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A NEIGHBORHOOD CONDITION FOR GRAPHS TO HAVE RESTRICTED FRACTIONAL (g, f)-FACTORS

SIZHONG ZHOU AND ZHIREN SUN

Abstract. Let h be a function defined on $E(G)$ with $h(e) \in [0,1]$ for any $e \in E(G)$. Set $d_G^h(x) = \sum_{e \ni x} h(e)$. If $g(x) \leq d_G^h(x) \leq f(x)$ for every $x \in V(G)$, then we call the graph F_h with vertex set $V(G)$ and edge set E_h a fractional (g, f)-factor of G with indicator function h, where $E_h = \{e : e \in E(G), h(e) > 0\}$. Let M and N be two sets of independent edges of G with $M \cap N = \emptyset$, $|M| = m$ and $|N| = n$. If G admits a fractional (g, f)-factor F_h such that $h(e) = 1$ for any $e \in M$ and $h(e) = 0$ for any $e \in N$, then we say that G has a fractional (g, f)-factor with the property $E(m, n)$. In this paper, we present a neighborhood condition for the existence of a fractional (g, f)-factor with the property $E(1, n)$ in a graph. Furthermore, it is shown that the neighborhood condition is sharp.

1. Introduction

The graphs considered here will be finite and undirected simple graphs. Let G be a graph. We denote by $V(G)$ the vertex set of G and by $E(G)$ the edge set of G. For a vertex x of G, we use $d_G(x)$ to denote the degree of x in G and use $N_G(x)$ to denote the neighborhood of x in G. For a vertex subset X of G, we denote by $G[X]$ the subgraph of G induced by X, and write $G - X = G[V(G) \setminus X]$ and $N_G(X) = \bigcup_{x \in X} N_G(x)$. If $G[X]$ does not admit edges, then we call X an independent set of G. For $E' \subseteq E(G)$, the graph obtained from G by deleting edges of E' is denoted by $G - E'$ The minimum degree of G is denoted by $\delta(G)$. Let c be a real number. Recall that $\lfloor c \rfloor$ is the greatest integer with $\lfloor c \rfloor \leq c$.

Let g and f be two integer-valued functions defined on $V(G)$ with $0 \leq g(x) \leq f(x)$ for every $x \in V(G)$. A (g, f)-factor of a graph G is defined as a spanning subgraph F of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for any

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$x \in V(G)$. A (g, f)-factor is called an $[a, b]$-factor if $g(x) \equiv a$ and $f(x) \equiv b$. A $[k, k]$-factor is simply called a k-factor.

Let h be a function defined on $E(G)$ with $h(e) \in [0, 1]$ for any $e \in E(G)$. Set $d_G^h(x) = \sum_{e \ni x} h(e)$. If $g(x) \leq d_G^h(x) \leq f(x)$ for every $x \in V(G)$, then we call the graph F_h with vertex set $V(G)$ and edge set E_h a fractional (g, f)-factor of G with indicator function h, where $E_h = \{e : e \in E(G), h(e) > 0\}$. A fractional (g, f)-factor is called a fractional f-factor if $g(x) = f(x)$ for each $x \in V(G)$.

Let M and N be two sets of independent edges of G with $M \cap N = \emptyset$, $|M| = m$, and $|N| = n$. If G admits a fractional (g, f)-factor F_h such that $h(e) = 1$ for any $e \in M$ and $h(e) = 0$ for any $e \in N$, then we say that G has a fractional (g, f)-factor with the property $E(m, n)$. A fractional (g, f)-factor with the property $E(m, n)$ is called a fractional f-factor with the property $E(m, n)$ if $g(x) \equiv f(x)$. Similarly, we may define a (g, f)-factor with the property $E(m, n)$ of G and an f-factor with the property $E(m, n)$ of G.

Kano [5] showed a neighborhood condition for a graph to admit an $[a, b]$-factor. Zhou [14] improved and generalized Kano's result, and proved a theorem that is generally stronger than Kano's result. Porteous and Aldred [11] first introduced the concept of 1-factors with the property $E(m, n)$, and obtained some results on the existence of 1-factors with the property $E(m, n)$ in graphs. Plummer and Saito [10] presented a binding number condition for the existence of 1-factors with the property $E(m, n)$ in graphs, and put forward a toughness condition for the existence of 1-factors with the property $E(m, n)$ in graphs. Zhou [17] and Zhou, Sun, and Pan [21] obtained two sufficient conditions for a graph to admit a fractional (g, f)-factor with the property $E(1, n)$. More results on factors and fractional factors in graphs can be found in Piummer [9], Zhou and Sun [19, 20], Wang and Zhang [12], Lv [8], Zhou [16, 18, 15, 13], Cai, Wang and Yan [1], Zhou, Yang and Xu [24], Zhou, Zhang and Xu [25], Gao et al. [2, 3, 4], Zhou, Xu and Sun [23], Zhou, Sun and Ye [22], and Liu and Lu [7].

In this paper, we proceed to study fractional (g, f)-factors with the property $E(m, n)$, and show a neighborhood condition that guarantees a graph admitting a fractional (g, f)-factor with the property $E(1, n)$.

Theorem 1.1. *Let $r \geq 0$, $n \geq 0$ and $2 \leq a \leq b - r$ be four integers, let G be a graph of order p with*

$$p \geq \frac{(a + b - 1)(a + 2b - r - 4) + a + r + 2}{a + r} + \frac{2n}{a + r - 1},$$

and let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - r \leq b - r$ for every $x \in V(G)$. Suppose for any subset $X \subset V(G)$, we have

$$N_G(X) = V(G), \text{ if } |X| \geq \left\lfloor \frac{((a + r)(p - 1) - 2n - 2)p}{(a + b - 1)(p - 1)} \right\rfloor;$$

or

$$|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|, \text{ if } |X| < \left\lfloor \frac{((a+r)(p-1)-2n-2)p}{(a+b-1)(p-1)} \right\rfloor.$$

Then G contains a fractional (g,f)-factor with the property $E(1,n)$.

If $n=0$ in Theorem 1.1, then we have the following corollary.

Corollary 1.2. *Let $r \geq 0$ and $2 \leq a \leq b-r$ be three integers, let G be a graph of order p with*

$$p \geq \frac{(a+b-1)(a+2b-r-4)+a+r+2}{a+r},$$

and let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x)-r \leq b-r$ for every $x \in V(G)$. Suppose for any subset $X \subset V(G)$, we have

$$N_G(X) = V(G), \text{ if } |X| \geq \left\lfloor \frac{((a+r)(p-1)-2)p}{(a+b-1)(p-1)} \right\rfloor; \text{ or}$$

$$|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2}|X|, \text{ if } |X| < \left\lfloor \frac{((a+r)(p-1)-2)p}{(a+b-1)(p-1)} \right\rfloor.$$

Then G contains a fractional (g,f)-factor with the property $E(1,0)$.

If $n=1$ in Theorem 1.1, then we have the following corollary.

Corollary 1.3. *Let $r \geq 0$ and $2 \leq a \leq b-r$ be three integers, let G be a graph of order p with*

$$p \geq \frac{(a+b-1)(a+2b-r-4)+a+r+2}{a+r} + \frac{2}{a+r-1},$$

and let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x)-r \leq b-r$ for every $x \in V(G)$. Suppose for any subset $X \subset V(G)$, we have

$$N_G(X) = V(G), \text{ if } |X| \geq \left\lfloor \frac{((a+r)(p-1)-4)p}{(a+b-1)(p-1)} \right\rfloor; \text{ or}$$

$$|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-4}|X|, \text{ if } |X| < \left\lfloor \frac{((a+r)(p-1)-4)p}{(a+b-1)(p-1)} \right\rfloor.$$

Then G contains a fractional (g,f)-factor with the property $E(1,1)$.

If $r=0$ in Theorem 1.1, then we obtain the following corollary.

Corollary 1.4. *Let $n \geq 0$ and $2 \leq a \leq b$ be three integers, let G be a graph of order p with*

$$p \geq \frac{(a+b-1)(a+2b-4)+a+2}{a} + \frac{2n}{a-1},$$

and let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for every $x \in V(G)$. Suppose for any subset $X \subset V(G)$, we have

$$N_G(X) = V(G), \text{ if } |X| \geq \left\lfloor \frac{(a(p-1)-2n-2)p}{(a+b-1)(p-1)} \right\rfloor; \text{ or}$$

$$|N_G(X)| \geq \frac{(a+b-1)(p-1)}{a(p-1)-2n-2}|X|, \text{ if } |X| < \left\lfloor \frac{(a(p-1)-2n-2)p}{(a+b-1)(p-1)} \right\rfloor.$$

Then G contains a fractional (g, f)-factor with the property $E(1, n)$.

2. The proof of Theorem 1.1

The proof of Theorem 1.1 depends heavily on the following lemmas.

Lemma 2.1 (Li, Yan, and Zhang [6])**.** *Let G be a graph, and let g, f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for every $x \in V(G)$. Then G has a fractional (g, f)-factor with the property $E(1, 0)$ if and only if*

$$\gamma_G(S,T) = f(S) + d_{G-S}(T) - g(T) \geq \varepsilon(S,T)$$

for any $S \subseteq V(G)$, where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq g(x)\}$ and $\varepsilon(S,T)$ is defined as follows:

$$\varepsilon(S,T) = \begin{cases} 2, & \text{if } S \text{ is not independent,} \\ 1, & \text{if } S \text{ is independent and there is an edge joining } S \\ & \text{and } V(G) \setminus (S \cup T), \text{ or there is an edge } e = uv \\ & \text{joining } S \text{ and } T \text{ such that } d_{G-S}(v) = g(v) \text{ for } v \in T, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.2. *Let G be a graph of order p that satisfies the hypothesis of Theorem 1.1. Then*

$$\delta(G) \geq \frac{(b-r-1)p + a + r + 2n + 2}{a+b-1}.$$

Proof. Let $v \in V(G)$ with degree $\delta(G)$. Let $Q = V(G) \setminus N_G(v)$. Obviously, $v \notin N_G(Q)$, that is, $N_G(Q) \neq V(G)$. Thus we obtain

$$\begin{aligned} (a+b-1)(p-1)|Q| &\leq ((a+r)(p-1)-2n-2)|N_G(Q)| \\ &\leq ((a+r)(p-1)-2n-2)(p-1), \end{aligned}$$

which implies

$$(a+b-1)|Q| \leq (a+r)(p-1) - 2n - 2.$$

Note that $|Q| = p - \delta(G)$. Thus we have

$$(a+b-1)(p-\delta(G)) \leq (a+r)(p-1) - 2n - 2,$$

that is,

$$\delta(G) \geq \frac{(b-r-1)p+a+r+2n+2}{a+b-1}.$$

This finishes the proof of Lemma 2.2. □

Proof of Theorem 1.1. Suppose that a graph G satisfies the hypothesis of Theorem 1.1, but does not have a fractional (g,f)-factor with the property $E(1,n)$. Then there exist a set of independent edges $\{e_1,e_2,\ldots,e_n\}$ and an edge e of G such that G does not contain a fractional (g,f)-factor F_h with $h(e_i)=0$ for $1\leq i\leq n$ and $h(e)=1$. Set $N=\{e_1,e_2,\ldots,e_n\}$ and $H=G-N$. Obviously, H does not have a fractional (g,f)-factor with the property $E(1,0)$. By Lemma 2.1, there exists a subset $S\subseteq V(H)$ such that

$$\gamma_H(S,T)=f(S)+d_{H-S}(T)-g(T)\leq \varepsilon(S,T)-1, \tag{1}$$

where $T=\{x: x\in V(H)\setminus S, d_{H-S}(x)\leq g(x)\}$. It is easy to see that $T\neq\emptyset$ by (1). Hence, we may define

$$h=\min\{d_{G-S}(x): x\in T\}.$$

Claim 2.3. $0\leq h\leq b-r+1$.

Proof. Note that $N=\{e_1,e_2,\ldots,e_n\}$ is a set of independent edges of G and $H=G-N$. Combining these with the definition of T, we have

$$0\leq d_{G-S}(x)\leq d_{H-S}(x)+1\leq g(x)+1\leq b-r+1$$

for each $x\in T$. In terms of the definition of h, we get $0\leq h\leq b-r+1$. Claim 2.3 is proved. □

Claim 2.4. $d_{H-S}(T)\geq d_{G-S}(T)-\min\{2n,|T|\}$.

Proof. We write $D=V(G)\setminus(S\cup T)$ and $E_G(T)=\{e: e=xy\in E(G), x,y\in T\}$. It is obvious that $2|N\cap E_G(T)|+|N\cap E_G(T,D)|\leq\min\{2n,|T|\}$. Thus, we have

$$\begin{aligned} d_{H-S}(T) &= d_{G-N-S}(T)\\ &= d_{G-S}(T)-(2|N\cap E_G(T)|+|N\cap E_G(T,D)|)\\ &\geq d_{G-S}(T)-\min\{2n,|T|\}. \end{aligned}$$

Claim 2.4 is verified. □

We shall consider four cases.

Case 1: $h=0$.

Set $\lambda=|\{x: x\in T,\ d_{G-S}(x)=0\}|$. It is obvious that $\lambda\geq 1$ by $h=0$. We write $X=V(G)\setminus S$. Clearly, $N_G(X)\neq V(G)$ since $\lambda\geq 1$. It follows from the hypothesis of Theorem 1.1 that

$$\begin{aligned} p-\lambda &\geq |N_G(X)|\geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|\\ &= \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}(p-|S|), \end{aligned}$$

which implies

$$|S| \geq p - \frac{(p-\lambda)((a+r)(p-1)-2n-2)}{(a+b-1)(p-1)}. \tag{2}$$

It follows from (1), (2),

$$p \geq \frac{(a+b-1)(a+2b-r-4)+a+r+2}{a+r} + \frac{2n}{a+r-1},$$

$\lambda \geq 1$, $|S|+|T| \leq p$, and Claim 2.4 that

$$\begin{aligned}
\varepsilon(S,T)-1 &\geq \gamma_H(S,T) = f(S) + d_{H-S}(T) - g(T) \\
&\geq f(S) + d_{G-S}(T) - \min\{2n, |T|\} - g(T) \\
&\geq (a+r)|S| + |T| - \lambda - 2n - (b-r)|T| \\
&= (a+r)|S| - (b-r-1)|T| - \lambda - 2n \\
&\geq (a+r)|S| - (b-r-1)(p-|S|) - \lambda - 2n \\
&= (a+b-1)|S| - (b-r-1)p - \lambda - 2n \\
&\geq (a+b-1)\Big(p - \frac{(p-\lambda)((a+r)(p-1)-2n-2)}{(a+b-1)(p-1)}\Big) \\
&\quad -(b-r-1)p - \lambda - 2n \\
&= \frac{p(2n+2)}{p-1} + \Big(\frac{(a+r)(p-1)-2n-2}{p-1} - 1\Big)\lambda - 2n \\
&\geq \frac{p(2n+2)}{p-1} + \frac{(a+r)(p-1)-2n-2}{p-1} - 1 - 2n \\
&= a+r+1 > 2 \geq \varepsilon(S,T),
\end{aligned}$$

which is a contradiction.

Case 2: $h = 1$.

Subcase 2.1: $|T| > \left\lfloor \frac{((a+r)(p-1)-2n-2)p}{(a+b-1)(p-1)} \right\rfloor$.

It is easy to see that

$$|T| \geq \left\lfloor \frac{((a+r)(p-1)-2n-2)p}{(a+b-1)(p-1)} \right\rfloor + 1. \tag{3}$$

Note that $h = 1$. Hence, there exists $v \in T$ with $d_{G-S}(v) = h = 1$, and so

$$v \notin N_G(T \setminus N_G(v)),$$

which implies

$$N_G(T \setminus N_G(v)) \neq V(G).$$

Combining this with $d_{G-S}(v) = h = 1$ and the hypothesis of Theorem 1.1, we get

$$|T| - 1 \leq |T \setminus N_G(v)| < \left\lfloor \frac{((a+r)(p-1)-2n-2)p}{(a+b-1)(p-1)} \right\rfloor,$$

that is,

$$|T| < \left\lfloor \frac{((a+r)(p-1)-2n-2)p}{(a+b-1)(p-1)} \right\rfloor + 1,$$

which contradicts (3).

Subcase 2.2: $|T| \leq \left\lfloor \frac{((a+r)(p-1)-2n-2)p}{(a+b-1)(p-1)} \right\rfloor$.

In terms of $h = 1$ and Lemma 2.2, we obtain

$$\begin{aligned} |S| &\geq \delta(G) - 1 \geq \frac{(b-r-1)p + a + r + 2n + 2}{a+b-1} - 1 \\ &= \frac{(b-r-1)(p-1) + 2n + 2}{a+b-1}, \end{aligned}$$

that is,

$$|S| \geq \frac{(b-r-1)(p-1) + 2n + 2}{a+b-1}. \tag{4}$$

Claim 2.5. $|T| \leq \frac{(a+r)(p-1)-2n-2}{a+b-1}$.

Proof. We assume that

$$|T| > \frac{(a+r)(p-1)-2n-2}{a+b-1}.$$

In light of (4), we have

$$|S| + |T| > \frac{(b-r-1)(p-1)+2n+2}{a+b-1} + \frac{(a+r)(p-1)-2n-2}{a+b-1} = p - 1.$$

On the other hand, $|S| + |T| \leq p$. Thus, we obtain

$$|S| + |T| = p. \tag{5}$$

According to (4), (5),

$$|T| \leq \left\lfloor \frac{((a+r)(p-1)-2n-2)p}{(a+b-1)(p-1)} \right\rfloor \leq \frac{((a+r)(p-1)-2n-2)p}{(a+b-1)(p-1)},$$

and Claim 2.4, we obtain

$$\begin{aligned} \gamma_H(S,T) &= f(S) + d_{H-S}(T) - g(T) \\ &\geq f(S) + d_{G-S}(T) - \min\{2n, |T|\} - g(T) \\ &\geq (a+r)|S| + |T| - 2n - (b-r)|T| \\ &= (a+r)|S| - (b-r-1)|T| - 2n \\ &= (a+r)(p-|T|) - (b-r-1)|T| - 2n \\ &= (a+r)p - (a+b-1)|T| - 2n \\ &\geq (a+r)p - (a+b-1) \cdot \frac{((a+r)(p-1)-2n-2)p}{(a+b-1)(p-1)} - 2n \\ &= (a+r)p - \left((a+r)p - \frac{p(2n+2)}{p-1}\right) - 2n \end{aligned}$$

$$= \frac{p(2n+2)}{p-1} - 2n$$
$$> 2n + 2 - 2n = 2 \geq \varepsilon(S,T),$$

which contradicts (1). Claim 2.5 is verified. □

We write $\beta = |\{x : x \in T,\ d_{G-S}(x) = 1\}|$. It is obvious that $\beta \geq 1$ and $|T| \geq \beta$. In light of (4) and Claims 2.4–2.5, we have

$$\begin{aligned}
\gamma_H(S,T) &= f(S) + d_{H-S}(T) - g(T) \\
&\geq f(S) + d_{G-S}(T) - \min\{2n, |T|\} - g(T) \\
&\geq (a+r)|S| + 2|T| - \beta - 2n - (b-r)|T| \\
&= (a+r)|S| - (b-r-2)|T| - \beta - 2n \\
&\geq (a+r) \cdot \frac{(b-r-1)(p-1)+2n+2}{a+b-1} \\
&\quad -(b-r-2) \cdot \frac{(a+r)(p-1)-2n-2}{a+b-1} - \beta - 2n \\
&= \frac{(a+r)(p-1)+(a+b-2)(2n+2)}{a+b-1} - \beta - 2n \\
&= \frac{(a+r)(p-1)-2n-2}{a+b-1} + 2n + 2 - \beta - 2n \\
&\geq |T| - \beta + 2 \geq 2 \geq \varepsilon(S,T),
\end{aligned}$$

which contradicts (1).

Case 3: $2 \leq h \leq b-r$.

Note that $h = \min\{d_{G-S}(x) : x \in T\}$. Then there exists $x_1 \in T$ such that $d_{G-S}(x_1) = h$. Hence, we obtain

$$\delta(G) \leq d_G(x_1) \leq d_{G-S}(x_1) + |S| = h + |S|.$$

Combining this with Lemma 2.2, we have

$$|S| \geq \delta(G) - h \geq \frac{(b-r-1)p+a+r+2n+2}{a+b-1} - h. \tag{6}$$

In terms of (6), $|S| + |T| \leq p$, and Claim 2.4, we have

$$\begin{aligned}
\gamma_H(S,T) &= f(S) + d_{H-S}(T) - g(T) \\
&\geq f(S) + d_{G-S}(T) - \min\{2n, |T|\} - g(T) \\
&\geq (a+r)|S| + h|T| - 2n - (b-r)|T| \\
&= (a+r)|S| - (b-r-h)|T| - 2n \\
&\geq (a+r)|S| - (b-r-h)(p-|S|) - 2n \\
&= (a+b-h)|S| - (b-r-h)p - 2n \\
&\geq (a+b-h) \cdot \left(\frac{(b-r-1)p+a+r+2n+2}{a+b-1} - h\right) \\
&\quad -(b-r-h)p - 2n.
\end{aligned}$$

Let

$$\varphi(h)=(a+b-h)\cdot\Big(\frac{(b-r-1)p+a+r+2n+2}{a+b-1}-h\Big)-(b-r-h)p-2n.$$

Then we have

$$\gamma_H(S,T)\geq\varphi(h). \tag{7}$$

By $2\leq h\leq b-r$ and

$$p\geq\frac{(a+b-1)(a+2b-r-4)+a+r+2}{a+r}+\frac{2n}{a+r-1},$$

we get

$$\begin{aligned}\varphi'(h)&=-\Big(\frac{(b-r-1)p+a+r+2n+2}{a+b-1}-h\Big)-(a+b-h)+p\\&=\frac{(a+r)p-a-r-2n-2}{a+b-1}+2h-(a+b)\\&\geq a+2b-r-4+4-(a+b)\\&=b-r\geq a>0,\end{aligned}$$

and so, $\varphi(h)$ attains its minimum value at $h=2$ by $2\leq h\leq b-r$. Combining this with (7) and

$$p\geq\frac{(a+b-1)(a+2b-r-4)+a+r+2}{a+r}+\frac{2n}{a+r-1},$$

we obtain

$$\begin{aligned}\gamma_H(S,T)\geq\varphi(h)\geq\varphi(2)&\\&=(a+b-2)\cdot\Big(\frac{(b-r-1)p+a+r+2n+2}{a+b-1}-2\Big)\\&\quad-(b-r-2)p-2n\\&=\frac{(a+r)p}{a+b-1}+\frac{(a+b-2)(a+r+2n+2)}{a+b-1}\\&\quad-2(a+b-2)-2n\\&\geq\frac{(a+b-1)(a+2b-r-4)+a+r+2n+2}{a+b-1}\\&\quad+\frac{(a+b-2)(a+r+2n+2)}{a+b-1}-2(a+b-2)-2n\\&=2\geq\varepsilon(S,T),\end{aligned}$$

which contradicts (1).

Case 4: $h=b-r+1$.

It follows from (1) and Claim 2.4 that

$$\begin{aligned}\varepsilon(S,T)-1 &\geq \gamma_H(S,T) = f(S) + d_{H-S}(T) - g(T)\\ &\geq f(S) + d_{G-S}(T) - \min\{2n, |T|\} - g(T)\\ &\geq (a+r)|S| + h|T| - |T| - (b-r)|T|\\ &= (a+r)|S| - (b-r+1-h)|T|\\ &= (a+r)|S| \geq |S| \geq \varepsilon(S,T),\end{aligned}$$

which is a contradiction. This completes the proof of Theorem 1.1. □

3. Remark

In this section, we claim that the assumption on the neighborhood in Theorem 1.1 is the best possible, which cannot be replaced by $N_G(X) = V(G)$ or

$$|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|$$

for any $X \subseteq V(G)$.

Let a, b, r, n be four nonnegative integers such that $2 \leq a = b-r$, b is odd and $(2n+2)/b$ is an integer. We construct a graph

$$G = K_{(a-1)m+\frac{2n+2}{b}} \vee \Big(\frac{bm+1}{2}K_2\Big)$$

of order p, where m is an enough large positive integer, m is odd, and $\vee$ means "join". It is easy to see that $p = (a-1)m + bm + 1 + ((2n+2)/b)$. We write

$$S = V\Big(K_{(a-1)m+\frac{2n+2}{b}}\Big)$$

and

$$T = V\Big(\frac{bm+1}{2}K_2\Big).$$

We first show that the assumption $N_G(X) = V(G)$ or

$$|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|$$

for any $X \subseteq V(G)$ holds. Let any $X \subseteq V(G)$. It is easy to see that if $|X \cap S| \geq 2$, or $|X \cap S| = 1$ and $|X \cap T| \geq 1$, then $N_G(X) = V(G)$. Of course, if $|X| = 1$ and $X \subseteq S$, then we easily obtain

$$|N_G(X)| = |V(G)| - 1 = p - 1 > \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|.$$

Therefore, we may assume that $X \subseteq T$. Note that

$$|N_G(X)| = |S| + |X| = (a-1)m + \frac{2n+2}{b} + |X|.$$

Hence,

$$|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|$$

holds if and only if

$$(a-1)m+\frac{2n+2}{b}+|X|\geq\frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|.$$

This inequality is equivalent to $|X|\leq bm$. Thus if $X\neq T$ and $X\subset T$, then we have

$$|N_G(X)|\geq\frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|$$

for any $X\subset V(G)$. If $X=T$, then it is obvious that $N_G(X)=V(G)$. Consequently, $N_G(X)=V(G)$ or

$$|N_G(X)|\geq\frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|$$

for any $X\subseteq V(G)$ holds.

Let g,f be two integer-valued functions defined on $V(G)$ with $g(x)=a$ and $f(x)=b=a+r$ for every $x\in V(G)$. Let $N=\{e_1,e_2,\dots,e_n\}\subseteq E(\frac{bm+1}{2}K_2)$ be a set of independent edges of G. We write $H=G-N$. Next, we show that H does not have a fractional (g,f)-factor with the property $E(1,0)$. For above S and T, we have $|S|=(a-1)m+((2n+2)/b)$, $|T|=bm+1$, $d_{H-S}(T)=bm+1-2n$ and $\varepsilon(S,T)=2$. Thus, we have

$$\begin{aligned}\gamma_H(S,T)&=f(S)+d_{H-S}(T)-g(T)\\&=b|S|+d_{H-S}(T)-a|T|\\&=b\cdot\left((a-1)m+\frac{2n+2}{b}\right)+bm+1-2n-a\cdot(bm+1)\\&=3-a\leq 1<2=\varepsilon(S,T).\end{aligned}$$

In light of Lemma 2.1, H does not admit a fractional (g,f)-factor with the property $E(1,0)$, and so, G does not admit a fractional (g,f)-factor with the property $E(1,n)$.

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School of Science, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu 212100, China
E-mail address: zsz_cumt@163.com, zhousizhong@just.edu.cn

School of Mathematical Sciences, Nanjing Normal University, Nanjing, Jiangsu 210046, China
E-mail address: 05119@njnu.edu.cn