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CLOSED FORMULAS AND IDENTITIES FOR THE BELL POLYNOMIALS AND FALLING FACTORIALS

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Abstract. The authors establish a pair of closed-form expressions for special values of the Bell polynomials of the second kind for the falling factorials, derive two pairs of identities involving the falling factorials, find an equivalent expression between two special values for the Bell polynomials of the second kind, and present five closed-form expressions for the (modified) spherical Bessel functions.

1. Motivations

The Bell polynomials of the second kind, also known as partial Bell polynomials, denoted by $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ for $n\ge k\ge 0$, are defined in [1, p. 134, Theorem A] by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1}i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}. \tag{1.1}$$

The Faà di Bruno formula [1, p. 139, Theorem C] can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}f\circ h(x)=\sum_{k=0}^{n}f^{(k)}(h(x))\,\mathrm{B}_{n,k}\bigl(h'(x),h''(x),\dots,h^{(n-k+1)}(x)\bigr). \tag{1.2}$$

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By (1.1), we can easily deduce that, for $n \ge k \ge 0$,

$$\begin{aligned} \mathrm{B}_{n,k}(1,0,\dots,0) &= \mathrm{B}_{n,k}\left(\frac{\mathrm{d}}{\mathrm{d}x}x, \frac{\mathrm{d}^2}{\mathrm{d}x^2}x, \dots, \frac{\mathrm{d}^{n-k+1}}{\mathrm{d}x^{n-k+1}}x\right) \\ &= \binom{0}{n-k} = \begin{cases} 1, & n=k; \\ 0, & n \ne k. \end{cases} \end{aligned} \tag{1.3}$$

In [10, Theorem 5.1] and [16, Section 3], it was established that the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ satisfy

$$\mathrm{B}_{n,k}(x,1,0,\dots,0) = \frac{1}{2^{n-k}}\frac{n!}{k!}\binom{k}{n-k}x^{2k-n}, \tag{1.4}$$

where $n \ge k \ge 0$, $\binom{0}{0} = 1$, and $\binom{p}{q} = 0$ for $q > p \ge 0$. Since

$$\mathrm{B}_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n\, \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{1.5}$$

for $n \ge k \ge 0$, see [1, p. 135], we can rearrange (1.4) as

$$\begin{aligned} \mathrm{B}_{n,k}\left(\frac{\mathrm{d}}{\mathrm{d}x}x^2, \frac{\mathrm{d}^2}{\mathrm{d}x^2}x^2, \dots, \frac{\mathrm{d}^{n-k+1}}{\mathrm{d}x^{n-k+1}}x^2\right) &= \mathrm{B}_{n,k}(2x,2,0,\dots,0) \\ = 2^k\, \mathrm{B}_{n,k}(x,1,0,\dots,0) &= \frac{n!}{k!}\binom{k}{n-k}(2x)^{2k-n} \end{aligned}$$

for $n \ge k \ge 0$. This means that one can combine the Faà di Bruno formula (1.2) with the formula (1.4) to compute the nth derivative for functions of the type $f\big(ax^2+bx+c\big)$, such as

$$\begin{gathered} e^{\pm x^2}, \quad \sin\big(x^2\big), \quad \cos\big(x^2\big), \quad \ln\big(1 \pm x^2\big), \\ \big(1 \pm x^2\big)^\alpha, \quad \arcsin x, \quad \arccos x, \quad \arctan x, \end{gathered} \tag{1.6}$$

and to investigate the generating functions

$$\begin{gathered} \frac{1}{\sqrt{1-2xt+t^2}}, \quad \frac{1}{1-2xt+t^2}, \quad \frac{1}{\sqrt{1-6x+x^2}}, \\ \frac{1-x-\sqrt{x^2-6x+1}}{2x}, \quad \frac{1+x-\sqrt{x^2-6x+1}}{4}, \\ \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}, \quad e^{2xt-t^2}, \quad \frac{2e^z}{e^{2z}+1} \end{gathered}$$

of the Legendre polynomials, the Chebyshev polynomials of the second kind, central Delannoy numbers, the large and little Schröder numbers, the Motzkin numbers, the Hermite polynomials, the Euler numbers, and the Rodrigues formulas for the Chebyshev polynomials of the first and second kinds. This idea has been carried out and applied in [7, 8, 10, 14, 16] and closely related references therein.

Now it is natural and significant to ask the following question: how to compute the nth derivative of functions of the type $f(x^\alpha)$ for $\alpha \in \mathbb{R}$ and to apply the results? To answer such a question, by virtue of the Faà di Bruno formula (1.2), we need to calculate

$$\begin{aligned}
&\mathrm{B}_{n,k}\left(\frac{\mathrm{d}}{\mathrm{d}x}x^{\alpha},\frac{\mathrm{d}^2}{\mathrm{d}x^2}x^{\alpha},\dots,\frac{\mathrm{d}^{n-k+1}}{\mathrm{d}x^{n-k+1}}x^{\alpha}\right)\\
&=\mathrm{B}_{n,k}\big(\langle\alpha\rangle_1x^{\alpha-1},\langle\alpha\rangle_2x^{\alpha-2},\dots,\langle\alpha\rangle_{n-k+1}x^{\alpha-(n-k+1)}\big)\\
&=x^{k\alpha-n}\,\mathrm{B}_{n,k}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-k+1}),
\end{aligned}$$

where we used identity (1.5) and

$$\langle\alpha\rangle_n=\prod_{k=0}^{n-1}(\alpha-k)=\begin{cases}\alpha(\alpha-1)\cdots(\alpha-n+1), & n\ge1;\\ 1, & n=0\end{cases} \tag{1.7}$$

is called the falling factorial.

In recent years, the first author and his coauthors discovered and applied many closed-form expressions of special values for the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ in the papers [6, 7, 8, 9, 15] and closely related references therein.

In this paper, we will establish a pair of closed-form expressions for the Bell polynomials of the second kind $\mathrm{B}_{n,k}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-k+1})$, derive two pairs of identities involving the falling factorials $\langle\alpha\rangle_\ell$, find an equivalent expression of $\mathrm{B}_{n,k}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-k+1})$, and present five closed-form expressions for the (modified) spherical Bessel functions $\mathsf{j}_n(z)$, $\mathsf{y}_n(z)$, $\mathsf{i}_n^{(1)}(z)$, $\mathsf{i}_n^{(2)}(z)$, and $\mathsf{k}_n(z)$.

2. A pair of closed-form expressions for Bell polynomials

Now we state a pair of closed-form expressions for the Bell polynomials of the second kind $\mathrm{B}_{n,k}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-k+1})$ and its proof.

Theorem 2.1. *For $n\ge k\ge0$ and $\alpha\in\mathbb{R}$, the Bell polynomials of the second kind $\mathrm{B}_{n,k}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-k+1})$ can be computed by*

$$\mathrm{B}_{n,k}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-k+1})=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle\alpha\ell\rangle_n \tag{2.1}$$

and

$$\sum_{\ell=0}^{k}\frac{\mathrm{B}_{n,\ell}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-\ell+1})}{(k-\ell)!}=\frac{\langle\alpha k\rangle_n}{k!}. \tag{2.2}$$

Proof. In [1, p. 133], it is listed that

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty}x_m\frac{t^m}{m!}\right)^k=\sum_{n=k}^{\infty}\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})\frac{t^n}{n!} \tag{2.3}$$

for $k\ge0$. Taking $x_m=\langle\alpha\rangle_m x^{\alpha-m}$ for $\alpha\in\mathbb{R}$ in (2.3) leads to

$$
\begin{aligned}
\sum_{n=k}^{\infty}&\mathrm{B}_{n,k}\bigl(\langle\alpha\rangle_1x^{\alpha-1},\langle\alpha\rangle_2x^{\alpha-2},\dots,\langle\alpha\rangle_{n-k+1}x^{\alpha-(n-k+1)}\bigr)\frac{t^n}{n!}\\
&=\frac{x^{\alpha k}}{k!}\left[\sum_{m=1}^{\infty}\frac{\langle\alpha\rangle_m}{m!}\left(\frac{t}{x}\right)^m\right]^k=\frac{x^{\alpha k}}{k!}\left[\left(1+\frac{t}{x}\right)^\alpha-1\right]^k.
\end{aligned}
$$

Taking $a=x^\alpha$, $b=\frac{1}{x}$, and $x_i=\langle\alpha\rangle_i$ for $1\le i\le n-k+1$ in the identity (1.5), we obtain

$$
\begin{aligned}
\mathrm{B}_{n,k}&\bigl(\langle\alpha\rangle_1x^{\alpha-1},\langle\alpha\rangle_2x^{\alpha-2},\dots,\langle\alpha\rangle_{n-k+1}x^{\alpha-(n-k+1)}\bigr)\\
&=x^{k\alpha-n}\,\mathrm{B}_{n,k}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-k+1}).
\end{aligned}
$$

Consequently, it follows that

$$
\sum_{n=k}^{\infty}x^{k\alpha-n}\,\mathrm{B}_{n,k}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-k+1})\frac{t^n}{n!}=\frac{x^{\alpha k}}{k!}\left[\left(1+\frac{t}{x}\right)^\alpha-1\right]^k
$$

which can be rearranged as

$$
\begin{aligned}
\sum_{n=k}^{\infty}&\mathrm{B}_{n,k}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-k+1})\frac{s^n}{n!}=\frac{[(1+s)^\alpha-1]^k}{k!}\\
&=\frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}(1+s)^{\alpha\ell}=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}(1+s)^{\alpha\ell},
\end{aligned}
$$

where $s=\frac{t}{x}$. Differentiating $p\ge k\ge 0$ times with respect to s at both ends of the above equality gives

$$
\begin{aligned}
\sum_{n=k}^{\infty}&\mathrm{B}_{n,k}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-k+1})\frac{s^{n-p}}{(n-p)!}\\
&=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\langle\alpha\ell\rangle_p(1+s)^{\alpha\ell-p}.
\end{aligned}
$$

Further letting $s\to0$ on both sides of the above equality results in

$$
\mathrm{B}_{p,k}(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\dots,\langle\alpha\rangle_{n-k+1})=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\langle\alpha\ell\rangle_p.
$$

The formula (2.1) thus follows.

The binomial inversion theorem [1, pp. 143–144] reads

$$
s_n=\sum_{k=0}^{n}\binom{n}{k}S_k\quad\text{if and only if}\quad S_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}s_k \tag{2.4}
$$

for $n\ge0$, where $\{s_n,n\ge0\}$ and $\{S_n,n\ge0\}$ are sequences of complex numbers. Applying this theorem to (2.1) readily produces (2.2). The proof of Theorem 2.1 is complete. $\square$

Let

$$(x)_n=\prod_{\ell=0}^{n-1}(x+\ell)=\begin{cases}x(x+1)\cdots(x+n-1), & n\ge 1;\\ 1, & n=0\end{cases}$$

denote the rising factorial of $x\in\mathbb{R}$. Since

$$(-x)_n=(-1)^n\langle x\rangle_n \quad\text{and}\quad \langle -x\rangle_n=(-1)^n(x)_n,$$

we can now rewrite Theorem 2.1 in terms of the rising factorial $(x)_n$ as follows.

Corollary 2.1. *For $n\ge k\ge 0$ and $\alpha\in\mathbb{R}$, the Bell polynomials of the second kind $\mathrm{B}_{n,k}((\alpha)_1,(\alpha)_2,\dots,(\alpha)_{n-k+1})$ can be computed by*

$$\mathrm{B}_{n,k}((\alpha)_1,(\alpha)_2,\dots,(\alpha)_{n-k+1})=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}(\alpha\ell)_n$$

and

$$\sum_{\ell=0}^{k}\frac{\mathrm{B}_{n,\ell}((\alpha)_1,(\alpha)_2,\dots,(\alpha)_{n-\ell+1})}{(k-\ell)!}=\frac{(\alpha k)_n}{k!}.$$

3. Two pairs of identities involving falling factorials

We now recover formula (1.3) from Theorem 2.1 as follows. Replacing α by 1 in (2.1) leads to

$$\mathrm{B}_{n,k}(\langle 1\rangle_1,\langle 1\rangle_2,\dots,\langle 1\rangle_{n-k+1})=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle\ell\rangle_n$$

which is equivalent to

$$\begin{aligned}\mathrm{B}_{n,k}(1,0,\dots,0)&=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\sum_{m=0}^{n}s(n,m)\ell^m\\&=\sum_{m=0}^{n}s(n,m)\left[\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\ell^m\right]\\&=\sum_{m=0}^{n}s(n,m)S(m,k)=\binom{0}{n-k},\end{aligned}$$

where $s(n,k)$ and $S(n,k)$ denote the Stirling numbers of the first and second kinds and we used the formulas

$$\langle x\rangle_n=\sum_{k=0}^{n}s(n,k)x^k,\quad S(n,k)=\frac{1}{k!}\sum_{\ell=1}^{k}(-1)^{k-\ell}\binom{k}{\ell}\ell^n,$$

and

$$\sum_{\alpha=0}^{n}s(n,\alpha)S(\alpha,k)=\binom{0}{n-k}$$

in [1, p. 206] and [17, p. 171, Eq. (12.19)]. Formula (1.3) is thus recovered.

Now we state and prove two pairs of identities involving falling factorials.

Theorem 3.1. *For $n \ge k \ge 0$, we have*

$$\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\langle 2\ell\rangle_n = \frac{n!}{2^{n-2k}}\binom{k}{n-k} \tag{3.1}$$

and

$$\sum_{\ell=0}^{k}4^{\ell}\binom{k}{\ell}\binom{\ell}{n-\ell} = \frac{2^n}{n!}\langle 2k\rangle_n. \tag{3.2}$$

Proof. Replacing α by 2 in (2.1) and making use of (1.7) and (1.5) in sequence, we derive

$$\begin{aligned}\mathrm{B}_{n,k}(\langle 2\rangle_1, \langle 2\rangle_2, \dots, \langle 2\rangle_{n-k+1}) &= \mathrm{B}_{n,k}(2,2,0,\dots,0)\\ = 2^k\,\mathrm{B}_{n,k}(1,1,0,\dots,0) &= \frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\langle 2\ell\rangle_n.\end{aligned}$$

Comparing this with (1.4) for $x = 1$ yields (3.1).

Applying the binomial inversion theorem recited in (2.4) to (3.1) leads to (3.2). The proof of Theorem 3.1 is complete. □

Theorem 3.2. *For $n \ge k \ge 0$, we have*

$$\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\left\langle\frac{\ell}{2}\right\rangle_n = (-1)^n\frac{k![2(n-k)-1]!!}{2^n}\binom{2n-k-1}{2(n-k)} \tag{3.3}$$

and

$$\sum_{\ell=0}^{k}(-1)^{\ell}\frac{[2(n-\ell)-1]!!}{(k-\ell)!}\binom{2n-\ell-1}{2(n-\ell)} = (-1)^n\frac{2^n}{k!}\left\langle\frac{k}{2}\right\rangle_n, \tag{3.4}$$

where the double factorial of negative odd integers $-(2n+1)$ is defined by

$$(-2n-1)!! = \frac{(-1)^n}{(2n-1)!!} = (-1)^n\frac{2^n n!}{(2n)!}, \quad n \ge 0.$$

Proof. The first paragraph in [14, Theorem 4] reads that the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ satisfy

$$\begin{aligned}&\mathrm{B}_{n,k}\big(g'(x), g''(x), \dots, g^{(n-k+1)}(x)\big)\\ &\quad= (-1)^{n+k}[2(n-k)-1]!!\left(\frac{b}{2}\right)^n\binom{2n-k-1}{2(n-k)}\frac{1}{(a+bx)^{n-k/2}},\end{aligned} \tag{3.5}$$

where $n \in \mathbb{N}$ and $g(x) = \sqrt{a+bx}$ for $a, b \in \mathbb{R}$ and $b \ne 0$. Taking $a = 0$, $b = 1$, and $x \to 1$ in (3.5) results in

$$\begin{aligned}(3.6)\quad \mathrm{B}_{n,k}&\left(\left\langle\frac{1}{2}\right\rangle_1,\left\langle\frac{1}{2}\right\rangle_2,\dots,\left\langle\frac{1}{2}\right\rangle_{n-k+1}\right)\\ &= (-1)^{n+k}[2(n-k)-1]!!\left(\frac{1}{2}\right)^n\binom{2n-k-1}{2(n-k)}\end{aligned}$$

for $n\in\mathbb{N}$. On the other hand, taking $\alpha=\frac{1}{2}$ in (2.1) reduces to

$$\mathrm{B}_{n,k}\left(\left\langle\frac{1}{2}\right\rangle_1,\left\langle\frac{1}{2}\right\rangle_2,\dots,\left\langle\frac{1}{2}\right\rangle_{n-k+1}\right)=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\left\langle\frac{\ell}{2}\right\rangle_n.$$

Identity (3.3) thus follows.

Applying the above mentioned binomial inversion theorem [1, pp. 143–144] again to identity (3.3) gives

$$\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}(-1)^n\frac{\ell![2(n-\ell)-1]!!}{2^n}\binom{2n-\ell-1}{2(n-\ell)}=\left\langle\frac{k}{2}\right\rangle_n$$

which can be reformulated as identity (3.4). The proof of Theorem 3.2 is complete. □

4. An equivalent expression

Formula (2.1) in Theorem 2.1 has an equivalent expression.

Theorem 4.1. *For $n\ge k\ge 0$ and $\lambda\in\mathbb{R}$, formula* (2.1) *and*

$$\begin{aligned}(4.1)\quad \mathrm{B}_{n,k}&\left(1,1-\lambda,(1-\lambda)(1-2\lambda),\dots,\prod_{\ell=0}^{n-k}(1-\ell\lambda)\right)\\ &=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell-q\lambda)\end{aligned}$$

are equivalent to each other.

Proof. The closed-form expression (4.1) has been applied in [8, Lemma 2.2], [10, Remark 6.1], [11, Lemma 3], and [12, Lemma 2.6].

By identity (1.5), it is easy to see that

$$\begin{aligned}&\mathrm{B}_{n,k}\left(1,1-\lambda,(1-\lambda)(1-2\lambda),\dots,\prod_{\ell=0}^{n-k}(1-\ell\lambda)\right)\\ &=\mathrm{B}_{n,k}\left(\frac{1}{\lambda}\lambda,\frac{1}{\lambda}\lambda^2\left(\frac{1}{\lambda}-1\right),\frac{1}{\lambda}\lambda^3\left(\frac{1}{\lambda}-1\right)\left(\frac{1}{\lambda}-2\right),\right.\\ &\qquad\left.\dots,\frac{1}{\lambda}\lambda^{n-k+1}\prod_{\ell=1}^{n-k}\left(\frac{1}{\lambda}-\ell\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \lambda^n \mathrm{B}_{n,k}\left(\frac{1}{\lambda}, \frac{1}{\lambda}\left(\frac{1}{\lambda}-1\right), \frac{1}{\lambda}\left(\frac{1}{\lambda}-1\right)\left(\frac{1}{\lambda}-2\right), \dots, \prod_{\ell=0}^{n-k}\left(\frac{1}{\lambda}-\ell\right)\right) \\
&= \lambda^n \mathrm{B}_{n,k}\left(\left\langle\frac{1}{\lambda}\right\rangle_1, \left\langle\frac{1}{\lambda}\right\rangle_2, \left\langle\frac{1}{\lambda}\right\rangle_3, \dots, \left\langle\frac{1}{\lambda}\right\rangle_{n-k+1}\right).
\end{aligned}
$$

Substituting this into the closed-form expression (4.1) and simplifying yield

$$
\mathrm{B}_{n,k}\left(\left\langle\frac{1}{\lambda}\right\rangle_1, \left\langle\frac{1}{\lambda}\right\rangle_2, \left\langle\frac{1}{\lambda}\right\rangle_3, \dots, \left\langle\frac{1}{\lambda}\right\rangle_{n-k+1}\right) = \frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}\left(\frac{\ell}{\lambda}-q\right).
$$

Replacing $\frac{1}{\lambda}$ by α in the above equation leads to formula (2.1). Conversely, each equality and every step above are invertible. The proof of Theorem 4.1 is complete. □

5. Closed-form expressions for spherical Bessel functions

In [2, 3, 5, 18, 19] and closely related references therein, the nth derivatives of the functions

$$
\frac{1}{1-\lambda e^{\mu x}}, \quad \cot x, \quad \tan x, \quad \sec x, \quad \csc x, \quad \tanh x, \quad \coth x, \quad \operatorname{sech} x, \quad \operatorname{csch} x
$$

were computed by various approaches. If replacing x by x^α for $\alpha \in \mathbb{R}$ in these functions and those listed in (1.6), we obtain the functions

$$
\begin{gathered}
e^{\pm x^\alpha}, \quad \sin(x^\alpha), \quad \cos(x^\alpha), \quad \ln(1\pm x^\alpha), \quad (1\pm x^\alpha)^\beta, \quad \arcsin(x^\alpha), \\
\arccos(x^\alpha), \quad \arctan(x^\alpha), \quad \frac{1}{1-\lambda e^{\mu x^\alpha}}, \quad \cot(x^\alpha), \quad \tan(x^\alpha), \\
\sec(x^\alpha), \quad \csc(x^\alpha), \quad \tanh(x^\alpha), \quad \coth(x^\alpha), \quad \operatorname{sech}(x^\alpha), \quad \operatorname{csch}(x^\alpha).
\end{gathered}
$$

The nth derivatives for these functions and others of the type $f(x^\alpha)$ can be alternatively and explicitly calculated by combining the Faà di Bruno formula (1.2), the identity (2.1), and other techniques appeared in the above mentioned references.

In [4, p. 266, Section 10.56], it is listed that

$$
\begin{aligned}
\frac{\cos\sqrt{z^2-2zt}}{z} &= \frac{\cos z}{z} + \sum_{n=1}^{\infty}\frac{t^n}{n!}\mathsf{j}_{n-1}(z), \\
\frac{\sin\sqrt{z^2-2zt}}{z} &= \frac{\sin z}{z} + \sum_{n=1}^{\infty}\frac{t^n}{n!}\mathsf{y}_{n-1}(z), \\
\frac{\cosh\sqrt{z^2+2izt}}{z} &= \frac{\cosh z}{z} + \sum_{n=1}^{\infty}\frac{(it)^n}{n!}\mathsf{i}^{(1)}_{n-1}(z),
\end{aligned}
$$

$$\frac{\sinh\sqrt{z^2+2izt}}{z}=\frac{\sinh z}{z}+\sum_{n=1}^{\infty}\frac{(it)^n}{n!}\mathsf{i}_{n-1}^{(2)}(z),$$
$$\frac{\exp\left(-\sqrt{z^2+2izt}\,\right)}{z}=\frac{e^{-z}}{z}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-it)^n}{n!}\mathsf{k}_{n-1}(z)$$

for $2|t|<|z|$, where $\mathsf{j}_n(z)$ and $\mathsf{y}_n(z)$ denote the spherical Bessel functions of the first and second kinds and $\mathsf{i}_n^{(1)}(z)$, $\mathsf{i}_n^{(2)}(z)$, and $\mathsf{k}_n(z)$ denote the modified spherical Bessel functions.

Theorem 5.1. *For $n\in\mathbb{N}$ and $z\ne0$, we have*

(5.1)
$$\mathsf{j}_{n-1}(z)=(-1)^n\sum_{\ell=0}^{n}(-1)^\ell(2\ell-1)!!\binom{n+\ell-1}{2\ell}\cos\left(z+\frac{(n-\ell)\pi}{2}\right)\frac{1}{z^{\ell+1}},$$

(5.2)
$$\mathsf{y}_{n-1}(z)=(-1)^n\sum_{\ell=0}^{n}(-1)^\ell(2\ell-1)!!\binom{n+\ell-1}{2\ell}\sin\left(z+\frac{(n-\ell)\pi}{2}\right)\frac{1}{z^{\ell+1}},$$

$$\begin{aligned}\mathsf{i}_{n-1}^{(1)}(z)=&\sum_{\ell=0}^{n}(-1)^\ell(2\ell-1)!!\binom{n+\ell-1}{2\ell}\\&\times\left[\frac{1-(-1)^{n-\ell}}{2}\sinh z+\frac{1+(-1)^{n-\ell}}{2}\cosh z\right]\frac{1}{z^{\ell+1}},\end{aligned}\tag{5.3}$$

$$\begin{aligned}\mathsf{i}_{n-1}^{(2)}(z)=&\sum_{\ell=0}^{n}(-1)^\ell(2\ell-1)!!\binom{n+\ell-1}{2\ell}\\&\times\left[\frac{1-(-1)^{n-\ell}}{2}\cosh z+\frac{1+(-1)^{n-\ell}}{2}\sinh z\right]\frac{1}{z^{\ell+1}},\end{aligned}\tag{5.4}$$

$$\mathsf{k}_{n-1}(z)=(-1)^n\frac{\pi}{2}e^{-z}\sum_{\ell=0}^{n}(2\ell-1)!!\binom{n+\ell-1}{2\ell}\frac{1}{z^{\ell+1}}.\tag{5.5}$$

Proof. For $n\in\mathbb{N}$, by virtue of the formula (1.2), the identity (1.5), and the formula (3.6) in sequence, we have

$$\begin{aligned}\mathsf{j}_{n-1}(z)&=\frac{1}{z}\lim_{t\to0}\frac{\mathrm{d}^n\cos\sqrt{z^2-2zt}}{\mathrm{d}t^n}=\frac{1}{z}\lim_{t\to0}\sum_{k=0}^{n}(\cos u)^{(k)}\\&\times\mathrm{B}_{n,k}\Bigg((-2z)\left\langle\frac{1}{2}\right\rangle_1\left(z^2-2zt\right)^{-1/2},(-2z)^2\left\langle\frac{1}{2}\right\rangle_2\left(z^2-2zt\right)^{-3/2},\dots,\\&(-2z)^{n-k+1}\left\langle\frac{1}{2}\right\rangle_{n-k+1}\left(z^2-2zt\right)^{1/2-(n-k+1)}\Bigg)\\&=\frac{1}{z}\lim_{t\to0}\sum_{k=0}^{n}\cos\left(u+\frac{k\pi}{2}\right)(-2z)^n\left(z^2-2zt\right)^{k/2-n}\end{aligned}$$

$$
\begin{aligned}
&\times \mathrm{B}_{n,k}\left(\left\langle\frac{1}{2}\right\rangle_1,\left\langle\frac{1}{2}\right\rangle_2,\dots,\left\langle\frac{1}{2}\right\rangle_{n-k+1}\right)\\
&=\frac{1}{z}\sum_{k=0}^{n}\cos\left(z+\frac{k\pi}{2}\right)(-2z)^n z^{k-2n}\,\mathrm{B}_{n,k}\left(\left\langle\frac{1}{2}\right\rangle_1,\left\langle\frac{1}{2}\right\rangle_2,\dots,\left\langle\frac{1}{2}\right\rangle_{n-k+1}\right)\\
&=\frac{(-2)^n}{z^{n+1}}\sum_{k=0}^{n}(-1)^{n+k}[2(n-k)-1]!!\left(\frac{1}{2}\right)^n\binom{2n-k-1}{2(n-k)}\cos\left(z+\frac{k\pi}{2}\right)z^k\\
&=\frac{1}{z^{n+1}}\sum_{k=0}^{n}(-1)^{k}[2(n-k)-1]!!\binom{2n-k-1}{2(n-k)}\cos\left(z+\frac{k\pi}{2}\right)z^k,
\end{aligned}
$$

where $u=u_z(t)=\sqrt{z^2-2zt}$. Formula (5.1) is thus proved.

By the same arguments as above, we can derive the formulas (5.2), (5.3), (5.4), and (5.5) immediately. The proof Theorem 5.1 is complete. □

Remark 5.1. This paper is a revised version of the electronic preprint [13].

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