A NEIGHBORHOOD CONDITION FOR GRAPHS TO HAVE RESTRICTED FRACTIONAL \((g, f)\)-FACTORS

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Abstract. Let \(h\) be a function defined on \(E(G)\) with \(h(e) \in [0, 1]\) for any \(e \in E(G)\). Set \(d^h_G(x) = \sum_{e \ni x} h(e)\). If \(g(x) \leq d^h_G(x) \leq f(x)\) for every \(x \in V(G)\), then we call the graph \(F_h\) with vertex set \(V(G)\) and edge set \(E_h\) a fractional \((g, f)\)-factor of \(G\) with indicator function \(h\), where \(E_h = \{e : e \in E(G), h(e) > 0\}\). Let \(M\) and \(N\) be two sets of independent edges of \(G\) with \(M \cap N = \emptyset\), \(|M| = m\) and \(|N| = n\). If \(G\) admits a fractional \((g, f)\)-factor \(F_h\) such that \(h(e) = 1\) for any \(e \in M\) and \(h(e) = 0\) for any \(e \in N\), then we say that \(G\) has a fractional \((g, f)\)-factor with the property \(E(m, n)\). In this paper, we present a neighborhood condition for the existence of a fractional \((g, f)\)-factor with the property \(E(1, n)\) in a graph. Furthermore, it is shown that the neighborhood condition is sharp.

1. Introduction

The graphs considered here will be finite and undirected simple graphs. Let \(G\) be a graph. We denote by \(V(G)\) the vertex set of \(G\) and by \(E(G)\) the edge set of \(G\). For a vertex \(x\) of \(G\), we use \(d_G(x)\) to denote the degree of \(x\) in \(G\) and use \(N_G(x)\) to denote the neighborhood of \(x\) in \(G\). For a vertex subset \(X\) of \(G\), we denote by \(G[X]\) the subgraph of \(G\) induced by \(X\), and write \(G - X = G[V(G) \setminus X]\) and \(N_G(X) = \bigcup_{x \in X} N_G(x)\). If \(G[X]\) does not admit edges, then we call \(X\) an independent set of \(G\). For \(E' \subseteq E(G)\), the graph obtained from \(G\) by deleting edges of \(E'\) is denoted by \(G - E'\). The minimum degree of \(G\) is denoted by \(\delta(G)\). Let \(c\) be a real number. Recall that \(\lfloor c\rfloor\) is the greatest integer with \(\lfloor c\rfloor \leq c\).

Let \(g\) and \(f\) be two integer-valued functions defined on \(V(G)\) with \(0 \leq g(x) \leq f(x)\) for every \(x \in V(G)\). A \((g, f)\)-factor of a graph \(G\) is defined as a spanning subgraph \(F\) of \(G\) satisfying \(g(x) \leq d_F(x) \leq f(x)\) for any
\( x \in V(G) \). A \((g, f)\)-factor is called an \([a, b]\)-factor if \( g(x) \equiv a \) and \( f(x) \equiv b \). A \([k, k]\)-factor is simply called a \( k \)-factor.

Let \( h \) be a function defined on \( E(G) \) with \( h(e) \in [0, 1] \) for any \( e \in E(G) \). Set \( d^G_h(x) = \sum_{e \ni x} h(e) \). If \( g(x) \leq d^G_h(x) \leq f(x) \) for every \( x \in V(G) \), then we call the graph \( F_h \) with vertex set \( V(G) \) and edge set \( E_h \) a fractional \((g, f)\)-factor of \( G \) with indicator function \( h \), where \( E_h = \{ e : e \in E(G), h(e) > 0 \} \). A fractional \((g, f)\)-factor is called a fractional \( f \)-factor if \( g(x) = f(x) \) for each \( x \in V(G) \).

Let \( M \) and \( N \) be two sets of independent edges of \( G \) with \( M \cap N = \emptyset \), \(|M| = m\), and \(|N| = n\). If \( G \) admits a fractional \((g, f)\)-factor \( F_h \) such that \( h(e) = 1 \) for any \( e \in M \) and \( h(e) = 0 \) for any \( e \in N \), then we say that \( G \) has a fractional \((g, f)\)-factor with the property \( E(m, n) \). A fractional \((g, f)\)-factor with the property \( E(m, n) \) is called a fractional \( f \)-factor with the property \( E(m, n) \) if \( g(x) \equiv f(x) \). Similarly, we may define a \((g, f)\)-factor with the property \( E(m, n) \) of \( G \) and a \( f \)-factor with the property \( E(m, n) \) of \( G \).

Kano [5] showed a neighborhood condition for a graph to admit an \([a, b]\)-factor. Zhou [14] improved and generalized Kano’s result, and proved a theorem that is generally stronger than Kano’s result. Porteous and Aldred [11] first introduced the concept of \( 1 \)-factors with the property \( E(m, n) \), and obtained some results on the existence of \( 1 \)-factors with the property \( E(m, n) \) in graphs. Plummer and Saito [10] presented a binding number condition for the existence of \( 1 \)-factors with the property \( E(m, n) \) in graphs, and put forward a toughness condition for the existence of \( 1 \)-factors with the property \( E(m, n) \) in graphs. Zhou [17] and Zhou, Sun, and Pan [21] obtained two sufficient conditions for a graph to admit a fractional \((g, f)\)-factor with the property \( E(1, n) \). More results on factors and fractional factors in graphs can be found in Piummer [9], Zhou and Sun [19, 20], Wang and Zhang [12], Lv [8], Zhou [16, 18, 15, 13], Cai, Wang and Yan [1], Zhou, Yang and Xu [24], Zhou, Zhang and Xu [25], Gao et al. [2, 3, 4], Zhou, Xu and Sun [23], Zhou, Sun and Ye [22], and Liu and Lu [7].

In this paper, we proceed to study fractional \((g, f)\)-factors with the property \( E(m, n) \), and show a neighborhood condition that guarantees a graph admitting a fractional \((g, f)\)-factor with the property \( E(1, n) \).

**Theorem 1.1.** Let \( r \geq 0 \), \( n \geq 0 \) and \( 2 \leq a \leq b - r \) be four integers, let \( G \) be a graph of order \( p \) with

\[
p \geq \frac{(a + b - 1)(a + 2b - r - 4) + a + r + 2}{a + r} + \frac{2n}{a + r - 1},
\]

and let \( g, f \) be two integer-valued functions defined on \( V(G) \) such that \( a \leq g(x) \leq f(x) - r \leq b - r \) for every \( x \in V(G) \). Suppose for any subset \( X \subset V(G) \), we have

\[
N_G(X) = V(G), \quad \text{if} \quad |X| \geq \left\lfloor \frac{(a + r)(p - 1) - 2n - 2p}{(a + b - 1)(p - 1)} \right\rfloor;
\]
Let \( G \) and let \( g, f \) be two integer-valued functions defined on \( V(G) \) such that \( a \leq g(x) \leq f(x) \leq b - r \) for every \( x \in V(G) \). Suppose for any subset \( X \subset V(G) \), we have

\[
\begin{align*}
|N_G(X)| &\geq \frac{(a + b - 1)(p - 1)}{(a + r)(p - 1) - 2n - 2} |X|, \text{ if } n = 0 \\
|X| &< \frac{((a + r)(p - 1) - 2n - 2)p}{(a + b - 1)(p - 1)}.
\end{align*}
\]

Then \( G \) contains a fractional \((g, f)\)-factor with the property \( E(1, n) \).

If \( n = 0 \) in Theorem 1.1, then we have the following corollary.

**Corollary 1.2.** Let \( r \geq 0 \) and \( 2 \leq a \leq b - r \) be three integers, let \( G \) be a graph of order \( p \) with

\[
p \geq \frac{(a + b - 1)(a + 2b - r - 4) + a + r + 2}{a + r},
\]

and let \( g, f \) be two integer-valued functions defined on \( V(G) \) such that \( a \leq g(x) \leq f(x) \leq b - r \) for every \( x \in V(G) \). Suppose for any subset \( X \subset V(G) \), we have

\[
|N_G(X)| = V(G), \text{ if } |X| \geq \frac{((a + r)(p - 1) - 2)p}{(a + b - 1)(p - 1)}; \text{ or}
\]

\[
|N_G(X)| \geq \frac{(a + b - 1)(p - 1)}{(a + r)(p - 1) - 2} |X|, \text{ if } |X| < \frac{((a + r)(p - 1) - 2)p}{(a + b - 1)(p - 1)}.
\]

Then \( G \) contains a fractional \((g, f)\)-factor with the property \( E(1, 0) \).

If \( n = 1 \) in Theorem 1.1, then we have the following corollary.

**Corollary 1.3.** Let \( r \geq 0 \) and \( 2 \leq a \leq b - r \) be three integers, let \( G \) be a graph of order \( p \) with

\[
p \geq \frac{(a + b - 1)(a + 2b - r - 4) + a + r + 2}{a + r} + \frac{2}{a + r - 1},
\]

and let \( g, f \) be two integer-valued functions defined on \( V(G) \) such that \( a \leq g(x) \leq f(x) \leq b - r \) for every \( x \in V(G) \). Suppose for any subset \( X \subset V(G) \), we have

\[
|N_G(X)| = V(G), \text{ if } |X| \geq \frac{((a + r)(p - 1) - 4)p}{(a + b - 1)(p - 1)}; \text{ or}
\]

\[
|N_G(X)| \geq \frac{(a + b - 1)(p - 1)}{(a + r)(p - 1) - 4} |X|, \text{ if } |X| < \frac{((a + r)(p - 1) - 4)p}{(a + b - 1)(p - 1)}.
\]

Then \( G \) contains a fractional \((g, f)\)-factor with the property \( E(1, 1) \).

If \( r = 0 \) in Theorem 1.1, then we obtain the following corollary.

**Corollary 1.4.** Let \( n \geq 0 \) and \( 2 \leq a \leq b \) be three integers, let \( G \) be a graph of order \( p \) with

\[
p \geq \frac{(a + b - 1)(a + 2b - 4) + a + 2}{a} + \frac{2n}{a - 1},
\]
and let \( g, f \) be two integer-valued functions defined on \( V(G) \) such that \( a \leq g(x) \leq f(x) \leq b \) for every \( x \in V(G) \). Suppose for any subset \( X \subset V(G) \), we have
\[
N_G(X) = V(G), \quad \text{if } |X| \geq \left\lceil \frac{(a(p-1) - 2n - 2)p}{(a+b-1)(p-1)} \right\rceil; \quad \text{or}
\]
\[
|N_G(X)| \geq \frac{(a+b-1)(p-1)}{a(p-1) - 2n - 2}|X|, \quad \text{if } |X| < \left\lfloor \frac{(a(p-1) - 2n - 2)p}{(a+b-1)(p-1)} \right\rfloor.
\]
Then \( G \) contains a fractional \((g, f)\)-factor with the property \( E(1, n) \).

2. The proof of Theorem 1.1

The proof of Theorem 1.1 depends heavily on the following lemmas.

**Lemma 2.1** (Li, Yan, and Zhang [6]). Let \( G \) be a graph, and let \( g, f \) be two integer-valued functions defined on \( V(G) \) such that \( 0 \leq g(x) \leq f(x) \) for every \( x \in V(G) \). Then \( G \) has a fractional \((g, f)\)-factor with the property \( E(1, 0) \) if and only if
\[
\gamma_G(S, T) = f(S) + d_{G-S}(T) - g(T) \geq \varepsilon(S, T)
\]
for any \( S \subseteq V(G) \), where \( T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq g(x)\} \) and \( \varepsilon(S, T) \) is defined as follows:

\[
\varepsilon(S, T) = \begin{cases} 
2, & \text{if } S \text{ is not independent}, \\
1, & \text{if } S \text{ is independent and there is an edge joining } S \\
& \text{and } V(G) \setminus (S \cup T), \text{ or there is an edge } e = uv \\
& \text{joining } S \text{ and } T \text{ such that } d_{G-S}(v) = g(v) \text{ for } v \in T, \\
0, & \text{otherwise.}
\end{cases}
\]

**Lemma 2.2.** Let \( G \) be a graph of order \( p \) that satisfies the hypothesis of Theorem 1.1. Then
\[
\delta(G) \geq \frac{(b - r - 1)p + a + r + 2n + 2}{a + b - 1}.
\]

**Proof.** Let \( v \in V(G) \) with degree \( \delta(G) \). Let \( Q = V(G) \setminus N_G(v) \). Obviously, \( v \notin N_G(Q) \), that is, \( N_G(Q) \neq V(G) \). Thus we obtain
\[
(a + b - 1)(p - 1)|Q| \leq ((a + r)(p - 1) - 2n - 2)|N_G(Q)| \leq ((a + r)(p - 1) - 2n - 2)(p - 1),
\]
which implies
\[
(a + b - 1)|Q| \leq (a + r)(p - 1) - 2n - 2.
\]
Note that \( |Q| = p - \delta(G) \). Thus we have
\[
(a + b - 1)(p - \delta(G)) \leq (a + r)(p - 1) - 2n - 2,
\]
that is,
\[
\delta(G) \geq \frac{(b - r - 1)p + a + r + 2n + 2}{a + b - 1}.
\]
This finishes the proof of Lemma 2.2. \(\square\)

**Proof of Theorem 1.1.** Suppose that a graph \(G\) satisfies the hypothesis of Theorem 1.1, but does not have a fractional \((g, f)\)-factor with the property \(E(1, n)\). Then there exist a set of independent edges \(\{e_1, e_2, \ldots, e_n\}\) and an edge \(e\) of \(G\) such that \(G\) does not contain a fractional \((g, f)\)-factor \(F_h\) with \(h(e_i) = 0\) for \(1 \leq i \leq n\) and \(h(e) = 1\). Set \(N = \{e_1, e_2, \ldots, e_n\}\) and \(H = G - N\). Obviously, \(H\) does not have a fractional \((g, f)\)-factor with the property \(E(1, 0)\). By Lemma 2.1, there exists a subset \(S \subseteq V(H)\) such that
\[
\gamma_H(S, T) = f(S) + d_{H - S}(T) - g(T) \leq \varepsilon(S, T) - 1,
\]
where \(T = \{x : x \in V(H) \setminus S, d_{H - S}(x) \leq g(x)\}\). It is easy to see that \(T \neq \emptyset\) by (1). Hence, we may define
\[
h = \min\{d_{G - S}(x) : x \in T\}.
\]

**Claim 2.3.** \(0 \leq h \leq b - r + 1\).

**Proof.** Note that \(N = \{e_1, e_2, \ldots, e_n\}\) is a set of independent edges of \(G\) and \(H = G - N\). Combining these with the definition of \(T\), we have
\[
0 \leq d_{G - S}(x) \leq d_{H - S}(x) + 1 \leq g(x) + 1 \leq b - r + 1
\]
for each \(x \in T\). In terms of the definition of \(h\), we get \(0 \leq h \leq b - r + 1\). Claim 2.3 is proved. \(\square\)

**Claim 2.4.** \(d_{H - S}(T) \geq d_{G - S}(T) - \min\{2n, |T|\}\).

**Proof.** We write \(D = V(G) \setminus (S \cup T)\) and \(E_G(T) = \{e : e = xy \in E(G), x, y \in T\}\). It is obvious that \(2|N \cap E_G(T)| + |N \cap E_G(T, D)| \leq \min\{2n, |T|\}\). Thus, we have
\[
d_{H - S}(T) = d_{G - N - S}(T)
\]
\[
= d_{G - S}(T) - (2|N \cap E_G(T)| + |N \cap E_G(T, D)|)
\]
\[
\geq d_{G - S}(T) - \min\{2n, |T|\}.
\]
Claim 2.4 is verified. \(\square\)

We shall consider four cases.

**Case 1:** \(h = 0\).

Set \(\lambda = |\{x : x \in T, d_{G - S}(x) = 0\}|\). It is obvious that \(\lambda \geq 1\) by \(h = 0\). We write \(X = V(G) \setminus S\). Clearly, \(N_G(X) \neq V(G)\) since \(\lambda \geq 1\). It follows from the hypothesis of Theorem 1.1 that
\[
p - \lambda \geq |N_G(X)| \geq \frac{(a + b - 1)(p - 1)}{(a + r)(p - 1) - 2n - 2}|X|
\]
\[
= \frac{(a + b - 1)(p - 1)}{(a + r)(p - 1) - 2n - 2}(|X| - |S|),
\]
which implies
\[ |S| \geq p - \frac{(p-\lambda)((a+r)(p-1) - 2n - 2)}{(a+b-1)(p-1)}. \]  

(2)

It follows from (1), (2),
\[ p \geq \frac{(a+b-1)(a+2b - r - 4) + a + r + 2}{a + r} + \frac{2n}{a + r - 1}, \]

\( \lambda \geq 1, |S| + |T| \leq p, \) and Claim 2.4 that
\[ \varepsilon(S, T) - 1 \geq \gamma_H(S, T) = f(S) + d_{H-S}(T) - g(T) \]
\[ \geq f(S) + d_{G-S}(T) - \min\{2n, |T|\} - g(T) \]
\[ \geq (a+r)|S| + |T| - \lambda - 2n - (b-r)|T| \]
\[ = (a+r)|S| - (b-r-1)|T| - \lambda - 2n \]
\[ \geq (a+r)|S| - (b-r-1)(p - |S|) - \lambda - 2n \]
\[ = (a+b-1)|S| - (b-r-1)p - \lambda - 2n \]
\[ \geq (a+b-1)\left(p - \frac{(p-\lambda)((a+r)(p-1) - 2n - 2)}{(a+b-1)(p-1)}\right) \]
\[ - (b-r-1)p - \lambda - 2n \]
\[ = \frac{p(2n+2)}{p-1} + \left(\frac{(a+r)(p-1) - 2n - 2}{p-1} - 1\right)\lambda - 2n \]
\[ \geq \frac{p(2n+2)}{p-1} + \frac{(a+r)(p-1) - 2n - 2}{p-1} - 1 - 2n \]
\[ = a + r + 1 > 2 \geq \varepsilon(S, T), \]

which is a contradiction.

Case 2: \( h = 1. \)

Subcase 2.1: \( |T| > \left\lfloor \frac{(a+r)(p-1) - 2n - 2)p}{(a+b-1)(p-1)} \right\rfloor. \)

It is easy to see that
\[ |T| \geq \left\lfloor \frac{(a+r)(p-1) - 2n - 2)p}{(a+b-1)(p-1)} \right\rfloor + 1. \]  

(3)

Note that \( h = 1. \) Hence, there exists \( v \in T \) with \( d_{G-S}(v) = h = 1, \) and so
\[ v \notin N_G(T \setminus N_G(v)), \]

which implies
\[ N_G(T \setminus N_G(v)) \neq V(G). \]

Combining this with \( d_{G-S}(v) = h = 1 \) and the hypothesis of Theorem 1.1, we get
\[ |T| - 1 \leq |T \setminus N_G(v)| < \left\lfloor \frac{(a+r)(p-1) - 2n - 2)p}{(a+b-1)(p-1)} \right\rfloor, \]
that is,

\[ |T| < \left\lfloor \frac{(a + r)(p - 1) - 2n - 2)p}{(a + b - 1)(p - 1)} \right\rfloor + 1, \]

which contradicts (3).

**Subcase 2.2:**

\[ |T| \leq \left\lfloor \frac{(a + r)(p - 1) - 2n - 2)p}{(a + b - 1)(p - 1)} \right\rfloor. \]

In terms of \( h = 1 \) and Lemma 2.2, we obtain

\[ |S| \geq \delta(G) - 1 \geq (b - r - 1)(p - 1) + 2n + 2 \]

\[ \frac{a + r + 2n + 2}{a + b - 1}, \]

that is,

\[ |S| \geq \frac{(b - r - 1)(p - 1) + 2n + 2}{a + b - 1}. \]  \( \text{(4)} \)

**Claim 2.5.**

\[ |T| \leq \frac{(a + r)(p - 1) - 2n - 2}{a + b - 1}. \]

**Proof.** We assume that

\[ |T| > \frac{(a + r)(p - 1) - 2n - 2}{a + b - 1}. \]

In light of (4), we have

\[ |S| + |T| > \frac{(b - r - 1)(p - 1) + 2n + 2}{a + b - 1} + \frac{(a + r)(p - 1) - 2n - 2}{a + b - 1} = p - 1. \]

On the other hand, \( |S| + |T| \leq p \). Thus, we obtain

\[ |S| + |T| = p. \]  \( \text{(5)} \)

According to (4), (5),

\[ |T| \leq \left\lfloor \frac{(a + r)(p - 1) - 2n - 2)p}{(a + b - 1)(p - 1)} \right\rfloor \leq \frac{(a + r)(p - 1) - 2n - 2)p}{(a + b - 1)(p - 1)}, \]

and Claim 2.4, we obtain

\[ \gamma_H(S, T) = f(S) + d_{H-S}(T) - g(T) \]

\[ \geq f(S) + d_{G-S}(T) - \min\{2n, |T|\} - g(T) \]

\[ \geq (a + r)|S| + |T| - 2n - (b - r)|T| \]

\[ = (a + r)|S| - (b - r - 1)|T| - 2n \]

\[ = (a + r)(p - |T|) - (b - r - 1)|T| - 2n \]

\[ = (a + r)p - (a + b - 1)|T| - 2n \]

\[ \geq (a + r)p - (a + b - 1) \cdot \frac{(a + r)(p - 1) - 2n - 2)p}{(a + b - 1)(p - 1)} - 2n \]

\[ = (a + r)p - \left( (a + r)p - \frac{p(2n + 2)}{p - 1} \right) - 2n \]
which contradicts (1). Claim 2.5 is verified.

We write \( \beta = |\{x \in T : d_{G,T}(x) = 1\}| \). It is obvious that \( \beta \geq 1 \) and \( |T| \geq \beta \). In light of (4) and Claims 2.4–2.5, we have

\[
\gamma_H(S,T) = f(S) + d_{H,S}(T) - g(T)
\]

\[
\geq f(S) + d_{G,S}(T) - \min\{2n, |T|\} - g(T)
\]

\[
\geq (a + r)|S| + 2|T| - \beta - 2n - (b - r)|T|
\]

\[
= (a + r)|S| - (b - r - 2)|T| - \beta - 2n
\]

\[
\geq (a + r) \cdot \frac{(b - r - 1)(p - 1) + 2n + 2}{a + b - 1} - (b - r - 2) \cdot \frac{(a + r)(p - 1) - 2n - 2}{a + b - 1} - \beta - 2n
\]

\[
= \frac{(a + r)(p - 1) + (a + b - 2)(2n + 2)}{a + b - 1} - \beta - 2n
\]

\[
\geq |T| - \beta + 2 \geq 2 \geq \varepsilon(S,T),
\]

which contradicts (1).

Case 3: \( 2 \leq h \leq b - r \).

Note that \( h = \min\{d_{G,S}(x) : x \in T\} \). Then there exists \( x_1 \in T \) such that \( d_{G,S}(x_1) = h \). Hence, we obtain

\[
\delta(G) \leq d_G(x_1) \leq d_{G,S}(x_1) + |S| = h + |S|.
\]

Combining this with Lemma 2.2, we have

\[
|S| \geq \delta(G) - h \geq \frac{(b - r - 1)p + a + r + 2n + 2}{a + b - 1} - h.
\]

In terms of (6), \( |S| + |T| \leq p \), and Claim 2.4, we have

\[
\gamma_H(S,T) = f(S) + d_{H,S}(T) - g(T)
\]

\[
\geq f(S) + d_{G,S}(T) - \min\{2n, |T|\} - g(T)
\]

\[
\geq (a + r)|S| + h|T| - 2n - (b - r)|T|
\]

\[
= (a + r)|S| - (b - r - h)|T| - 2n
\]

\[
\geq (a + r)|S| - (b - r - h)(p - |S|) - 2n
\]

\[
= (a + b - h)|S| - (b - r - h)p - 2n
\]

\[
\geq (a + b - h) \cdot \left( \frac{(b - r - 1)p + a + r + 2n + 2}{a + b - 1} - h \right)
\]

\[-(b - r - h)p - 2n.\]
Let
\[ \varphi(h) = (a + b - h) \cdot \left( \frac{(b - r - 1)p + a + r + 2n + 2}{a + b - 1} - h \right) - (b - r - h)p - 2n. \]

Then we have
\[ \gamma_H(S, T) \geq \varphi(h). \quad (7) \]

By \(2 \leq h \leq b - r\) and
\[ p \geq \frac{(a + b - 1)(a + 2b - r - 4) + a + r + 2}{a + r} + \frac{2n}{a + r - 1}, \]
we get
\[ \varphi'(h) = -\left( \frac{(b - r - 1)p + a + r + 2n + 2}{a + b - 1} - h \right) - (a + b - h) + p \]
\[ = \frac{(a + r)p - a - r - 2n - 2}{a + b - 1} + 2h - (a + b) \]
\[ \geq a + 2b - r - 4 + 4 - (a + b) \]
\[ = b - r \geq a > 0, \]
and so, \(\varphi(h)\) attains its minimum value at \(h = 2\) by \(2 \leq h \leq b - r\). Combining this with (7) and
\[ p \geq \frac{(a + b - 1)(a + 2b - r - 4) + a + r + 2}{a + r} + \frac{2n}{a + r - 1}, \]
we obtain
\[ \gamma_H(S, T) \geq \varphi(h) \geq \varphi(2) \]
\[ = (a + b - 2) \cdot \left( \frac{(b - r - 1)p + a + r + 2n + 2}{a + b - 1} - 2 \right) \]
\[ - (b - r - 2)p - 2n \]
\[ = \frac{(a + r)p}{a + b - 1} + \frac{(a + b - 2)(a + r + 2n + 2)}{a + b - 1} \]
\[ - 2(a + b - 2) - 2n \]
\[ \geq \frac{(a + b - 1)(a + 2b - r - 4) + a + r + 2n + 2}{a + b - 1} \]
\[ + \frac{(a + b - 2)(a + r + 2n + 2)}{a + b - 1} - 2(a + b - 2) - 2n \]
\[ = 2 \geq \varepsilon(S, T), \]
which contradicts (1).

Case 4: \(h = b - r + 1\).
It follows from (1) and Claim 2.4 that
\[
\varepsilon(S,T) - 1 \geq \gamma_H(S,T) = f(S) + d_{H-S}(T) - g(T)
\geq f(S) + d_{G-S}(T) - \min\{2n,|T|\} - g(T)
\geq (a + r)|S| + h|T| - |T| - (b - r)|T|
= (a + r)|S| - (b - r + 1 - h)|T|
= (a + r)|S| \geq |S| \geq \varepsilon(S,T),
\]
which is a contradiction. This completes the proof of Theorem 1.1. \hfill \square

3. Remark

In this section, we claim that the assumption on the neighborhood in Theorem 1.1 is the best possible, which cannot be replaced by \(N_G(X) = V(G)\) or
\[
|N_G(X)| \geq \frac{(a + b - 1)(p - 1)}{(a + r)(p - 1) - 2n - 2}|X|
\]
for any \(X \subseteq V(G)\).

Let \(a, b, r, n\) be four nonnegative integers such that \(2 \leq a = b - r\), \(b\) is odd and \((2n + 2)/b\) is an integer. We construct a graph
\[
G = K_{(a-1)m + \frac{2n+2}{b}} \vee \left( \frac{bm + 1}{2} K_2 \right)
\]
of order \(p\), where \(m\) is an enough large positive integer, \(m\) is odd, and \(\vee\) means “join”. It is easy to see that \(p = (a - 1)m + bm + 1 + ((2n + 2)/b)\).

We write
\[
S = V\left( K_{(a-1)m + \frac{2n+2}{b}} \right)
\]
and
\[
T = V\left( \frac{bm + 1}{2} K_2 \right).
\]

We first show that the assumption \(N_G(X) = V(G)\) or
\[
|N_G(X)| \geq \frac{(a + b - 1)(p - 1)}{(a + r)(p - 1) - 2n - 2}|X|
\]
for any \(X \subseteq V(G)\) holds. Let any \(X \subseteq V(G)\). It is easy to see that if \(|X \cap S| \geq 2\), or \(|X \cap S| = 1\) and \(|X \cap T| \geq 1\), then \(N_G(X) = V(G)\). Of course, if \(|X| = 1\) and \(X \subseteq S\), then we easily obtain
\[
|N_G(X)| = |V(G)| - 1 = p - 1 > \frac{(a + b - 1)(p - 1)}{(a + r)(p - 1) - 2n - 2}|X|.
\]

Therefore, we may assume that \(X \subseteq T\). Note that
\[
|N_G(X)| = |S| + |X| = (a - 1)m + \frac{2n + 2}{b} + |X|.
\]
Hence,
\[
|N_G(X)| \geq \frac{(a + b - 1)(p - 1)}{(a + r)(p - 1) - 2n - 2}|X|
\]
holds if and only if
\[(a-1)m + \frac{2n+2}{b} + |X| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|.
\]
This inequality is equivalent to \(|X| \leq bm\). Thus if \(X \neq T\) and \(X \subset T\), then we have
\[|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|\]
for any \(X \subset V(G)\). If \(X = T\), then it is obvious that \(N_G(X) = V(G)\). Consequently, \(N_G(X) = V(G)\) or
\[|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|\]
for any \(X \subseteq V(G)\) holds.

Let \(g, f\) be two integer-valued functions defined on \(V(G)\) with \(g(x) = a\) and \(f(x) = b = a + r\) for every \(x \in V(G)\). Let \(N = \{e_1, e_2, \ldots, e_n\} \subseteq E(K_2)\) be a set of independent edges of \(G\). We write \(H = G - N\). Next, we show that \(H\) does not have a fractional \((g, f)\)-factor with the property \(E(1, 0)\). For above \(S\) and \(T\), we have \(|S| = (a-1)m + ((2n+2)/b)\), \(|T| = bm + 1\), \(d_{H-S}(T) = bm + 1 - 2n\) and \(\varepsilon(S, T) = 2\). Thus, we have
\[
\gamma_H(S, T) = f(S) + d_{H-S}(T) - g(T)
= b|S| + d_{H-S}(T) - a|T|
= b \cdot \left((a-1)m + \frac{2n+2}{b}\right) + bm + 1 - 2n - a \cdot (bm + 1)
= 3 - a \leq 1 < 2 = \varepsilon(S, T).
\]
In light of Lemma 2.1, \(H\) does not admit a fractional \((g, f)\)-factor with the property \(E(1, 0)\), and so, \(G\) does not admit a fractional \((g, f)\)-factor with the property \(E(1, n)\).

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**References**

5. M. Kano, *A sufficient condition for a graph to have \([a, b]\)-factors*, Graphs Combin. 6 (1990), 245–251.
6. Z. Li, G. Yan, and X. Zhang, *On fractional $(g, f)$-covered graphs*, OR Transactions (China) 6 (2002), no. 4, 65–68.

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