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HYPERBALL PACKINGS RELATED TO TRUNCATED CUBE AND OCTAHEDRON TILINGS IN HYPERBOLIC SPACE

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Abstract. In this paper, we study congruent and noncongruent hyperball (hypersphere) packings to the truncated regular cube and octahedron tilings. These are derived from the Coxeter truncated orthoscheme tilings $\{4,3,p\}$ $(6 < p \in \mathbb{N})$ and $\{3,4,p\}$ $(4 < p \in \mathbb{N})$, respectively, by their Coxeter reflection groups in hyperbolic space $\mathbb{H}^3$. We determine the densest hyperball packing arrangement and its density with congruent and noncongruent hyperballs.

We prove that the locally densest (noncongruent half) hyperball configuration belongs to the truncated cube with a density of approximately 0.86145 if we allow $6 < p \in \mathbb{R}$ for the dihedral angle $2\pi/p$. This local density is larger than the Böröczky–Florian density upper bound for balls and horoballs. But our locally optimal noncongruent hyperball packing configuration cannot be extended to the entire hyperbolic space $\mathbb{H}^3$. We determine the extendable densest noncongruent hyperball packing arrangement to the truncated cube tiling $\{4,3,p=7\}$ with a density of approximately 0.84931.

1. Introduction

In n-dimensional hyperbolic space $\mathbb{H}^n$ $(n \geq 2)$, there are 3 kinds of "balls (spheres)": the classical balls (spheres), horoballs (horospheres) and hyperballs (hyperspheres).

In this paper we consider the hyperballs and their packings in 3-dimensional hyperbolic space $\mathbb{H}^3$. First we briefly survey previous results related to this topic.

In the hyperbolic plane $\mathbb{H}^2$, I. Vermes proved the universal upper bound of the congruent hypercycle packing density is $3/\pi$ in [21]. He initiated this topic and also determined the universal lower bound of the congruent hypercycle covering density is equal to equal to $\sqrt{12}/\pi$ in [22],

In [13] and [14] we analysed regular prism tilings (simple truncated Coxeter orthoscheme tilings) and the corresponding optimal hyperball packings

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in $\mathbb{H}^n$ $(n = 3, 4, 5)$. Recently (to the best of author's knowledge) these have been the densest packings with congruent hyperballs.

In [15] we studied the n-dimensional hyperbolic regular prism honeycombs and the corresponding coverings by congruent hyperballs where we determined their least dense covering densities. Furthermore, we formulated conjectures for the candidates of the least dense covering by congruent hyperballs in the 3 and 5-dimensional hyperbolic space.

In [18] we discussed congruent and noncongruent hyperball packings to the truncated regular tetrahedron tilings. These are derived from the truncated Coxeter simplex tilings $\{3, 3, p\}$ $(7 \leq p \in \mathbb{N})$ and $\{3, 3, 3, 3, 5\}$ in 3 and 5-dimensional hyperbolic space, respectively. We determined the densest packing arrangement and its density with congruent hyperballs in $\mathbb{H}^5$ and determined the smallest density upper bounds of noncongruent hyperball packings generated by the above tilings.

In [17] we deal with such packings by horoballs and hyperballs (briefly hyp-hor packings) in $\mathbb{H}^n$ $(n = 2, 3)$ which form a new class of classical packing problems.

In [16] we studied a large class of hyperball packings in $\mathbb{H}^3$ that can be derived from truncated tetrahedron tilings (see e.g. [12]). We proved that if the truncated tetrahedron is regular $\{3, 3, p\}$, but we also allow $6 < p \in \mathbb{R}$, then the density of the locally densest packing is approximately 0.86338. This is larger than the Böröczky–Florian density upper bound but our locally optimal hyperball packing configuration cannot be extended to the entirety of $\mathbb{H}^3$. However, we described a hyperball packing construction by the regular truncated tetrahedron tiling under the extended Coxeter group $\{3, 3, 7\}$ with maximal density of approximately 0.82251.

In [19] we developed a decomposition algorithm that for each saturated hyperball packing provides a decomposition of $\mathbb{H}^3$ into truncated tetrahedra. Therefore, to get a density upper bound for hyperball packings, it is sufficient to determine the density upper bound of hyperball packings in truncated simplices.

In [20] we proved that the density upper bound of the saturated congruent hyperball packings, related to the corresponding truncated tetrahedron cells, is locally realized in a regular truncated tetrahedron with a density of approximately 0.86338, but then $6 < p \in \mathbb{R}$ is allowed as well. Furthermore, we proved that the density of locally optimal congruent hyperball arrangement in a regular truncated tetrahedron is not a monotonically increasing function of the height of the corresponding optimal hyperball, contrary to the ball radius in the ball packings.

Now, we consider hyperball packings related to truncated regular cube and octahedron tilings that are derived from the Coxeter truncated orthoscheme tilings $\{4, 3, p\}$ $(6 < p \in \mathbb{N})$ and $\{3, 4, p\}$ $(4 < p \in \mathbb{N})$ in hyperbolic space $\mathbb{H}^3$. If we allow $p \in \mathbb{R}$ as well, then the locally densest (noncongruent half) hyperball configuration belongs to the truncated cube with a density of approximately 0.86145. This is larger than the Böröczky–Florian density upper bound for balls and horoballs, but our locally optimal noncongruent hyperball packing configuration cannot be extended to the entire $\mathbb{H}^3$. We determine the extendable densest noncongruent hyperball packing arrangement related to the truncated cube tiling $\{4, 3, 7\}$ with a density of approximately 0.84931.

The main results are summarized in Theorem 3.2, 3.6–3.9, and Corollary 3.3 for truncated cube tilings, and in Theorem 3.12 and Corollary 3.13 for truncated octahedron tilings.

2. Basic notions

We use for $\mathbb{H}^3$ (and analogously for $\mathbb{H}^n$, $n \geq 3$) the projective model in the Lorentz space $\mathbb{E}^{1,3}$ that denotes the real vector space $\mathbf{V}^4$ equipped with the bilinear form of signature $(1, 3)$, $\langle \mathbf{x}, \mathbf{y} \rangle = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3$, where the nonzero vectors $\mathbf{x} = (x^0, x^1, x^2, x^3) \in \mathbf{V}^4$ and $\mathbf{y} = (y^0, y^1, y^2, y^3) \in \mathbf{V}^4$ are determined up to real factors, for representing points of $\mathcal{P}^3(\mathbb{R})$. Then $\mathbb{H}^3$ can be interpreted as the interior of the conical quadric $Q = \{(\mathbf{x}) \in \mathcal{P}^3 | \langle \mathbf{x},\ \mathbf{x} \rangle = 0\} =: \partial\mathbb{H}^3$ in the real projective space $\mathcal{P}^3(\mathbf{V}^4, \boldsymbol{V}_4)$ (here $\boldsymbol{V}_4$ is the dual space of $\mathbf{V}^4$). Namely, for an interior point $\mathbf{y}$ there holds $\langle \mathbf{y},\ \mathbf{y} \rangle < 0$. Restricting this model to the hyperplane $x^0 = 1$, we obtain the usual collinear, i.e., Beltrami–Cayley–Klein model also in Euclidean space, as we shall use later on.

Points of the boundary $\partial\mathbb{H}^3$ in $\mathcal{P}^3$ are called points at infinity, or at the absolute of $\mathbb{H}^3$. Points lying outside $\partial\mathbb{H}^3$ are said to be outer points of $\mathbb{H}^3$ relative to Q. Let $(\mathbf{x}) \in \mathcal{P}^3$, a point $(\mathbf{y}) \in \mathcal{P}^3$ is said to be conjugate to $(\mathbf{x})$ relative to Q if $\langle \mathbf{x},\ \mathbf{y} \rangle = 0$ holds. The set of all points which are conjugate to $(\mathbf{x})$ form a projective (polar) hyperplane $\mathrm{pol}(\mathbf{x}) := \{(\mathbf{y}) \in \mathcal{P}^3 | \langle \mathbf{x},\ \mathbf{y} \rangle = 0\}$. Thus, the quadric Q induces a bijection (linear polarity $\mathbf{V}^4 \to \boldsymbol{V}_4$) from the points of $\mathcal{P}^3$ onto their polar hyperplanes.

Point $X(\mathbf{x})$ and hyperplane $\alpha(\boldsymbol{a}) = \{(x^0, x^1, x^2, x^3) | \sum_{i=0}^{3} x^i a_i = 0\}$ are incident if $\mathbf{x}\boldsymbol{a} = 0$ ($\mathbf{x} \in \mathbf{V}^4 \setminus \{\mathbf{0}\},\ \boldsymbol{a} \in \boldsymbol{V}_4 \setminus \{\mathbf{0}\}$). Or, with the above scalar product, $\langle \mathbf{x},\ \mathbf{a} \rangle = 0$ with the pole $(\mathbf{a})$ of plane $(\boldsymbol{a})$

The hypersphere (or equidistant surface) is a quadratic surface at a constant distance from a plane (base plane) in both halfspaces. The infinite body bounded by the hypersphere, containing the base plane, is called a *hyperball*.

The *half hyperball* (i.e., the part of the hyperball lying on one side of its base plane) with distance h to a base plane β is denoted by $_{+}\mathcal{H}^h$. The volume of the intersection of $_{+}\mathcal{H}^h(\mathcal{A})$ and the right prism with base a 2-polygon $\mathcal{A} \subset \beta$ can be determined by the classical formula of J. Bolyai [2].

$$\mathrm{Vol}(_{+}\mathcal{H}^h(\mathcal{A})) = \frac{1}{4}\mathrm{Area}(\mathcal{A}) \left[k \sinh \frac{2h}{k} + 2h \right]. \tag{2.1}$$

The constant $k = \sqrt{-1/K}$ is the natural length unit in $\mathbb{H}^3$, where K denotes the constant negative sectional curvature. In the following we may assume that $k = 1$.

2.1. **Complete orthoschemes.** The orthoschemes in $\mathbb{H}^3$ are bounded by 4 (hyper)planes $H^0, H^1, \dots, H^3$ such that $H^i \perp H^j$ for $j \neq i-1,\ i,\ i+1$. For a usual (classical) orthoscheme we denote the face opposite to the vertex A_i by H^i $(0 \leq i \leq 3)$. An orthoscheme $\mathcal{O}$ has 3 dihedral angles which are not right angles. Let β^{ij} denote the dihedral angle of $\mathcal{O}$ between the faces H^i and H^j. Then we have $\beta^{ij} = \pi/2$, if $0 \leq i < j-1 \leq 3$. The 3 remaining dihedral angles $\beta^{i,i+1},\ (0 \leq i \leq 2)$ are called the essential angles of $\mathcal{O}$. Geometrically, complete orthoschemes of degree m can be described as follows:

(1) For $m = 0$, they coincide with the class of classical orthoschemes introduced by Schläfli. The initial and final vertices, A_0 and A_3 of the orthogonal edge-path $A_iA_{i+1},\ i = 0, 1, 2$, are called principal vertices of the orthoscheme.
(2) A complete orthoscheme of degree $m = 1$ can be interpreted as an orthoscheme with one outer principal vertex, say A_0, which is truncated by its polar plane $\mathrm{pol}(A_0)$ (see Fig. 1 and 3).
(3) A complete orthoscheme of degree $m = 2$ can be interpreted as an orthoscheme with two outer principal vertices, $A_0,\ A_3$, which is truncated by its polar planes $\mathrm{pol}(A_0)$ and $\mathrm{pol}(A_3)$. In this case, the orthoscheme is called doubly truncated. We distinguish two different types of orthoschemes but will not enter into the details (see [7], [8]).

The ordered set $\{k_{01}, k_{12}, k_{23}\}$ of natural numbers, bigger than 2, is said to be the Coxeter–Schläfli symbol of the simplex tiling $\mathcal{P}$ generated by the Coxeter reflection group of $\mathcal{O}$. To every scheme there is a corresponding symmetric matrix (b^{ij}) of size 4×4 where $b^{ii} = 1$ and, for $i \neq j \in \{0, 1, 2, 3\}$, b^{ij} equals $-\cos(\pi/k_{ij})$ with all angles between the facets i,j of $\mathcal{O}$.

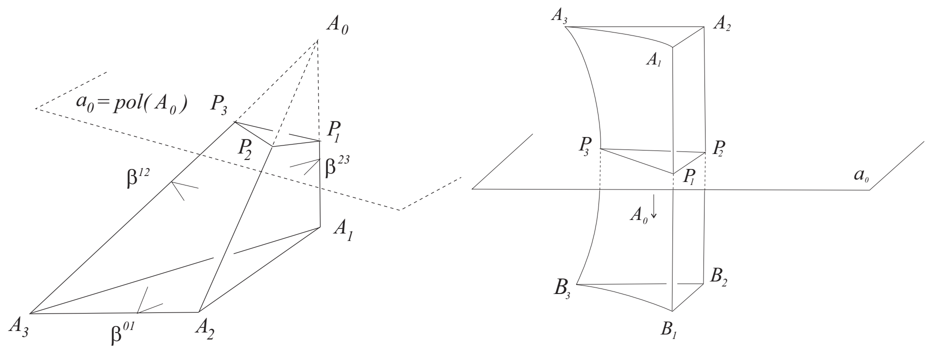
For example, (b^{ij}) below is the so called Coxeter–Schläfli matrix of the orthoscheme O in hyperbolic space $\mathbb{H}^3$ with parameters $k_{01} = u$, $k_{12} = v$, $k_{23} = w$:

$$(b^{ij}) = \langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle := \begin{pmatrix} 1 & -\cos\frac{\pi}{u} & 0 & 0 \\ -\cos\frac{\pi}{u} & 1 & -\cos\frac{\pi}{v} & 0 \\ 0 & -\cos\frac{\pi}{v} & 1 & -\cos\frac{\pi}{w} \\ 0 & 0 & -\cos\frac{\pi}{w} & 1 \end{pmatrix}. \tag{2.2}$$

In general, the complete Coxeter orthoschemes were classified by Im Hof in [7] by generalizing the method of Coxeter and Böhm, who showed that they exist only for dimensions up to 9. From this classification, it follows that the complete orthoschemes of degree $m = 1$ exist up to 5 dimensions.

In this paper we consider some tilings generated by Coxeter reflection groups of orthoschemes of degree 1, where the initial vertex A_0 is an outer point regarding the quadric Q. That is, we allow $w = p > 6$ for the truncated cube or $w = p > 4$ for the truncated octahedron, but we examine $p \in \mathbb{R}$ as well.

In hyperbolic space $\mathbb{H}^3$ it can be seen (Fig. 1) that if $\mathcal{O} = A_1A_2A_3\ P_1P_2P_3$ is a complete orthoscheme with degree $m = 1$ where A_0 is an outer vertex of $\mathbb{H}^3$, then the points P_1, P_2, P_3 lie on the polar hyperplane a_0 of A_0 (see Fig. 1). The images of $\mathcal{O}$ under reflections on its side facets generate a tiling



a. b.

FIGURE 1. a. A 3-dimensional complete orthoscheme of degree $m = 1$ (simple frustum or truncated orthoscheme) with outer vertex A_0. This orthoscheme is truncated by its polar plane $a_0 = \mathrm{pol}(A_0)$. b. Two reflected simple orthoschemes for a hyperball with base plane $a_0 = P_1P_2P_3$.

in $\mathbb{H}^3$. Our polyhedron $A_1A_2A_3P_1P_2P_3$ is a simple truncated orthoscheme with outer vertex A_0 (see Fig. 1) whose volume can be calculated by the following theorem of R. Kellerhals [8]:

Theorem 2.1. *The volume of a three-dimensional hyperbolic complete orthoscheme $\mathcal{O}$ is expressed with the essential angles $\beta^{01}, \beta^{12}, \beta^{23}$, $(0 \le \beta^{ij} \le \pi/2)$ (Fig. 1) in the following form:*

$$\begin{gathered}\mathrm{Vol}_3(\mathcal{O}) = \frac{1}{4}\{\mathcal{L}(\beta^{01}+\theta) - \mathcal{L}(\beta^{01}-\theta) + \mathcal{L}(\frac{\pi}{2}+\beta^{12}-\theta) + \\ + \mathcal{L}(\frac{\pi}{2}-\beta^{12}-\theta) + \mathcal{L}(\beta^{23}+\theta) - \mathcal{L}(\beta^{23}-\theta) + 2\mathcal{L}(\frac{\pi}{2}-\theta)\},\end{gathered} \tag{2.3}$$

where $\theta \in [0, \pi/2)$ is defined by the following formula:

$$\tan(\theta) = \frac{\sqrt{\cos^2\beta^{12} - \sin^2\beta^{01}\sin^2\beta^{23}}}{\cos\beta^{01}\cos\beta^{23}}$$

and where $\mathcal{L}(x) := -\int_0^x \log|2\sin t|\,dt$ denotes the Lobachevsky function (in J. Milnor's interpretation).

In (2.2) we have: $\beta^{01} = \pi/u$, $\beta^{12} = \pi/v$, $\beta^{23} = \pi/w$.

3. On hyperball packings to truncated cube and octahedron tilings

Similar to the truncated tetrahedral cases (see [18], [19]) it is interesting to examine and to construct at least locally optimal *congruent and non-congruent* hyperball packings and coverings relating to suitable truncated polyhedron tilings in 3- and higher dimensions as well.

In this paper we consider the 3-dimensional regular truncated cube and octahedron tilings that are derived from the Coxeter truncated simplex tilings $\{4,3,p\}$ for $6 < p \in \mathbb{N}$ and $\{3,4,p\}$ for $4 < p \in \mathbb{N}$, extended to $p \in \mathbb{R}$ as well.

3.1. **Packings with congruent hyperballs to truncated cube tilings $\{4,3,p\}$.** We consider a truncated cube tiling $\mathcal{T}(\mathcal{C}^r(p))$ with Schläfli symbol $\{4,3,p\}$ $(7 \le p \in \mathbb{N})$. These can be considered by duality as Coxeter tilings $\{p,3,4\}$ and in a complete analogy of p-gonal prism tiling as Fig. 1 shows (compare also with Fig. 2.a.b). One tile of $\mathcal{C}^r(p)$ (a truncated cube) is illustrated in Fig. 2.a.b which can also be derived by truncation from a regular Euclidean cube centred at the origin with vertices C_i $(i \in \{1,\dots,8\})$. The truncating planes γ_i are the polar planes of outer vertices C_i that can be the ultraparallel base planes of hyperballs $\mathcal{H}_i^s$ $(i \in \{1,\dots,8\})$ with height s. The nonorthogonal dihedral angles of $\mathcal{C}^r(p)$ are equal to $2\pi/p$. The distances between two base planes are equal to $d(\gamma_i,\gamma_j) =: e_{ij} = 2h(p)$ $(i < j,\ i,j \in \{1,\dots 8\})$ (d is the hyperbolic distance function). Therefore the height of a hyperball is at most $h(p)$ (see Fig. 2.a.b). It is clear, that in the congruent, densest case, the heights of the hyperballs are $h(p)$, i.e., the neighbouring hyperballs touch each other.

We consider a saturated congruent hyperball packing $\mathcal{B}^{h(p)}$ of hyperballs $\mathcal{H}_i^{h(p)}$ related to the above truncated cube $\{4,3,p\}$ for $6 < p \in \mathbb{R}$ as well. The volume of the truncated cube $\mathcal{C}^r(p)$ is denoted by $\mathrm{Vol}(\mathcal{C}^r(p))$ and we introduce the local density function $\delta(\mathcal{C}^r(h(p)))$ related to $\mathcal{C}^r(p)$:

Definition 3.1.

$$\delta(\mathcal{C}^r(h(p))) := \frac{\sum_{i=1}^{8} \mathrm{Vol}({}_{+}\mathcal{H}_i^{h(p)} \cap \mathcal{C}^r(p))}{\mathrm{Vol}(\mathcal{C}^r(p))} = \frac{8 \cdot \mathrm{Vol}({}_{+}\mathcal{H}_i^{h(p)} \cap \mathcal{C}^r(p))}{\mathrm{Vol}(\mathcal{C}^r(p))}.$$

If the parameter p is given, then the common length of the common perpendiculars $2h(p) = e_{ij}$ $(i < j,\ i,j \in \{1,\dots,8\})$ can be determined by the machinery of projective geometry (see e.g. [10]).

$$\cosh h(p) = \cosh A_1P_1 = \frac{-\langle \mathbf{a}_1, \mathbf{p}_1 \rangle}{\sqrt{\langle \mathbf{a}_1, \mathbf{a}_1 \rangle \langle \mathbf{p}_1, \mathbf{p}_1 \rangle}} = \sqrt{1 - \frac{a_{01}^2}{a_{00}a_{11}}}. \tag{3.1}$$

where a_{ij} $(i,j = 0,1,2,3))$ is the inverse of the corresponding Coxeter–Schläfli matrix (see (2.2), where $u = 4$, $v = 3$, and $w = p$) of the orthoscheme $P_1P_2P_3A_1A_2A_3$ (see Fig. 2.b).

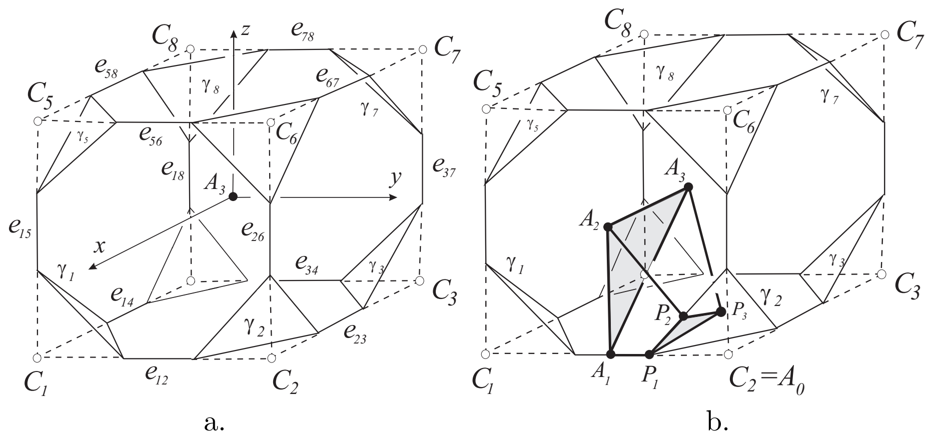


FIGURE 2. a. Regular truncated cube. b. Truncated cube with complete orthoscheme of degree $m = 1$ (simple frustum orthoscheme) with outer vertex $C_2 = A_0$. This orthoscheme is truncated by its polar plane $\gamma_2 = \mathrm{pol}(C_2)$.

The volume $\mathrm{Vol}(\mathcal{C}^r(p))$ can be calculated by Theorem 2.1 and the volume of the hyperball pieces lying in $\mathcal{C}^r(p)$ can be computed by the formula (2.1) for each given parameter p, therefore the maximal height $h(p)$ of the congruent hyperballs and the $\sum_{i=1}^{8} \mathrm{Vol}({}_+\mathcal{H}_i^{h(p)} \cap \mathcal{C}^r(p)))$ depend only on the parameter p of the truncated regular cube $\mathcal{C}^r(p)$.

Thus, the density $\delta(\mathcal{C}^r(h(p)))$ depends only on the parameter p for $6 < p \in \mathbb{R}$.

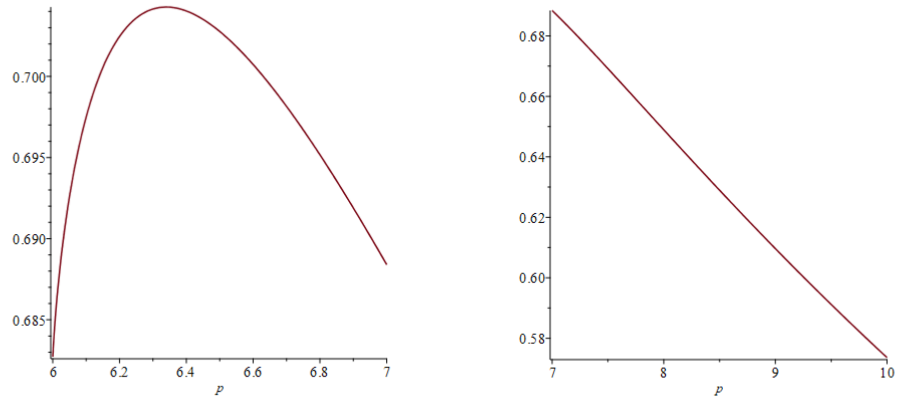
Finally, we obtain after careful analysis of the smooth density function (see [16] and Fig. 3) the following

Theorem 3.2. *The density function $\delta(\mathcal{C}^r(h(p)))$ $(p \in (6, \infty))$ attains its maximum at $p^{opt} \approx 6.33962$. It is strictly increasing on the interval $(6, p^{opt})$ and strictly decreasing on the interval (p^{opt}, ∞). Moreover, the optimal density $\delta^{opt}(\mathcal{C}^r(h(p^{opt}))) \approx 0.70427$, however these hyperball packing configurations are only locally optimal and cannot be extended to the entirety of $\mathbb{H}^3$ (see Fig. 3).*

Corollary 3.3. *The density function $\delta(\mathcal{C}^r(h(p)))$ $(7 \le p \in \mathbb{N})$ attains its maximum at the parameter $p = 7$. The congruent hyperball packing $\mathcal{B}^{h(7)}$ related to the truncated cube tilings can be extended to the entire $\mathbb{H}^3$. The maximal density is $\delta(\mathcal{C}^r(h(7))) \approx 0.68839$.*

Remark: We note here that these results coincide with the hyperball packings to the regular prism tilings in $\mathbb{H}^3$ with Schläfli symbols $\{p, 3, 4\}$ which are discussed in [13].

For completeness, in the following table we summarize the data of the hyperball packings for some parameters p $(7 \ge p \in \mathbb{N})$.



a. b.

FIGURE 3. a. Density function $\delta(\mathcal{C}^r(h(p)))$, $(p \in [6,7])$.
b. Density function $\delta(\mathcal{C}^r(h(p)))$, $(p \in [7,10])$.

Table 1, $\{4,3,p\}$

p	$h(p)$	$Vol(\mathcal{C}^r)/48$	$Vol({}_+\mathcal{H}^{h(p)} \cap \mathcal{C}^r(p))/6$	$\delta^{opt}(\mathcal{C}^r(h(p)))$
7	1.03799291	0.16297337	0.11218983	0.68839367
8	0.76428546	0.18789693	0.12193107	0.64892530
9	0.62216938	0.20295023	0.12372607	0.60963750
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	0.23086908	0.24206876	0.08613744	0.35583872
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	0.08938872	0.24956032	0.04129724	0.16547999
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	0.04449475	0.25061105	0.02191401	0.08744233
$p \to \infty$	0	0.25096025	0	0

3.2. **Packings with noncongruent hyperballs in truncated cube.** We consider, similar to the above subsection, a truncated cube tiling $\mathcal{T}(\mathcal{C}^r(p))$ with Schläfli symbol $\{4,3,p\}$ $(7 \le p \in \mathbb{N})$ and we will use the same notation (see Fig. 2.a.b).

The distances of the plane γ_i $(i \in \{1,\dots 8\})$ from rectangular octagon faces of the cube $\mathcal{C}^r(p)$ whose planes do not contain the vertex C_i are equal. This distance is denoted by $w(p)$ and can be calculated by an analogous formula to (3.1) (see Fig. 2.a.b).

We would like to construct noncongruent hyperball packings to $\mathcal{T}(\mathcal{C}^r(p))$ tilings. Therefore the hyperballs have to satisfy the following requirements:

(1) The base plane γ_i of the hyperball $\mathcal{H}_i^{h_i(p)}$ (with height $h_i(p)$) is the polar plane of the vertex C_i (see Fig. 2).
(2) The hyperballs have disjoint interiors.
(3) For the above distances, $w(p) \ge h_i(p)$ holds.

If the hyperballs satisfy the above requirements then we obtain congruent or noncongruent hyperball packings in the cube $\mathcal{C}^r(p)$ and if we extend them by the structure of the considered Coxeter cube tilings $\mathcal{T}(\mathcal{C}^r(p))$, then we get hyperball packings $\mathcal{B}(p)$ in hyperbolic 3-space.

We introduce the local density function $\delta(\mathcal{C}^r(p))$ related to above packings:

Definition 3.5.

$$\delta(\mathcal{C}^r(p)) := \frac{\sum_{i=1}^{8} \mathrm{Vol}({}_+\mathcal{H}_i^{h_i(p)} \cap \mathcal{C}^r(p))}{\mathrm{Vol}(\mathcal{C}^r(p))}.$$

We will use that if a packing is locally optimal (i.e., its density is locally maximal), then it is locally stable (i.e., each hyperball is fixed by the other ones so that no hyperball of packing can be moved alone without overlapping another one, or by other requirements of the tiling).

To get the locally optimal (noncongruent half) hyperball packing arrangement for packing a fixed truncated cube, we distinguish three essential cases:

Case 1. We set up from the optimal congruent half hyperball arrangement (see former subsection), where the edge adjacent congruent hyperballs touch each other at the "edge midpoints" of $\mathcal{C}^r(p)$.

We choose two opposite hyperballs (e.g., $\mathcal{H}_2^{h(p)}$ and $\mathcal{H}_8^{h(p)}$) and blow up these hyperballs keeping the hyperballs $\mathcal{H}_i^{h_i(p)}(p)$ $(i \in \{1,3,\ 4,5,6,7\})$ tangent to them upto their heigths $h_2(p) = h_8(p) = \min\{2h(p), w(p), t(p), s(p)\}$ where $s(p) = d(A_2, P_2)$ and $t(p) = d(A_3, P_3)$ (see Fig. 2.a.b). During this expansion the heights of hyperballs $\mathcal{H}_j^{h_j(p)}$ $(j = 2, 8)$ are $h_j(p) = h(p) + x$ where $x \in [0,\ \min\{2h(p), w(p), t(p)\} - h(p)]$. The heights of further hyperballs are $h_1(p) = h_3(p) = h_4(p) = h_5(p) = h_6(p) = h_7(p) = h(p) - x$. (If $x = 0$ then the hyperballs are congruent.)

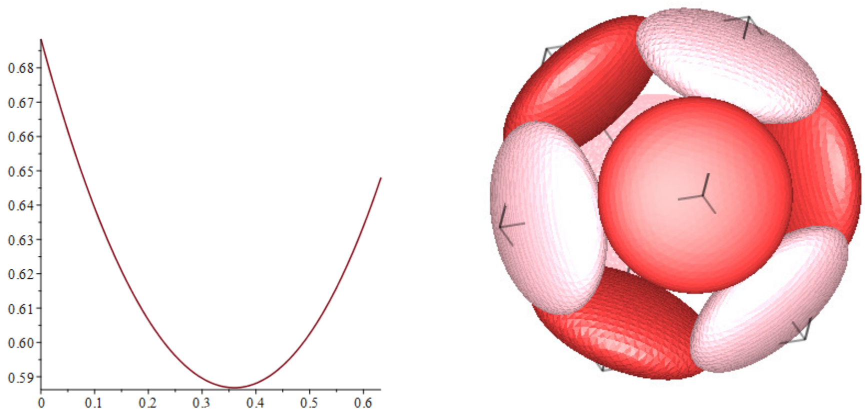
We extend this procedure to images of the hyperballs or half hyperballs ${}_+\mathcal{H}_i^{h_i(p)}$ $(i \in \{1, \dots, 8\})$ under the considered Coxeter reflection group and obtain noncongruent hyperball arrangements $\mathcal{B}_1^x(p)$.

Applying Definition 3.5 we obtain the density function $\delta_1(\mathcal{C}^r(x,p))$:

$$\delta_1(\mathcal{C}^r(x,p)) = \frac{2 \cdot \mathrm{Vol}({}_+\mathcal{H}^{h(p)+x} \cap \mathcal{C}^r(p)) + 6 \cdot \mathrm{Vol}({}_+\mathcal{H}^{h(p)-x} \cap \mathcal{C}^r(p))}{\mathrm{Vol}(\mathcal{C}^r(p))}, \tag{3.2}$$

where $x \in [0, \min\{2h(p), w(p), t(p)\} - h(p)]$.

Case 2. Now, we start from the noncongruent ball arrangement where two opposite "larger hyperballs" with base planes γ_2 and γ_8 are tangent at the centre A_3 of the cube, while hyperballs at the remaining six vertices touch the corresponding "larger" hyperball. The point of tangency of the above two larger hyperballs on line C_2C_8 is denoted by $A_3 = I(0)$. We blow up the hyperball $\mathcal{H}_2^{t(p)}$ with base plane γ_2 keeping the hyperballs $\mathcal{H}_i^{h_i(p)}$ $(i = 1,3,6,8)$ tangent to it while the hyperballs $\mathcal{H}_i^{h_i(p)}$ $(i = 4,5,7)$ are blown up to touch the hyperball $\mathcal{H}_8^{h_8(p)}$. The point of tangency of the hyperballs



a. b.

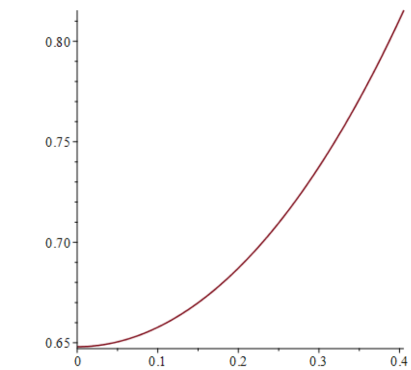
FIGURE 4. a. Density function $\delta_1(x,7))$ $(x \in [0, t(7) - h(7)], 7)$. If $x = 0$ then the density is approximately 0.68839. The density in the endpoint of the above interval is approximately 0.64805 less than before. b. The congruent hyperball arrangement with the first data.

$\mathcal{H}_2^{h_2(p)}$ and $\mathcal{H}_8^{h_8(p)}$ along the line C_2C_8 is denoted by $I(x)$, where x is the hyperbolic distance between $A_3 = I(0)$ and $I(x)$. During this expansion the height of hyperball $\mathcal{H}^{h_2(p)}$ is $h_2(p) = t(p) + x$ where $x \in [0, \min\{2h(p) - t(0), t(p), w(p) - t(p), s(p) - h(p)\}$. We note here, that $w(p) \geq 2h(p)$ and $t(p) \geq s(p) \geq h(p)$. Therefore the hyperball $\mathcal{H}_2^{h_2(p)}$ can be blown up at most to the hypersphere touching the planes γ_i $(i = 1, 2, 3, 4)$. We extend this procedure to images of the hyperballs $\mathcal{H}_i^{h_i(p)}$ $(i \in \{1, \dots, 8\})$ by the considered Coxeter group and obtain noncongruent hyperball arrangements $\mathcal{B}_2^x(p)$. Applying Definition 3.5 we obtain the density function $\delta_2(\mathcal{C}^r(x,p))$:

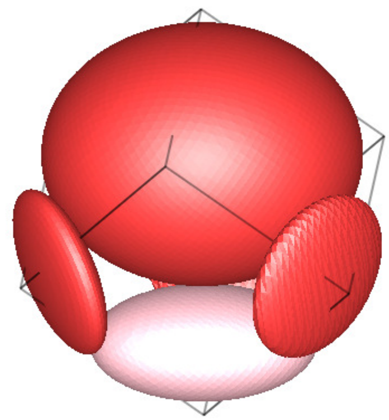
$$\begin{gathered}\delta_2(\mathcal{C}^r(x,p)) = (\mathrm{Vol}({}_{+}\mathcal{H}^{t(p)+x} \cap \mathcal{C}^r(p)) + \\ + \mathrm{Vol}({}_{+}\mathcal{H}^{t(p)-x} \cap \mathcal{C}^r(p)) + 3 \cdot \mathrm{Vol}({}_{+}\mathcal{H}^{2h(p)-t(p)-x} \cap \mathcal{C}^r(p)) + \\ +3 \cdot \mathrm{Vol}({}_{+}\mathcal{H}^{2h(p)-t(p)+x} \cap \mathcal{C}^r(p)))/\,\mathrm{Vol}(\mathcal{C}^r(p)),\end{gathered} \tag{3.3}$$

where $x \in [0, \min\{2h(p) - t(0), t(p), w(p) - t(p), s(p) - h(p)\}$.

Case 3. We set up from the congruent ball arrangement (see the previous section) where the neighbouring congruent hyperballs touch each other at the "midpoints" of the edges of $\mathcal{C}^r(p)$. We distinguish two different classes of hyperballs related to two complementary tetrahedral sublattices of cube vertices. E.g. the hyperballs $\mathcal{H}_i^{h_i(p)}$ $(i \in \{1, 3, 6, 8\})$ form the first class and the remaining hyperballs form the second class. We blow up the hyperballs of the first class keeping the remaining hyperballs tangent to them up to their heights $h_1(p) = h_3(p) = h_4(p) = h_7(p) = \min\{2h(p), s(p)\}$ (see Fig. 2.b).



a.



b.

FIGURE 5. a. The density function $\delta_2(x,7))$ $(x \in [0, w(7) - t(7)], 7)$. If $x = w(7) - t(7) \approx 0.40530$ then the density is approximately 0.81542. The density in the starting point is approximately 0.64805 less than before. b. The noncongruent hyperball arrangement with the first data.

During this expansion, the heights of hyperballs $\mathcal{H}_i^{h_i(p)}$ $(i \in \{1,3,6,8\})$ are $h_i(p) = h(p) + x$ and $\mathcal{H}_j^{h_j(p)}$ $(j \in \{2,4,5,7\})$ are $h_j(p) = h(p) - x$, where $x \in [0, \min\{h(p), s(p) - h(p)]$.

We extend this procedure to images of $\mathcal{H}_i^{h_i(p)}$ $(i \in \{1,\dots,8\})$ by the considered Coxeter group and obtain noncongruent hyperball arrangements $\mathcal{B}_3^x(p)$. Applying Definition 3.5 we get the density function $\delta_3(\mathcal{C}^r(x,p))$:

$$\delta_3(\mathcal{C}^r(x,p)) = \frac{4\,\mathrm{Vol}({}_+\mathcal{H}^{h(p)+x} \cap \mathcal{C}^r(p)) + 4 \cdot \mathrm{Vol}({}_+\mathcal{H}^{h(p)-x} \cap \mathcal{C}^r(p))}{\mathrm{Vol}(\mathcal{C}^r(p))}, \tag{3.4}$$

where $x \in [0, \min\{h(p), s(p) - h(p)]$.

The main problem is: what is the maximum of the above density functions $\delta_i(\mathcal{C}^r(x,p))$ $(i \in \{1,2,3\})$ for given integer parameters $p \geq 7$, where $x \in \mathbb{R}$ belongs to the corresponding intervals during these expansion processes?

3.2.1. *Computations.* Every 3-dimensional hyperbolic truncated cube can be derived from a 3-dimensional regular Euclidean cube (see Fig. 2). We introduce a projective coordinate system (see Section 2 and Fig. 2.a) and a unit sphere $\mathbb{S}^2$ centred at the origin which is interpreted as the ideal boundary of $\overline{\mathbb{H}}^3$ in Beltrami–Cayley–Klein's ball model.

Besides the computations for (2.2), the next ones will also be important for the visualization.

Now, we consider a 3-dimensional regular Euclidean cube centred at the origin with outer vertices regarding the Beltrami–Cayley–Klein's ball model.

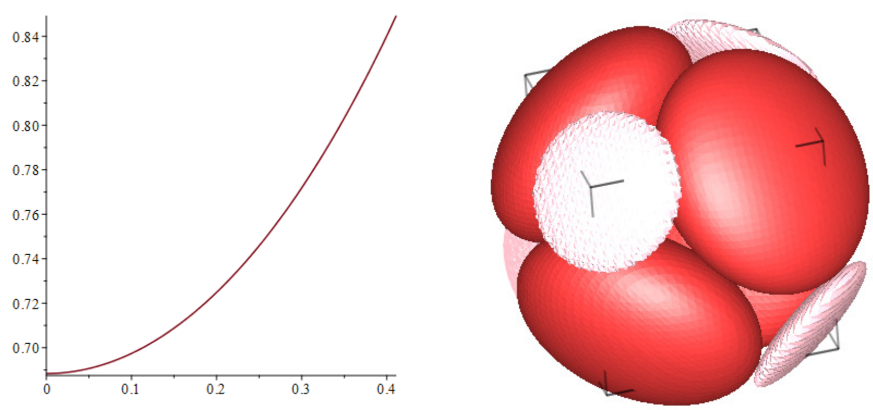
The projective coordinates of the vertices of this cube are
(3.5)

$$C_1 = (1, y, -y, -y);\ C_2 = (1, y, y, -y);\ B_3 = (1, -y, y, -y);$$
$$B_C = (1, -y, -y, -y);\ C_5 = (1, y, -y, y);$$
$$C_6 = (1, y, y, y);\ C_7 = (1, -y, y, y);\ C_8 = (1, -y, -y, y) \text{ where } \frac{1}{\sqrt{3}} < y \in \mathbb{R}.$$

The truncated cube $\mathcal{C}^r(p)$ can be derived from the above cube by cuttings with the polar planes of vertices C_i ($i \in \{1, \dots, 8\}$). The images of $\mathcal{C}^r(p)$ under reflections on its side facets generate a tiling in $\mathbb{H}^3$ if its nonright dihedral angles are $2\pi/p$ for $6 < p \in \mathbb{N}$ and $6 < p \in \mathbb{R}$, respectively. It is straightforward to see that if the parameter p is given, then

$$y = \sqrt{\frac{\cos\frac{2\pi}{p}}{\cos\frac{2\pi}{p}+1}}. \tag{3.6}$$

We have to determine for any parameter p the distances $h(p)$, $t(p)$, $s(p)$, and $w(p)$. The values of $h(p)$ can be derived from formula (3.1). The distances $t(p)$ and $s(p)$ can be determined similarly to (3.1). Then $w(p)$ follows from the next formula (similarly to (3.1)):



a. b.

FIGURE 6. a. The density function $\delta_3(x, 7))$ ($x \in [0, s(7) - h(7)], 7)$. If $s(7) - h(7) \approx 0.41108$ then the density is approximately 0.84931. The density in the starting point is ≈ 0.68839. b. The noncongruent hyperball arrangement with the above data.

$$\sinh w(p) = \sqrt{\frac{3y^4+1}{(1-3y^2)(y^2-1)}}. \tag{3.7}$$

If $p = 7$ then we obtain the following results:

$$2h(7) \approx 2.07599; w(7) \approx 2.07599 \Rightarrow \; w(7) = 2h(7);$$
$$t(7) \approx 1.67069, \; s(7) \approx 1.03799.$$

During the expansion processes we can compute the densities $\delta_i(\mathcal{C}^r(x,\,7))$ $(i \in \{1,2,3\})$ (see Definition 3.5) of the considered packings as functions of x using formulas (2.1), (3.1)–(3.4), (3.6), (3.7), and Theorem 2.1.

We find $\delta_1(\mathcal{C}^r(x,7))$ by (3.2) where

$$x \in [0, \min\{2h(7), w(7), t(7)\} - h(7)\} \approx 0.63270]. \tag{3.8}$$

The graph of $\delta_1(\mathcal{C}^r(x,7))$ is described in Fig. 4.a.

We find $\delta_2(\mathcal{C}^r(x,7))$ by (3.3), where

$$x \in [0, \min\{2h(7) - t(0), t(7), w(7) - t(7), s(7) - h(7)\} \approx 0.40530]. \tag{3.9}$$

The graph of $\delta_2(\mathcal{C}^r(x,7))$ is described in Fig. 5.a.

We find $\delta_3(\mathcal{C}^r(x,7))$ by (3.4), where

$$x \in [0, \min\{h(7), s(7) - h(7)\} \approx 0.41108]. \tag{3.10}$$

The graph of $\delta_3(\mathcal{C}^r(x,7))$ is described in Fig. 6.a.

Similar to the above discussions, we obtain that if $p = 8$, then the maximal density belongs to $\delta_3(\mathcal{C}^r(x,8))$ $(x \in [0, s(p) - h(p) \approx 0.45994])$. Analysing the above density function, we get the maximal density at the endpoint of the above interval with density $\delta_3(\mathcal{C}^r(s(7) - h(p) \approx 0.45994, 8)) \approx 0.82259$.

If $p > 8$, then $2h(p) < t(p) < w(p)$. We have to examine only the density functions $\delta_i(\mathcal{C}^r(x,p))$ $(i \in \{1,3\})$. Similar to the above computations, we can analyse the density functions and their maxima of noncongruent hyperball packings generated by considered truncated cube tilings (or Coxeter tilings $\{4,3,p\}$) for all possible integer parameters $p > 8$. Using Theorem 3.2 and Corollary 3.3 we can summarize our results in the following

Theorem 3.6.

(1) *The maximum of the density function* $\delta_1(\mathcal{C}^r(x,p))$ *attains at the starting point of the corresponding interval* $x \in [0, \min\{h(p),\, t(p) - h(p)]$ *depending on the given integer parameter* $p \geq 7$, *i.e., the congruent hyperball packing provides the densest hyperball packing.*

(2) *If* $p = 7, 8$, *then the maximum of* $\delta_2(\mathcal{C}^r(x,p))$ *is achieved at the endpoint of the interval* $[0, 2h(p) - t(p)]$. *If* $p > 8$ *then this case does not occur because* $2h(p) < t(p) < w(p)$.

(3) *The maximum of* $\delta_3(\mathcal{C}^r(x,p))$ *is attained at the endpoint of the corresponding intervals* $x \in [0, s(p) - h(p)]$, *where* $p \geq 7$ *is a given integer parameter.*

Theorem 3.7. *The maximum of the density functions* $\delta_i(\mathcal{C}^r(x,p))$, $p \geq 7 \in \mathbb{N}$, $i \in \{1,2,3\}$ *is achieved at the parameters* $x = s(p) - h(p) \approx 0.41108$, $p = 7$. *Therefore, the density upper bound of the congruent and noncongruent hyperball packings, related to the truncated cube tilings* $\{4,3,p\}$ $(\mathbb{N} \ni p \geq 7)$, *is approximately* 0.84931.

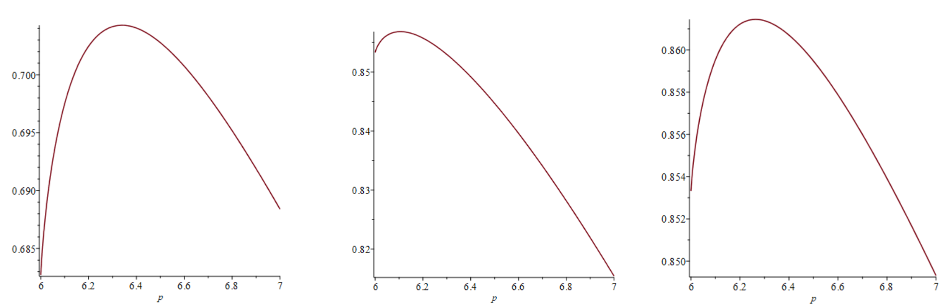
3.2.2. *On nonextendable noncongruent hyperball packings* $(6 < p < 7)$. The computation method described in the previous sections are suitable to determine the densities of congruent and noncongruent hyperball packings related to truncated cubes with parameters $6 < p < 7,\ p \in \mathbb{R}$. For any p we can determine the corresponding densities of their optimal half hyperball packings. But these packings cannot be extended to 3-dimensional space. Analysing these nonextendable packings for parameters $6 < p < 7,\ p \in \mathbb{R}$ we obtain the following

Theorem 3.8. (1) *The maximum of the density function* $\delta_1(\mathcal{C}^r(x,p))$ $(6 < p < 7,\ p \in \mathbb{R})$ *is attained at the starting point of the corresponding interval* $x \in [0, t(p) - h(p)]$, *i.e., the congruent hyperball packing provides the densest hyperball packing (Fig. 7.a shows the graph of* $\delta_1(\mathcal{C}^r(0,p))$ *if* $p \in (6,7)$*). This function attains its maximum at* $p_1 \approx 6.33962$ *where* $\delta_1(\mathcal{C}^r(0,p_1)) \approx 0.70427$.

(2) *The maximum of* $\delta_2(\mathcal{C}^r(x,p))$ $(6 < p < 7,\ p \in \mathbb{R})$ *is attained at the endpoint of the interval* $[0, w(p) - t(p)]$ *(Fig. 7.b shows the graph of the function* $\delta_2(\mathcal{C}^r(w(p) - t(p), p))$ *if* $p \in (6,7)$*). This function attains its maximum at* $p_2 \approx 6.10563$*, where* $\delta_2(\mathcal{C}^r(0,p_2)) \approx 0.85684$.

(3) *The density function* $\delta_3(\mathcal{C}^r(x,p))$ *attains its maximum at the endpoint of the corresponding interval* $x \in [0, s(p) - h(p)]$ *where* $6 < p < 7$ *(Fig. 7.c shows the graph of the function* $\delta_3(\mathcal{C}^r(s(p) - h(p), p))$ *if* $p \in (6,7)$*). Thus we get the maximum at* $p_3 \approx 6.26384$ *where* $\delta_3(\mathcal{C}^r(s(p_3) - h(p_3), p_3)) \approx 0.86145$.

Theorem 3.9. *The maximum of the density functions* $\delta_i(\mathcal{C}^r(x,p))$ $(6 < p < 7,\ p \in \mathbb{R},\ i \in \{1,2,3\})$ *is achieved at the parameters* $x = s(p) - h(p) \approx 0.36563$, $p_3 \approx 6.26384$. *Therefore, the density upper bound of the congruent and noncongruent hyperball packings related to the truncated cube* $\{4,3,p\}$ $(6 < p < 7,\ p \in \mathbb{R})$ *is approximately* 0.86145.



a. b. c.

FIGURE 7. The graphs of the densities $\delta_1(\mathcal{C}^r(0,p))$, $\delta_2(\mathcal{C}^r(w(p) - t(p), p))$, $\delta_3(\mathcal{C}^r(s(p) - h(p), p))$, respectively, described in Theorems 3.8 and 3.9.

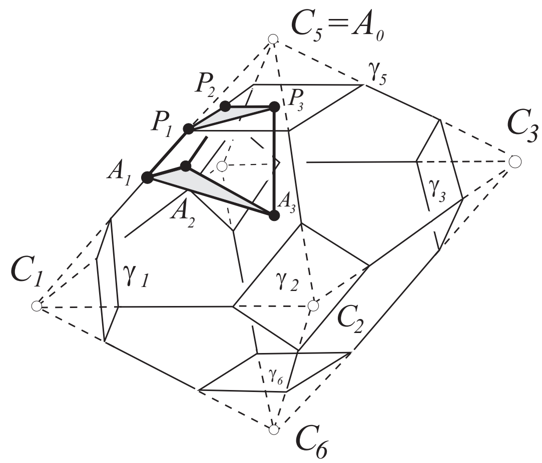


FIGURE 8. Regular truncated octahedron with complete orthoscheme of degree $m = 1$ (simple frustum orthoscheme) and outer vertex $C_5 = A_0$. This orthoscheme is truncated by its polar plane $\gamma_5 = \text{pol}(C_5)$.

Remark: In our case $\lim_{p\to 6}(\delta_i(\mathcal{C}^r(x,p)))$ $(i \in \{2,3\})$ is equal to the Böröczky–Florian upper bound of the ball and horoball packings in $\mathbb{H}^3$ (see [4], [9]).

The locally optimal hyperball configurations $\delta_2(\mathcal{C}^r(0, p_2 \approx 6.10563)) \approx 0.85684$ and $\delta_3(\mathcal{C}^r(s(p_3) - h(p_3), p_3 \approx 6.26384)) \approx 0.86145$ provide larger densities that the Böröczky–Florian density upper bound $\delta_{BF} \approx 0.85328$ for ball and horoball packings ([4]) but these hyperball packing configurations are only locally optimal and cannot be extended to the entire hyperbolic space $\mathbb{H}^3$.

3.3. **Hyperball packings to regular truncated octahedron tiling** $\{3, 4, p\}$**.** We consider a regular truncated octahedron tiling $\mathcal{T}(\mathcal{O}^r(p))$ with Schläfli symbol $\{3,4,p\}$, $4 < p \in \mathbb{N}$, and $4 < p \in \mathbb{R}$, respectively. One tile of $\mathcal{O}^r(p)$ is illustrated in Fig. 9. This truncated octahedron can also be derived by truncation from a regular Euclidean octahedron centred at the origin with vertices C_i $(i \in \{1,\dots,6\})$. The truncating planes γ_i are the polar planes of outer vertices C_i that can be the ultraparallel base planes of hyperballs $\mathcal{H}_i^{h_i(p)}$ $(i \in \{1,\dots,6\})$ with heights $h_i(p)$.

The distances between two adjacent base planes $d(\gamma_i,\gamma_j) =: e_{ij}$ are equal $(i < j,\ i,j \in \{1,\dots 6\})$. Moreover, the volume of the truncated simplex $\mathcal{O}^r(p)$ is denoted by $\text{Vol}(\mathcal{O}^r(p))$, similar to the above section.

The distances of the plane γ_i $(i \in \{1,\dots 6\})$ from rectangular hexagon faces of the octahedron $\mathcal{O}^r(p)$, whose planes do not contain the vertex C_i, are equal and denoted by $w(p)$ (and computed by (3.1), see Fig. 9).

We construct noncongruent hyperball packings to $\mathcal{T}(\mathcal{O}^r(p))$ tilings, therefore the hyperballs have to satisfy the following requirements:

(1) The base plane γ_i of the hyperball $\mathcal{H}_i^{h_i(p)}$ (with height $h_i(p)$) is the polar plane of the vertex C_i (see Fig. 2 and 9).
(2) The hyperballs have disjoint interiors.
(3) For the above distances, $w(p) \geq h_i(p)$ holds.

If the hyperballs satisfy the above requirements, then we obtain congruent or noncongruent hyperball packings $\mathcal{B}(\mathcal{O}^r(p))$ in hyperbolic 3-space derived by the structure of $\{3, 4, p\}$.

We introduce the local density function $\delta(\mathcal{O}^r(p))$ related to $\mathcal{O}^r(p)$:

Definition 3.11.

$$\delta(\mathcal{O}^r(p)) := \frac{\sum_{i=1}^{6} \mathrm{Vol}({}_{+}\mathcal{H}_i^{h_i(p)} \cap \mathcal{O}^r(p))}{\mathrm{Vol}(\mathcal{O}^r(p))}.$$

The main problem is: what is the maximum of density functions $\delta_i(\mathcal{O}^r(p))$ $(i = 1, 2)$ for given integer parameters $p \geq 5$?

Similar to the above "truncated cube case", after careful analysis of the density function $\delta(\mathcal{O}^r(h(p)))$ we get the following:

Theorem 3.12. *The density function $\delta(\mathcal{O}^r(h(p)))$ $(p \in (4, \infty))$ attains its maximum if the hyperballs are congruent and $p^{opt} \approx 4.11320$. In the congruent case, the density function is strictly increasing on the interval $(4, p^{opt})$ and strictly decreasing on the interval (p^{opt}, ∞). Moreover, the optimal density is $\delta^{opt}(\mathcal{O}^r(h(p^{opt}))) \approx 0.83173$. However, these hyperball packing configurations are only locally optimal and cannot be extended to the entirety of $\mathbb{H}^3$.*

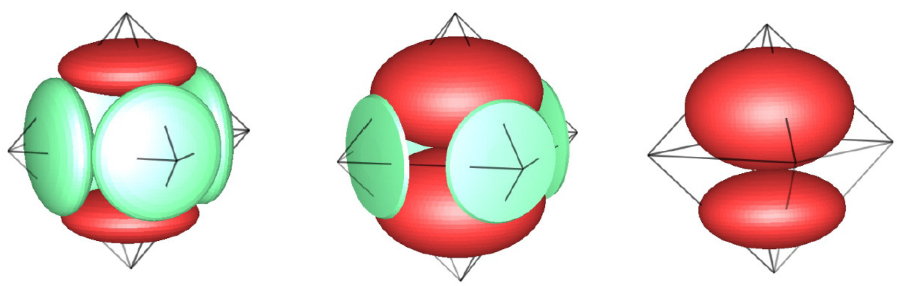
Corollary 3.13. *The density function $\delta(\mathcal{O}^r(h(p)))$ $(\mathbb{N} \ni p \geq 5)$ attains its maximum if the hyperballs are congruent and $p = 5$. The corresponding congruent hyperball packing $\mathcal{B}^{h(5)}$ related to the regular truncated octahedra can be extended to $\mathbb{H}^3$. The maximal density is $\delta(\mathcal{O}^r(h(5))) \approx 0.76893$ (see Fig. 9.a.)*

Remark: These coincide with the hyperball packings to the regular prism tilings in $\mathbb{H}^3$ with Schläfli symbols $\{p, 4, 3\}$ which are discussed in [13].

For completeness, in the following Table we summarize the data of the hyperball packings for some parameters p, $5 \leq p \in \mathbb{N}$.

Table 2, $\{3,4,p\}$

p	$h(p)$	$\mathrm{Vol}(\mathcal{O}^r(p))/48$	$\mathrm{Vol}(\mathcal{H}^{h(p)} \cap \mathcal{O}^r(p))/8$	$\delta^{opt}(\mathcal{O}^r(h(p)))$
5	0.69128565	0.16596371	0.12761435	0.76892924
6	0.48121183	0.19616337	0.13616563	0.69414405
7	0.37938071	0.21217704	0.13400462	0.63156984
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	0.11318462	0.24655736	0.07142045	0.28967074
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	0.04456095	0.25026133	0.03221956	0.12874366
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	0.02223088	0.25078571	0.01676445	0.06684770
$p \to \infty$	0	0.25096025	0	0



a. b. c.

FIGURE 9. a. The densest packing configuration with an approximate density of 0.76893. Here the hyperballs are congruent. b. The packing arrangement of parameters $p = 5$, with an approximate density of 0.72624 where the two opposite "larger hyperballs" with base planes γ_5 and γ_6 are tangent at the centre A_3 of the octahedron. c. The largest hypersphere $\mathcal{H}_5^{h_5(5)}(5)$ touches the planes γ_i $(i = 1,2,3,4)$ and the opposite hypersphere $\mathcal{H}_6^{h_6(5)}(5)$.

The problem of finding the densest hyperball (hypersphere) packing with congruent or noncongruent hyperballs in n-dimensional hyperbolic space $(n \geq 3)$ is not settled yet. For $\mathbb{H}^3$ at this time, the densest hyperball packing with congruent hyperballs is derived by the regular truncated tetrahedron tiling $\{3,3,7\}$ with an approximate density of 0.82251 and with noncongruent hyperballs the packing is derived by the truncated cube tiling $\{4,3,7\}$ with an approximate density of 0.84931 as in the present paper.

But, as we have seen, locally there are hyperball packings with larger density than the Böröczky–Florian density upper bound for ball and horoball packings (see e.g. [9], [10], [11], [17]).

References

1. K. Bezdek, *Sphere Packings Revisited*, Eur. J. Combin., **27** (2006) no. 6, 864–883.
2. J. Bolyai, *The science of absolute space*, Austin, Texas (1891).
3. K. Böröczky, *Packing of spheres in spaces of constant curvature*, Acta Math. Acad. Sci. Hungar., **32** (1978), 243–261.
4. K. Böröczky and A. Florian, *Über die dichteste Kugelpackung im hyperbolischen Raum*, Acta Math. Acad. Sci. Hungar., **15** (1964), 237–245.
5. G. Fejes Tóth, G. Kuperberg and W. Kuperberg, *Highly Saturated Packings and Reduced Coverings*, Monatsh. Math., **125** (1998), no. 2, 127–145.
6. L. Fejes Tóth, *Regular Figures*, Macmillian (New York), 1964.
7. H-C. Im Hof, *Napier cycles and hyperbolic Coxeter groups*, Bull. Soc. Math. Belgique, **42** (1990), 523–545.
8. R. Kellerhals. On the volume of hyperbolic polyhedra, *Math. Ann.*, **245** (1989), 541–569.
9. R. T. Kozma and J. Szirmai, *Optimally dense packings for fully asymptotic Coxeter tilings by horoballs of different types*, Monatsh. Math., **168** (2012), no. 1, 27–47.
10. E. Molnár and J. Szirmai, *Top dense hyperbolic ball packings and coverings for complete Coxeter orthoscheme groups*, Publications de l'Institut Mathématique, **103** (2018), no. 117, 129–146, DOI: 10.2298/PIM1817129M.
11. L. Németh, *On the hyperbolic Pascal pyramid*, Beitr. Algebra Geom., **57** (2016), no. 4, 913–927.
12. M. Stojanović, *Hyperbolic space groups with truncated simplices as fundamental domains*, Filomat, **33** (2019), no. 4 (to appear).
13. J. Szirmai, *The p-gonal prism tilings and their optimal hypersphere packings in the hyperbolic 3-space*, Acta Math. Hungar. **111** (2006), 65–76.
14. ———, *The regular prism tilings and their optimal hyperball packings in the hyperbolic n-space*, Publ. Math. Debrecen **69** (2006), no. 1, 195–207.
15. ———, *The least dense hyperball covering to the regular prism tilings in the hyperbolic n-space*, Ann. Mat. Pur. Appl., **195** (2016), 235-248, DOI: 10.1007/s10231-014-0460-0.
16. ———, *Hyperball packings in hyperbolic* 3*-space*, Mat. Vesnik **70** (2018), no. 3, 211–221.
17. ———, *Packings with horo- and hyperballs generated by simple frustum orthoschemes*, Acta Math. Hungar., **152** (2017), no. 2, 365–382, DOI:10.1007/s10474-017-0728-0.
18. ———, *Density upper bound of congruent and noncongruent hyperball packings generated by truncated regular simplex tilings*, Rend. Circ. Mat. Palermo (2), **67** (2018), 307–322, DOI: 10.1007/s12215-017-0316-8.
19. ———, *Decomposition method related to saturated hyperball packings*, Ars Math. Contemp., **16** (2019), 349–358, DOI: 10.26493/1855-3974.1485.0b1.
20. ———, *Upper bound of density for packing of congruent hyperballs in hyperbolic* $3-$*space*, Submitted manuscript, (2019).
21. I. Vermes, *Ausfüllungen der hyperbolischen Ebene durch kongruente Hyperzykelbereiche*, Period. Math. Hungar. **10** (1979), no. 4, 217–229.
22. ———, *Über reguläre Überdeckungen der Bolyai-Lobatschewskischen Ebene durch kongruente Hyperzykelbereiche*, Period. Polytechnica Mech. Eng. (1981) **25**, no. 3, 249–261.

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