions [9]. Research on properties of Sq has drawn our attention to study other lattices similar to Sq.

Let Hex be the *Hexagonal lattice* defined by its Hasse diagram, see Fig. 2 and [4]. Then Hex has a unique minimal element and in general, $\lfloor (i+2)/2 \rfloor$ elements of rank i for $i \geq 0$. Fig. 1 and Fig. 2 show that Hex is a kind of "stretched" version of Sq. Since there has been a considerable amount of research on Sq, this motivates further research on Hex. In [4], Propp investigated the generating function of the numbers of n-element order ideals of Hex by using Ferrers diagrams. By rewriting the generating function, the n-element order ideals of Hex can be enumerated by the partitions of n into parts not congruent to $2 \pmod 4$. In this note, we will give another combinatorial interpretation of the n-element order ideals of Hex. Furthermore, we will study the flag h-vector of Hex.

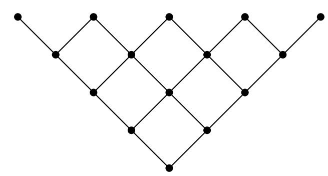

FIGURE 1. The square lattice Sq

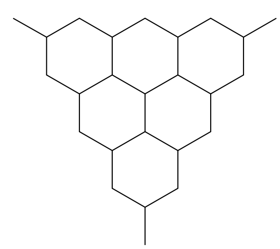

FIGURE 2. The hexagonal lattice Hex

2. MAIN RESULTS

We start our main work from enumerating the number of n-element order ideals in Hex by showing a bijection between the order ideals and Schröder paths. Commonly, a *Schröder path* is a lattice path in the xy-plane from $(0,0)$ to (n,n), $n \geq 0$, consisting of three kinds of elementary steps, $(1,0)$, $(0,1)$, and $(1,1)$ without traveling above the line $y = x$. The number of such paths are counted by the large Schröder numbers. Let p be a Schröder path. Define its weight $w(p)$ to be twice the area between p and the line

$y = x$. We double the area so that $w(p)$ will always be an integer. Let $\bar{w}(p) = n^2 - w(p)$ be the reversed weight of p.

Let $e(n)$ be the number of n-element order ideals in Hex. Then $e(n)$ can be given by a bijection between the Schröder paths p with weights $\bar{w}(p)$ in the xy-plane and the order ideals in Hex.

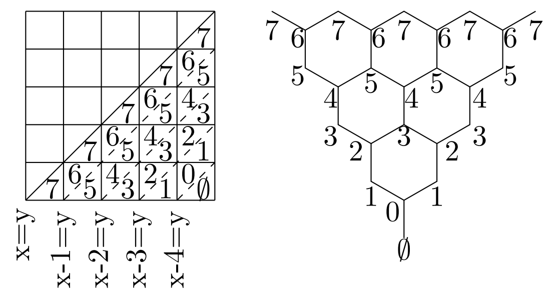

FIGURE 3. Labelling of the xy-plane and Hex lattice

Theorem 2.1. *Let $e(n)$ and $\bar{w}(p)$ be defined as above. Then $e(n)$ is the number of Schröder paths with reversed weight $\bar{w}(p) = n$ in the xy-plane.*

Proof. To define the bijection between Schröder paths with reversed weight $\bar{w}(p) = n$ in the xy-plane and n-element order ideals of Hex, we first present a labelling of the elements of Hex and some triangles between the line $y = x$ and the x-axis in the xy-plane. Label Hex by letting $l(A) = i - 1$ (or $\emptyset$) if $A \in$ Hex in level $i \geq 1$ (or $i = 0$). Next we label the triangles in the xy-plane. Consider the axis-parallel unit squares with the property that their vertices are nonnegative integers and one of their diagonals lies on the line $y = x - i$, where $i = 0, 1, \ldots, n - 1$. The line $y = x - i$ dissects each such square into two triangles; we call the triangle above the line $y = x - i$ an up-triangle, and the triangle below the line a down-triangle. Let $l(a) = 2n - 2i - 2$ if a is an up-triangle of the line $y = x - i$, $1 \leq i \leq n - 1$. Let $l(a) = 2n - 2i - 3$ if a is a down-triangle of the line $y = x - i$, $0 \leq i \leq n - 2$. Let $l(a) = \emptyset$ if a is a down-triangle of the line $y = x - n + 1$. See Fig. 3 as an example for labelling. Then every element of the Hex lattice corresponds to one triangle of the xy-plane by the same number of labels. It is easy to check that two triangles a and b share a common side if and only if their corresponding elements A and B in Hex are comparable.

We now show that every Schröder path of the xy-plane corresponds to an ideal of Hex. Let p be a Schröder path from $(0, 0)$ to (n, n). By the definition of a Schröder path, p only contains northeast steps $(0, 1)$, east steps $(1, 0)$, and north steps $(1, 1)$. This means that if the triangle a is under path p, then any triangle either adjacent to, or under a, must be under path p when their labels are less than a. Hence, the elements in Hex corresponding to the triangles under path p form an ideal. We show that every ideal of Hex corresponds to a Schröder path of the xy-plane. Let $\mathcal{I}$ be an n-element ideal in Hex. Then define the path p as follows. The first step of p depends on whether the element A corresponding to the leftmost down-triangle with label $2n-3$ is in $\mathcal{I}$ or not. If $A \in \mathcal{I}$, then the first step is northeast $(0,1)$. Otherwise the first step is east $(1,0)$. The second step is decided by the elements B and C corresponding to the leftmost triangles with label $2n-4$ and $2n-5$, respectively. If $B \in \mathcal{I}$, then the second step is north $(1,1)$. If $B \notin \mathcal{I}$ but $C \in \mathcal{I}$, then the second step is northeast $(0,1)$. If $B \notin \mathcal{I}$ and $C \notin \mathcal{I}$, then the second step is east $(1,0)$. Continuing this process we then get a Schröder path from $(0,0)$ to (n,n). Hence, the number of elements in the ideal in Hex is equal to the number of triangles under its corresponding Schröder path p. Therefore, $e(n)$ is equal to the number of Schröder paths with reversed weight $\bar{w}(p) = n$. □

Define $f(\mathrm{Hex}; q) = \sum_{n=0}^{\infty} e(n) q^n$, and let $r_n(q) = \sum q^{\bar{w}(p)}$, where the sum is over all Schröder paths from $(0,0)$ to (n,n). Then by [2, Cor. 2], we have the following result which also appears in [4, Thm. 2].

Theorem 2.2.

$$f(\mathrm{Hex}; q) = \lim_{n\to\infty} r_n(q) = \prod_{n=1}^{\infty} \frac{1+q^{2n-1}}{1-q^{2n}}.$$

In the following, we consider the formula for the flag f- and h-vector of Hex. For simplicity, we denote $\tilde{f}_{Hex}(S)$ and $\tilde{h}_{Hex}(S)$ by $\tilde{f}(S)$ and $\tilde{h}(S)$. Throughout this paper we suppose that $i_1, \ldots, i_t$ are integers, and $1 \le i_1 < i_2 < \cdots < i_t \le n$.

Theorem 2.3. *Let $S = \{i_1, \ldots, i_t\}$ and $i_j = 2n_j + m_j$, where $m_j \le 1$. Then*

$$\begin{aligned} \tilde{f}(S) &= (n_1+1)(n_2-n_1+1)(n_3-n_2+1)\cdots(n_t-n_{t-1}+1), \\ \tilde{h}(S) &= n_1(n_2-n_1-1)(n_3-n_2-1)\cdots(n_t-n_{t-1}-1). \end{aligned} \tag{2.1}$$

Proof. Assume that $i_1 = 2n_1 + m_1$, where $m_1 \le 1$. There are n_1+1 elements with rank i_1 in Hex, i.e., $\tilde{f}(i_1) = n_1 + 1$. The formula for $\tilde{f}(S)$ is true for $t = 1$. We use induction on $t-1$, i.e., $\tilde{f}(i_1, \ldots, i_{t-1}) = (n_1+1)(n_2-n_1+1)\cdots(n_{t-1}-n_{t-2}+1)$, and consider the induction step from $t-1$ to t. For every element A with rank i_{t-1}, there are $n_t - n_{t-1} + 1$ elements with rank i_t which are comparable with A. Hence, we get the formula for $\tilde{f}(S)$ by the induction hypothesis.

Since

$$\tilde{h}(i_1) = \tilde{f}(i_1) - 1 = n_1,$$

and

$$\tilde{h}(i_1, i_2) = 1 - \tilde{f}(i_1) - \tilde{f}(i_2) + \tilde{f}(i_1, i_2) = n_1(n_2 - n_1 - 1),$$

it remains to show that the formula of $\tilde{h}(S)$ holds for $t > 2$. Let $S = S_0 = \{i_1, \ldots, i_t\}$ and $S_k = \{i_1, \ldots, i_{t-k}\}$, where $t > 2$. We use induction on k and assume that

$$\tilde{h}(S_k) = n_1(n_2 - n_1 - 1)(n_3 - n_2 - 1)\cdots(n_{t-k} - n_{t-k-1} - 1), \tag{2.2}$$

for $k > 0$. Then we need to prove that this equality is true for $k = 0$. By the definition of the flag h-vector,

$$\begin{aligned}\tilde{h}(S) &= (-1)^t + (-1)^{t-1}\sum_{j=1}^{t}\tilde{f}(i_j) + (-1)^{t-2}\sum_{j<p}\tilde{f}(i_j, i_p) + \cdots + \tilde{f}(S)\\ &= -\tilde{h}(S_1) + (-1)^{t-1}\tilde{f}(i_t) + (-1)^{t-2}\sum_{j=1}^{t-1}\tilde{f}(i_j, i_t) + \cdots + \tilde{f}(S).\end{aligned}$$

If we could prove the following identity,

(2.3)

$$\begin{aligned}&(-1)^{t-1}\tilde{f}(i_t) + (-1)^{t-2}\sum_{j=1}^{t-1}\tilde{f}(i_j, i_t) + (-1)^{t-3}\sum_{j<p}\tilde{f}(i_j, i_p, i_t) + \cdots + \tilde{f}(S)\\ &= \tilde{h}(S_1)(n_t - n_{t-1}),\end{aligned}$$

then the theorem follows from the fact that

$$\tilde{h}(S) = -\tilde{h}(S_1) + \tilde{h}(S_1)(n_t - n_{t-1}) = \tilde{h}(S_1)(n_t - n_{t-1} - 1).$$

We next use induction on t to prove (2.3). Note that $t > 2$, so first consider the case $t = 3$. The LHS of (2.3) becomes

$$\tilde{f}(i_3) - \tilde{f}(i_1, i_3) - \tilde{f}(i_2, i_3) + \tilde{f}(i_1, i_2, i_3) = n_1(n_2 - n_1 - 1)(n_3 - n_2),$$

which is equal to the RHS of (2.3). Before considering the induction step, we prove two necessary identities.

Claim 2.4.

$$\tilde{f}(i_t) - \tilde{f}(i_m, i_t) = -n_m(n_t - n_m)$$

and

$$\tilde{f}(i_1, \ldots, i_r, i_t) - \tilde{f}(i_1, \ldots, i_r, i_{t-1}, i_t) = -\tilde{f}(i_1, \ldots, i_r)(n_t - n_{t-1})(n_{t-1} - n_r).$$

Proof of Claim 2.4. By (2.1), it is easy to check that

$$\tilde{f}(i_t) - \tilde{f}(i_m, i_t) = n_t + 1 - (n_m + 1)(n_t - n_m + 1) = -n_m(n_t - n_m)$$

and

$$\begin{aligned}&\tilde{f}(i_1, \ldots, i_r, i_t) - \tilde{f}(i_1, \ldots, i_r, i_{t-1}, i_t)\\ &= \tilde{f}(i_1, \ldots, i_r)(n_t - n_r + 1) - \tilde{f}(i_1, \ldots, i_r)(n_{t-1} - n_r + 1)(n_t - n_{t-1} + 1)\\ &= -\tilde{f}(i_1, \ldots, i_r)(n_t - n_{t-1})(n_{t-1} - n_r).\end{aligned}$$

□

By rearranging the sum on the LHS of (2.3) and using Claim 2.4, the LHS of (2.3) equals

$$\begin{aligned}
&(-1)^{t-1}\tilde{f}(i_t) + (-1)^{t-2}\sum_{j=1}^{t-1}\tilde{f}(i_j, i_t) + \cdots + \tilde{f}(S) \\
&= (-1)^{t-1}\left(\tilde{f}(i_t) - \tilde{f}(i_{t-1}, i_t)\right) + (-1)^{t-2}\sum_{j<t-1}\left(\tilde{f}(i_j, i_t) - \tilde{f}(i_j, i_{t-1}, i_t)\right) \\
&\quad + \cdots + (-1)\left(\tilde{f}(i_1, \ldots, i_{t-2}, i_t) - \tilde{f}(S)\right) \\
&= (-1)^t n_{t-1}(n_t - n_{t-1}) + (-1)^{t-1}\sum_{j<t-1}\tilde{f}(i_j)(n_{t-1} - n_j)(n_t - n_{t-1}) \\
&\quad + \cdots + \tilde{f}(i_1, \ldots, i_{t-2})(n_{t-1} - n_{t-2})(n_t - n_{t-1}) \\
&= (n_t - n_{t-1})((-1)^t n_{t-1} + (-1)^{t-1}\sum_{j<t-1}\tilde{f}(i_j)(n_{t-1} - n_j) \\
&\quad + \cdots + \tilde{f}(i_1, \ldots, i_{t-2})(n_{t-1} - n_{t-2})).
\end{aligned}$$

Hence the final step is to prove the following identity:

$$\begin{aligned}
(2.4) \qquad &(-1)^t n_{t-1} + (-1)^{t-1}\sum_{j<t-1}\tilde{f}(i_j)(n_{t-1} - n_j) + \cdots \\
&+ \tilde{f}(i_1, \ldots, i_{t-2})(n_{t-1} - n_{t-2}) = \tilde{h}(S_1).
\end{aligned}$$

By simplifying the LHS of (2.4), it is not difficult to get

$$\begin{aligned}
&n_{t-1}\tilde{h}(S_2) + (-1)^{t-2}\sum_{j<t-1}\tilde{f}(i_j)n_j + (-1)^{t-3}\sum_{j<p<t-1}\tilde{f}(i_j, i_p)n_p \\
&\quad + \cdots + (-1)\tilde{f}(i_1, \ldots, i_{t-2})n_{t-2}.
\end{aligned}$$

We will use the following claim to finish the proof of (2.4).

Claim 2.5.

$$\begin{aligned}
&(-1)^{t-1}\sum_{j<t-1}\tilde{f}(i_j)n_j + (-1)^{t-2}\sum_{j<p<t-1}\tilde{f}(i_j, i_p)n_p + \cdots \\
&+ \tilde{f}(i_1, \ldots, i_{t-2})n_{t-2} = \tilde{h}(S_2)(n_{t-2} + 1).
\end{aligned}$$

Proof of Claim 2.5. We use induction on t. As above, we only need to consider $t > 2$. It is not difficult to check that the case when $t = 3$ is true. For the induction step, assume that the result is true for $t - 1$, i.e.,

$$\begin{aligned}
&(-1)^{t-2}\sum_{j<t-2}\tilde{f}(i_j)n_j + (-1)^{t-3}\sum_{j<p<t-2}\tilde{f}(i_j, i_p)n_p + \cdots \\
&+ \tilde{f}(i_1, \ldots, i_{t-3})n_{t-3} = \tilde{h}(S_3)(n_{t-3} + 1).
\end{aligned}$$

Then

$$(-1)^{t-1}\sum_{j<t-1}\tilde{f}(i_j)n_j+(-1)^{t-2}\sum_{j<p<t-1}\tilde{f}(i_j,i_p)n_p+\cdots$$
$$+\tilde{f}(i_1,\ldots,i_{t-2})n_{t-2}=-\tilde{h}(S_3)(n_{t-3}+1)+$$
$$n_{t-2}\left((-1)^{t-1}\tilde{f}(i_{t-2})+(-1)^{t-2}\sum_{j<t-2}\tilde{f}(i_j,i_{t-2})+\cdots+\tilde{f}(i_1,\ldots,i_{t-2})\right).$$

Applying the induction hypothesis on (2.3), we have

$$\begin{aligned}&(-1)^{t-3}\tilde{f}(i_{t-2})+(-1)^{t-4}\sum_{j<t-2}\tilde{f}(i_j,i_{t-2})+\cdots+\tilde{f}(i_1,\ldots,i_{t-2})\\&=\tilde{h}(S_3)(n_{t-2}-n_{t-3}).\end{aligned}$$

Then

$$\begin{aligned}&(-1)^{t-1}\sum_{j<t-1}\tilde{f}(i_j)n_j+(-1)^{t-2}\sum_{j<p<t-1}\tilde{f}(i_j,i_p)n_p+\cdots\\&\quad+\tilde{f}(i_1,\ldots,i_{t-2})n_{t-2}\\&=-\tilde{h}(S_3)(n_{t-3}+1)+n_{t-2}\tilde{h}(S_3)(n_{t-2}-n_{t-3})\\&=\tilde{h}(S_2)(n_{t-2}+1),\end{aligned}$$

where the last equality follows from (2.2). □

From Claim 2.5, the RHS of (2.4) equals

$$n_{t-1}\tilde{h}(S_2)-\tilde{h}(S_2)(n_{t-2}+1)=\tilde{h}(S_2)(n_{t-1}-n_{t-2}-1)=\tilde{h}(S_1).$$

This completes the proof. □

The formula for the values of the flag h-vector shows that these values are no longer necessarily nonnegative. Nevertheless, we can determine their signs from the formula.

Corollary 2.6. *Let $S=\{i_1,\ldots,i_t\}$ and $i_j=2n_j+m_j$, where $m_j\le 1$. Suppose that n is the number of pairs satisfying $n_i=n_j$. Then $\tilde{h}(S)\ge 0$ if n is even and $\tilde{h}(S)\le 0$ if n is odd.*

3. Remark

As previously mentioned, Hex is a kind of stretched version of Sq. What if we "stretch" Hex again, i.e., we put a single vertex on each vertical line of Hex? Furthermore, what if we put k vertices on each vertical line of Hex? Denote such a lattice by H'. We can also consider the flag f- and h- vectors in the new lattice H'. Then we have the same result for the new lattice H'.

Theorem 3.1. *Let $S = \{i_1, \ldots, i_t\}$ and $i_j = (k+2)n_j + m_j$, where $m_j \leq k+1$. Then*

$$\tilde{f}_{H'}(S) = (n_1+1)(n_2-n_1+1)(n_3-n_2+1)\cdots(n_t-n_{t-1}+1),$$
$$\tilde{h}_{H'}(S) = n_1(n_2-n_1-1)(n_3-n_2-1)\cdots(n_t-n_{t-1}-1).$$

From the above theorem, it is easy to determine the sign of the flag h-vectors of H'.

Corollary 3.2. *Let $S = \{i_1, \ldots, i_t\}$ and $i_j = (k+2)n_j + m_j$, where $m_j \leq k+1$. Suppose that n is the number of pairs satisfying $n_i = n_j$. Then $\tilde{h}_{H'}(S) \geq 0$ if n is even and $\tilde{h}_{H'}(S) \leq 0$ if n is odd.*

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